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PREFACE TO THE SEVENTH EDITION

The general response to the fifth edition of the book was very encouraging. The authors feel that their work has been amply rewarded and wish to express their deep sense of gratitude to the large number of readers who have used it and in particular to those of them who have sent helpful suggestions from time to time for the improvement of the book.

In the present edition, authors have made sincere efforts to make the book up-to-date. There are many notable features of this edition including **Answers to very short answer questions, Answers to short answer questions and solution of NCERT Textbook exercises.** Keeping in view the growing demands of fast developing science and technology, a large number of practical problems have been added to make the book practically oriented. It is hoped that these features will make the book more useful to the students.

We are thankful to the Management Team and the Editorial Department of S. Chand And Company Limited, New Delhi for all help and support in the publication of this book.

Errors might have crept in despite utmost care to avoid them and authors shall be grateful if these are pointed out alongwith other suggestions for the improvement of the book.

V.K. MEHTA
ROHIT MEHTA

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PHYSICS (Code No. 042)
COURSE STRUCTURE
Class XI (Theory)

Time : 3 hrs **Max. Marks : 70**

		No. of Periods	Marks
Unit–I	**Physical World and Measurement**	**10**	**23**
	Chapter–1: Physical World		
	Chapter–2: Units and Measurements		
Unit–II	**Kinematics**	**24**	
	Chapter–3: Motion in a Straight Line		
	Chapter–4: Motion in a Plane		
Unit–III	**Laws of Motion**	**14**	
	Chapter–5: Laws of Motion		
Unit–IV	**Work, Energy and Power**	**12**	**17**
	Chapter–6: Work, Energy and Power		
Unit–V	**Motion of System of Particles and Rigid Body**	**18**	
	Chapter–7: System of Particles and Rotational Motion		
Unit–VI	**Gravitation**	**12**	
	Chapter–8: Gravitation		
Unit–VII	**Properties of Bulk Matter**	**24**	**20**
	Chapter–9: Mechanical Properties of Solids		
	Chapter–10: Mechanical Properties of Fluids		
	Chapter–11: Thermal Properties of Matter		
Unit–VIII	**Thermodynamics**	**12**	
	12 Chapter–12: Thermodynamics		
Unit–IX	**Behaviour of Perfect Gases and Kinetic Theory of Gases**	**08**	
	Chapter–13: Kinetic Theory		
Unit–X	**Oscillations and Waves**	**26**	**10**
	Chapter–14: Oscillations		
	Chapter–15: Waves		
	Total	**160**	**70**

QUESTION PAPER DESIGN (Class: XI/XII)
Board Examination –Theory

Maximum Marks : 70 Duration 3 hrs.

S. No.	Typology of Questions	VSA Objective Type (1 mark)	SA (2 marks)	LA-I (3 marks)	LA-II (5 marks)	Total Marks	Percentage
1	**Remembering:** Exhibit memory of previously learned material by recalling facts, terms, basic concepts, and answers.	2	2	1	–	9	12%
2	**Understanding:** Demonstrate understanding of facts and ideas by organizing, comparing, translating, interpreting, giving descriptions, and stating main ideas	6	2	2	1	21	30%

3	**Applying:** Solve problems to new situations by applying acquired knowledge, facts, techniques and rules in a different way.	6	2	1	2	23	33%
4	**Analysing and Evaluating:** Examine and break information into parts by identifying motives or causes. Make inferences and find evidence to support generalizations Present and defend opinions by making judgments about information, validity of ideas, or quality of work based on a set of criteria.	6	1	2	—	14	20%
5	**Creating:** Compile information together in a different way by combining elements in a new pattern or proposing alternative solutions.	—	—	1	—	3	5%
	Total	20×1=20	7×2=14	7×2=21	3×5=15	70	100

Practical: 30 Marks

Note:

1. **Internal Choice:** *There is no overall choice in the paper. However, there will be at least 33% internal choice.*
2. *The above template is only a sample. Suitable internal variations may be made for generating similar templates keeping the overall weightage to different form of questions and typology of questions same.*

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COURSE STRUCTURE
Class XI (Theory)

Time : 3 hrs **Max. Marks : 70**

		No. of Periods	Marks
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2	**Understanding:** Demonstrate understanding of facts and ideas by organizing, comparing, translating, interpreting, giving descriptions, and stating main ideas	6	2	2	1	21	30%

SYLLABUS
Class–XI

Unit I: Physical World and Measurement **(Periods 10)**

Physics - scope and excitement; nature of physical laws; Physics, technology and society.

Need for measurement: Units of measurement; systems of units; SI units, fundamental and derived units. Length, mass and time measurements; accuracy and precision of measuring instruments; errors in measurement; significant figures.

Dimensions of physical quantities, dimensional analysis and its applications.

Unit II: Kinematics **(Periods 24)**

Frame of reference, Motion in a straight line: Position-time graph, speed and velocity.

Elementary concepts of differentiation and intergration for describing motion. Uniform and nonuniform motion, average speed and instantaneous velocity. Uniformly accelerated motion, velocitytime and position-time graphs.

Relations for uniformly accelerated motion (graphical treatment).

Scalar and vector quantities; Position and displacement vectors, general vectors and their notations; equality of vectors, multiplication of vectors by a real number; addition and subtraction of vectors. Relative velocity.

Unit vector; Resolution of a vector in a plane - rectangular components. Scalar and Vector product of vectors.

Motion in a plane. Cases of uniform velocity and uniform acceleration-projectile motion. Uniform circular motion.

Unit III: Laws of Motion **(Periods 14)**

Intuitive concept of force. Inertia, Newton's first law of motion; momentum and Newton's second law of motion; impulse; Newton's third law of motion.

Law of conservation of linear momentum and its applications.

Equilibrium of concurrent forces. Static and kinetic friction, laws of friction, rolling friction, lubrication.

Dynamics of uniform circular motion: Centripetal force, examples of circular motion (vehicle on a level circular road, vehicle on banked road).

Unit IV: Work, Energy and Power **(Periods 12)**

Work done by a constant force and a variable force; kinetic energy, work-energy theorem, power.

Notion of potential energy, potential energy of a spring, conservative forces: conservation of mechanical energy (kinetic and potential energies); non-conservative forces: motion in a vertical circle; elastic and inelastic collisions in one and two dimensions.

Unit V: Motion of System of Particles and Rigid Body **(Periods 18)**

Centre of mass of a two-particle system, momentum conservation and centre of mass motion.

Centre of mass of a rigid body; centre of mass of a uniform rod.

Moment of a force, torque, angular momentum, laws of conservation of angular momentum and is applications.

Equilibrium of rigid bodies, rigid body rotation and equations of rotational motion, comparison of linear and rotational motions.

Moment of inertia, radius of gyration. Values of moments of inertia, for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorems and their applications.

(Periods 12)

Unit VI: Gravitation

Keplar's laws of planetary motion. The universal law of gravitation.

Acceleration due to gravity and its variation with altitude and depth.

Gravitational potential energy and gravitational potential. Escape velocity. Orbital velocity of a satellite. Geo-stationary satellites. (Periods 24)

Unit VII: Properties of Bulk Matter

Elastic behaviour, Stress-strain relationship, Hooke's law, Young's modulus, bulk modulus, shear modulus of rigidity, Poisson's ratio; elastic energy.

Pressure due to a fluid column; Pascal's law and its applications (hydraulic lift and hydraulic brakes). Effect of gravity on fluid pressure.

Viscosity, Stokes' law, terminal velocity, streamline and turbulent flow, critical velocity. Bernoulli's theorem and its applications.

Surface energy and surface tension, angle of contact, excess of pressure accros a curved surface, application of surface tension ideas to drops, bubbles and capillary rise.

Heat, temperature, Thermal expansion; thermal expansion of solids, liquids and gases, anomalous expansion of water; specific heat capacity; Cp, Cv - calorimetry; change of state - latent heat capacity. Heat transfer-conduction, convection and radiation, thermal conductivity, Qualitative ideas of Blackbody radiation , Wein's displacement Law, Stefan's law Green house effect. (Periods 12)

Unit VIII: Thermodynamics

Thermal equilibrium and definition of temperature (zeroth law of thermodynamics). Heat, work and internal energy. First law of thermodynamics. Isothermal and adiabatic processes.

Second law of thermodynamics: reversible and irreversible processes. Heat engine and refrigerator. (Periods 08)

Unit IX: Behaviour of Perfect Gases and Kinetic Theory of Gases

Equation of state of a perfect gas, work done in compressing a gas.

Kinetic theory of gases - assumptions, concept of pressure. Kinetic interpretation of temperature; rms speed of gas molecules; degrees of freedom, law of equipartition of energy (statement only) and application to specific heat capacities of gases; concept of mean free path, Avogadro's number. (Periods 26)

Unit X: Oscillations and Waves

Periodic motion - time period, frequency, displacement as a function of time. Periodic functions. Simple harmonic motion (S.H.M) and its equation; phase; oscillations of a loaded spring–restoring force and force constant; energy in S.H.M. Kinetic and potential energies; simple pendulum–derivation of expression for its time period.

Free, forced and damped oscillations (qualitative ideas only), resonance.

Wave motion. Transverse and longitudinal waves, speed of travelling wave. Displacement relation for a progressive wave. Principle of superposition of waves, reflection of waves, standing waves in strings and organ pipes, fundamental mode and harmonics, Beats, Doppler effect.

SYLLABUS
Class–XI

Unit I: Physical World and Measurement **(Periods 10)**

Physics - scope and excitement; nature of physical laws; Physics, technology and society.

Need for measurement: Units of measurement; systems of units; SI units, fundamental and derived units. Length, mass and time measurements; accuracy and precision of measuring instruments; errors in measurement; significant figures.

Dimensions of physical quantities, dimensional analysis and its applications.

Unit II: Kinematics **(Periods 24)**

Frame of reference, Motion in a straight line: Position-time graph, speed and velocity.

Elementary concepts of differentiation and intergration for describing motion. Uniform and nonuniform motion, average speed and instantaneous velocity. Uniformly accelerated motion, velocitytime and position-time graphs.

Relations for uniformly accelerated motion (graphical treatment).

Scalar and vector quantities; Position and displacement vectors, general vectors and their notations; equality of vectors, multiplication of vectors by a real number; addition and subtraction of vectors. Relative velocity.

Unit vector; Resolution of a vector in a plane - rectangular components. Scalar and Vector product of vectors.

Motion in a plane. Cases of uniform velocity and uniform acceleration-projectile motion. Uniform circular motion.

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Law of conservation of linear momentum and its applications.

Equilibrium of concurrent forces. Static and kinetic friction, laws of friction, rolling friction, lubrication.

Dynamics of uniform circular motion: Centripetal force, examples of circular motion (vehicle on a level circular road, vehicle on banked road).

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Work done by a constant force and a variable force; kinetic energy, work-energy theorem, power.

Notion of potential energy, potential energy of a spring, conservative forces: conservation of mechanical energy (kinetic and potential energies); non-conservative forces: motion in a vertical circle; elastic and inelastic collisions in one and two dimensions.

Unit V: Motion of System of Particles and Rigid Body **(Periods 18)**

Centre of mass of a two-particle system, momentum conservation and centre of mass motion.

Centre of mass of a rigid body; centre of mass of a uniform rod.

Moment of a force, torque, angular momentum, laws of conservation of angular momentum and is applications.

Equilibrium of rigid bodies, rigid body rotation and equations of rotational motion, comparison of linear and rotational motions.

Moment of inertia, radius of gyration. Values of moments of inertia, for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorems and their applications.

Unit VI: Gravitation (Periods 12)

Keplar's laws of planetary motion. The universal law of gravitation.

Acceleration due to gravity and its variation with altitude and depth.

Gravitational potential energy and gravitational potential. Escape velocity. Orbital velocity of a satellite. Geo-stationary satellites.

Unit VII: Properties of Bulk Matter (Periods 24)

Elastic behaviour, Stress-strain relationship, Hooke's law, Young's modulus, bulk modulus, shear modulus of rigidity, Poisson's ratio; elastic energy.

Pressure due to a fluid column; Pascal's law and its applications (hydraulic lift and hydraulic brakes). Effect of gravity on fluid pressure.

Viscosity, Stokes' law, terminal velocity, streamline and turbulent flow, critical velocity. Bernoulli's theorem and its applications.

Surface energy and surface tension, angle of contact, excess of pressure accros a curved surface, application of surface tension ideas to drops, bubbles and capillary rise.

Heat, temperature, Thermal expansion; thermal expansion of solids, liquids and gases, anomalous expansion of water; specific heat capacity; Cp, Cv - calorimetry; change of state - latent heat capacity. Heat transfer-conduction, convection and radiation, thermal conductivity, Qualitative ideas of Blackbody radiation , Wein's displacement Law, Stefan's law Green house effect.

Unit VIII: Thermodynamics (Periods 12)

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Second law of thermodynamics: reversible and irreversible processes. Heat engine and refrigerator.

Unit IX: Behaviour of Perfect Gases and Kinetic Theory of Gases (Periods 08)

Equation of state of a perfect gas, work done in compressing a gas.

Kinetic theory of gases - assumptions, concept of pressure. Kinetic interpretation of temperature; rms speed of gas molecules; degrees of freedom, law of equipartition of energy (statement only) and application to specific heat capacities of gases; concept of mean free path, Avogadro's number.

Unit X: Oscillations and Waves (Periods 26)

Periodic motion - time period, frequency, displacement as a function of time. Periodic functions.

Simple harmonic motion (S.H.M) and its equation; phase; oscillations of a loaded spring–restoring force and force constant; energy in S.H.M. Kinetic and potential energies; simple pendulum–derivation of expression for its time period.

Free, forced and damped oscillations (qualitative ideas only), resonance.

Wave motion. Transverse and longitudinal waves, speed of travelling wave. Displacement relation for a progressive wave. Principle of superposition of waves, reflection of waves, standing waves in strings and organ pipes, fundamental mode and harmonics, Beats, Doppler effect.

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Introduction

GENERAL

In the present century, we find that science is held in high esteem. Although the word "science" has its origin in a Latin verb meaning "to know", science has come to mean not merely knowledge but a body of knowledge of natural world in an organised and rational way. The scientific developments have exerted influence on every phase of human activity. The intelligent man recognises that the material progress of technology is a by-product of the discoveries of science. In all fairness, honest credit is being given to science for its contribution to knowledge, comfort, efficiency and to the general well-being of mankind.

Physics is the fundamental science. *The main goal of physics is to give correct and precise description of the material universe.* The study of physics is an adventure. It is challenging, sometimes frustrating, occasionally painful and often richly rewarding and satisfying. Although physics has earned the reputation as a difficult subject, yet it is inherently simple. It is because there are only a few fundamental laws that you need to lodge in your memory bank. If you really understand these laws, you can readily use them to deduce how nature behaves in a variety of situations. As you read text and work physics problems, ask yourself how each problem you approach is really similar to other problems and to the text examples. And you will find that similarity because most of the problems and examples really involve only a few underlying laws. So physics is simple provided if you understand its fundamental laws and have the experience in reasoning out how these laws apply to the world about you. In this chapter, we shall deal with some fundamental aspects of physics.

1.1. PHYSICS

The branch of science which deals with correct and precise description of the material universe is known as **physics.**

The word physics comes from a Greek word meaning "knowledge of nature". Physics attempts to describe the fundamental nature of the universe and how it works. For example, physics explains why the sky is blue, why rainbows have colours, why steel ships float, what atoms and nuclei are made of etc. To all these and many more questions, physics explains using as few laws as possible, revealing their underlying simplicity and beauty. The amount of knowledge available on physics is so vast that as a matter of convenience, it has been divided into two classes :

(*i*) Classical (old) physics (*ii*) Modern physics

(*i*) **Classical physics.** *The physics prior to 1890 is known as* **classical physics.**

By 1890, the various laws governing the material universe were discovered and it appeared that knowledge of this subject was complete. Newton's laws of motion and gravitation had provided

a solid foundation on which science of mechanics had been raised to impressive heights. The laws of thermodynamics and kinetic theory of matter gave a satisfactory account of the entire subject of heat. Maxwell developed the theory of electromagnetic radiation and explained all electrical, magnetic and optical phenomena. It appeared that the laws and theories of classical physics would hold the field unchallenged. However, startling discoveries after 1890 gave a death blow to the so-called classical physics.

(*ii*) Modern physics. *The physics after 1890 is known as* **modern physics.**

After 1890, there had been enormous advances in physics which marked the beginning of revolutionary concepts. The thrilling discoveries posed new problems unknown to classical physics. In 1895, Roentgen discovered **X-rays** and a few weeks later, Henry Becquerel announced the discovery of **radioactivity**. In the year 1897, J.J. Thomson discovered **electron**, a fundamental constituent of all matter. In 1901, Max Planck developed **quantum theory** according to which energy is not radiated continuously but in discrete packets of energy called quanta. These *quanta* eventually came to be known as *photons*. The **special theory of relativity** proposed by Albert Einstein in 1905 gave new dimensions to atomic and nuclear physics. In short, the rapid startling discoveries after 1890 gave a new shape to the entire concept of physics. The study of these concepts has been placed under the head of modern physics.

The reader may wonder about the difference between classical physics and modern physics. The answer is that classical physics does not work for microscopic particles (*e.g.*, atoms) or for objects travelling at very high speed. In fact, these two aspects gave birth to what is now called *modern physics. Indeed modem physics may be considered as mere 'fine-tuning" of classical physics.* Except for the microscopic particles and for motion at speeds near the speed of light, classical physics correctly and precisely describes the behaviour of the physical world. Although modern physics has made many important contributions to technology, great bulk of technical knowledge and skill is still based squarely on classical physics. It is through the applications of classical physics that we have been able to exploit natural resources successfully. *In this book, we shall discuss classical physics only.*

1.2. SOME SCIENTIFIC NOTIONS

Some important scientific notions are: (*i*) scientific law (*ii*) theory (*iii*) model (*iv*) scientific method.

(*i*) Scientific law. *The general statement of happenings in nature after repeated observations and experiments is known as* **scientific law**.

A scientific law is a concise statement how nature behaves. It is actually the *essence* of a large number of observations and experiments. For example, *the law of gravitation* states that the two objects attract each other. Similarly, "unlike charges attract each other and like charges repel" is also a law and is based on observations and experiments. A law has no theoretical or other basis. In other words, we cannot ask "why like charges repel and not attract? Why two bodies attract each other and not repel?" The only answer to such questions is that this is how nature behaves. And the matter ends there. No "buts" and "ifs".

Physical laws are usually expressed as mathematical equations which are then used to make predictions about other phenomena. For example, the law of gravitation states that the two objects attract each other with a force which is proportional to the mass of each object and inversely proportional to the square of the distance between them. It is written in the mathematical form as $F = G\, m_1 m_2 / r^2$. Because this expresses the law in the form of an equation, it is far preferable to the qualitative statement. *It may be noted that a scientific law has only experimental proof.*

(*ii*) Theory. *The speculation on the basis of determined laws is known as* **theory**.

When several laws are determined, then one may make a guess about the way the universe must be constituted in order that such things happen ; this guess is called *theory*. It may be noted that there is a basic difference between a theory and law. *Theory is given by man and hence may be wrong but law is made by nature and is always right.*

Theories are never derived from observations — they are *created* to explain observations. They are inspirations that come from the minds of intelligent people. For example, the idea that matter is made up of atoms (atomic theory) was certainly not arrived at because someone observed atoms. Rather the idea was concieved by a creative mind. The theory of relativity and the electromagnetic theory were likewise the result of inspiration. Nevertheless, the history of physics tells us that theories come and go ; that long-held theories are replaced by new ones. However, a new theory is accepted only if it explains a *greater range* of phenomena than does the older one.

(*iii*) Model. *A* **model** *is a visual picture of the phenomenon under study.*

When scientists are trying to understand a particular phenomenon, they often make use of a model. The purpose of a model is to give us a visual picture — something to hold on — when we cannot see what actually is happening. One example is the wave model of light. We cannot see waves of light as we can see water waves. But it is valuable to think of light as if it were made up of waves because experiments on light show that it behaves in many respects as water waves do.

(*iv*) Scientific method. *To understand any natural process by a combination of logical reasoning and controlled experimentation is called the* **scientific method**.

Prior to about 1600, questions of truth and falsehood were most often determined by political or religious dictates. But great scientists like Isaac Newton introduced scientific methods to the world. The two main tools of scientific method are logic and experimentation. The result is that the experimental results are *reproducible*. That is, the same set of circumstances will always produce the same observed results in the same experiment, no matter who the observer is.

Notes. (*i*) Scientific laws are different from political laws in that the latter tell us how we must behave. Scientific laws do not say how nature *must* behave but rather describe how nature *does* behave.

(*ii*) To be called a law, a statement must be found experimentally valid over a wide range of observed phenomena. For less general statements, the term "principle" is often used (such as Archimede's principle). Where to draw the line between laws and principles is, of course, arbitrary and there is not always complete consistency.

1.3. SCOPE AND EXCITEMENT OF PHYSICS

The scope of physics is very wide. It covers an immense range of natural phenomena. Distances extend from incredibly small dimensions of subatomic particles to thousands of million metres that separate galaxies of the universe. For example, the radius of atomic nucleus is about 10^{-14} m while the radius of the universe is about 10^{25} m. The time intervals encountered in the physical world vary over an extremely wide range. For instance, the time taken by light to cross distance of nuclear size is about 10^{-22} s while the life of sun is about 10^{18} s. Similarly, the mass of an electron is approximately 10^{-30} kg whereas the mass of universe is about 10^{55} kg. All these wide range of measurements are in the domain of physics.

The subject of physics has wide applications. It includes medical physics, computers, meteorology, material science, geophysics, engineering communication, environmental physics *etc.* Physics is a very exciting subject. It provides answers to exciting questions such as :

(*i*) Why is the sky blue? (*ii*) Why is sunset red?

(*iii*) Why is glass transparent? (*iv*) Why are atoms held together?

These are only a few exciting questions. In fact, there are a very large number of such questions to which physics provides answers.

1.4. FIELDS OF PHYSICS

Physics is the *physical science which deals with matter and energy and their transformations. Knowledge of physics will help you understand both natural phenomena and the technologies that increasingly prevade our lives. There are numerous problems that physics faces. Yet we can describe and understand any problem with the help of the following five fields of physics :

(*i*) **Classical mechanics.** *It deals with objects from molecules to galaxies that are moving at speeds small compared with the speed of light.* It deals with such ideas as inertia, motion, force and energy. Mechanics also includes the properties and laws of both solids and fluids. You use the laws of classical mechanics (also known as Newtonian mechanics) when you drive a car, ride a skateboard, build a skyscrapper *etc.* Indeed many physics problems can be solved through the application of laws of classical mechanics. You will see that classical mechanics is but an approximation to a more comprehensive set of physical laws that includes Einstein's theory of relativity and the theory of quantum mechanics.

(*ii*) **Thermodynamics.** *It deals with the study of heat and its interaction with matter.* It includes effects of temperature on the properties of materials, heat flow and transformations involving heat and work.

(*iii*) **Electromagnetism.** *It deals with electricity, magnetism and electromagnetic radiation.* Today electromagnetic technology dominates our civilisation as evidenced by radio, television, taperecorders, microwave ovens and computers.

(*iv*) **Relativity.** In 1905, Albert Einstein gave the theory of relativity. This theory has changed our notions of space and time and of mass and energy. Space and time are seen to be intimately connected with time being the fourth dimension in addition to space's three dimensions. Although theory of relativity provides a more correct description of physical reality than does classical mechanics, the two theories differ significantly only at speeds approaching that of light. *In our everyday existence, we need not worry about relativity.*

(*v*) **Quantum mechanics.** *It deals with the behaviour of objects on the extremely small scale of atomic dimensions.* Indeed classical physics fails to explain the behaviour of such microscopic particles. The quantum mechanics touches every aspect of modem physics and most of classical physics. Indeed one requires an understanding and the application of quantum mechanics for a satisfactory explanation of most of the phenomena.

1.5. THE SIMPLICITY OF PHYSICS

There is a common notion that physics is incomprehensible to the people of average ability. If you think that, you are missing the point because it is so fundamental, physics is inherently simple. There are only a few basic laws to learn ; if you really understand those laws, you can readily apply them in a wide variety of situations. *The laws and theories of physics describe the working of the universe at the most basic level.* For that reason, physics is extremely powerful ; the same laws describe the behaviour of molecules, aeroplanes and galaxies. Scientists believe that it would be possible to describe the operation of a living cell or organism using only the fundamental laws of physics. So physics is simple — challenging too — but with an underlying simplicity that reflects the scope and power of this fundamental science.

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* A physical science is one which primarily deals with non-living things. On the other hand, biological science deals with living things.

1.6. PHYSICS IS BASIC SCIENCE

Physics is the basic science. It is about the nature of basic things such as motion, force, energy, matter, heat, sound, light and the inside of atoms. An understanding of science begins with an understanding of physics. There is physics in everything you see, hear, smell, taste and touch.

(*i*) **Physics and Mathematics.** The laws of physics are usually expressed as *mathematical equations which are then used to make predictions about other phenomena and to test the range of validity of the laws. In order to understand physics on any level beyond qualitative description, you require a considerable knowledge of mathematics. The mathematical description of the laws and theories of physics gives new insight to the subject. Indeed, the primary goal of mathematics is to aid physics.

(*ii*) **Physics and Chemistry.** Chemistry is about how matter is put together, how atoms combine to form molecules and how molecules combine to make up the many kinds of matter around us. In order to explain these aspects, we have to apply the laws and theories of physics. So underneath chemistry is physics.

(*iii*) **Physics and Biology.** Biology is more complex and involves matter that is alive. So underneath biology is chemistry and underneath chemistry is physics. That is why physics is the most basic science.

(*iv*) **Physics and other sciences.** Since physics is a fundamental science, its laws and theories are applied to various physical sciences. The knowledge of physics is required in many fields and the list below shows some of these :

(*a*) Medical physics (*b*) Meterology (*c*) Geophysics (*d*) Environmental physics (*e*) Communication (*f*) Material science.

1.7. PHYSICS AND TECHNOLOGY

Physics and technology are different from each other. *Physics is the knowledge of material universe for its own sake, whereas technology transfers the knowledge of physics into practical shape for the general well-being of human race.* In other words, physicists provide the knowledge of material universe, whereas the engineers provide the tools, techniques and procedures for putting this knowledge to practical use. For example, physics has provided the knowledge of semiconductors but technology made of them transistor radios. Many technologies have been developed on the basis of the knowledge of physics and a few of them are mentioned below by way of illustration :

(*a*) The discovery of Faraday's laws of electromagnetic induction enabled the engineers to develop electric generator, electric motor, transformer and a host of other equipment.

(*b*) The discovery of laws of thermodynamics led to the development of many types of heat engines.

(*c*) The discovery of electromagnetic radiation, vacuum tubes and semiconductors led to wireless communication *e.g.* radio and television *etc.*

(*d*) The discovery of nuclear fission led to the development of nuclear power plants which are generating huge amount of electric power all over the world.

The above short list indicates how important is the knowledge of physics in developing new technologies. *Indeed, technology may be regarded as applied physics.*

* When laws of physics are expressed in mathematical form, they do not have the double meanings that so often confuse the discussion of the ideas expressed in common language. When laws are expressed mathematically, they are easier to verify or disprove by experiments.

1.8. PHYSICS AND SOCIETY

The advancement of technology as a result of discoveries of knowledge of physics has made a significant impact on human lives. We travel faster than ever before in aeroplanes. We enjoy colour television. We use electricity to operate a host of electrical appliances. We can solve complex problems at amazing speed with the help of computers. We send satellites into orbit to learn about the conditions in outer space. The development of radio and television has led to instant communication. Indeed, the uses of physics have exerted considerable influence on every phase of human activity.

We are all familiar with the abuses of applied physics or technology. Many people blame technology itself for widespread pollution, resource depletion and even social decay in general. It is much wiser to combat the misuse of technology with knowledge than with ignorance. Wise applications of physics can lead to a better world.

1.9. SOME DISCOVERIES AND INVENTIONS

The discoveries and inventions in the field of physics were made by brilliant, logical and creative minds. The table below gives some discoveries and inventions in physics.

Some Inventions and Discoveries

S.No.	Invention or Discovery	Inventor or Discoverer	Year
1.	Compound microscope	Janssen	1590
2.	Refracting telescope	Lippershey	1608
3.	Laws of refraction	Snell	1621
4.	Spectrum (light)	Newton	1666
5.	Reflecting telescope	Newton	1669
6.	Capacitor	Kleist	1745
7.	Gold-leaf electroscope	Bennet	1787
8.	Simple cell	Volta	1800
9.	Double-slit interference	Young	1801
10.	Diffraction	Fraunhoffer	1819
11.	Magnetic effect	Oersted	1819
12.	Electric motor	Faraday	1821
13.	Ohm's law	Ohm	1826
14.	Transformer	Faraday	1831
15.	Dynamo	Faraday	1831
16.	Laws of Electrolysis	Faraday	1834
17.	Semiconductors	Rosenschold	1834
18.	Cathode rays	(several)	1859
19.	Electric lamp	Edison	1879
20.	Electric iron	Seeley	1882
21.	Radioactivity	Becquerel	1896
22.	Nuclear reactor	Fermi	1942
23.	Transistor	Shockley, Bardeen and Brattain	1947
24.	Integrated circuits	Kilby	1958
25.	Optical fibre	Kao, Newnes and Beales	1966

1.10. BASIC FORCES IN NATURE

There are four basic forces that operate in nature. These are (*i*) the gravitational force (*ii*) the electromagnetic force (*iii*) the strong nuclear force and (*iv*) the weak nuclear force. For detailed discussion, the reader may refer to Art. 10.8.

1.11. TOWARDS UNIFICATION OF FORCES

In physics, we attempt to explain different physical phenomena in terms of a few concepts and laws. This thrust of unification in physics has greatly simplified its study. We give here a few examples to illustrate this important point.

(*i*) Newton unified terrestrial and celestial domains under a common law of gravitation.

(*ii*) The experimental discoveries of Oersted and Faraday showed that electric and magnetic phenomena are in general inseparable.

(*iii*) Maxwell's equations form the basis of all electromagnetic phenomena.

(*iv*) Maxwell unified electromagnetism and optics with the discovery that light is an electromagnetic wave.

1.12. CONSERVATION LAWS

In physics, there are some physical quantities which remain the same before and after an event or interaction. Such quantities are called **conserved quantities.* For example, electric charge is a conserved quantity. In every event, the principle of *conservation of charge* has always been found to apply. Thus when we produce a free negative charge by tearing an electron loose from an atom, an equal positive charge is left behind, thereby causing no change in total charge of the system. The laws relating to conserved quantities are called conservation laws.

Those laws of nature that state that some physical quantity is the same before and after an event or interaction are called **conservation laws.**

Some important physical quantities that are conserved in an event or interaction are :

(*i*) Electric charge _____ **Law of conservation of electric charge**

(*ii*) Energy _____ **Law of conservation of energy**

(*iii*) **Momentum _____ **Law of conservation of momentum**

We do not know the reason for it. But scientists have performed a very large number of experiments over the years to test the conserved nature of the above physical quantities. In all the experiments, the conserved nature of the above physical quantities is proved beyond any doubt. Nevertheless, conservation laws reflect one of the most basic ways of describing nature. Because conservation laws allow us to consider quantities, such as energy, that do not change during an event, we do not need to know the details of the interaction. We simply deal with the value of the conserved quantity before and after the interaction or event. Moreover, the fact that some physical quantities are conserved and some are not, tells us something about nature itself. Note that conservation laws are applicable to only isolated systems. The reader will find detailed discussion on these conservation laws in the text at apporopriate places.

1.13. SYSTEM, ENVIRONMENT AND ISOLATED SYSTEM

Since conservation laws are valid in isolated systems, it is desirable to know, what is an isolated system?

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* A physical quantity that is the same before and after an event or interaction is called a *conserved quantity.*

** Linear momentum as well as angular momentum.

(*i*) **System and Environment.** *A* **system** *is some part of the universe in which we have a special interest.* All the rest of the universe outside the system is called **environment.**

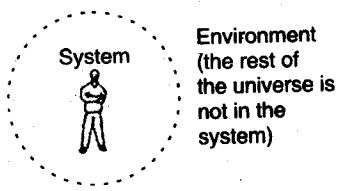

Fig. 1.1

The system can be any part of the universe we choose—a human body, the whole earth, a single atom *etc.* Fig. 1.1 shows the dashed line around a person. The part of the universe within the dashed line is the system. All the rest of the universe outside our chosen system is the environment. The dashed line can be an actual boundary between the system and its environment, or it can be an imaginary boundary that we use to help identify the system.

(*ii*) **Isolated system.** *The system is said to be* **isolated** *if nothing (energy, matter, light) crosses the boundary of the system to or from the environment and if the net sum of all the external forces on the system is zero.*

Within an isolated system, the amount of conserved quantity (*e.g.* electric charge, energy, momentum *etc.*) will not change during an event or interaction. Often we can consider a system approximately isolated by simply choosing its boundaries in an appropriate manner. Then we can apply conservation laws to learn more about elements within the system. For instance, when two cars collide, we can choose the system to be the region of space that includes both cars. The interaction of each car with the other is much greater than any other force affecting the cars. As a result, we consider the system as being approximately isolated. Therefore, we can use conservation laws to learn more about the motions of the cars during their collision.

CONCEPTUAL QUESTIONS

Q.1. Which are the two main tools of physics?

Ans. The two main tools of physics are logic and experimentation.

Q.2. What is the drawback of classical physics?

Ans. Some aspects of classical physics do not work for the tiny world of the atoms or objects travelling at very high speed.

Q.3. What is the importance of mathematics for physics?

Ans. The laws of physics are generalisations from observations and experiments. When these laws are expressed in mathematical forms, they are unambiguous. They do not have double meanings that so often confuse the discussion of ideas expressed in common language. When findings of nature are expressed mathematically, they are easier to verify or disprove by experiments.

Q.4. What is the difference between a theory and a model?

Ans. Usually, a model is fairly simple and provides a structural similarity to the phenomenon being studied. However, a theory is broader, more detailed, and attempts to solve a set of problems often with mathematical precision.

Q.5. The theories of science are not fixed but undergo change. Discuss.

Ans. The history of science shows that theories come and go ; that long-held theories are replaced by new ones. For example, during the last hundred years, the theory of atom has been repeatedly refined as new evidence on atomic behaviour has been gathered. A new theory is accepted only if it explains a *greater range* of phenomena than does the older one.

Q.6. What are the laws of physics?

Ans. Among all the sciences, physics is the most basic science. *It is a quantitative science.* Its goal is to describe all phenomena in the physical world in terms of a few relationships between measured properties of matter and energy. These fundamental relationships are called *laws* of physics. They are statements of great generality derived from a wide range of phenomena.

Q.7. Why is physics inherently simple?

Ans. Physics is inherently simple. It is because there are only a few basic laws to learn ; if you really understand these *laws,* you can readily apply them in a wide variety of situations.

Q.8. Explain the terms (*i*) fact (*ii*) hypothesis.

Ans. (*i*) A *fact* is a phenomenon about which competent observers who have made a series of observations are in agreement.

(*ii*) A *hypothesis* is an educated guess; a reasonable explanation of an observation or experimental result that is not fully accepted as factual until tested over and over again by experiments.

VERY SHORT ANSWER QUESTIONS

Q.1. What is physics?

Ans. Physics is a branch of science which deals with the study of nature and natural phenomena.

Q.2. What is a scientific law?

Ans. The general statement of happenings in nature after repeated observations and experiments is known as a scientific law.

Q.3. What is theory?

Ans. The speculation on the basis of determined scientific laws is known as theory.

Q.4. What is a model?

Ans. A model is a visual picture of the phenomenon under study.

Q.5. What is the basic difference between science and technology?

Ans. Science is the study of laws of nature. Technology is the application of these laws for the general well-being of human race.

Q.6. What are the five main branches of physics?

Ans. The five main branches of physics are : Classical mechanics, Thermodynamics, Electromagnetism, Theory of relativity and Quantum mechanics.

Q.7. What is a scientific method?

Ans. To understand any natural process by a combination of logical reasoning and controlled experimentation is called the scientific method.

Q.8. What is classical physics?

Ans. The physics prior to 1890 is known as classical physics.

Q.9. What is modern physics?

Ans. The physics after 1890 is known as modern physics.

Q.10. What are conservation laws?

Ans. Those laws of nature which state that some physical quantity is the same before and after an event or interaction are called conservation laws.

SHORT ANSWER QUESTIONS

Q.1. What is the difference between classical physics and modern physics?

Ans. The classical physics does not work for microscopic particles (*e.g.,* atoms) or for objects travelling at very high speed (near the speed of light). These two aspects can only be explained on the basis of modern physics. Indeed modern physics may be considered as mere "*fine-tuning*" of classical physics. Although modern physics has made many important contributions to technology, great bulk of technical knowledge and skill is still based squarely on classical physics.

Q.2. Why do we express laws of physics in mathematical form?

Ans. The laws of physics are generally expressed as mathematical equations which are then used to make predictions about other phenomena. When a physics law is expressed in mathematical form, it is far preferable to the qualitative statement. Moreover, mathematical form permits us to verify the various laws experimentally.

Q.3. Name two advancements made in technology on the basis of physics.

Ans. (*i*) The discovery of Faraday's laws of electromagnetic induction has enabled the engineers to develop electric generator, electric motor, transformer *etc.*

(*ii*) The discovery of nuclear fission led to the development of nuclear power plants which are generating huge amount of electric power all over the world.

Q.4. Why is physics called science of measurement?

Ans. Physics is an experimental science. In physics, we are to deal with a large number of physical quantities like length, mass, time, velocity, pressure *etc.* These quantities can give clear understanding only if we can measure them and express our conclusions into these measurements. For this reason, physics is sometimes called the science of measurement.

Q.5. Discuss relation of physics to chemistry.

Ans. The knowledge of physics has enabled us to study the structure of atom, radioactivity, X-ray, diffraction *etc.* These concepts of physics find wide applications in chemistry *e.g.*, determining the nature of valency and chemical bonding, to detect traces of substances in a given sample *etc.*

Q.6. Discuss relation of physics to biology.

Ans. Biology involves matter that is alive. The knowledge of physics has provided us a powerful tool to study the biological sciences. For example, electron microscope has made it possible to see even the structure of a cell. Similarly, optical microscope is widely used in the study of biology.

Q.7. What do you understand by system, environment and isolated system?

Ans. A *system* is some part of the universe in which we have a special interest. All the rest of the universe outside the system is the *environment.* The system is said to be *isolated* if nothing (energy, matter, light) crosses the boundary of the system to or from the environment and if the net sum of all external forces on the system is zero. Note that conservation laws are valid only in an isolated system.

LONG ANSWER QUESTIONS

Q.1. State the laws of conservation of charge, energy and linear momentum. **[Refer to Art. 1.12]**

Q.2. How is physics related to society? **[Refer to Art. 1.8]**

Q.3. Discuss the basic forces in nature. **[Refer to Art. 1.10]**

Q.4. Give a short note on physics and technology. **[Refer to Art. 1.7]**

Q.5. Give three examples where physics has been used in technology. **[Refer to Art. 1.7]**

Q.6. Explain briefly the scope and excitement of physics. **[Refer to Art. 1.3]**

MCQ FOR COMPETITIVE EXAMINATIONS

1. The physicist who discovered the law of gravitation was
   (*a*) Newton (*b*) Faraday
   (*c*) Chadwick (*d*) Curie
2. The theory of relativity was proposed by
   (*a*) Newton (*b*) Bohr
   (*c*) Einstein (*d*) None of the above
3. Of all the sciences, the most basic science is
   (*a*) chemistry (*b*) biology
   (*c*) physics (*d*) none of the above
4. Which of the following activities involve the utmost expression of passion, talent and intelligence?
   (*a*) literature (*b*) science
   (*c*) music (*d*) all the above
5. The classical physics is applicable to
   (*a*) microscopic world
   (*b*) macroscopic world
   (*c*) both microscopic and macroscopic world
   (*d*) cannot say
6. Electron was discovered by
   (*a*) Einstein (*b*) Faraday
   (*c*) Bohr (*d*) J.J. Thomson
7. Who is the greatest physicist of the twentieth century?
   (*a*) Chadwick (*b*) J.J. Thomson
   (*c*) Bohr (*d*) Einstein
8. J.J. Thomson belonged to
   (*a*) England (*b*) America
   (*c*) France (*d*) India
9. Physics is
   (*a*) experimental science
   (*b*) quantitative science
   (*c*) the most basic science
   (*d*) all the above
10. The laws of electromagnetic induction were discovered by
    (*a*) Faraday (*b*) Newton
    (*c*) Bohr (*d*) J.J. Thomson

ANSWERS TO MCQ FOR COMPETITIVE EXAMINATIONS

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*d*)	**5.** (*b*)
6. (*d*)	**7.** (*d*)	**8.** (*a*)	**9.** (*d*)	**10.** (*a*)

PAULO MWERA CHAMA.

2

Systems of Units

INTRODUCTION

Physics is an experimental science. To understand physics or any experimental science, we must be able to connect our theoretical description of nature with our experimental observations of nature. This connection is made through quantitative measurements. In physics, we are to deal with a large number of physical quantities like length, mass, time, velocity, force, pressure *etc.* These quantities can give clear understanding only if we can measure them and express our conclusions into these measurements. For this reason, physics is sometimes called the science of measurement. There were several systems of units in use over the years. For example, the unit of length was foot in British system, metre in metric system and so on. *But today, all scientists and engineers have agreed to state their observations of nature in terms of single universal system of units called Systeme International (French for International system) which is abbreviated as SI.* In this chapter, we shall discuss the various aspects of systems of units.

2.1. MEASUREMENT

For the measurement of a physical quantity, we consider a constant quantity as a standard and then find the number which expresses how many times the standard quantity is contained in the physical quantity. This standard quantity is called the **unit.**

Measurement *is thus the comparison of unknown physical quantity with a known fixed unit quantity.*

For example, if the length of a rod is 3 metres, it means that the unit of length is the *metre* and this unit is contained 3 times in the length of that rod. Similarly, if mass of a ball is 2 kilogrammes, it means that unit of mass is *kilogramme* and this unit is contained 2 times in the mass of the ball. Hence to express the magnitude of a physical quantity, we should know two things *viz.*

(*i*) The *unit* in which the quantity is measured.

(*ii*) The *numerical value* — which expresses how many times the above mentioned unit is contained in the given physical quantity.

$\therefore$ *Magnitude of physical quantity = Numerical value of physical quantity* $\times$ *size of its unit*

If n is the numerical value of a physical quantity and u is the size of the unit, then, magnitude Q of the physical quantity is

$$Q = n\,u$$

Thus when we say that the length of the rod is 3 m *i.e.* $Q = 3$ m, it means $n = 3$ and $u = 1$ m.

The size of unit is different in different systems of units. Suppose the magnitude of a physical quantity is Q. If n_1 is the numerical value of the physical quantity for a unit u_1 and n_2 for a unit u_2 then.

$$Q = n_1 u_1 \quad \text{for first system of units}$$

$$Q = n_2 u_2 \quad \text{... for second system of units}$$

$$\therefore \quad n_1 u_1 = n_2 u_2 \quad \ldots(i)$$

Two things are obvious from eq. (*i*) :

(*a*) The magnitude of a given physical quantity is always the same irrespective of the system of units adopted *i.e.*

$$n\,u = \text{Constant}$$

(*b*) If $u_1 > u_2$, then $n_1 < n_2$. Therefore, the larger the magnitude of the unit selected, the smaller is the numerical value of the physical quantity.

2.2. NEED FOR MEASUREMENT

In physics, we are to deal with a large number of physical quantities like length, mass, time, speed, acceleration, force, pressure *etc*. These quantities can give clear understanding if we can measure them. The need for measurement is clear from the following examples :

(*i*) $$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

We can find the speed if we can measure the distance travelled and the time taken to travel that distance.

(*ii*) $$\text{Acceleration} = \frac{\text{Force}}{\text{Mass}}$$

In order to find the acceleration produced in the body, we should know the magnitudes (*i.e.* measurement) of force acting on the body and mass of the body.

(*iii*) $$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

Again we can find pressure if we measure the force acting and the area over which the force acts.

2.3. UNIT

The quantity used as a standard of measurement is called the **unit.**

For example, when we say that the length of a rod is 3 metres, we compare the length of the rod with a standard quantity of length called *metre*. Hence metre is the unit of length. Several systems of units have been in use over the years. Today all scientists and engineers have agreed to use Systeme International (French for International system) which is abbreviated as SI. *In SI units, the standard of length is* **metre** *and that for time is the* **second** *and that for mass is* **kilogramme.**

2.4. FUNDAMENTAL UNITS AND DERIVED UNITS

Physical quantities can be divided into two categories : *base quantities* and *derived quantities*. The corresponding units for these quantities are called **fundamental units** (or **base units**) and **derived units.** Scientists in the interest of simplicity want the smallest number of base quantities possible to give full description of the physical world.

1. Fundamental units. In mechanics, the units of only three base quantities *viz. mass, length* and *time* need to be selected. All other physical quantities in mechanics (*e.g.* speed, acceleration, force, power, energy *etc*.) can be defined in terms of the units of these three base quantities. For example, speed is defined as the distance travelled divided by the time taken to travel that distance. Therefore, speed is related to distance (*i.e.* length) and time. The units selected for these base quantities (*i.e.* mass, length and time) are called fundamental units.

The units selected for measuring mass, length and time are called **fundamental units.**

These three fundamental units are independent of each other. However, the units of other physical quantities encountered in mechanics can be expressed in terms of the units of these three fundamental units. This eliminates the need for assigning a separate unit for each physical quantity in mechanics.

Notes. (*i*) *The fundamental units are not definable in terms of other quantities. For this reason, they are called* **base** *or* **fundamental units.**

(*ii*) In order to cover all physical measurements in science and engineering, we require the units of seven base quantities *viz.* units of mass, length, time, temperature, current, amount of substance and luminous intensity. However, to express the units of physical quantities in mechanics, the units of mass, length and time are adequate.

2. Derived units. *The units of physical quantities which can be expressed in terms of fundamental units (mass, length and time) are called* **derived units.**

(*a*) The unit of velocity is a derived unit.

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{\text{Length}}{\text{Time}}$$

Thus the unit of velocity can be expressed in the fundamental units of length and time.

(*b*) The unit of acceleration is a derived unit.

$$\text{Acceleration} = \frac{\text{Velocity}}{\text{Time}} = \frac{\text{Displacement}}{(\text{Time})^2} = \frac{\text{Length}}{(\text{Time})^2}$$

Thus the unit of acceleration can be expressed in the fundamental units of length and time.

(*c*) The unit of force is a derived unit.

$$\text{Force} = \text{Mass} \times \text{Acceleration} = \text{Mass} \times \frac{\text{Length}}{(\text{Time})^2}$$

Thus the unit of force can be expressed in the fundamental units of mass, length and time.

2.5. CHARACTERISTICS OF A STANDARD UNIT

We have seen that units of all physical quantities encountered in mechanics can be expressed in terms of fundamental units of mass, length and time. In SI units, the standard of length is *metre,* that for mass is *kilogramme* and that for time is *second.* The question arises how we should define these fundamental units. Obviously, the standard unit must be both accessible and invariable. Indeed, the fundamental unit selected must have the following features :

(*i*) It should be of convenient size.
(*ii*) It should be well defined.
(*iii*) It should be easily available so that as many laboratories as possible can duplicate it.
(*iv*) It should not change with time and place.
(*v*) It should not change with the change in physical conditions *e.g.* temperature, pressure *etc.*
(*vi*) It should be universally agreed upon so that results obtained in different countries are comparable.

If we define the length standard as the distance between one's nose and the index finger on an outstretched arm, we certainly have an accessible standard. But it will vary from person to person and cannot be accepted as the standard of length. This calls for precise and accurate definition of the standard quantity selected for fundamental units. The standard quantity must meet the above requirements. *There is international agreement on both the definitions of each of the seven standard quantities and the design of experiments used to measure them.* It is reminded that for mechanics, we only require standard units of length, mass and time.

2.6. OLD SYSTEMS OF FUNDAMENTAL UNITS

Different units were assigned for mass, length and time in different countries. However, the following two systems of fundamental units need brief description.

(*i*) C.G.S. (centimetre, gramme, second) system.

(*ii*) M.K.S. (metre, kilogramme, second) system.

In C.G.S. system, the units of length, mass and time are centimetre (cm), gramme (g) and second (s) respectively. However, in M.K.S. system, the units of length, mass and time are metre (m), kilogramme (kg) and second (s) respectively. Note that M.K.S. and C.G.S. units are related to each other by multiples of ten. It is, therefore, easier to convert M.K.S. units into C.G.S. units and *vice-versa*. Although M.K.S. system held ground for a long time, it did not quite solve the problem of coherence. A system is said to be coherent when calculations involving one unit of *A*, one unit of *B*, one unit of *C* and so forth, yield one unit of *X*.

2.7. SI UNITS

The SI units (Systeme International) were formally introduced in 1960 and have been accepted by all the countries for scientific work. In this system, seven base quantities *viz* *mass, length, time, electric current, temperature, luminous intensity* and *quantity of matter* have been assigned proper and standard units. The units of all other physical quantities in science and engineering can be derived from the units of these seven fundamental quantities. The units assigned to these seven fundamental quantities are given below in the tabular form.

Base Units

Quantity	Symbol	Unit name	Unit symbol
Length	*l*, *L*	metre	m
Mass	*m*	kilogramme	kg
Time	*t*	second	s
Electric current	*I*	ampere	A
Temperature	*T*	degree kelvin	K (not °K)
Luminous intensity	*I*	candela	cd
Quantity of matter		mole	mol

In addition to these seven base units, two more supplementary units of plane angle and solid angle have been introduced in this system.

Supplementary Units

Plane angle	radian	rad
Solid angle	steradian	sr

Note. In mechanics, we use only the first three of the base units – metre, kilogramme and second – along with measures of angle.

2.8. DEFINITIONS OF SI BASE UNITS

The following are the official definitions of SI base units. Although not required for an understanding of SI, they illustrate the extraordinary precision associated with this modern system of units.

1. **Metre.** In the past, the metre has been defined in various ways.

   (*i*) In 1889, the **metre** *was defined to be the distance between two fine lines engraved near the ends of a bar of platinum-iridium alloy when the bar was maintained at 0°C.* The original international prototype metre is still kept at the International Bureau of Weights and Measures in Paris.

(*ii*) As new methods developed, the above definition of metre was not sufficiently accurate. In 1960, the **metre** *was redefined as 1,650,763.73 wavelengths of orange-red light emitted by the isotope krypton-86.*

(*iii*) Even the above definition became insufficient. In 1983, the metre was redefined in terms of speed of light in vacuum. The speed of light is now one of the most accurately determined quantities. Its value in vacuum has been measured at 299,79,2458 m/s. Therefore, the **metre** *is defined as the length of the path travelled by light in a vacuum during a time interval of 1/299,792,458 of a second.*

2. **Second.** The **second** *is the duration of 9192 631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atoms.*

3. **Kilogram.** The **kilogram** *is the mass equal to the mass of a standard platinum-iridium alloy cylinder (90% platinum and 10% iridium) kept at the International Bureau of Weights and Measures in Paris.* This cylinder is about 4 cm high and 4 cm in diameter. The standard of mass is least satisfactory. Unlike the standards of length and time which are based on measurement procedures that can be repeated by scientists anywhere, the unit of mass is still defined in terms of a particular cylinder kept in Paris.

4. **Ampere.** The **ampere** *is that constant current which, if maintained in two parallel straight conductors of infinite length and of negligible circular cross-section placed 1 metre apart in vacuum, would produce a force between them of* 2×10^{-7} *N per metre of length.*

5. **Kelvin.** The **kelvin** *is 1/273.16 of the thermodynamic temperature of the *triple point of water.*

6. **Candela.** The **candela** *is the luminous intensity in a given direction of a source which emits monochromatic radiation of frequency* 540×10^{12} *hertz and whose radiant intensity is 1/683 watt per steradian.*

7. **Mole.** The **mole** *is the amount of substance of a system that contains as many elementary particles as there are in 0.012 kg of carbon-12.*

SUPPLEMENTARY UNITS

8. **Radian.** The mathematics of circular motion is much simpler if we use the *radian* for angular measure. *We define the angle in radians* (θ) *to be the ratio of the arc length to the radius i.e.,*

$$\theta = \frac{l}{r} \text{ radians} \quad \text{[See Fig. 2.1]}$$

If $l = r$, $\theta = 1$ radian.

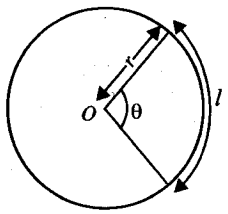

Fig. 2.1

One radian (*rad*) *is defined as the angle subtended by an arc whose length is equal to the radius.*

9. **Steradian.** *It is the measure of solid angle and is defined as the ratio of the subtended area to radius squared.* Referring to Fig. 2.2 (a part of sphere), we have, solid angle, $\omega = \frac{A}{r^2}$.

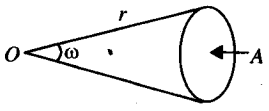

Fig. 2.2

If $A = 1$ m^2 and $r = 1$ m, then $\omega = 1$ steradian.

Hence **one steradian** *(sr) is the angle subtended at the centre of a sphere of radius 1 m by a part of its surface having an area of 1 m^2.*

* Pure water in an evacuated cell is cooled until ice begins to form. The resulting temperature where ice, water and water-vapour co-exist is called the triple point of water and is equal to 273.16 kelvin, by definition.

2.9. SOME DERIVED SI UNITS

The units of all physical quantities in science and engineering can be derived from the seven base SI units *viz.* units of mass, length, time, electric current, temperature, luminous intensity and mole. A few SI derived units are given below by way of illustration.

(*i*) $$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}} = \text{m/s or } ms^{-1}$$

(*ii*) $$\text{Acceleration} = \frac{\text{Velocity}}{\text{Time}} = \frac{\text{Displacement}}{(\text{Time})^2} = \text{m/s}^2 \text{ or } ms^{-2}$$

(*iii*) $$\text{Force} = \text{Mass} \times \text{Acceleration} = \text{kg m/s}^2 = \text{newton}$$

Thus the derived SI unit of force is kg m/s^2. This derived unit has been given the special name *viz.* newton.

(*iv*) $$\text{Work} = \text{Force} \times \text{Distance} = \text{Nm or joule}$$

Thus the derived SI unit of work is Nm. This derived unit has been given the special name *viz.* joule.

(*v*) $$\text{Power} = \frac{\text{Work}}{\text{Time}} = \frac{\text{Nm or joule}}{\text{s}} = \text{joule/s or Nm/s}$$

Thus the derived SI unit of power is Nm/s or joule/s. This derived unit has been given the special name *viz.* watt.

The reader will find a large number of derived SI units in the text.

2.10. ADVANTAGES OF SI UNITS

The SI possesses a number of remarkable advantages over other systems of units.

(*i*) **Coherent system of units.** All the derived SI units can be obtained by dividing and multiplying the base SI units and no numerical factors are involved as used to be the case with units of C.G.S. and M.K.S. systems. For example, the unit of force is obtained when unit of mass is multiplied by unit of acceleration.

(*ii*) **Rational system of units.** One of the important advantages of SI is that it is a rational system of units. It is evident from the following discussion.

(*a*) It uses one unit for one physical quantity. For example, all types of energies (*e.g.* mechanical energy, heat energy, electrical energy *etc.*) are measured in joules. However, in C.G.S. and M.K.S. systems, different units are used for different types of energies. Thus in M.K.S. system, mechanical energy is measured in joules, heat energy in calories and electrical energy in watt-hours.

(*b*) In C.G.S. and M.K.S. systems, both gravitational and absolute units of a physical quantity are used. For example, in the M.K.S. system, the absolute unit of force is newton and gravitational unit is kilogram force. This often leads to confusion. However, in SI, only newton is used as the unit of force.

(*iii*) **Decimal system.** Like C.G.S. and M.K.S. system, SI is also a decimal system. This makes the calculation work a simple affair.

(*iv*) **Blend of practical and theoretical work.** It can be used by research scientists, the technicians, the practising engineers and by the layman, thereby blending the theoretical and practical world.

Despite these advantages, SI is not the answer to everything. In specialised areas of atomic physics and even in day to day work, other units may be convenient. Thus we continue to measure

plane angle in degrees even though the SI unit is the radian. Furthermore, "day" and "hour" are still used despite the fact that the SI unit of time is the second.

2.11. PRACTICAL UNITS FOR LARGE DISTANCES

In order to measure very large distances, we use the following three units :

(*i*) **Astronomical Unit** (A.U.). *It is the mean distance between the centres of earth and sun.*

$$1 \text{ A.U.} = 1.496 \times 10^{11} \text{ m} \simeq 1.5 \times 10^{11} \text{ m}$$

(*ii*) **Light Year** (ly). *It is the distance travelled by light in vacuum in one year.*

$$1 \text{ ly} = (3 \times 10^8) \times (24 \times 60 \times 60 \times 365) = 9.46 \times 10^{15} \text{ m}$$

(*iii*) **Parsec** (*parallactic second*). *It is the distance at which an arc of length 1 A.U. subtends an angle of* 1".

$$\theta = \frac{l}{r} \quad \text{or} \quad r = \frac{l}{\theta} = \frac{1 \text{A.U.}}{1''} = \frac{1.496 \times 10^{11}}{\pi/60 \times 60 \times 180} = 3.08 \times 10^{16} \text{ m}$$

2.12. PRACTICAL UNITS FOR VERY SMALL DISTANCES

The following units of length are commonly used for measuring very small distances in scientific work :

(*i*) 1 micron = 1 μm = 10^{-6} m (*ii*) 1 nanometre = 1 nm= 10^{-9} m

(*iii*) 1 angstrom = 1 Å = 10^{-10} m (*iv*) 1 fermi = 1 fm = 10^{-15} m

2.13. PRACTICAL UNITS FOR HEAVY MASSES

The following units of mass are commonly used for measuring heavy masses :

(*i*) 1 tonne or 1 metric ton = 1000 kg

(*ii*) 1 quintal = 100 kg

(*iii*) 1 Chandra Shekhar Limit = 1 C.S.L. = 1.4 × Mass of Sun

Note. For measuring very small masses, we use atomic mass unit (a.m.u.).

$$1 \text{ a.m.u.} = 1.66 \times 10^{-27} \text{ kg}$$

The atomic and nuclear masses are measured in this unit.

2.14. PRACTICAL UNITS FOR LARGE TIME

(*i*) Solar day = Time taken by earth to complete one rotation about its axis *w.r.t.* sun

(*ii*) Solar year (or year) = Time taken by earth to complete one revolution around the sun in its orbit.

1 solar year = 365.25 average solar days

(*iii*) Lunar month = Time taken by moon to complete one revolution around the earth in its orbit.

1 Lunar month = 27.3 days

Note. The smallest unit of time is *shake*. Note that 1 shake = 10^{-8} s.

2.15. MULTIPLES AND SUB-MULTIPLES OF SI UNITS

Multiples and sub-multiples of SI units are generated by adding appropriate prefixes to the units. Thus, prefixes such as kilo, mega, nano, centi and so on multiply the value of the unit by the factors listed in the table below. For example, 1 *kilo*watt = 1000 watts, 1 *nano*second = 10^{-9} second and 1 *mega*watt = 10^6 watts.

Multiplication factor	Prefix	Symbol
1 000 000 000 000	10^{12} tera	T
1 000 000 000	10^{9} giga	G
1 000 000	10^{6} mega	M
1 000	10^{3} kilo	k
100	10^{2} hecto	h
10	10^{1} deca	da
0.1	10^{-1} deci	d
0.01	10^{-2} centi	c
0.001	10^{-3} milli	m
0.000 001	10^{-6} micro	μ
0.000 000 001	10^{-9} nano	n
0.000 000 000 001	10^{-12} pico	p

2.16. CONVERSIONS

It is sometimes necessary to convert SI units into C.G.S. or M.K.S. units and *vice-versa*. A few cases are discussed below by way of illustration.

(*i*) 1 km = 1000 m = 10^5 cm = 10^6 mm

(*ii*) 1 km^2 = 10^6 m^2 = 10^{10} cm^2 = 10^{12} mm^2 = 10^{18} μ m^2

(*iii*) 1 tonne/m^3 = 1000 kg/m^3 = 1 g/cm^3 = 1 kg/litre

(*iv*) 1 km/hr = $\frac{1000}{3600}$ m/s = 0.278 m/s = 27.8 cm/s

(*v*) 1 MW = 10^3 kW = 10^6 W = 10^9 mW (*vi*) 1 kWh = 3.6 × 10^6 J

CONCEPTUAL QUESTIONS

Q.1. In what three types of units can all physical measurements encountered in mechanics be expressed?

Ans. All physical measurements encountered in mechanics can be expressed in the units of mass, length and time.

Q.2. Why is metre now defined in terms of speed of light?

Ans. The speed of light is one of the most accurately determined quantities and its value in vacuum is 299,792,458 kms^{-1}.

Q.3. Why is the unit of mass least satisfactory?

Ans. Unlike the units of length and time, which are based on measurement procedures that can be repeated by scientists anywhere, the unit of mass is still defined in terms of a particular object — the international prototype kilogram kept at the International Bureau of Weights and Measures in France. Although kept under controlled conditions, it could conceivably change in value. Further, comparison with such a standard is less convenient than a procedural standard that can be checked in a laboratory.

Q.4. Why is the unit of mass not based on atomic standards?

Ans. The reason that unit of mass is not yet based on atomic standards is that the measurement of atomic masses and their comparison with large-scale masses is not yet as precise as the measurement that can be made on large-scale objects such as the standard kilogram.

Q.5. What is the current thinking of scientists about the redefinition of unit of mass?

Ans. The scientists are now working on techniques to measure the spacing of atoms in a crystal of silicon, essentially counting the atoms in a given volume and therefore scale up from the mass of a single silicon atom to a new definition of the kilogram.

Q.6. Why are the units of mass, length and time called fundamental units?

Ans. The fundamental quantities mass, length and time are not definable in terms of other quantities. For this reason, the units of these quantities are called fundamental units.

Q.7. What is the importance of base units?

Ans. There are only seven base units in SI. The unit of every physical quantity encountered in science and engineering can be expressed in terms of these base units. This eliminates the need for assigning a separate unit for each physical quantity.

Q.8. In physics, the quantities used to describe nature are defined very precisely. Why?

Ans. In physics, the quantities used to describe nature are defined so that they have the same meaning for every person using them. These quantities differ from the descriptive words used in everyday language. For example, you might ask "I will force him to work for my election". Here the words "force" and "work" are used to create an impression. In physics, the precise scientific meanings of force and work have nothing to do with their meanings in the quoted sentence.

Q.9. Why have scientists and engineers all over the world agreed to use SI units?

Ans. The scientists and engineers all over the world have agreed to use SI units so that results of experiments conducted in different countries are comparable.

Q.10. Why is sometimes a special name given to derived SI unit ?

Ans. Often a special name is given to some derived SI units, usually to honour a noteworthy physicist. For example, the derived SI unit of force—kg ms^{-2}—is called *newton* in honour of Isaac Newton.

Q.11. In defining the standard of length, we have to specify the temperature at which the measurement should be made. Are we justified in calling length a fundamental quantity if another physical quantity (temperature) has to be specified in choosing a standard?

Ans. The length of an object changes with temperature. But the unit of length (*i.e.* metre) is now defined in terms of wavelength. This is not affected by *temperature. Therefore, we are justified in calling length a fundamental quantity.

VERY SHORT ANSWER QUESTIONS

Q.1. What is a physical quantity?

Ans. A quantity that can be measured is called a physical quantity.

Q.2. Does the magnitude of a physical quantity change with the change in the system of units?

Ans. The magnitude of a physical quantity does not change with the change in the system of units.

Q.3. What is a unit?

Ans. The unit of a physical quantity is a standard quantity of the same kind chosen in order to measure that quantity.

Q.4. Can a body have zero mass and zero weight?

Ans. The mass of a body cannot be zero. However, weight of a body may be zero.

Q.5. What is the unit for measuring wavelength of light?

Ans. Angstrom (Å) ; 1 Å = 10^{-10} m.

Q.6. Is light year a unit of time?

Ans. No. Light year is a unit of distance and is defined as the distance travelled by light in vacuum in one year. Note that 1 light year = 9.46×10^{15} m.

---

* Now metre is defined in terms of speed of light in vacuum.

Q.7. How many light years make 1 parsec?

Ans. 1 parsec = 3.08×10^{16} m and 1 light year = 9.46×10^{15} m.

$$\therefore \text{ 1 parsec } = \frac{3.08 \times 10^{16}}{9.46 \times 10^{15}} = 3.256 \text{ light years}$$

Q.8. The mass of a proton is 1.67×10^{-27} kg. How many protons would make 1g?

Ans. No. of protons in 1g $= \frac{1\text{g}}{m_p} = \frac{10^{-3}\text{kg}}{1.67 \times 10^{-27}} = 5.99 \times 10^{23}$

Q.9. How many light years are there in 1 m?

Ans. 1 light year $= 9.46 \times 10^{15}$ m

$$\therefore \quad 1 \text{ m} = \frac{1}{9.46 \times 10^{15}} = 1.057 \times 10^{-16} \text{ light year}$$

Q.10. How many times a kg is larger than a mg?

Ans. 1 kg $= 10^3$ g and 1 mg $= 10^{-3}$ g so that 1 kg/1mg $= 10^6$. Therefore, 1 kg is 10^6 times larger than 1 mg.

Q.11. How many quintals are there in a gigagram?

Ans. $$\frac{1 \text{ gigagram}}{1 \text{ quintal}} = \frac{10^9 \text{ g}}{10^2 \text{ kg}} = \frac{10^6 \text{ kg}}{10^2 \text{ kg}} = 10^4$$

Q.12. Human heart is a clock. Comment.

Ans. It is true because human heart beats at a regular rate.

Q.13. What is a coherent system of units?

Ans. A coherent system of units is one in which all derived units are obtained by simply multiplying or dividing without introducing numerical factors.

Q.14. How much are a nanometre, a light year and an angstrom?

Ans. 1 nanometre $= 10^{-9}$ m ; 1 light year $= 9.46 \times 10^{15}$ m ; 1Å $= 10^{-10}$ m

Q.15. How many metres are there in 1 fermi?

Ans. 1 fermi $= 10^{-15}$ m

SHORT ANSWER QUESTIONS

Q.1. The average wavelength of yellow light emitted from a sodium lamp is 5893 Å. Express it in nm.

Ans. Wavelength, $\lambda = 5893$ Å $= 5893 \times 10^{-10}$ m. Since 1 nm $= 10^{-9}$ m,

$$\therefore \quad \lambda = \frac{5893 \times 10^{-10}}{10^{-9}} = 589.3 \text{ nm}$$

Q.2. Give the approximate ratio of 1 A.U. and 1 light year.

Ans. 1 A.U. $= 1.496 \times 10^{11}$ m and 1 light year $= 9.46 \times 10^{15}$ m.

$$\therefore \quad \frac{1 \text{A.U.}}{1 \text{ light year}} = \frac{1.496 \times 10^{11}}{9.46 \times 10^{15}} \simeq 10^{-5}$$

Q.3. Are there more nanoseconds in a second than number of seconds in a year?

Ans. Number of nanoseconds in a second $= 1\text{s}/10^{-9}\text{s} = 10^9$. The number of seconds in a year $= 24 \times 60 \times 60 \times 365 = 3.1536 \times 10^7$ s. Therefore, the number of nanoseconds in a second is more than the number of seconds in a year.

Q.4. What are the advantages in choosing the wavelength of light radiation as a standard of length?

Ans. The following are the advantages of choosing the wavelength of light radiation as a standard of length:

(*i*) The wavelength of light is not affected by change in place, time and physical conditions (*e.g.*, temperature).

(*ii*) It is easily reproducible in any good laboratory.

Q.5. Which type of phenomenon can be used as a measure of time? Give three examples.

Ans. Any phenomenon which repeats itself at regular intervals of time can be used as a measure of time. The three examples are :

(*i*) Oscillations of a pendulum (*ii*) revolution of earth around the sun (*iii*) rotation of earth around its axis.

Q.6. Derive the SI unit of joule (J) in terms of fundamental units.

Ans. Joule is the unit of work.

Now, Work = Force × Distance = Mass × Acceleration × Distance

= Mass × Distance/(Time)^2 × Distance

= kg × m^2 × 1/s^2 = kgm^{2}s^{-2}

LONG ANSWER QUESTIONS

1. What is the need for measurement in physics? **[Refer to Art. 2.2]**
2. What are the characteristics of a standard unit? **[Refer to Art. 2.5]**
3. What are the advantages of SI units over other systems of units? **[Refer to Art. 2.10]**
4. Explain fundamental and derived units. **[Refer to Art. 2.4]**
5. Define fundamental units of SI. **[Refer to Art. 2.8]**

MCQ FOR COMPETITIVE EXAMINATIONS

1. A physical quantity is one which
   (*a*) cannot be measured
   (*b*) can be measured
   (*c*) is a scalar quantity
   (*d*) is a vector quantity
2. The number of base units required in mechanics is
   (*a*) 2 (*b*) 4
   (*c*) 5 (*d*) 3
3. The number of base units required to cover all physical quantities in science and engineering is
   (*a*) 7 (*b*) 5
   (*c*) 6 (*d*) 3
4. The unit of speed is a
   (*a*) fundamental unit (*b*) derived unit
   (*c*) neither of above (*d*) cannot say
5. The least satisfactory standard unit is that of
   (*a*) length (*b*) time
   (*c*) mass (*d*) none of the above
6. In SI units, candela is the unit of
   (*a*) current
   (*b*) temperature
   (*c*) luminous intensity
   (*d*) none of the above
7. The prefix nano stands for
   (*a*) 10^{-12} (*b*) 10^{-9}
   (*c*) 10^{-6} (*d*) 10^{-3}
8. In SI units, the prefix micro stands for
   (*a*) 10^{-3} (*b*) 10^{-19}
   (*c*) 10^{-6} (*d*) 10^{12}
9. SI is
   (*a*) decimal system
   (*b*) not decimal system
   (*c*) the same as M.K.S. system
   (*d*) none of the above
10. In SI units, all energies are measured in
    (*a*) joules (*b*) watts
    (*c*) newtons (*d*) none of the above

ANSWERS TO MCQ FOR COMPETITIVE EXAMINATIONS

1. (*b*)	**2.** (*d*)	**3.** (*a*)	**4.** (*b*)	**5.** (*c*)
6. (*c*)	**7.** (*b*)	**8.** (*c*)	**9.** (*a*)	**10.** (*a*)

numbers. In such a case, the knowledge of the order of magnitude of the physical quantity under calculation is quite helpful.

(*iv*) Sometimes the correct power of 10 may be lost on a calculator and the knowledge of the order of the physical quantity can be used to obtain it.

Tables below give some typical order-of-magnitude values for some lengths, masses and time intervals encountered in the physical world.

Table 1

Length	Order of Magnitude (m)
Radius of Proton	10^{-15}
Radius of atom	10^{-10}
Height of man	10^{0}
Radius of earth	10^{7}
Earth-Sun distance	10^{11}
Distance of nearest star	10^{10}

Table 2

Mass	Order of Magnitude (kg)
Electron	10^{-30}
Proton	10^{-27}
Ant	10^{-2}
Man	10^{2}
Earth	10^{24}
Sun	10^{30}

Table 3

Time Interval	Order of Magnitude (s)
Period of visible light	10^{-15}
Period of micro waves	10^{-10}
Period of human heart	10^{0}
Period of earth rotation	10^{5}
Life	10^{9}

3.2. MEASUREMENT OF PHYSICAL QUANTITIES

There are a large number of physical quantities encountered in the physical world. However, three very important quantities are distances (lengths), masses and time intervals. The technique to measure any of these quantities would depend upon two principal factors, *viz.*

(*i*) the magnitude of the physical quantity to be measured.

(*ii*) the accuracy or precision required in the measurement.

Sometimes a physical quantity can be measured by ***direct method** while in some cases, we have to employ ***indirect methods*** for its measurement. For example, the length of a table can be measured *directly* with a metre rod, whereas the diameter of a wire can be determined *directly* with the help of a vernier calliper or screw gauge. Sometimes the magnitude of a physical quantity is too large or too small or there are other reasons that direct methods of measurement cannot be used. In such cases, we use indirect methods of measurement. For example, the distance of moon from earth cannot be measured by direct method and requires the use of indirect method. Similarly, in order to measure the height of an inaccessible object (*e.g.* height of a mountain, height of a tree on the other bank of a river *etc.*), we need the help of indirect methods of measurement. Indeed the choice of a particular method depends upon the situation and accuracy of measurement required.

3.3. INDIRECT METHODS FOR MEASURING LARGE DISTANCES

There are several indirect methods of measuring large distances (*e.g.* distance of moon or star from earth) and heights of accessible and inaccessible objects. A few methods are discussed below by way of illustration.

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* In the direct method, we compare the unknown physical quantity with a certain standard *e.g.* we use a metre rod for measuring the length of a table.

(*i*) *Echo method. *This method is based on the reflection of sound* and can be used to find the distance of a hill. In this method, a gun is fired towards the hill and the time interval between the production of sound and the reception of its echo after reflection from hill is noted. If S is the distance of the hill from the observer, t is the total time interval and v is the velocity of sound, then,

$$v \times t = 2S \quad \text{or} \quad S = \frac{vt}{2}.$$

Since the values of v and t are known, the distance S of the hill from the observer can be determined.

(*ii*) **SONAR.** *This method is based on the reflection of* **ultrasonics* and can be used to locate the position of a submerged submarine, sunken ship, depth of sea *etc.* In this method, ultrasonics from a transmitter are sent towards the bottom of sea. The ultrasonic beam is reflected from the submerged object and is received by a detector. The time interval between the emission of ultrasonics and their arrival back after reflection is noted. If S is the distance of the submerged object, then,

$$S = \frac{vt}{2}$$

where v = velocity of ultrasonics in water and t = time interval.

Knowing the values of v and t, the value of S can be determined. Note that SONAR stands for SOund Navigation And Ranging.

(*iii*) **LASER Method.** *A LASER is a very intense, monochromatic and unidirectional beam of light.* This method uses echo principle except that a LASER beam is used in place of sound waves. The laser beam transmitted from earth is received back on earth after reflection from a heavenly body *e.g.*, moon. If t is the time taken by the LASER beam in going to and returning from the moon, then distance S of the moon from the earth is given by;

$$S = \frac{c \times t}{2} \quad \text{where } c = \text{velocity of LASER beam} = 3 \times 10^8 \text{ ms}^{-1}$$

(*iv*) **RADAR.** *This method is based on the reflection of* † *radiowaves* and can be used to find the distance of an astronomical object or location of an aeroplane. In this method, radiowaves are sent into space from a transmitter. If an aeroplane comes in the path of radiowaves, they are reflected back. The time interval between the sending of radiowaves and their reception after reflection is noted. Then distance S of the aeroplane from the observation point is

$$S = \frac{ct}{2}$$

where c = speed of radiowaves = 3×10^8 m s^{-1}

t = time interval

Note that RADAR stands for RAdio Detection And Ranging.

(*v*) **Measurement of size of heavenly bodies.** The size of a heavenly body (*e.g.* diameter of moon) can be found by measuring angle θ subtended by its diameter AB at a point E on the earth (See Fig. 3.1). For this purpose, we use a telescope. If AB $(= d)$ is the diameter of the moon and D is its distance from earth, then,

---

* If a person stands in front of a distant hill and produces a sharp sound, he will hear two distinct sounds ; one original sound and other echo (*i.e.* sound reflected from the hill).

** Sound waves having frequency greater than 20 kHz are called ultrasonics. These cannot be heard by human beings. However, these can be detected by various devices.

† Radiowaves are electromagnetic waves which travel at a speed of 3×10^8 ms^{-1} .

$$\theta = \frac{d}{D} \text{ or } d = D\theta$$

Since D (distance of the moon from earth) is known and θ is measured, the diameter of the moon can be found. Note that if θ is in radian and D in metres, then diameter d of the moon will also be in metres. For example, the angular diameter of the sun is 0.0093 radian and sun-earth distance is 1.496×10^{11} m *i.e.*

$$\theta = 0.0093 \text{ radian} ; \quad D = 1.496 \times 10^{11} \text{ m}$$

$\therefore$ Diameter of sun, $d = D\theta = 1.496 \times 10^{11} \times 0.0093 = 1.39 \times 10^{9}$ m

Note that diameter of the sun is about 10^2 times the diameter of the earth.

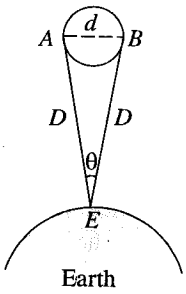

Fig. 3.1

(*vi*) **Measurement of distances of heavenly bodies.** There are several methods available to determine the distance of a heavenly body from earth. Two methods are discussed by way of illustration.

(*a*) **Parallax method.** In this method, a planet P is observed simultaneously from two distant places A and B (See Fig. 3.2). The angle $\theta (= \angle APB)$ is measured. If $AB = b$ and $S (= AP = BP)$ is the distance of the planet from the surface of earth, then,

$$\theta = \frac{\overset{\frown}{AB}}{S} = \frac{b}{S} \quad \therefore \quad S = \frac{b}{\theta}$$

If θ is in radian and b in A.U., then distance S of the planet from earth will be in *A.U. Note that 1 A.U. = 1.496×10^{11} m.

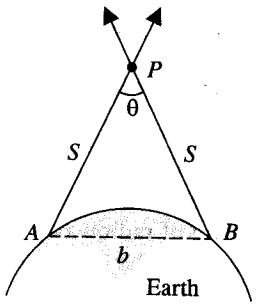

Fig. 3.2

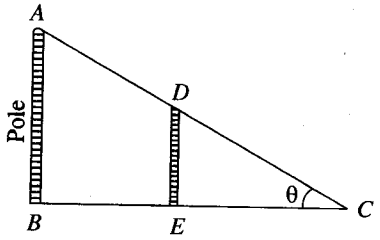

Fig. 3.3

(*b*) **Radio-echo method.** In this method, a radio signal is sent towards the heavenly body (*e.g.* a planet) and the reflected signal is received. The time interval between the transmission and reception of the signal is accurately determined. If time interval is t seconds, then distance S of the planet from earth is given by ;

$$S = \frac{ct}{2} \quad \text{where } c = 3 \times 10^8 \text{ m s}^{-1}$$

(*vii*) **Determination of height of accessible objects.** The height of an accessible object (*e.g.* electric pole) on a sunny day can be determined by *triangulation method i.e.* by using the geometrical properties of a triangle. Suppose AB is the pole whose height is to be determined. Since it is a sunny day, the pole will cast a shadow BC on the ground. Now take a stick DE and place it perpendicular on BC in such a way that top of the shadow of the stick is at C. Clearly ADC will be a straight line (See Fig. 3.3).

---

* The average distance of sun to earth is called *astronomical unit* (A.U.).
$\therefore$ 1 A.U. = 1.496×10^{11} m

$$\text{In } \Delta ABC,\ \tan\theta = \frac{AB}{BC} \ \ldots (i)\ ;\ \text{In } \Delta DEC,\ \tan\theta = \frac{DE}{EC} \ \ldots (ii)$$

From eqs. (*i*) and (*ii*), $\frac{AB}{BC} = \frac{DE}{EC}$ $\therefore$ Height of pole, $AB = \frac{DE}{EC} \times BC$

Since lengths *DE*, *EC* and *BC* can be measured, we can find the height of the pole.

Another method. If it is not a sunny day, the height of the accessible object (*e.g.*, electric pole, tower, etc.) can be determined by a sextant. In this method, the angle θ subtended by the top of tower *AB* at point *C* on the ground is measured. In ΔABC (See Fig. 3.4),

$$\tan\theta = \frac{AB}{BC} \qquad \therefore \text{ Height of tower, } AB = BC\tan\theta$$

Since the value of θ and *BC* are known, we can find the height of the tower. Note that angle θ is called elevation of tower at point *C*.

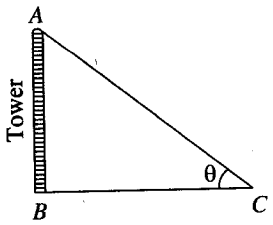

Fig. 3.4

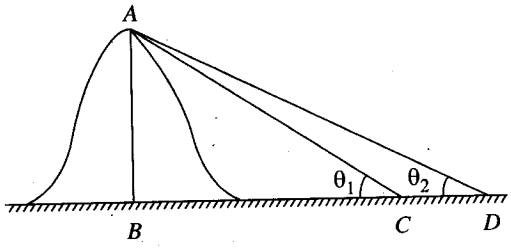

Fig. 3.5

(*viii*) **Determination of height of inaccessible objects.** The height of an inaccessible object (*e.g.* a mountain) can be determined by triangulation method. Suppose *AB* is the height of the mountain. The angles of elevation of the peak of the mountain are measured at two convenient points *C* and *D* on the ground as shown in Fig. 3.5. Let θ_1 and θ_2 be the angles of elevation of peak of mountain at points *C* and *D* respectively.

$$\text{In } \Delta ABC,\ \cot\theta_1 = \frac{BC}{AB}\ ;\ \text{In } \Delta ABD,\ \cot\theta_2 = \frac{BD}{AB}$$

$$\therefore \qquad \cot\theta_2 - \cot\theta_1 = \frac{BD}{AB} - \frac{BC}{AB} = \frac{BD - BC}{AB} = \frac{CD}{AB}$$

or Height of mountain, $AB = \dfrac{CD}{\cot\theta_2 - \cot\theta_1}$

Since the values of θ_1 and θ_2 are known and the distance *CD* can be measured, we can find the height *AB* of the mountain.

(*ix*) **Determination of width of a river.** The width of a river can be determined by triangulation method as shown in Fig. 3.6. Suppose *AB* is the width of the river; point *B* being some reference object (*e.g.* a tree). A pole is fixed (perpendicular to the ground) at some point *C* on the bank of the river so that reference object and the pole at *C* are in the same line. Now proceeding along the line *BC*, fix another pole *at E* in such a way that reference object at *B* and the poles at *C* and *E* are in the same line. From *E*, draw a perpendicular *ED* on the bank of the river.

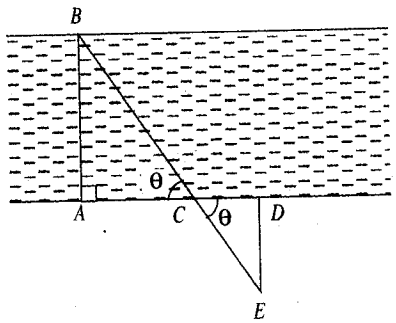

Fig. 3.6

In ΔBAC, $\tan\theta = \frac{AB}{AC}$ ... (*i*) ; In ΔEDC, $\tan\theta = \frac{ED}{CD}$ ... (*ii*)

From eqs. (*i*) and (*ii*), $\frac{AB}{AC} = \frac{ED}{CD}$ $\therefore$ Width of river, $AB = \frac{ED}{CD} \times AC$

Since DE, CD and AC can be measured, we can find the width AB of the river.

Example 3.1. **A boy hears an echo of his own voice from a distant hill after one second. The velocity of sound in air is 350 ms^{-1}. What is the distance of the hill from the boy?**

Solution. Suppose the distance of the hill from the boy is S. Then total distance travelled by sound in going and coming back is 2 S.

$$\therefore \quad 2S = v \times t \text{ or } S = \frac{vt}{2} = \frac{350 \times 1}{2} = 175\text{m}$$

Example 3.2. **The angle of elevation of a tree at a place 90 m away is 15°. What is the height of the tree?**

Solution. Fig. 3.7 shows the conditions of the problem. Here $\theta = 15°$; $BC = 90$ m ; $AB = ?$

$$\frac{AB}{BC} = \tan\theta \text{ or } AB = BC\tan\theta = 90\tan 15° = 24.3 \text{ m}$$

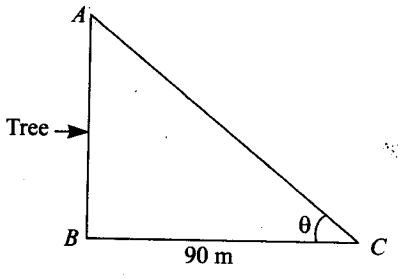

Fig. 3.7

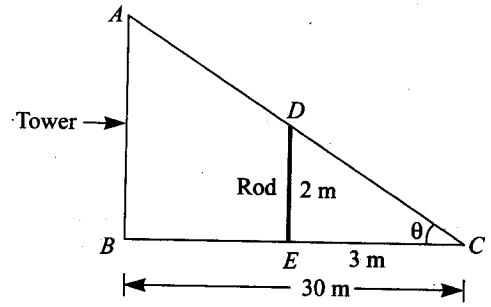

Fig. 3.8

Example 3.3. **On a sunny day, a tower casts 30 m long shadow. An iron rod dug vertically also casts its shadow as far as that of tower. If the height of rod above ground is 2 m and it is dug 3 m from the tip of the shadow towards the tower, find the height of the tower.**

Solution. Fig. 3.8 shows the conditions of the problem. Here $BC = 30$ m ; $EC = 3$ m ; $DE = 2$ m ; $AB = ?$

In ΔABC, $\tan\theta = \frac{AB}{BC}$; In ΔDEC, $\tan\theta = \frac{DE}{EC}$

$$\therefore \quad \frac{AB}{BC} = \frac{DE}{EC} \text{ or } AB = \frac{DE}{EC} \times BC = \frac{2}{3} \times 30 = 20 \text{ m}$$

Example 3.4. **Find the distance of moon from earth if the parallax angle as measured from two places at a distance of 6.4 × 10^6 m on earth is 57 minutes of an arc.**

Solution. Distance of moon from earth, $S = \frac{b}{\theta}$

Here $b = 6.4 \times 10^6$ m ; $\theta = 57' = 57 \times 60'' = \left(\frac{57 \times 60}{3600}\right) \times \frac{\pi}{180}$ rad

$$\therefore \quad S = \frac{6.4 \times 10^6}{(57 \times 60)\pi} \times 3600 \times 180 = 3.858 \times 10^8 \text{ m}$$

Example 3.5. A planet is at a distance of 8.247×10^8 km from earth. Its angular diameter is 35.72" of an arc. Calculate the diameter of the planet.

Solution. Diameter of the planet, $d = D\theta$

$$\text{Here } D = 8.247\times10^8 \text{ km} ; \quad \theta = 35.72'' = \frac{35.72}{3600}\times\frac{\pi}{180}\text{ radian}$$

$$\therefore \quad d = \left(8.247\times10^8\right)\times\frac{35.72}{3600}\times\frac{\pi}{180} = \mathbf{1.426 \times 10^5 \text{ km}}$$

PROBLEMS FOR PRACTICE

1. A boy standing in front of a high steep rock claps once and hears the echo of his clap after 2.5 seconds. If the velocity of sound is 340 ms^{-1} , find the distance of the boy from the rock. **[425 m]**
2. A radar signal is beamed towards a planet from earth and its echo is received seven minutes later. The distance between the planet and earth is 6.3×10^{10} m. Find the velocity of the signal. **[3×10^8 ms^{-1}]**
3. The angle of elevation of a tree at a place 80 m away is 12°5′. What is the height of the tree? **[71.2 m]**
4. A distant cliff has 10° and 12° angles of elevation at two places 100 m apart. What is the height of the cliff? **[3.63 m]**
5. A parsec is a convenient unit of length on the astronomical scale. It is the distance at which the arc of length 1 A.U. subtends an angle of one second of an arc. Show that 1 parsec = 3.08×10^{16} m.
6. Light year is a convenient unit of length on the astronomical scale. It is the distance travelled by light in vacuum in one year. Show that 1 light year = 9.5×10^{15} m.
7. The parallax angle of a planet on two places of earth is 20 seconds. If the two places are situated diametrically on the equator, find the distance of the planet from earth. Radius of earth is 6400 km. **[1.32×10^{11} m]**

3.4. MEASUREMENT OF VERY SMALL (ATOMIC) DISTANCES

Several methods are available to measure very small distances such as inter-molecular distances, size of molecules, radius of atoms etc. However, we shall discuss only three methods by way of illustration.

(*i*) **Avogadro's hypothesis.** We can find the radius of atom by making use of Avogadro's hypothesis. According to Avogadro's hypothesis, the gram-atomic weights of all elements contain 6.02×10^{23} atoms and gram-molecular weights of all substances contain 6.02×10^{23} molecules. This number is called *Avogadro's number* N_A. Thus in 108 g Ag, there are 6.02×10^{23} silver atoms ; in 23 g Na are 6.02×10^{23} sodium atoms; 18 g of water contains 6.02×10^{23} water molecules ; 22.4 litres of any gas at 0°C and 1 atm pressure contains 6.02×10^{23} molecules.

Suppose A is the gram-atomic weight of an element. According to Avogadro's hypothesis, A gm of the element will contain N_A ($= 6.02 \times 10^{23}$) atoms. If r is the radius of the atom (spherical in shape),

$$\text{Volume of one atom} = \frac{4}{3}\pi r^3$$

$$\text{Volume occupied by atoms in 1 g of element} = \frac{N_A}{A}\times\frac{4}{3}\pi r^3$$

There are empty spaces between the atoms in a substance. According to Avogadro's hypothesis, *the actual volume occupied by the atoms in 1 g of a substance is two-third of the volume of 1 g of the substance.* Thus if V is the actual volume of 1 g of the element, then,

$$\frac{N_A}{A}\times\frac{4}{3}\pi r^3=\frac{2}{3}V \quad \text{or} \quad r^3=\frac{VA}{2\pi N_A}$$

$$\therefore \qquad r=\left(\frac{VA}{2\pi N_A}\right)^{1/3}$$

Since the values of V, A and N_A are known, the value of radius of the atom can be calculated.

(ii) **α-particle scattering experiment.** We can find the radius of nucleus by firing α particles for head on collision towards the nucleus. Suppose an α-particle is directed towards the centre of the nucleus of an atom as shown in Fig. 3.9. The α-particle slows down and then comes to a momentary stop at a distance r_0 from the nucleus before it begins to move back along its original path. The distance r_0 is called the *distance of closest approach.* When the α-particle is outside the atom, the electric potential energy of the system is zero because the atom is neutral and does not produce any external electric field. But once the α-particle passes through the region of electron orbits on its way towards the nucleus, it experiences a repulsive force due to the positive charge on the nucleus. Consequently, the α-particle starts slowing down. As the α-particle slows down, its kinetic energy starts transferring into electric potential energy of the system. The transfer is complete when the α-particle stops momentarily. At this point, the distance of the α-particle from the nucleus is r_0 and its entire kinetic energy is transferred to electric potential energy of the system. The α-particle now retraces its path.

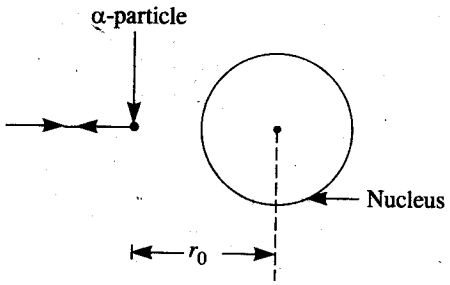

Fig. 3.9

Let m = mass of the α-particle ; v = initial velocity of the α-particle

q_1 = charge on α-particle (+ $2e$) ; q_2 = charge on the nucleus (+ Ze)

Initial K.E. of α-particle, $K = \frac{1}{2}mv^2$

Electric potential energy (U) of the system when the α-particle is at a distance r_0 from the nucleus is

$$U=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_0}=\frac{1}{4\pi\varepsilon_0}\frac{(2e)(Ze)}{r_0}=\frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{r_0}$$

According to law of conservation of energy, $K = U$.

$$\therefore \qquad \frac{1}{2}mv^2=\frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{r_0} \quad \text{or} \quad r_0=\frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{K} \quad \text{where } K=\frac{1}{2}mv^2$$

The radius of the nucleus will be less than r_0. The magnitude of r_0 can be estimated by considering a typical case. Consider the case of a gold nucleus bombarded by an α-particle having an initial energy of 24.6 MeV.

$$r_0=\frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{K}$$

Here K = 24.6 MeV = 24.6 × 10^6 × 1.6 × 10^{-19} J; Z = 79; e = 1.6 × 10^{-19} C; $1/4\pi\varepsilon_0 = 9\times10^9$

$$\therefore \qquad r_0=(9\times10^9)\times\frac{2\times79\times(1.6\times10^{-19})^2}{24.6\times10^6\times1.6\times10^{-19}}=9.25\times10^{-15}\text{ m}$$

The radius of the gold nucleus will be less than this value.

(iii) **Size of molecule of a liquid.** To measure a very small size like that of a liquid molecule, we have to adopt special methods. A simple method for determining the molecular size of **oleic acid** is given below. Oleic acid is a soapy liquid with large molecular size ($\simeq 10^{-9}$ m). To find the size of molecule of oleic acid, we form mono-molecular layer (*i.e.,* single-molecule layer) of oleic acid on water surface.

Procedure. First, we prepare a solution of known concentration of oleic acid in alcohol. To prepare a solution of 1/500 concentration (say), 1 c.c. of oleic acid is mixed with 500 c.c. of alcohol. We then pour n drops of this solution on the surface of water contained in a flat and broad vessel. The oleic acid solution spreads very fast on the surface of water and forms a very thin layer of thickness t (say). The alcohol evaporates and oleic acid layer is left on the surface of water. Using a trace paper and a graph paper, we carefully measure the surface area A of the oleic acid layer.

Calculations. If v is the volume of each of the n drops on water surface, then,

Volume of oleic acid in n drops of solution $= n\ v \times \dfrac{1}{500}$

Volume of oleic acid layer $= A \times t$

$$\therefore \qquad nv \times \frac{1}{500} = A \times t \quad \text{or} \quad t = \frac{nv}{500\,A}$$

If the thin layer of oleic acid is assumed to be one molecule thick, the value of t gives the size (*i.e.,* diameter) of a molecule of oleic acid.

Example 3.6. When 96500 C of electricity is passed through silver voltameter, 108 g (1 gram-atomic mass) of silver is deposited on the cathode. What is Avogadro's number N_A ?

Solution. Each silver ion (Ag^+) accepts one electron to become silver atom (Ag) and gets deposited on the cathode. The question becomes: How many electron charges are there in 96500 C? We know that charge on an electron is 1.6×10^{-19} C.

$$\therefore \qquad \text{Avogardo's Number, } N_A = \frac{96500}{1.6 \times 10^{-19}} = \mathbf{6.02 \times 10^{23}}$$

Thus in 108 g of silver, there are 6.02×10^{23} atoms.

Example 3.7. How many water molecules are there in 1g of water?

Solution. The molecular weight of water (H_2O) is that of two hydrogen atoms plus one oxygen atom *i.e.* $2 \times 1.0 + 16.0 = 18.0$. Hence 18 g of water contains N_A ($= 6.02 \times 10^{23}$) molecules.

$\therefore$ Number of molecules in 1g of water

$$= \frac{N_A}{18} = \frac{6.02 \times 10^{23}}{18} = \mathbf{3.34 \times 10^{22}\ molecules}$$

Example 3.8. How many copper atoms are there in a solid cube of copper which is 1.00 cm on each side ? What is the approximate size of atom? The atomic mass of copper is 63.6 g/mol and density is 8.96 g/cm^3.

Solution.

Mass of copper in the cube = Density × Volume = $8.96 \times 1.00 = 8.96$ g

$$\text{Number of moles} = \frac{\text{Mass}}{\text{Atomic mass}} = \frac{8.96\,\text{g}}{63.6\,\text{g/mol}} = 0.141\,\text{mol}$$

Number of atoms = Number of moles × N_A

$$= 0.141 \times 6.02 \times 10^{23} = \mathbf{8.49 \times 10^{22}\ atoms}$$

We see that 1 cm^3 of copper contains 8.49×10^{22} atoms, a very large number indeed.

$$\text{Volume of one atom} = \frac{\text{Total volume}}{\text{Number of atoms}} = \frac{1 cm^3}{8.49 \times 10^{22}}$$

$$= 1.18 \times 10^{-23} cm^3 = 1.18 \times 10^{-29} m^3$$

If, for simplicity, we assign each atom a small cubic volume of that size, then cube edge has a length (d) given by ;

$$d = \sqrt[3]{1.18 \times 10^{-29}} \simeq 10^{-10} \text{ m}$$

Thus the atomic size is of the order of 10^{-10} m.

Example 3.9. Calculate the percentage empty space in one mole of N_2 gas at STP. Given that the radius of N_2 molecule is 2Å.

Solution. Volume of one molecule $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (2 \times 10^{-10})^3 = 3.2 \times 10^{-29}$ m^3

Now 1 mol of N_2 gas at STP contains 6.02×10^{23} molecules.

$\therefore$ Total volume occupied by 1 mol of N_2 molecules

= Volume of one molecule $\times$ N_A

= $3.2 \times 10^{-29} \times 6.02 \times 10^{23} = 19.2 \times 10^{-6}$ m^3

We know that 1 mole of a gas at STP occupies 22.4 litres = 22.4×10^{-3} m^3.

$\therefore$ Volume of empty space = $22.4 \times 10^{-3} - 19.2 \times 10^{-6} = 22.38 \times 10^{-3}$ m^3

$\therefore$ Percentage empty space $= \frac{22.38 \times 10^{-3}}{22.4 \times 10^{-3}} \times 100 = 99.9\%$

Note that there is a large empty space in a gas. This is not surprising because intermolecular separation in a gas is very large. Indeed, it is about 10 times the separation of molecules in solids or liquids.

Example 3.10. Find the average separation of gas molecules at S.T.P.

Solution. We know that 1 mole of any gas occupies 22.4 litres or 22.4×10^{-3} m^3 at S.T.P. Since one mole of a gas contains 6.02×10^{23} molecules,

$\therefore$ Volume of one molecule $= \frac{22.4 \times 10^{-3}}{6.02 \times 10^{23}}$ m^3

The volume occupied by one molecule is of the order of d^3 where d is the separation of molecules.

$\therefore$ $d^3 = \frac{22.4 \times 10^{-3}}{6.02 \times 10^{23}}$ or Average separation, $d = \sqrt[3]{\frac{22.4 \times 10^{-3}}{6.02 \times 10^{23}}} \simeq 33 \times 10^{-10}$ m

Note. The order of separation of molecules in liquids is about the same as in solids.

Example 3.11. An α-particle whose kinetic energy is 7.68 MeV is fired for head on collision towards a copper atom. The atomic number of copper atom is 29. Find the approximate radius of copper nucleus.

Solution. $r_0 = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{K}$

Here K = 7.68 MeV = $7.68 \times 10^6 \times 1.6 \times 10^{-19}$ J ; Z = 29 ; e = 1.6×10^{-19} C ;

$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$$

$\therefore$ $r_0 = (9 \times 10^9) \times \frac{2 \times 29 \times (1.6 \times 10^{-19})^2}{7.68 \times 10^6 \times 1.6 \times 10^{-19}} = 1.1 \times 10^{-14}$ m

Note. The value of r_0 depends upon the initial kinetic energy K of the α - particle. If the value of K is very large, then α - particle will reach extremely close to the nucleus. Under this condition, the nuclear forces will become active and α - particles will penetrate into the nucleus. Therefore, α -particles with kinetic energy greater than certain value cannot be scattered back by the nucleus.

PROBLEMS FOR PRACTICE

1. Calculate the number of molecules/m^3 of an ideal gas at STP. **[2.69 × 10^{25} molecules]**
2. Silicon has a density of 2.23 g/cm^3 and an atomic mass of 28.09 g/mol. Calculate the number of atoms in 1 cm^3 of silicon. **[4.78 × 10^{22}/cm^3]**
3. Estimate the order of separation of atoms in aluminium metal. Given the density of aluminium is 2700 kg m^{-3}, the relative atomic mass is 27. **[2.6 × 10^{-10} m]**
4. Estimate the average density of sodium. Given that radius of sodium atom is 1.25Å. **[4.67 × 10^3 kg m^{-3}]**

3.5. MEASUREMENT OF MASS

Mass is the basic property of matter. We normally define the mass of a body as the quantity of matter contained in the body. This notion of the mass of the body is not precise because the concept "quantity of matter" is itself not well-defined. Therefore, we define the mass of a body in two different ways depending upon the method of its measurement.

(*i*) We can say that **mass** *is a measure of *inertia of a body.* The more mass a body has, the harder is to change its state of motion; it is harder to start it moving from rest or to stop it when it is moving or to change its motion out of straight line. This fact directly follows from Newton's second law of motion. If a force F acts on a body of mass m_i and produces an acceleration 'a', then according to Newton's second law of motion,

$$F = m_i a \quad \text{or} \quad m_i = F / a \qquad ...(i)$$

The mass defined in this way is called **inertial mass** (m_i) because it makes use of inertia of the body. Eq. (*i*) tells us that more mass a body has, the larger the force necessary to give it a given acceleration and vice-versa.

When a body is in linear motion, the ratio of resultant force applied on the body to the acceleration produced in it is called **inertial mass** (m_i) *of the body.*

The inertial mass of a body can be measured by *inertial balance.*

(*ii*) Another way to measure the mass of a body is by a physical balance. In this method, we place a 1 kg (say) standard mass in one of the pans and claim any mass placed in the other pan that causes the pans to balance precisely is by definition also 1 kg. This method depends upon the fact that the balance of the two pans of the equal-arm balance occurs when the gravitational pull on the two masses is equal. Because of this, mass defined in this way is often called **gravitational mass** (m_G).

When the mass of a body is measured under the effect of gravity and the body is not in motion, it is called **gravitational mass** (m_G).

Now it is not at all obvious that the inertial mass of a body should be equal to its gravitational mass. However, careful experiments show that they are equal. Therefore, we need not maintain the distinction between m_G and m_i and can put $m_G = m_i$.

Note. If the gravitational and inertial masses were not equal, then the conclusion that all bodies fall to earth with the same acceleration in the absence of air resistance would not be valid. For suppose a heavier object had the same inertia as a lighter body. Then the heavier body would accelerate faster than the lighter one since $a = F/m$ *i.e.* F would be greater for the heavier body but m would

* The tendency of a body to maintain its state of rest or of uniform motion along a straight line is called inertia.

be the same as for the lighter body. Since the inertial mass is equal to the gravitational mass, the heavier body — inspite of greater force on it has greater inertia than the lighter one in exact proportion so its acceleration is the same.

3.6. MEASUREMENT OF INERTIAL MASS

The inertial mass of a body can be determined with the help of inertial balance.

Principle. It is based on the principle that when a long metal strip fixed at one end and carrying an object at the other end is displaced horizontally, it executes vibratory motion. The time period of vibrations depends upon the mass of the object.

Theory. Fig. 3.10 shows the essential parts of inertial balance. It consists of two long metal strips. One end of each strip is clamped to a wooden board in such a way that the flat faces of the strip are vertical; this arrangement permits easy horizontal vibrations. The other ends of the strip support a pan. The object whose inertial mass is to be determined is placed in the pan. When the pan is displaced *horizontally, it executes vibratory motion. It is found that time period is directly proportional to the square root of the mass of the pan and the object *i.e.* $T \propto \sqrt{m+\Delta m}$ where m is the mass of the object and Δm is the mass of the pan.

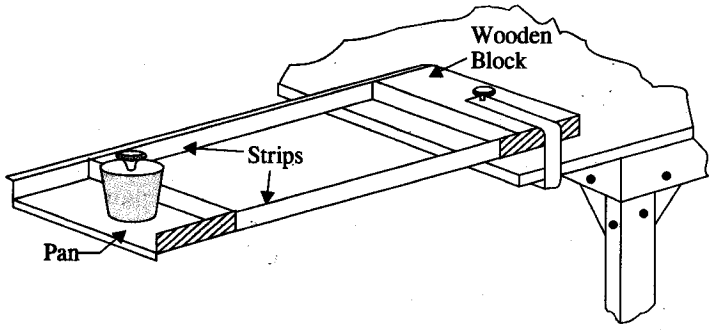

Fig. 3.10

If m_1 and m_2 are the masses of the two objects and T_1 and T_2 are the corresponding time periods of vibration, then,

$$\frac{m_1+\Delta m}{m_2+\Delta m}=\frac{T_1^2}{T_2^2} \quad \text{where } \Delta m = \text{mass of the pan}$$

Generally, the value of one of the masses (say m_1) is known. Therefore, the inertial mass m_2 of the object can be determined from the above relation.

3.7. MEASUREMENT OF GRAVITATIONAL MASS

A physical balance is used to measure the gravitational mass of a body by comparing it with a known standard mass.

Construction. Fig. 3.11 shows the various parts of a physical balance. It consists of a rigid light beam of aluminium or brass. The beam is balanced at its centre on a knife edge which rests on a flat piece. The flat piece is fixed on the top of a vertical pillar. Two scale pans of equal masses are suspended at the two ends of the beam. A pointer is attached at the centre of the beam which moves on a scale fixed at the base of the pillar. When the physical balance is in equilibrium, the pointer remains either at the zero of the scale or it swings equally on both sides of the zero mark. The ends of the pointer are provided with screws for zero mark adjustment. A lever arrangement is provided at the base for raising or lowering the beam. Levelling screws are provided at the base board to make it horizontal. A plumb line is suspended near the vertical pillar to ascertain whether the base board is horizontal or not.

* Due to horizontal vibratory motion, the effect of gravity is absent.

Procedure. In order to find the mass of a body with a physical balance, the following procedure is adopted :

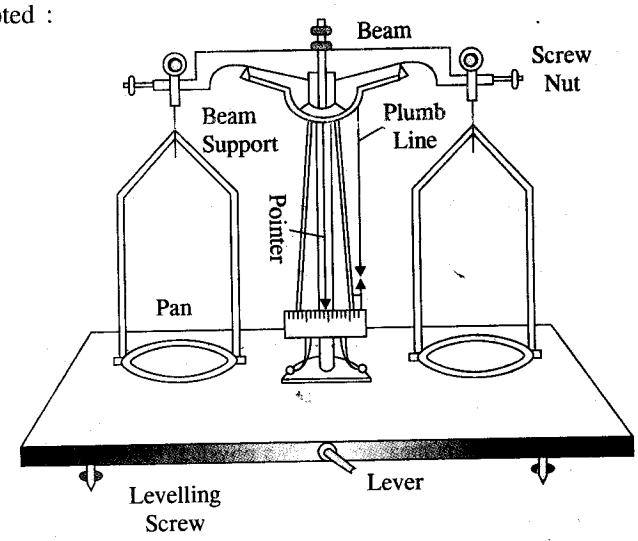

Fig. 3.11

(*i*) The first step is to make the base board horizontal. This can be done by bringing the plumb line just above the pointed mark through the adjustment of levelling screws fixed at the base board.

(*ii*) The next step is to ensure that the beam is horizontal when the pans are empty. For this purpose, the beam is raised gently by the lever and it is ensured that the pointer swings equally on both sides of the zero mark. This can be achieved through the adjustment of screws provided at the ends of the beam.

(*iii*) The beam is now lowered and the body whose mass *(m)* is to be found out is placed on the left pan of the balance. The known weights are placed on the right pan till the pointer remains at zero of the scale or it swings equally on both sides of the zero mark. Under this condition, the balance is in equilibrium. Consequently, the total mass (*M*) of weights on the right pan gives the mass of the body. It is because weight of the object is equal to weight on the right pan *i.e.*

$$m\,g = M\,g \quad \text{or} \quad m = M$$

Two things are worth noting. First, a physical balance measures the mass of a body at a place and not its weight. Secondly, a physical balance is used for comparing masses of the bodies.

Weight. *The* **weight** *of a body is defined as the gravitational force exerted on it due to earth i.e.,*

$$\text{Weight, } W = \text{gravitational force on the body due to earth}$$
$$= \text{mass of body} \times \text{acceleration due to gravity}$$
$$= m \times g$$
$$\therefore \quad W = m\,g$$

Thus, if a body has a mass of 10 kg, then its weight is $W = m\,g = 10 \times 9.8 = 98$ N. It means that earth is attracting the body with a force of 98 N. The reader may note that mass of an object is different from its weight.

Difference between mass and weight

Following are the differences between mass and weight :

(*i*) The mass of a body is the same whether it is on the earth, on the moon or somewhere in space. But the weight of body (= *m g*) depends upon acceleration due to gravity. For example, on the moon, an object would have the same mass as on earth but it would weigh about one-sixth of its weight on earth as acceleration due to gravity on the moon is about *g*/6.

(*ii*) The mass has magnitude only whereas weight has magnitude as well as direction. We say that *mass is a scalar quantity and weight is a vector quantity.* The direction of weight is always vertically downward.

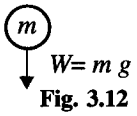

Fig. 3.12

(*iii*) The mass is a measure of body's inertia *i.e.* it is a measure of body's resistance to changes in motion. However, weight of a body is the gravitational force.

(*iv*) The SI unit of mass is kilogram (kg), whereas that of weight is newton (N).

Note. Suppose the mass of a body is measured with a physical balance on earth and on the moon. Will the two masses be equal?

3.8. SPRING BALANCE

A spring balance is used to measure the weight of a body. The working of spring balance is based on Hooke's law stated below :

The extension produced in a spring is directly proportional to the applied force provided the spring is not loaded beyond elastic limit i.e.

$$F = k\,x$$

where F = force applied to the spring

x = extension produced in the spring

k = spring constant

Suppose a spring has a spring constant k = 200 N/m. It means a force of 2 N applied to the spring will produce an extension of 1 cm. An applied force of 4 N will cause an extension of 2 cm.

Fig. 3.13 shows the various parts of a spring balance. It consists of steel helical spring enclosed in a metal case having a slit along its length. The upper end of the spring is attached to a ring provided at the top of the case. The lower end of the spring is attached to a pan on which the body whose weight is to be determined is placed. A pointer *P* attached to the spring moves in the slit and rests against a scale graduated in newtons. The marks on the scale are equally spaced because of Hooke's law.

When there is no load on the pan, the pointer reads zero. When the body to be weighed is placed on the pan, the spring is stretched due to the weight of the body. When the object hangs at rest, the upward force due to the spring balances the downward force due to gravitational attraction of earth (*i.e.* weight of the body). Therefore, the position of the pointer on the scale gives the weight of the body.

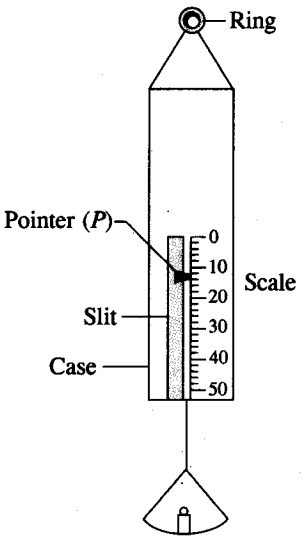

Fig. 3.13

Note. The weight of a body is not constant at all places because the value of *g* (acceleration due to gravity) is slightly different at different places. However, the difference is so small that for all practical purposes, we take g = 9.8 ms^{-2}. Therefore, a body of mass m kg has weight, $W = mg$ = 9.8 m newtons.

3.9. MEASUREMENT OF TIME INTERVALS

In scientific work, we want to know how long an event lasts. It is called time interval. Thus time interval is tagged with two questions : "At what time does an event occur?" and "how long does the event last ?" The question arises how to measure time interval. Any phenomenon that repeats itself in a regular fashion serves as a clock for the measurement of time.

Thus **clock** *is a device that counts the number of repetitions (including the fractions) of some periodic event.*

For example, we can measure time intervals by counting the number of swings of a pendulum or the number of cycles of an alternating current. Our general concept of clock will not depend on the details of a mechanical or electrical device but on the operational definition of a device that counts the number of repetitions of some periodic event. The instruments commonly used for measuring short time intervals are spring-driven stopwatches or electric stopclocks. Other methods for measuring time intervals are *pendulums of suitable lengths, oscillating quartz crystals and oscillating atoms. The SI unit of time is *second.* There are 60 seconds in a *minute,* 60 minutes in an *hour* and 24 hours in a *day.*

We now briefly discuss the various techniques to measure time.

(*i*) **Electrical oscillators.** The mains a.c. supply has a frequency of 50 Hz. The rotations of synchronous motor run on a.c. supply can be used to obtain a time scale.

(*ii*) **Electronic oscillators.** An electronic oscillator uses a transistor and other auxiliary circuitry to produce electromagnetic waves of very high frequencies *i.e.,* of very small time period. These oscillations can be used to measure small time intervals.

(*iii*) **Quartz-crystal clocks.** A quartz crystal shows *piezo electric effect i.e.* when fluctuating pressure is applied across one pair of faces of the crystal, an oscillatory e.m.f. is developed across another pair of perpendicular faces and vice-versa. The oscillations so produced can be used to measure time intervals.

(*iv*) **Atomic clock.** Such clocks make use of periodic vibrations taking place within the atom. These are usually cesium –133 clocks and can have time intervals of the order of picosecond (10^{-12} s).

(*v*) **Decay of elementary particles.** The life times of many unstable elementary particles range form 10^{-16} sec to 10^{-24} sec. By studying the decay of elementary particles, very small time intervals can be measured.

(*vi*) **Radioactive dating.** The radioactive dating technique is used to measure long intervals of time of the order of 10^{17} seconds. Carbon dating is used to determine the age of fossils while uranium dating is used to determine the age of a rock or earth.

CONCEPTUAL QUESTIONS

Q.1. What is triangulation method of measuring distances?

Ans. This method is based on the geometrical properties of a triangle.

Q.2. What is the principle of measuring very large distances?

Ans. In order to measure very large distances such as the distance of moon from earth or the distance of a star from earth, we make angular measurements.

Q.3. What is the working principle of RADAR?

Ans. The RADAR is based on the reflection of radiowaves from an aeroplane or distant astronomical object. Note that RADAR stands for RAdio Detection And Ranging.

Q.4. How many atoms are there in 108 g of silver?

Ans. There are 6.02×10^{23} atoms (Avogadro's number) in 108 g of silver.

* Suppose we have a pendulum that makes 3600 oscillations in one hour. Then we shall call the time period of pendulum as one second.

VERY SHORT ANSWER QUESTIONS

Q.1. What do you mean by order of magnitude of a physical quantity?

Ans. The actual value of a physical quantity rounded to the nearest power of 10 is called order of magnitude of that physical quantity.

Q.2. What does RADAR mean?

Ans. RADAR is an abbreviation for RAdio Detection And Ranging.

Q.3. What does LASER stand for?

Ans. LASER is an abbreviation for Light Amplification Stimulated Emission Radiation.

Q.4. What does SONAR stand for?

Ans. SONAR is an abbreviation for SOund Navigation And Ranging.

Q.5. How do you find the order of magnitude of a physical quantity?

Ans. If a given number is less than 5, it is taken as 1. If it lies between 5 and 10, it is taken as 10.

Q.6. What is the order of magnitude of velocity of light in vacuum?

Ans. The velocity of light in vacuum is 3×10^8 ms^{-1}. Here 3 is regarded as 1 so that order of magnitude of velocity of light is 10^8 ms^{-1}.

Q.7. Name two types of mass.

Ans. Inertial mass and gravitational mass.

Q.8. Are inertial and gravitational mass of an object different from one another?

Ans. No. Inertial mass and gravitational mass of an object are the same.

Q.9. What is the smallest mass measured indirectly so far?

Ans. The smallest mass measured so far is that of an electron ($\simeq 10^{-30}$ kg).

Q.10. What is the shortest distance measured indirectly so far?

Ans. The shortest distance measured indirectly so far is the radius of a proton ($\simeq 10^{-15}$m).

Q.11. What is the shortest time interval measured indirectly so far?

Ans. The shortest time interval measured indirectly so far is the time taken by light to cross a distance of nuclear size ($\simeq 10^{-22}$s).

Q.12. What is the mass of a body that weighs 1 N at a place where g = 9.8 ms^{-2}?

Ans. Weight = mg or $1 = m \times 9.8$ $\therefore$ $m = 1/9.8$ kg

Q.13. What is meant by angular diameter of moon?

Ans. The angular diameter of moon is the angle subtended at a point on earth by two diametrically opposite ends of the moon. Its value is about 0.5°.

Q.14. Which is the most accurate clock?

Ans. A cesium clock is most accurate. Two cesium clocks may differ by only 1 s after running for 5000 years.

Q.15. What is the order of human life expectancy?

Ans. The human life expectancy is about 100 years which is of the order of 10^9 s.

SHORT ANSWER QUESTIONS

Q.1. For a given base line, will a distant star show greater parallax or a near star?

Ans. Parallactic angle, $\theta = b/S$

For a given base b, θ is inversely proportional to the distance (S) of the star. Therefore, the near star will show greater parallax.

3.9. MEASUREMENT OF TIME INTERVALS

In scientific work, we want to know how long an event lasts. It is called time interval. Thus time interval is tagged with two questions : "At what time does an event occur?" and "how long does the event last ?" The question arises how to measure time interval. Any phenomenon that repeats itself in a regular fashion serves as a clock for the measurement of time.

Thus **clock** *is a device that counts the number of repetitions (including the fractions) of some periodic event.*

For example, we can measure time intervals by counting the number of swings of a pendulum or the number of cycles of an alternating current. Our general concept of clock will not depend on the details of a mechanical or electrical device but on the operational definition of a device that counts the number of repetitions of some periodic event. The instruments commonly used for measuring short time intervals are spring-driven stopwatches or electric stopclocks. Other methods for measuring time intervals are *pendulums of suitable lengths, oscillating quartz crystals and oscillating atoms. The SI unit of time is *second.* There are 60 seconds in a *minute,* 60 minutes in an *hour* and 24 hours in a *day.*

We now briefly discuss the various techniques to measure time.

(*i*) **Electrical oscillators.** The mains a.c. supply has a frequency of 50 Hz. The rotations of synchronous motor run on a.c. supply can be used to obtain a time scale.

(*ii*) **Electronic oscillators.** An electronic oscillator uses a transistor and other auxiliary circuitry to produce electromagnetic waves of very high frequencies *i.e.,* of very small time period. These oscillations can be used to measure small time intervals.

(*iii*) **Quartz-crystal clocks.** A quartz crystal shows *piezo electric effect i.e.* when fluctuating pressure is applied across one pair of faces of the crystal, an oscillatory e.m.f. is developed across another pair of perpendicular faces and vice-versa. The oscillations so produced can be used to measure time intervals.

(*iv*) **Atomic clock.** Such clocks make use of periodic vibrations taking place within the atom. These are usually cesium –133 clocks and can have time intervals of the order of picosecond (10^{-12} s).

(*v*) **Decay of elementary particles.** The life times of many unstable elementary particles range form 10^{-16} sec to 10^{-24} sec. By studying the decay of elementary particles, very small time intervals can be measured.

(*vi*) **Radioactive dating.** The radioactive dating technique is used to measure long intervals of time of the order of 10^{17} seconds. Carbon dating is used to determine the age of fossils while uranium dating is used to determine the age of a rock or earth.

CONCEPTUAL QUESTIONS

Q.1. What is triangulation method of measuring distances?

Ans. This method is based on the geometrical properties of a triangle.

Q.2. What is the principle of measuring very large distances?

Ans. In order to measure very large distances such as the distance of moon from earth or the distance of a star from earth, we make angular measurements.

Q.3. What is the working principle of RADAR?

Ans. The RADAR is based on the reflection of radiowaves from an aeroplane or distant astronomical object. Note that RADAR stands for RAdio Detection And Ranging.

Q.4. How many atoms are there in 108 g of silver?

Ans. There are 6.02×10^{23} atoms (Avogadro's number) in 108 g of silver.

* Suppose we have a pendulum that makes 3600 oscillations in one hour. Then we shall call the time period of pendulum as one second.

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VERY SHORT ANSWER QUESTIONS

Q.1. What do you mean by order of magnitude of a physical quantity?

Ans. The actual value of a physical quantity rounded to the nearest power of 10 is called order of magnitude of that physical quantity.

Q.2. What does RADAR mean?

Ans. RADAR is an abbreviation for RAdio Detection And Ranging.

Q.3. What does LASER stand for?

Ans. LASER is an abbreviation for Light Amplification Stimulated Emission Radiation.

Q.4. What does SONAR stand for?

Ans. SONAR is an abbreviation for SOund Navigation And Ranging.

Q.5. How do you find the order of magnitude of a physical quantity?

Ans. If a given number is less than 5, it is taken as 1. If it lies between 5 and 10, it is taken as 10.

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Ans. Parallactic angle, $\theta = b/S$

For a given base b, θ is inversely proportional to the distance (S) of the star. Therefore, the near star will show greater parallax.

Q.2. Why parallax method cannot be used for measuring distances of stars more than 100 light years away?

Ans. When a star is more than 100 light years away, the parallax angle is so small that it cannot be measured accurately.

Q.3. What is the difference between mass and weight?

Ans. Refer to Art. 3.7.

Q.4. What is the basic principle of alpha particle scattering method for estimating the size of the nucleus?

Ans. When an alpha particle (positively charged) is projected towards the nucleus, it experiences a repulsive force from the positively charged nucleus. Consequently, α-particle starts slowing down. As the α particle starts slowing down, its kinetic energy starts converting into electric potential energy of the system. At the distance of closest approach (r_0), the entire K.E. of α particle is converted into electric potential energy of the system. The value of r_0 gives us the order of the size of the nucleus.

Q.5. Suggest a distance corresponding to each of the following orders of length: (*i*). 10^7 m (*ii*) 10^4 m (*iii*) 10^3 m (*iv*) 10^2 m (*v*) 10^{-3} m (*vi*) 10^{-6} m (*vii*) 10^{-14} m.

Ans. (*i*) 10^7 m : radius of earth

(*ii*) 10^4 m : height of Mount Everest

(*iii*) 10^3 m : distance travelled by sound in water in 1 s

(*iv*) 10^2 m : length of a college playground

(*v*) 10^{-3} m : thickness of cardboard

(*vi*) 10^{-6} m : mean free path of an air molecule

(*vii*) 10^{-14} m : size of atomic nucleus

Q.6. How will you measure the width of a river?

Ans. Refer to Art. 3.3.

Q.7. When circular scale of a screw gauge carrying 100 divisions is given four complete rotations, the head of the screw moves through 2 mm. Calculate the pitch and least count of the screw gauge.

Ans. No. of divisions on circular scale = 100 ; No. of complete rotations = 4 ; Distance moved = 2 mm.

$$\text{Pitch} = \frac{\text{Distance moved}}{\text{No. of complete rotations}} = \frac{2\,\text{mm}}{4} = 0.5\ \text{mm}$$

$$\text{Least count} = \frac{\text{Pitch}}{\text{No. of divisions on circular scale}} = \frac{0.5\,\text{mm}}{100} = 0.005\ \text{mm}$$

Q.8. What is meant by order of magnitude? Illustrate with examples.

Ans. Refer to Art. 3.1.

Q.9. Mention some repetitive phenomena in nature which could serve as the time standards. Which one is most suitable?

Ans. The following repetitive phenomena occurring in nature can serve as standard for measuring time :

(*i*) Beating of human heart (*ii*) Rotation of earth about its axis

(*iii*) Oscillations of a simple pendulum (*iv*) Vibrations of atoms

Earlier, the rotation of earth was found to be the most appropriate. Now, the vibrations of an atom (cesium–133 atom) is being used to measure time.

Q.10. What is the basic difference between inertial mass and gravitational mass?

Ans. The inertial mass of a body is a measure of its inertia. It is equal to the force required to produce unit acceleration in the body. On the other hand, gravitational mass of a body is the force experienced by the body in a gravitational field of unit intensity. It has been found experimentally that these two masses of a body are equal.

LONG ANSWER QUESTIONS

1. Explain parallax method for measuring the distance of a nearby star. **[Refer to Art. 3.3]**
2. How will you measure large distances by (*i*) SONAR and (*ii*) RADAR? **[Refer to Art. 3.3]**
3. How will you measure the inertial mass of a body? **[Refer to Art. 3.6]**
4. How will you measure the gravitational mass of a body? **[Refer to Art. 3.7]**
5. How will you find the radius of an atom making use of Avogadro's number? **[Refer to Art. 3.4]**
6. How will you find the radius of a nucleus by α-scattering experiment? **[Refer to Art. 3.4]**

MCQ FOR COMPETITIVE EXAMINATIONS

1. The speed of light in vacuum is 3×10^8 ms^{-1}. The order of magnitude is
   (*a*) 10^7 ms^{-1} (*b*) 10^9 ms^{-1}
   (*c*) 10^8 ms^{-1} (*d*) none of the above
2. The order of a magnitude of a physical quantity is correct within a factor of
   (*a*) 10 (*b*) 1
   (*c*) 2 (*d*) none of the above
3. The speed of radio waves is
   (*a*) 3×10^8 ms^{-1} (*b*) 1500 ms^{-1}
   (*c*) 10^7 ms^{-1} (*d*) 5×10^6 ms^{-1}
4. 1 A.U. is equal to
   (*a*) 3×10^8 m (*b*) 10^{10} m
   (*c*) 1.5×10^{11} m (*d*) 10^7 m
5. A spring balance is used to measure
   (*a*) the weight of a body
   (*b*) the mass of a body
   (*c*) both mass and weight of a body
   (*d*) none of the above
6. 23 g of sodium contains
   (*a*) 6.02×10^{23} atoms (*b*) 10^7 atoms
   (*c*) 10^{11} atoms (*d*) 5×10^6 atoms
7. The atomic size is of the order of
   (*a*) 10^{-2} m (*b*) 10^{-10} m
   (*c*) 10^{-15} m (*d*) 10^{-20} m
8. One mole of any gas at S.T.P. occupies
   (*a*) 22.4×10^{-3} m^3 (*b*) 10^{-4} m^3
   (*c*) 11.2×10^{-2} m^3 (*d*) 8×10^{-6} m^3
9. The inertial mass of a body is 1.5 kg. Its gravitational mass is
   (*a*) 3 kg (*b*) 15 kg
   (*c*) 4.5 kg (*d*) 1.5 kg
10. One object is weighed on moon and the other on the earth. If their weights are equal, then their masses will be
    (*a*) equal (*b*) unequal
    (*c*) may be equal or unequal
    (*d*) none of the above

ANSWERS TO MCQ FOR COMPETITIVE EXAMINATIONS

1. (*c*)	**2.** (*a*)	**3.** (*a*)	**4.** (*c*)	**5.** (*a*)
6. (*a*)	**7.** (*b*)	**8.** (*a*)	**9.** (*d*)	**10.** (*b*)

4

Dimensions and Dimensional Analysis

INTRODUCTION

By the dimensions of a physical quantity in mechanics we mean the way it is related to the fundamental quantities mass, length and time; these are usually denoted by [M], [L] and [T] respectively. We use square brackets round the letter to indicate dimensions. For example, area is obtained by multiplying one length by another (terms such as breadth, width, distance, radius are merely convenient ways of saying length) and therefore dimensions of area are those of length squared. In other words, the dimensions of area are 2 in length. We write it as [Area] = $[L^2]$. As area has the dimensions of $[L^2]$, its SI unit is m^2. Similarly, the dimensions of volume are $[L^3]$ and hence its SI unit is m^3. The salient feature of dimensional analysis is that the formula of a physical quantity may be different in different cases but the dimensions remain the same. For example, the area of a triangle of base b and height h is $A = bh/2$ whereas the area of a circle of radius r is $A = \pi r^2$. The formulas are different in the two cases but the dimensions in the two cases are the same *i.e.*, $[L^2]$. In this chapter, we shall discuss the importance of dimensional analysis.

4.1. DIMENSIONS OF PHYSICAL QUANTITIES

When we are speaking of dimensions of a physical quantity in mechanics, we are referring to the type of fundamental quantities (mass, length and time) that make it up. The dimensions of a physical quantity can be determined from its definition. As an example, velocity is defined as displacement divided by time *i.e.*,

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}}$$

$$\therefore \quad \text{Dimensions of velocity} = \frac{\text{Length dimensions}}{\text{Time dimensions}}$$

In order to write this dimensional equation in compact form, we use square bracket [ ]. The use of square bracket indicates that we are talking only about dimensions and not the actual values. In this notation, the above dimensional equation becomes :

$$[\text{Velocity}] = \frac{[L]}{[T]} = [L]\,[T^{-1}] = [M^0LT^{-1}]$$

$$\therefore \quad [\text{Velocity}] = [M^0LT^{-1}]$$

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In order to write this dimensional equation in compact form, we use square bracket [ ]. The use of square bracket indicates that we are talking only about dimensions and not the actual values. In this notation, the above dimensional equation becomes :

$$[\text{Velocity}] = \frac{[L]}{[T]} = [L][T^{-1}] = [M^0LT^{-1}]$$

$$\therefore \quad [\text{Velocity}] = [M^0LT^{-1}]$$

We say that dimensions of velocity are *zero* in mass, 1 in length and –1 in time.

Hence the **dimensions** *of a physical quantity may be defined as the powers to which the fundamental quantities (M, L and T) must be raised to represent the physical quantity.*

The fundamental quantity which does not occur in the physical quantity is represented by raised to the power zero. Thus, the dimensions of velocity above are independent of mass and hence zero dimensions in mass (*i.e.*, M^0). The dimensions of a physical quantity can be determined from the formula of that quantity.

Discussion. The following points are worth noting about the dimensions of a physical quantity:

(*i*) The formula of a physical quantity may be different in different cases, but the dimensions remain the same. For example, the formulas for the area of a square and that of circle are different but dimensions in both cases are the same *i.e.*, $[L^2]$.

(*ii*) The dimensions of a physical quantity are independent of the system of units used.

(*iii*) If we know the dimensions of a physical quantity, we can at once write its unit. For example, if some physical quantity has the dimensions $[M\ L^{-1}\ T^{-1}]$, then its SI unit will be kg $m^{-1}s^{-1}$.

Examples (*i*) $$\text{Acceleration} = \frac{\text{Velocity}}{\text{Time}}$$

$$\therefore \quad \text{Dimensions of acceleration} = \frac{\text{Velocity dimensions}}{\text{Time dimensions}}$$

or $$[a] = \frac{[M^0LT^{-1}]}{[T]} = [M^0LT^{-2}]$$

Hence, the dimensions of acceleration are zero in mass, 1 in length and –2 in time. Clearly, the SI unit of acceleration is m s^{-2}.

(*ii*) $$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$\therefore \quad \text{Dimensions of force} = \text{Mass dimensions} \times \text{Acceleration dimensions}$$

or $$[F] = [M][LT^{-2}] = [M\ LT^{-2}]$$

Hence, the dimensions of force are 1 in mass, 1 in length and –2 in time. Obviously, the SI unit of force is kg m s^{-2}. This unit has been given the special name *viz.* newton.

4.2. DIMENSIONAL FORMULA

When we express a physical quantity in terms of its dimensions, it is called **dimensional formula** *of that quantity.* For example, the dimensional formula of force is

$$[F] = [M\ LT^{-2}]$$

Two things are quite apparent from the dimensional formula of force.

(*i*) The unit of force depends upon the units of mass, length and time.

(*ii*) The unit of force varies directly as the units of mass and length and inversely as the square of time.

4.3. DIMENSIONAL EQUATION

If in an equation containing physical quantities, each quantity is represented by its dimensional formula, the resulting equation is known as **dimensional equation.**

Consider the formula, $v = u + a\ t$

Here, u is the initial velocity of the body, a is acceleration and t is the time taken to attain the final velocity v. Writing this formula in the dimensional form, we have,

$$[M^0LT^{-1}] = [M^0LT^{-1}] + [M^0LT^{-2}][M^0L^0T] \qquad ...(i)$$

Eq. (*i*) is known as *dimensional equation*. Note that dimensions in every term of the formula must be the same. It is not surprising because we can *add or equate like quantities. Thus in the above formula, the three quantities (*viz. v, u* and *at*) have the same dimensions *i.e.*, $[M^0LT^{-1}]$

4.4. DIMENSIONAL FORMULAS OF PHYSICAL QUANTITIES

Dimensional formula of a physical quantity can be easily written if we know its relation with other physical quantities. Dimensional formulas of a few physical quantities encountered in mechanics are given below in tabular form :

S. No.	Physical Quantity	Relation with other physical quantities	Dimensional Formula	SI Unit
1.	**Area**	length × breadth	$[L][L] = [L^2]$	m^2
2.	**Volume**	length × breadth × height	$[L][L][L] = [L^3]$	m^3
3.	**Density**	$\frac{\text{mass}}{\text{volume}}$	$\frac{[M]}{[L^3]} = [ML^{-3}T^0]$	$kg\ m^{-3}$
4.	**Speed or Velocity**	$\frac{\text{distance}}{\text{time}}$	$\frac{[L]}{[T]} = [M^0LT^{-1}]$	ms^{-1}
5.	**Acceleration**	$\frac{\text{Velocity}}{\text{time}}$	$\frac{[M^0LT^{-1}]}{[T]} = [M^0LT^{-2}]$	ms^{-2}
6.	**Force**	mass × acceleration	$[M][M^0LT^{-2}] = [MLT^{-2}]$	N
7.	**Momentum**	mass × velocity	$[M][M^0LT^{-1}] = [MLT^{-1}]$	$kg\ ms^{-1}$
8.	**Work**	force × distance	$[MLT^{-2}][L] = [ML^2T^{-2}]$	Nm
9.	**Power**	$\frac{\text{work}}{\text{time}}$	$\frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$	W
10.	**Pressure**	$\frac{\text{force}}{\text{area}}$	$\frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$	Nm^{-2}
11.	**Kinetic energy**	$\frac{1}{2}$ × mass × (velocity)^2	$[M][M^0LT^{-1}]^2 = [ML^2T^{-2}]$	Nm
12.	**Potential energy**	mass × g × distance	$[M][M^0LT^{-2}][L] = [ML^2T^{-2}]$	Nm
13.	**Impulse**	force × time	$[MLT^{-2}][T] = [MLT^{-1}]$	Ns
14.	**Torque**	force × distance	$[MLT^{-2}][L] = [ML^2T^{-2}]$	Nm
15.	**Stress**	$\frac{\text{force}}{\text{area}}$	$\frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$	Nm^{-2}
16.	**Strain**	$\frac{\text{extension in length}}{\text{original length}}$	$\frac{[L]}{[L]} = [M^0L^0T^0]$	Number
17.	**Elasticity**	$\frac{\text{stress}}{\text{strain}}$	$\frac{[ML^{-1}T^{-2}]}{[M^0L^0T^0]} = [ML^{-1}T^{-2}]$	Nm^{-2}

* Suppose three quantities *A*, *B* and C are related as :

$$A = B + C$$

We can add *B* and C if they are like quantities *i.e.*, have the same dimensions. Similarly the sum of *B* and *C* can be equated to *A* if they are like quantities.

18.	**Surface tension**	$\frac{\text{force}}{\text{length}}$	$\frac{[MLT^{-2}]}{[L]}=[ML^0T^{-2}]$	Nm^{-1}
19.	**Force constant of spring**	$\frac{\text{applied force}}{\text{extension in length}}$	$\frac{[MLT^{-2}]}{[L]}=[ML^0T^{-2}]$	Nm^{-1}
20.	**Gravitational constant**	$\frac{\text{force} \times (\text{distance})^2}{(\text{mass})^2}$	$\frac{[MLT^{-2}][L^2]}{[M^2]}=[M^{-1}L^3T^{-2}]$	$Nm^2\ kg^{-2}$
21.	**Frequency**	$\frac{1}{\text{time period}}$	$\frac{1}{[T]}=[M^0L^0T^{-1}]$	s^{-1}
22.	**Angle**	$\frac{\text{arc}}{\text{radius}}$	$\frac{[L]}{[L]}=[M^0L^0T^0]$	rad
23.	**Angular velocity**	$\frac{\text{angle}}{\text{time}}$	$\frac{[M^0L^0T^0]}{[T]}=[M^0L^0T^{-1}]$	rad s^{-1}
24.	**Angular acceleration**	$\frac{\text{angular velocity}}{\text{time}}$	$\frac{[M^0L^0T^{-1}]}{[T]}=[M^0L^0T^{-2}]$	rad s^{-2}
25.	**Moment of Inertia**	mass × $(\text{distance})^2$	$[M][L^2]=[ML^2T^0]$	kg m^2
26.	**Angular momentum**	momentum of inertia × angular velocity	$[ML^2T^0][M^0L^0T^{-1}]=[ML^2T^{-1}]$	kg m^2 s^{-1}
27.	**Heat**	energy	$[ML^2T^{-2}]$	J
28.	**Planck's constant**	$\frac{\text{energy}}{\text{frequency}}$	$\frac{[ML^2T^{-2}]}{[M^0L^0T^{-1}]}=[ML^2T^{-1}]$	Js
29.	**Velocity gradient**	$\frac{\text{change in velocity}}{\text{distance}}$	$\frac{[M^0LT^{-1}]}{[L]}=[M^0L^0T^{-1}]$	s^{-1}
30.	**Radius of gyration**	distance	$[M^0LT^0]$	m

4.5. FOUR CATEGORIES OF PHYSICAL QUANTITIES

A careful study of the dimensional formulas of the physical quantities reveals that these can be divided into four categories *viz.*

(*i*) Dimensional variables (*ii*) Dimensional constants
(*iii*) Dimensionless variables (*iv*) Dimensionless constants

(*i*) **Dimensional variables.** *Those physical quantities which have dimensions but do not have fixed value are known as dimensional variables e.g.*, force, velocity, work, power *etc.* Thus one force may have a magnitude of 4N and the other may have a magnitude of 10N. Note that force has dimensions $[MLT^{-2}]$ but not a fixed value.

(*ii*) **Dimensional constants.** *Those physical quantities which possess dimensions and have fixed value are called dimensional constants e.g.*, gravitational constant, Planck's

constant, velocity of light *etc*. Thus velocity of light in vacuum is 3×10^8 ms^{-1}. Note that velocity of light has dimensions $[LT^{-1}]$ but has a fixed value in vacuum.

(*iii*) **Dimensionless variables.** *Those physical quantities which have neither dimensions nor fixed value are called dimensionless variables e.g.*, specific gravity, strain, angle *etc*. Thus specific gravity of water is different from that of mercury. Note that specific gravity has no dimensions $[M^0L^0T^0]$ *i.e.*, it is dimensionless. However, the specific gravity has different value for different substances.

(*iv*) **Dimensionless constants.** *Those quantities which do not possess dimensions but have fixed value are called dimensionless constants e.g.*, 1, 2, 3, π, e *etc.* Note that π has no dimensions but has a fixed value.

4.6. PRINCIPLE OF HOMOGENEITY OF DIMENSIONS

According to this principle, *a physical relation is dimensionally correct if the dimensions of fundamental quantities (mass, length and time) are the same in each and every term on either side of the equation.*

This principle is based on the fact that only quantities of the same kind (or dimensions) can be added or subtracted. For example, if we have an equation like $A = B + C$, the quantities A, B and C must have the same dimensions.

Illustration. Consider the following physical relation :

$$S = ut + \frac{1}{2}at^2$$

Here, S = displacement ; u = initial velocity ; a = uniform acceleration ; t = time interval.

At first sight, it may appear that dimensions of each term on both sides are not the same. But such a conclusion is wrong as discussed below.

$$\text{Dimensions of } 'S' = [M^0LT^0]$$

$$\text{Dimensions of } 'ut' = [M^0LT^{-1}][M^0L^0T] = [M^0LT^0]$$

$$\text{Dimensions of } 'at^2' = [M^0LT^{-2}][M^0L^0T^2] = [M^0LT^0]$$

$$\therefore \quad [M^0LT^0] = [M^0LT^0] + [M^0LT^0]$$

$$\text{or} \quad [M^0LT^0] = *[M^0LT^0]$$

Since dimensions on R.H.S. are the same as on L.H.S., the relation is dimensionally correct. Note that 1/2 is a pure number having no dimensions and is, therefore, ignored in the dimensional equation. If a physical relation is not dimensionally homogeneous (*i.e.*, the dimensions of each term on the two sides of the equation are not the same), then equation is incorrect. This means there is some error in the relation.

4.7. USES OF DIMENSIONAL EQUATIONS

The following are the uses of dimensional equations :

(*i*) To check the correctness of a physical relation. (*ii*) To recapitulate a forgotten formula. (*iii*) To derive relationship between different physical quantities. (*iv*) To convert one system of units to another. (*v*) To find the dimensions of constants in a given relation.

4.8. TO CHECK THE CORRECTNESS OF A PHYSICAL RELATION

To check the correctness of a physical relation, we apply the principle of homogeneity of dimensions *i.e.*, whether the dimensions of each term on both sides of the relation are same or not. If the dimensions of M, L and T are same on both the sides, then the relation is correct, otherwise not.

* Note that length on the L.H.S. of the equation is obtained by adding together two lengths on the R.H.S.

Illustration. Let us check the correctness of the relation :

$$t = 2\pi\sqrt{\frac{l}{g}}$$

Here, t = time period ; l = length of pendulum ; g = acceleration due to gravity

Dimensions of 't' = $[T] = [M^0L^0T]$; Dimensions of 'l' = $[L] = [M^0LT^0]$;

Dimensions of 'g' = $[LT^{-2}] = [M^0LT^{-2}]$

$$\therefore \quad [M^0L^0T] = \sqrt{\frac{[M^0LT^0]}{[M^0LT^{-2}]}} = [T] = [M^0L^0T]$$

Since the dimensions on the two sides of the relation are the same, the relation is dimensionally correct.

Note. The dimensional consistency is a necessary condition for a physical relation to be correct. However, it is not sufficient because of the following reasons :

(*i*) A physical relation may be dimensionally correct but dimensionless numerical factors (*e.g.*, 2π in the above case) could be wrong. Thus a dimensional check can tell you when a relationship is wrong ; it cannot tell that it is completely right.

(*ii*) A given relation may be dimensionally correct without describing any physical situation.

4.9. TO RECAPITULATE THE FORGOTTEN FORMULA

Suppose a person is sure of the physical quantities involved in a relation but is doubtful about the powers or a person is reading a book in which an equation has been written whose powers are not legible. In such situations, we can use the principle of homogeneity of dimensions to obtain the correct relation.

Illustration. Suppose a relation is written as $F = 4\pi^2 f m r$ where F = force; f = frequency ; m = mass ; r = radius.

A person wants to be sure of the powers of f, m and r. For this purpose, the principle of homogeneity of dimensions can be applied. Suppose f, m and r have the powers a, b and c respectively. By equating the dimensions of M, L and T of the two sides of the relation, we can find the values of a, b and c.

Dimensions of L.H.S. $= [F] = [MLT^{-2}]$

On the R.H.S., $4\pi^2$ has no dimensions and will not enter into the dimensional equation.

$\therefore$ Dimensions of R.H.S. $= [M]^b[L]^c[T^{-1}]^a$

For the relation to be correct, the dimensions of the two sides must be the same *i.e.*

$$[MLT^{-2}] = [M]^b[L]^c[T^{-1}]^a$$

Equating the powers of M, L and T on both sides, we have,

$$a = 2 \; ; \; b = 1 \; ; \; c = 1$$

$\therefore$ Correct relation is $F = 4\pi^2 f^2 m r$

4.10. DERIVATION OF FORMULAS

If we know the factors upon which a given physical quantity depends, we can find the relation between the physical quantity and those factors. Again, we shall use the principle of homogeneity of dimensions for this purpose.

Illustration. Suppose we want to derive the relation for the time period of a simple pendulum. Experiments show that time period of a simple pendulum is independent of the amplitude of oscillation provided it is small. Therefore, it may reasonably be supposed that for small oscillations, the time period of simple pendulum depends upon (*i*) mass m of the bob

(*ii*) length l of the string and (*iii*) acceleration due to gravity g. We can express the relation as :

$$T = k\, m^a l^b g^c \qquad ...(i)$$

where k is a dimensionless constant and a, b and c are the unknown numbers. If we equate the powers of M, L and T on both sides of eq. (*i*), we can find the values of a, b and c and hence the formula for the time period of a simple pendulum.

$$\therefore \quad [T] = [M]^a [L]^b [LT^{-2}]^c \text{ or } [M^0 L^0 T] = [M^a][L^{b+c}][T^{-2c}]$$

Equating the powers of M, L and T on both sides of equation, we have,

$$a = 0\,; \quad b + c = 0 \text{ and } -2\ c = 1$$

$$\therefore \qquad a = 0\,;\ c = -\frac{1}{2} \text{ and } b = \frac{1}{2}$$

Putting the values of a, b and c in eq. (*i*), we have,

$$T = k m^0\ l^{1/2} g^{-1/2} \text{ or } T = k\sqrt{\frac{l}{g}}$$

Dimensional analysis does not provide the value of k since it is a number. However, we can find the value of k by experiment or through mathematical analysis. It is found that $k = 2\pi$.

$$\therefore \qquad \text{Time period, } T = 2\pi\sqrt{\frac{l}{g}}$$

Although we have not been able to find the value of k through dimensional analysis, yet we have been able to relate the time period of simple pendulum to the major variables of the situation *viz.* l and g. Thus dimensional analysis can be used to find the relation between quantities when the mathematical analysis is too difficult.

4.11. TO CONVERT ONE SYSTEM OF UNITS TO ANOTHER

The magnitude of a physical quantity remains the same irrespective of the system of measurements. This fact enables us to convert one system of units into another.

Illustration. Suppose we have a physical quantity Q. Let n_1 and n_2 be numeric values of the physical quantity when measured in two systems having units of size u_1 and u_2 respectively. Then,

$$Q = n_1 u_1 \qquad \text{... on first system of units}$$

$$Q = n_2 u_2 \qquad \text{... on second system of units}$$

$$\text{Clearly,} \quad Q = n_1 u_1 = n_2 u_2$$

Suppose the physical quantity Q has the dimensional formula $M^a L^b T^c$. Let the fundamental units be M_1, L_1 and T_1 on the first system and M_2, L_2 and T_2 on the second system. Then,

$$Q = n_1\left[M_1^a L_1^b T_1^c\right] \qquad \text{... on first system of units}$$

$$Q = n_2\left[M_2^a\ L_2^b T_2^c\right] \qquad \text{... on second system of units}$$

$$\therefore \quad n_1\left[M_1^a L_1^b T_1^c\right] = n_2\left[M_2^a L_2^b T_2^c\right]$$

$$\therefore \qquad n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

Thus if we know the value of a physical quantity in one system of units, we can find its value in the other system of units.

Illustration. Suppose we want to convert 1 joule into ergs. Now joule and erg are the units of work in SI and C.G.S. systems respectively. The dimensional formula of work is ML^2T^{-2} *i.e.*,

$$[\text{Work}] = [ML^2T^{-2}]$$

Here $a = 1\ ;\ b = 2\ ;\ c = -2$

Now, $$*\frac{M_1}{M_2} = \frac{1\text{kg}}{1\text{g}} = \frac{1000}{1} = 10^3\ ;\ \frac{L_1}{L_2} = \frac{1\text{m}}{1\text{cm}} = \frac{100}{1} = 10^2\ ;\ \frac{T_1}{T_2} = \frac{1\text{s}}{1\text{s}} = 1$$

Now, $$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c = 1\,[10^3]^1\,[10^2]^2\,[1]^{-2} = 10^7$$

$\therefore$ $1\text{ J} = 10^7$ ergs

4.12. TO FIND DIMENSIONS OF CONSTANTS IN A RELATION

Sometimes a physical relation contains constants. In order to find the dimensions of these constants, we can use the principle of homogeneity of dimensions.

Illustration. The relation between pressure (P), volume (V) and temperature (T) of a gas is given by :

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

Here a and b are constants. It is desired to find the dimensions of a and b.

The quantity a/V^2 must represent a pressure since it is added to P.

$$\text{Dimensions of '}P\text{'} = \frac{[\text{Force}]}{[\text{Area}]} = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]\ ;\ \text{Dimensions of '}V^2\text{'} = [L^3]^2 = [L^6]$$

$$\therefore \quad \frac{[a]}{[L^6]} = [ML^{-1}T^{-2}] \quad \text{or} \quad [a] = [ML^5T^{-2}]$$

Hence, the dimensions of a are $[ML^5T^{-2}]$.

The constant b must represent a volume since it is subtracted from V. Therefore, $[b] = [L^3]$.

Hence the dimensions of b are $[L^3]$.

4.13. LIMITATIONS OF DIMENSIONAL ANALYSIS

Although dimensional analysis is very helpful in the science of measurements, it suffers from the following drawbacks:

(*i*) Dimensionless constants occurring in a physical formula cannot be determined by dimensional analysis. They have to be determined either by experimental or mathematical analysis.

(*ii*) Dimensional analysis cannot be used to derive relations involving trigonometrical and exponential functions.

(*iii*) Dimensional analysis cannot be used to derive the exact form of a physical relation if it consists of more than one term.

(*iv*) Dimensional analysis cannot be used to derive the exact form of a physical relation if the physical quantity depends upon more than three physical quantities (*viz.* M, L and T).

(*v*) Dimensional analysis does not indicate whether a physical quantity is a scalar or vector.

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* Note that in SI units, the units of M, L and T are 1 kg, 1 m and 1s respectively. However in C.G. S. system, the units of M, L and T are 1 g , 1 cm and 1 s respectively.

Example 4.1. **Check the dimensional consistency of the following relation :**

$$FS = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \qquad ...(i)$$

Here, F = force ; S = distance ; m = mass ; u = initial velocity ; v = final velocity.

Solution. The relation will be dimensionally consistent if the dimensions of each term are the same.

Dimensions of 'F' = $[MLT^{-2}]$; Dimensions of 'S' = $[L]$

Dimensions of 'u' or 'v' = $[LT^{-1}]$; Dimensions of 'm' = $[M]$

Since the constant 1/2 has no dimensions, it will not enter into the dimensional equation. Putting the dimensions of various physical quantities in eq. (i), we have,

$$[MLT^{-2}][L] = [M][LT^{-1}]^2 - [M][LT^{-1}]^2$$

or $$[ML^2T^{-2}] = [ML^2T^{-2}] - [ML^2T^{-2}]$$

or $$[ML^2T^{-2}] = *[ML^2T^{-2}]$$

Hence the given relation is dimensionally correct.

Example 4.2. **The equation of a wave is given by :**

$$y = r\sin\omega\left(\frac{x}{v} - k\pi\right)$$

where the symbols have their usual meanings. What are the dimensions of x and k?

Solution. Since $\omega\left(\frac{x}{v} - k\pi\right)$ is an angle, it is a dimensionless quantity. It means that $\omega x / v$ as well as $\omega k\pi$ is dimensionless.

$$\therefore \quad \frac{\omega x}{v} = [M^0L^0T^0] = 1 \quad \text{or} \quad [x] = \frac{[v]}{[\omega]} = \frac{[LT^{-1}]}{[T^{-1}]} = [L]$$

i.e., x has the dimensions of distance.

Also, $\omega k\pi = [M^0L^0T^0] = 1 \qquad \therefore \quad [k] = \left[\frac{1}{\omega}\right] = [T]$

i.e., k has the dimensions of time.

Example 4.3. **Find the value of 10 joules on a system which has 10 cm, 100 g and 30 s as the fundamental units.**

Solution. Joule is the SI unit of work.

S.I. units	New system of units
$n_1 = 10$ J	$n_2 = ?$
$M_1 = 1$ kg	$M_2 = 100$ g
$L_1 = 1$ m	$L_2 = 10$ cm
$T_1 = 1$ s	$T_2 = 30$ s

$$[\text{Work}] = [ML^2T^{-2}]$$

$$\therefore \quad a = 1 ; \quad b = 2 ; \quad c = -2$$

Now, $$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

* Note that this result is obtained by subtracting the quantities having the same dimensions.

$$= 10\left[\frac{1\text{kg}}{100\text{g}}\right]^1\left[\frac{1\text{m}}{10\text{cm}}\right]^2\left[\frac{1\text{s}}{30\text{s}}\right]^{-2}$$

$$= 10\left[\frac{1000\text{g}}{100\text{g}}\right]^1\left[\frac{100\text{cm}}{10\text{cm}}\right]^2\left[\frac{1\text{ s}}{30\text{ s}}\right]^{-2}$$

$$= 10[10]^1[10]^2\left[\frac{1}{30}\right]^{-2} = 9\times10^6 \text{ new units}$$

$\therefore$ $10\text{ J} = 9 \times 10^6$ **new units**

Example 4.4. The frequency *n* of vibration of a stretched string is a function of its tension *F*, the length *l* and the mass per unit length *m*. From the knowledge of dimensions, prove that,

$$n \propto \frac{1}{l}\sqrt{\frac{F}{m}}$$

Solution. We can express the frequency of vibration as

$$n \propto F^a l^b m^c \quad \text{or} \quad n = kF^a l^b m^c \qquad ...(i)$$

where k is a dimensionless constant and a, b and c are numbers we want to find.

The dimensions of frequency are T^{-1}, the dimensions of F are MLT^{-2}, the dimensions of length are L and that of mass per unit length (m) are ML^{-1}. Putting the dimensions of various physical quantities in eq. (i), we have,

$$\left[T^{-1}\right] = \left[MLT^{-2}\right]^a [L]^b \left[ML^{-1}\right]^c$$

or $$\left[T^{-1}\right] = \left[M^{a+c}\right]\left[L^{a+b-c}\right]\left[T^{-2a}\right]$$

Equating the powers of M, L and T on both sides, we have,

$a + c = 0$; $a + b - c = 0$ and $-2a = -1$

$\therefore$ $$a = \frac{1}{2}\ ;\ b = -1\ ;\ c = -\frac{1}{2}$$

Putting the values of a, b and c in eq. (i), we have,

$$n = k F^{1/2} l^{-1} m^{-1/2} = \frac{k}{l}\sqrt{\frac{F}{m}} \quad \therefore \quad n \propto \frac{1}{l}\sqrt{\frac{F}{m}}$$

Example 4.5. A wave is set up in a stretched string by plucking it. The velocity *v* of the wave depends upon tension *F* in the string, its length *l* and its mass *m*. Making use of knowledge of dimensions, prove that,

$$v \propto \sqrt{\frac{\textbf{Tension}}{\textbf{Mass per unit length}}}$$

Solution. The velocity v of the wave can be expressed as :

$$v \propto F^a l^b m^c \quad \text{or} \quad v = k F^a l^b m^c \qquad ...(i)$$

where k is a dimensionless constant and a, b and c are numbers we want to find.

The dimensions of velocity v, tension F, length l and mass m are LT^{-1}, MLT^{-2}, L and M respectively. Putting the dimensions of various physical quantities in eq. (i), we have,

$$[LT^{-1}] = [MLT^{-2}]^a [L]^b [M]^c$$

or $$[LT^{-1}] = [M^{a+c}]\,[L^{a+b}]\,[T^{-2a}]$$

Equating the powers of M, L and T on both sides, we have,

$$a + c = 0\ ;\ a + b = 1\ ;\ -2a = -1 \quad \therefore \quad a = \frac{1}{2}\ ;\ b = \frac{1}{2}\ ;\ c = -\frac{1}{2}$$

Putting the values of a, b and c in eq. (i), we have,

$$v = k F^{1/2} l^{1/2} m^{-1/2}$$

or
$$v = k\sqrt{\frac{Fl}{m}} = k\sqrt{\frac{F}{m/l}} = k\sqrt{\frac{\text{Tension}}{\text{Mass per unit length}}}$$

$\therefore$
$$v \propto \sqrt{\frac{Tension}{Mass\ per\ unit\ length}}$$

Example 4.6. The density ρ of earth is given by :

$$\rho = \frac{3g}{4\pi RG}$$

where R is the radius of earth and G is gravitational constant. Check the dimensional consistency of this relation.

Solution. Dimensions of L.H.S. $= \dfrac{[M]}{[L^3]} = [ML^{-3}]$

Dimensions of R.H.S. $= \dfrac{[LT^{-2}]}{[L]*[M^{-1}L^3T^{-2}]} = [ML^{-3}]$

Since the dimensions on the two sides of the equation are the same, the relation is dimensionally consistent.

Example 4.7. **Convert 1 km/hr into ms^{-1}.**

Solution. $n_2 = n_1\left[\dfrac{L_1}{L_2}\right]\left[\dfrac{T_1}{T_2}\right]^{-1} = 1\left[\dfrac{1\text{km}}{1\text{m}}\right]\left[\dfrac{1\text{hr}}{1\text{s}}\right]^{-1} = 1[10^3][3600]^{-1} = \dfrac{5}{18}\ \text{m s}^{-1}$

$\therefore$
$$1\text{km/hr} = \frac{5}{18}\text{ms}^{-1}$$

Alternatively, $1\text{km/hr} = \dfrac{1\text{km}}{1\text{hr}} = \dfrac{1000\text{ m}}{60\times 60\text{ s}} = \dfrac{5}{18}\ \text{ms}^{-1}$

Example 4.8. **Find the value of 60 joule/minute on a system which has 100 g, 100 cm and 1 minute as the fundamental units.**

Solution. 60 J/min. = 1 J/s = 1 watt

Now watt is the SI unit of power.

S.I. units	Units of new system
$n_1 = 1$ W	$n_2 = ?$
$M_1 = 1$ kg	$M_2 = 100$ g
$L_1 = 1$ m	$L_2 = 100$ cm
$T_1 = 1$ s	$T_2 = 1$ minute

$$[\text{Power}] = [ML^2T^{-3}] \quad \therefore \quad a = 1\ ;\ b = 2\ ;\ c = -3$$

* 3 and 4 π are dimensionless and $[G] = [M^{-1}L^3T^{-2}]$.

Now, $n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c = 1\left[\frac{1\text{kg}}{100\text{ g}}\right]^1\left[\frac{1\text{m}}{100\text{ cm}}\right]^2\left[\frac{1\text{s}}{1\text{min}}\right]^{-3}$

$= 1\left[\frac{1000\text{g}}{100\text{g}}\right]^1\left[\frac{100\text{cm}}{100\text{ cm}}\right]^2\left[\frac{1\text{s}}{60\text{s}}\right]^{-3} = 1\,[10]^1\,[1]^2\left[\frac{1}{60}\right]^{-3}$

$= 10\times60\times60\times60 = 2.16\times10^6$ new units

$\therefore$ 60 J/min = 2.16×10^6 new units

Example 4.9. **If the velocity of light, acceleration due to gravity and the normal pressure are choosen as the fundamental units, find the units of mass, length and time. Given that velocity of light, $c = 3 \times 10^8$ ms^{-1}; acceleration due to gravity, $g = 10$ ms^{-2} and normal atmospheric pressure, $P = 10^5$ N/m^2.**

Solution. $[c] = [LT^{-1}] = 3 \times 10^8$ m s^{-1} ..(*i*)

$[g] = [LT^{-2}] = 10$ m s^{-2} ...(*ii*)

$[P] = [ML^{-1}T^{-2}] = 10^5$ N/m^2

Dividing (*i*) by (*ii*), $\frac{LT^{-1}}{LT^{-2}} = \frac{3\times10^8}{10} = 3\times10^7$ $\therefore$ $T = 3 \times 10^7$ s

Now $LT^{-1} = 3\times10^8$ $\therefore$ $L = \frac{3\times10^8}{T^{-1}} = 3\times10^8 T = 3\times10^8\times3\times10^7 = \mathbf{9\times10^{15}\,m}$

Again, $ML^{-1}T^{-2} = 10^5$ $\therefore$ $M = \frac{10^5}{L^{-1}T^{-2}} = 10^5\,LT^2 = 10^5\times9\times10^{15}\times\left(3\times10^7\right)^2 = \mathbf{8.1\times10^{35}\,kg}$

PROBLEMS FOR PRACTICE

1. Check the dimensional consistency in the following relation :
$$S = ut + \frac{1}{2}at^2$$
where the symbols have their usual meanings. **[Correct]**

2. Consider the equation $\frac{1}{2}mv^2 = mgh$. Here m is the mass of the body, v its velocity, g acceleration due to gravity and h is the height. Check the dimensional consistency of this equation. **[Correct]**

3. Find the value of force of 100 dynes on a system based on metre, kg and minute as fundamental units. **[3.6 new units]**

4. If the escape velocity depends upon (*i*) acceleration due to gravity of the planet and (*ii*) the radius of the planet, establish dimensional relation between them. **[$v \propto \sqrt{rg}$]**

5. The wavelength λ associated with a moving electron depends upon its mass m, its velocity v and Planck's constant h. Prove dimensionally that :
$$\lambda \propto \frac{h}{mv}$$

6. A body moving through air at a speed v experiences a retarding force F given by $F = k A \rho v^x$ where A is the surface area of the body, ρ is the density of air and k is a dimensionless constant. Deduce the value of x. **[$x = 2$]**

Now, $n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c = 1\left[\frac{1\text{kg}}{100\text{ g}}\right]^1\left[\frac{1\text{m}}{100\text{ cm}}\right]^2\left[\frac{1\text{s}}{1\text{min}}\right]^{-3}$

$= 1\left[\frac{1000\text{g}}{100\text{g}}\right]^1\left[\frac{100\text{cm}}{100\text{ cm}}\right]^2\left[\frac{1\text{s}}{60\text{s}}\right]^{-3} = 1\,[10]^1\,[1]^2\left[\frac{1}{60}\right]^{-3}$

$= 10\times60\times60\times60 = 2.16\times10^6$ new units

$\therefore$ 60 J/min $= 2.16 \times 10^6$ new units

Example 4.9. **If the velocity of light, acceleration due to gravity and the normal pressure are choosen as the fundamental units, find the units of mass, length and time. Given that velocity of light, $c = 3 \times 10^8$ ms^{-1}; acceleration due to gravity, $g = 10$ ms^{-2} and normal atmospheric pressure, $P = 10^5$ N/m^2.**

Solution. $[c] = [LT^{-1}] = 3 \times 10^8$ m s^{-1} ...(*i*)

$[g] = [LT^{-2}] = 10$ m s^{-2} ...(*ii*)

$[P] = [ML^{-1}T^{-2}] = 10^5$ N/m^2

Dividing (*i*) by (*ii*), $\frac{LT^{-1}}{LT^{-2}} = \frac{3\times10^8}{10} = 3\times10^7 \quad \therefore \quad T = 3 \times 10^7$ s

Now $LT^{-1} = 3\times10^8 \quad \therefore \quad L = \frac{3\times10^8}{T^{-1}} = 3\times10^8 T = 3\times10^8\times3\times10^7 = \mathbf{9\times10^{15}}$ **m**

Again, $ML^{-1}T^{-2} = 10^5 \quad \therefore \quad M = \frac{10^5}{L^{-1}T^{-2}} = 10^5\,LT^2 = 10^5\times9\times10^{15}\times\left(3\times10^7\right)^2 = \mathbf{8.1\times10^{35}}$ **kg**

PROBLEMS FOR PRACTICE

1. Check the dimensional consistency in the following relation :

$$S = ut + \frac{1}{2}at^2$$

where the symbols have their usual meanings. **[Correct]**

2. Consider the equation $\frac{1}{2}mv^2 = mgh$. Here m is the mass of the body, v its velocity, g acceleration due to gravity and h is the height. Check the dimensional consistency of this equation. **[Correct]**
3. Find the value of force of 100 dynes on a system based on metre, kg and minute as fundamental units. **[3.6 new units]**
4. If the escape velocity depends upon (*i*) acceleration due to gravity of the planet and (*ii*) the radius of the planet, establish dimensional relation between them. $[v \propto \sqrt{rg}]$
5. The wavelength λ associated with a moving electron depends upon its mass m, its velocity v and Planck's constant h. Prove dimensionally that :

$$\lambda \propto \frac{h}{mv}$$

6. A body moving through air at a speed v experiences a retarding force F given by $F = k A \rho v^x$ where A is the surface area of the body, ρ is the density of air and k is a dimensionless constant. Deduce the value of x. **[$x = 2$]**

7. The frequency of vibration f of a mass m at the end of a spring is given by $f = k^b m^a$ where k is spring constant. Use dimensional analysis to find a and b. It is known that :
$$[f]=[T^{-1}] \text{ and } [k]=[MT^{-2}].$$
$$\left[a=\frac{-1}{2};\ b=\frac{1}{2}\right]$$

8. It is found that the volume V of water that passes any point of a canal during t seconds is connected with the cross-sectional area A of the canal and the velocity u of water by the relation :
$$V = k\ u\ t\ A \qquad \textbf{[correct]}$$
Using dimensional analysis check if the relation is correct.

9. Assuming that the period of vibration of a tuning fork depends upon the length of the prongs and on the density and Young's modulus of the material, use dimensional analysis to obtain the time period of vibration.
$$\left[T \propto l\sqrt{\frac{\rho}{Y}}\right]$$

10. Convert 5 dynes to newton. **[5×10^{-5} N]**

CONCEPTUAL QUESTIONS

Q.1. Give two drawbacks of dimensional analysis.

Ans. The two salient drawbacks of dimensional analysis are :

(*i*) The method does not tell us where the equation is wrong.

(*ii*) If the dimensions had been the same on each side of the equation, we would know only that it might be correct, for the method does not provide a check on numerical factors.

Q.2. What is the importance of dimensional analysis inspite of its drawbacks?

Ans. In many physical situations, it is very difficult to obtain the formula of a physical quantity. It is because the mathematical analysis involved is too difficult. In such situations, dimensional analysis can be a powerful tool.

Q.3. What are the basic rules of dimensional analysis?

Ans. There are two simple rules upon which dimensional analysis is based.

(*i*) We can add or subtract quantities only if they have the same dimensions. For example, we cannot add an area to a speed to obtain a meaningful sum.

(*ii*) An equation is correct if each and every term on the two sides of an equal sign has the same dimensions. For example, in an equation $A = B + C$, the dimensions of A, B and C must be the same.

Q.4. Can dimensional analysis tell you that a physical relation is completely right?

Ans. Even if a physical relation is dimensionally correct, it does not prove that the relation is completely correct. It is because the numerical factors in the relation can be wrong. Thus a dimensional check can tell you when a relation is wrong; it cannot tell you that it is completely right.

Q.5. Is the following relation dimensionally correct?
$$x = x_0 + v\,t + \frac{1}{2}a\,t$$
where t = time ; x_0 = initial displacement at $t = 0$, v = velocity and a = acceleration.

Ans. We can see immediately that this formula is not correct. It is because x has the dimensions of length and each term on the right side of the equation must have the dimensions of length. Both x_0 and vt have the dimensions of length. But the dimensions of the last term

are $[at] = [LT^{-2}][T] = L/T$. Since the last term does not have the correct dimensions, the above relation is wrong.

Q.6. Why do we use square brackets round *M*, *L* and *T*?

Ans. We use square brackets round letter to show that we are dealing with the dimensions of a physical quantity.

Q.7. Can we tell the unit of a physical quantity from its dimensions?

Ans. We can immediately tell the unit of a physical quantity from its dimensions. For example, if some physical quantity has the dimensions $ML^{-1}T^{-1}$, its SI unit will be kg m^{-1} s^{-1}.

Q.8. What is the basic requirement for a physical relation to be correct?

Ans. Dimensional consistency is the basic requirement for a physical relation to be correct. It is, of course, not sufficient.

Q.9. While deriving the relationship between physical quantities by dimensional analysis, dimensionless constant enters into the relationship. Can you find its magnitude by the method of dimensions?

Ans. We cannot find the magnitude of the dimensionless constant in the relation by the method of dimensions because it is a number. We find its value by experiment or through mathematical investigation.

Q.10. Should an equation describe a physical situation ?

Ans. It is very important that every equation must describe some physical situation. An equation can have the correct dimensions in each term without describing any physical situation.

VERY SHORT ANSWER QUESTIONS

Q.1. Can there be a physical quantity which has no units and no dimensions?

Ans. Yes. For example, strain has no units and no dimensions.

Q.2. Can a physical quantity have units but still be dimensionless?

Ans. Yes. For example, the unit of angle is radian but it is a dimensionless quantity.

Q.3. Can a physical quantity have dimensions but still have no units ?

Ans. No.

Q.4. What is the unit of pressure in SI ?

Ans. Pressure $= \frac{\text{Force}}{\text{Area}} = \frac{\text{N}}{\text{m}^2} = \text{Nm}^{-2}$

Q.5. What are the dimensions of kinetic energy?

Ans. Kinetic Energy $= \frac{1}{2}mv^2 = M\ (LT^{-1})^2 = [ML^2\ T^{-2}]$

Q.6. What are the dimensions of angular displacement ?

Ans. Angular displacement $= \frac{\text{Arc length}}{\text{Radius}}$

Therefore, angular displacement is dimensionless.

Q.7. Does a quantity have different dimensions in different systems of units?

Ans. No. A quantity has same dimensions in different systems of units.

Q.8. Are all constants dimensionless?

Ans. No, there are dimensional constants as well *e.g.*, Gravitational constant, Planck's constant, Velocity of light etc.

Q.9. Name two physical quantities which have the same dimensions as that of work.

Ans. Torque and Energy.

Q.10. Name a scalar and a vector quantity which have same dimensions.

Ans. Speed is a scalar quantity and velocity is a vector quantity. Both have the same dimensions.

Q.11. Name two quantities each whose dimensions are (*i*) $[M\ L^{-1}T^{-2}]$ (*ii*) $[M\ L^2T^{-1}]$.

Ans. (*i*) Pressure and stress.

(*ii*) Planck's constant and angular momentum.

Q.12. What are the dimensions of a and b in the relation $F = a + bx$ where F is force and x is distance ?

Ans. Force = Mass × Acceleration = $[M^1LT^{-2}]$. The dimensions of a must be that of force *i.e.*, $[M^1LT^{-2}]$. The dimensions of bx must be that of force *i.e.*, dimensions of b must be $[M^1L^0T^{-2}]$.

Q.13. Prove that energy density and pressure have the same dimensions.

Ans. Energy density $= \frac{\text{Energy}}{\text{Volume}} = \frac{M\,L^2T^{-2}}{L^3} = [M^1L^{-1}T^{-2}]$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{M\,LT^{-2}}{L^2} = [M^1L^{-1}T^{-2}]$$

Q.14. What type of quantity is Avogadro's number?

Ans. Avogadro's number is a dimensionless constant and is measured in mole^{-1}.

Q.15. What is the dimension of time in power?

Ans. Power $= \frac{\text{Work}}{\text{Time}} = \frac{ML^2T^{-2}}{T} = [M^1L^2T^{-3}]$. Therefore, the dimension of time in power is -3.

Q.16. Write the dimensional formula of heat.

Ans. [Heat] = Energy = $[M\ L^2T^{-2}]$.

Q.17. Give the dimensions of (*i*) latent heat (*ii*) angular momentum (*iii*) resistivity (*iv*) capacitance (*v*) inductance.

Ans. (*i*) Latent heat $= \frac{\text{Heat}}{\text{Mass}} = \frac{\text{Energy}}{\text{Mass}} = \frac{ML^2T^{-2}}{M} = [M^0L^2T^{-2}]$

(*ii*) Angular momentum = Moment of inertia × Angular velocity

$$= ML^2 \times T^{-1} = [ML^2\ T^{-1}]$$

(*iii*) Resistivity $= \frac{\text{Resistance} \times \text{Area}}{\text{Length}} = \frac{\text{Voltage}}{\text{Current}} \times \frac{\text{Area}}{\text{Length}}$

$$= \frac{\text{Work done/Charge}}{\text{Charge/Time}} \times \frac{\text{Area}}{\text{Length}} = \frac{M\,L^2T^{-2}Q^{-1}}{Q/T} \times \frac{L^2}{L}$$

$$= [ML^3T^{-1}Q^{-2}]$$

(*iv*) Capacitance $= \frac{\text{Charge}}{\text{Potential}} = \frac{\text{Charge}}{\text{W.D./Charge}} = \frac{Q}{ML^2T^{-2}/Q} = [M^{-1}L^{-2}T^2Q^2]$

(*v*) Inductance $= \frac{\text{Induced e.m.f.}}{\text{Current/Time}} = [ML^2Q^{-2}]$

SHORT ANSWER QUESTIONS

Q.1. Calculate the dimensions of universal gravitational constant.

Ans. $F = G\frac{m_1 m_2}{r^2}$ or $G = \frac{Fr^2}{m_1 m_2} = \frac{(MLT^{-2})(L^2)}{(M)(M)} = [M^{-1}L^3T^{-2}]$.

Q.2. If $x = a + bt + ct^2$ where x is in metres and t in seconds, find the units of b.

Ans. As L.H.S. represents distance, every term on R.H.S. must represent distance.

$\therefore \quad bt = x$ or $b = \frac{x}{t} = \frac{L}{T} = [LT^{-1}]$

Therefore, b represents velocity and hence its unit is ms^{-1}.

Q.3. What are dimensional variables and dimensional constants?

Ans. Refer to Art. 4.5.

Q.4. Show that the change in momentum (Δp) of a body is equal to the product of the time interval (Δt) and the force F acting during this time interval.

Ans. The linear momentum of a body is $p = mv$. Since mass m is constant, change in momentum $= m\Delta v$.

$\therefore$
$$[\Delta p] = [m\Delta v] = [M]\ [LT^{-1}] = [MLT^{-1}]$$
$$[F\Delta t] = [ML\ T^{-2}]\ [T] = [MLT^{-1}]$$

Therefore, the two quantities are equal.

Q.5. Energy and Young's modulus have the same dimensions. Is it true?

Ans. It is not true as proved below:

$$\text{Energy} = [M^1L^2T^{-2}]$$

$$\text{Young's Modulus} = \frac{\text{Stress}}{\text{Strain}} = \frac{F/A}{\Delta L/L} = \frac{MLT^{-2}/L^2}{1} = [M^1L^{-1}T^{-2}]$$

Q.6. The rotational kinetic energy of a body is given by $I\omega^2/2$ where ω is the angular velocity of the body. Use this equation to get dimensions of I.

Ans. $E = \frac{1}{2}I\omega^2$ or $I = \frac{2E}{\omega^2}$

$$\therefore [I] = \frac{2 \times \text{Energy}}{(\text{Angular velocity})^2} = \frac{[ML^2T^{-2}]}{[T^{-2}]} = [ML^2T^0]$$

Q.7. Write four pairs of physical quantities which have the same dimensions.

Ans. (*i*) Work and energy.

(*ii*) Frequency and velocity gradient.

(*iii*) Pressure and stress.

(*iv*) Angular momentum and Planck's constant.

Q.8. Write the dimensions of a/b in the relation $F = a\sqrt{x} + bt^2$ where F is force, x is distance and t is time.

Ans. As L.H.S. represents force, every term on R.H.S. must represent force.

$$\therefore \quad [a] = \frac{[F]}{[\sqrt{x}]} = \frac{[MLT^{-2}]}{[L^{1/2}]} = [ML^{1/2}\ T^{-2}]$$

$$\text{Also} \quad [b] = \frac{[F]}{[t^2]} = \frac{[MLT^{-2}]}{[T^2]} = [MLT^{-4}]$$

$$\therefore \quad \frac{[a]}{[b]} = \frac{[ML^{1/2}\ T^{-2}]}{[MLT^{-4}]} = [M^0L^{-1/2}\ T^2]$$

Q.9. Write down the limitations of dimensional analysis.

Ans. Refer to Art. 4.13.

Q.10. Pressure is defined as momentum per unit volume. Is it true?

Ans. No, because this is not dimensionally true.

$$\frac{\text{Momentum}}{\text{Volume}} = \frac{\text{Mass} \times \text{Velocity}}{\text{Volume}} = \frac{MLT^{-1}}{L^3} = [ML^{-2}T^{-1}]$$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{MLT^{-2}}{L^2} = [ML^{-1}T^{-2}]$$

Q.11. Explain the principle of homogeneity of dimensions.

Ans. According to this principle, a physical relation is dimensionally correct if the dimensions of fundamental quantities (mass, length and time) are the same in each and every term on either side of the equation. This principle is based on the fact that only quantities of the same kind (or dimensions) can be added or subtracted. For example, consider the equation, $A = B + C$. Here the quantities A, B and C must have the same dimensions.

LONG ANSWER QUESTIONS

1. What do you understand by the dimensions and dimensional formula of a physical quantity? Give examples. **[Refer to Arts. 4.1 and 4.2]**
2. What is a dimensional equation? Give uses of dimensional equations. **[Refer to Arts. 4.3 and 4.7]**
3. How will you convert one system of units into another by dimensional analysis? **[Refer to Art. 4.11]**
4. How will you check the correctness of a physical relation by dimensional analysis? **[Refer to Art. 4.8]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The formula of a physical quantity may be different but its dimensions are the same. Thus the area of a circle is πr^2 while the area of a triangle is $bh/2$. In both cases, the dimensions are the same.
2. Work, all types of energy, torque, couple, moment of force, all have the same dimensions *viz* $[ML^2T^{-2}]$.
3. The physical quantity which is the ratio of two similar physical quantities (*e.g.* strain, angle, relative density, refractive index, relative permeability etc.) has no dimensions.
4. Some important relations between various physical quantities are :

(*i*) Latent heat $= \dfrac{\text{Heat energy}}{\text{Mass}}$ (*ii*) Specific heat $= \dfrac{\text{Heat energy}}{\text{Mass} \times \text{Temperature}}$

(*iii*) Charge = Current × Time (*iv*) Electric potential $= \dfrac{\text{Electric work}}{\text{Charge}}$

(*v*) Capacitance $= \dfrac{\text{Charge}}{\text{Electric potential}}$ (*vi*) Electric intensity $= \dfrac{\text{Force}}{\text{Charge}}$

(*vii*) Resistance $= \dfrac{\text{Voltage}}{\text{Current}}$ (*viii*) Gas constant $= \dfrac{\text{Pressure} \times \text{Volume}}{\text{Temperature}}$

5. Some important formulas from which the dimesions of a physical quantity can be readly derived are :

(*i*) $F = BI\,l$ $\quad\therefore\quad B = \dfrac{F}{I\,l}$

(*ii*) $F = G\dfrac{m_1 m_2}{r^2}$ $\quad\therefore\quad G = \dfrac{Fr^2}{m_1 m_2}$

(*iii*) $R = \rho\dfrac{l}{a}$ $\quad\therefore\quad \rho = \dfrac{Ra}{l}$

(*iv*) $F = \dfrac{q_1 q_2}{4\pi\varepsilon_0 r^2}$ $\quad\therefore\quad \varepsilon_0 = \dfrac{q_1 q_2}{4\pi F r^2}$

(*v*) $F = \dfrac{\mu_0}{4\pi}\dfrac{m_1 m_2}{r^2}$ $\quad\therefore\quad \mu_0 = \dfrac{4\pi F r^2}{m_1 m_2}$

6. Some important physical quantities having the same dimensions are :

(*i*) $\left[M^0L^0T^{-1}\right]$ = frequency, angular frequency, angular velocity, velocity gradient.

(*ii*) $\left[ML^{-1}T^{-2}\right]$ = pressure, stress, modulus of elasticity, energy density.

(*iii*) $\left[ML^2T^{-1}\right]$ = angular momentum, Planck's constant.

(*iv*) $\left[MLT^{-2}\right]$ = thrust, force, weight, energy gradient.

(*v*) $\left[MLT^{-1}\right]$ = momentum and impulse.

(*vi*) $\left[ML^0T^0\right]$ = mass and inertia.

(*vii*) $\left[M^0L^0T\right]$ = $L/R, \sqrt{LC}$ and RC.

7. The dimensions of an equation (or formula) must be the same on the two sides. Consider the equation : $v = u + at$

   Here the dimensions of v, u and at must be the same *i.e.* those of velocity (M^0LT^{-1}) in this case.
8. The important dimensional constants are Planck's constant (h), gravitational constant (G) and velocity (c) of light in vacuum.
9. During dimensional analysis, dimensionless constants (*e.g.* π , 1, 2, 3 ....) are omitted.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The dimensions of electric current are **[Pb PMT 2001]**

   (*a*) $[M^0L^0T^{-1}Q]$ (*b*) $[ML^2T^{-1}Q]$

   (*c*) $[M^2LT^{-1}Q]$ (*d*) $[M^2L^2T^{-2}]$

2. The SI unit of gravitational potential is **[AFMC–Pune 2001]**

   (*a*) J (*b*) Jkg^{-1}

   (*c*) J-kg (*d*) $J\text{-}kg^2$

3. Which of the following pair does not have similar dimensions? **[AIIMS 2001]**

   (*a*) stress and pressure

   (*b*) angle and strain

   (*c*) tension and surface tension

   (*d*) Planck's constant and angular momentum

4. The dimensions of Planck's constant equal to that of **[CBSE PMT 2001]**

   (*a*) energy (*b*) momentum

   (*c*) angular momentum (*d*) power

5. A force F is given by $F = at + bt^2$ where t is time. What are the dimensions of a and b? **[AFMC – Pune 2000]**

   (*a*) $[MLT^{-3}]$ and $[MLT^{-4}]$

   (*b*) $[MLT^{-3}]$ and $[ML^2T^{-4}]$

   (*c*) $[MLT^{-1}]$ and $[MLT^0]$

   (*d*) $[MLT^{-4}]$ and $[MLT^4]$

6. The dimensions of Hubble's constant are **[Pb PMT 2000]**

   (*a*) $[T^{-1}]$ (*b*) $[M^0L^0T]$

   (*c*) $[MLT^4]$ (*d*) $[MLT^{-1}]$

7. The dimensions of magnetic flux are **[CBSE PMT 1999]**

   (*a*) $ML^2T^{-2}A^{-1}$ (*b*) $ML^0T^{-2}A^{-2}$

   (*c*) $M^0L^{-2}T^{-2}A^{-2}$ (*d*) $ML^2T^{-1}A^3$

8. Which of the following is a dimensional constant? **[AIIMS 1998]**

   (*a*) refractive index

   (*b*) Poisson's ratio

   (*c*) relative density

   (*d*) gravitational constant

9. A sphere of radius a moves with velocity v in a medium and the force F acting on it is given by; **[CBSE PMT 1997]**

   $$F = 6\pi\eta a v$$

   The dimensions of η will be

   (*a*) $ML^{-1}T^{-1}$ (*b*) MT^{-1}

   (*c*) MLT^{-2} (*d*) ML^{-3}

10. The velocity v of a particle at time t is given by : **[CPMT 1990]**

    $$v = at + \frac{b}{t+c}$$

    The dimensions of a, b, c are respectively

    (*a*) LT^{-2}, L, T (*b*) L^2, T, LT^2

    (*c*) LT^2, LT, L (*d*) L, LT, T^2

11. The time dependence of a physical quantity p is given by $p = p_0 \exp\left(-\alpha t^2\right)$ where α is a constant and t is the time. The constant α **[CBSE PMT 1993]**

    (*a*) is dimensionless

    (*b*) has dimensions T^{-2}

    (*c*) has dimensions T^2

    (*d*) none of the above.

12. Two quantities A and B have different dimensions. Which mathematical operation given below is physically meaningful? **[CPMT 1997]**

    (*a*) A/B (*b*) $A + B$

    (*c*) $A - B$ (*d*) none

13. Turpentine oil is flowing through a tube of length l and radius r. The pressure difference between the two ends of the tube is P. The viscosity of the oil is given by ; **[CBSE PMT 1993]**

$$\eta = \frac{p(r^2 - x^2)}{4vl}$$

where v is the velocity of oil at a distance x from the axis of the tube. The dimensions on η are

(a) $[M^0L^0T^0]$ (b) $[MLT^{-1}]$

(c) $[ML^2T^{-2}]$ (d) $[ML^{-1}T^{-1}]$

14. In the standard equation **[DPMT 1994]**

$$S_{nth} = u + \frac{a}{2}(2n-1)$$

what dimensions do you view for S_{nth}?

(a) $[M^0LT^0]$ (b) $[M^0LT^{-1}]$

(c) $[M^0L^{-1}T]$ (d) $[M^0L^0T]$

15. Which physical quantities have the same dimensions ? **[CPMT 1997]**

(a) moment of couple and work

(b) force and power

(c) latent heat and specific heat

(d) work and power

16. The frequency of vibration f of mass m suspended from a spring of spring constant k is given by the relation $f = cm^x k^y$ where c is a dimensionless constant. The values of x and y are : **[CBSE PMT 1990]**

(a) $x = \frac{1}{2}, y = \frac{1}{2}$ (b) $x = -\frac{1}{2}, y = -\frac{1}{2}$

(c) $x = \frac{1}{2}, y = -\frac{1}{2}$ (d) $x = -\frac{1}{2}, y = \frac{1}{2}$

17. Consider the following equation of Bernoulli's theorem. **[AFMC 2005]**

$$P + \frac{1}{2}\rho v^2 + \rho gh = K \text{ (constant)}$$

The dimensions of K/P are same as that of which of the following?

(a) Thrust (b) Pressure

(c) Angle (d) Viscosity

18. The ratio of the dimensions of Planck's constant and that of the moment of inertia are the dimensions of : **[CBSE 2005]**

(a) frequency (b) velocity

(c) angular momentum (d) time

19. The dimensions of light year are: **[Manipal 2003]**

(a) $[L^{-1}]$ (b) $[T^{-1}]$

(c) $[L]$ (d) $[T]$

20. The dimensions of electrical resistance are: **[AIIMS 2005]**

(a) $[ML^2T^{-3}A^{-1}]$ (b) $[ML^2T^{-3}A^{-2}]$

(c) $[ML^3T^{-3}A^{-2}]$ (d) $[ML^{-1}T^3A^2]$

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (a)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (a)
6. (a)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (a)
11. (b)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (a)
16. (d)	**17.** (c)	**18.** (a)	**19.** (c)	**20.** (b)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Electric current, $I = \frac{\text{charge}}{\text{time}} = \frac{q}{t}$

$$\therefore [I] = \frac{\text{Dimensions of charge}}{\text{Dimensions of time}} = \frac{[Q]}{[M^0L^0T]} = [M^0L^0T^{-1}Q]$$

2. Gravitational potential $= \frac{\text{Work}}{\text{Mass}}$

$\therefore$ SI unit of gravitational potential $= \frac{\text{SI unit of work}}{\text{SI unit of mass}} = \frac{\text{J}}{\text{kg}} = \text{J kg}^{-1}$

3. Surface tension $= \frac{\text{Force}}{\text{Length}}$

$\therefore$ Dimensions of surface tension $= \dfrac{\text{Dimensions of force}}{\text{Dimensions of length}} = \dfrac{[MLT^{-2}]}{[L]} = [MT^{-2}]$

The dimensions of tension are the same as that of force *i.e.* $[MLT^{-2}]$. Therefore, dimensions of tension and surface tension are not similar.

4. Planck's constant, $h = \dfrac{E\,(\text{energy})}{f\,(\text{frequency})}$

$\therefore$ Dimensions of Planck's constant are

$$[h] = \frac{\text{Dimensions of energy}}{\text{Dimensions of frequency}} = \frac{[ML^2T^{-2}]}{[T^{-1}]} = [ML^2T^{-1}]$$

Now $[ML^2T^{-1}]$ are the dimensions of angular momentum.

5. The dimensions of force are $[MLT^{-2}]$. Therefore, dimensions of at and bt^2 must be $[MLT^{-2}]$.

$\therefore$ Dimensions of $a = \dfrac{[MLT^{-2}]}{[T]} = [MLT^{-3}]$ and Dimensions of $b = \dfrac{[MLT^{-2}]}{[T^2]} = [MLT^{-4}]$

6. Hubble's constant, $H = \dfrac{\text{Velocity}}{\text{Distance}}$

$\therefore$ Dimensions of Hubble's constant $= \dfrac{\text{Dimensions of velocity}}{\text{Dimensions of distance}} = \dfrac{[LT^{-1}]}{[L]} = [T^{-1}]$

7. $F = BIl$ or $B = \dfrac{F}{Il}$

$\therefore$ Dimensions of B (magnetic flux density) $= \dfrac{\text{Dimensions of } F}{\text{Dimensions of } I \times \text{Dimensions of } l}$

$$= \frac{[MLT^{-2}]}{[A][L]} = [MT^{-2}A^{-1}]$$

Magnetic flux, $\phi = B \times$ area

$\therefore$ Dimensions of magnetic flux = Dimensions of $B \times$ Dimensions of area

$$= [MT^{-2}A^{-1}][L^2] = [ML^2T^{-2}A^{-1}]$$

8. Refractive index, Poisson's ratio and relative density are ratio of two similar physical quantities and, therefore, these are dimensionless constants. However, G is dimensional constant having dimensions $= M^{-1}L^3T^{-2}$.

9. $F = 6\pi\eta a v \;\therefore\; \eta = \dfrac{F}{6\pi a v} \;\therefore$ Dimensions of $\eta = \dfrac{[MLT^{-2}]}{[L][LT^{-1}]} = [ML^{-1}T^{-1}]$

10. The dimensions of $a\,t$ as well as that of $b/t+c$ must be that of velocity *i.e.* $[LT^{-1}]$.

$\therefore$ Dimensions of $a = \dfrac{[LT^{-1}]}{[T]} = [LT^{-2}]$

In the term $b/t+c$, the dimensions of c must be that of time.

$\therefore$ Dimensions of $c = [T]$; Dimensions of $b = [LT^{-1}]\,[T] = [L]$

11. The quantity αt^2 must be a number. $\therefore$ Dimensions of $\alpha = [T^{-2}]$

12. The physical quantities having different dimensions can only be divided or multiplied but cannot be added or subtracted.

13. Dimensions of $\eta = \dfrac{[ML^{-1}T^{-2}][L^2]}{[LT^{-1}][L]} = [ML^{-1}T^{-1}]$

14. The two terms on R.H.S. have the dimensions of velocity *i.e.*, $[M^0LT^{-1}]$. Therefore, S_{nth} (*i.e.* distance travelled in one particular second) must have the dimensions of velocity.

15. **Moment of Couple** = Force × Distance = Dimensions of **work**

16. $f = c\, m^x k^y$

The spring constant k = force/length. Therefore, putting the dimensions of various quantities, we have,

$$[M^0L^0T^{-1}] = [M]^x[MT^{-2}]^y = [M^{x+y}][T^{-2y}]$$

$\therefore \quad x+y=0$ and $-2y=-1 \quad \therefore x = -\dfrac{1}{2},\ y = \dfrac{1}{2}$

17. $P+\dfrac{1}{2}\rho v^2+\rho gh = K$ (constant)

According to principle of homogeneity,

Dimensions of K = Dimensions of P

Therefore, K/P is dimensionless. Out of the given options, only angle is dimensionless. Therefore, the dimensions of K/P are the same as that of **angle**.

18. $E = hf$ or Planck's constant, $h = \dfrac{E}{f}$

$\therefore \quad [h] = \dfrac{[E]}{[f]} = \dfrac{[ML^2T^{-2}]}{[T^{-1}]} = [ML^2T^{-1}]$

Moment of inertia, $I = MR^2$

$\therefore \quad [I] = [M]\ [L^2] = [ML^2]$

$\therefore \quad \dfrac{[h]}{[I]} = \dfrac{[ML^2T^{-1}]}{[ML^2]} = [T^{-1}] = \dfrac{1}{[T]}$ = **Dimensions of frequency**

19. We know that light year is the distance travelled by light in one year. Therefore, the dimensions of light year are that of length *i.e.*, $[L]$.

20. Electrical resistance, $R = \dfrac{\text{Potential difference } (V)}{\text{Current } (I)}$

$\therefore \quad [R] = \dfrac{[V]}{[I]} = \dfrac{[ML^2T^{-3}A^{-1}]}{[M^0L^0T^0A]} = [ML^2T^{-3}A^{-2}]$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. The SI unit of gravitational constant G is **[Haryana CET 2001]**

(*a*) Nm kg^{-1} (*b*) Nm^2 kg^{-1}

(*c*) Nm kg^{-2} (*d*) Nm^2 kg^{-2}

2. Which of the following pair does not have the same dimensions? **[BHU 1997]**

(*a*) angular momentum and Planck's constant

(*b*) moment of inertia and moment of force

(*c*) work and torque

(*d*) impulse and momentum

3. A quantity X is defined by the equation $X = 3CB^2$ where C is the capacitance in farad and B represents magnetic field in tesla.The dimensions of X are **[AMU 1998]**

(*a*) $[ML^{-2}]$ (*b*) $[ML^{-2}T^{-2}A]$

(*c*) $[ML^{-2}T^{-2}A^2]$ (*d*) $[L^{-1}A^{-1}]$

4. A quantity X is given by ;

$$X = \varepsilon_0 L \frac{\Delta V}{\Delta t}$$

where ε_0 is the permittivity of free space, L is length, ΔV is the potential difference and Δt is the time interval. The dimensions of X are the same as for **[IIT 2001]**

(a) resistance (b) charge

(c) voltage (d) current

5. If speed of light (c), acceleration due to gravity (g) and pressure (p) are taken as fundamental units, the dimensions of gravitational constant (G) are **[AMU 1999]**

(a) $c^0 g p^{-3}$ (b) $c^2 g^3 p^{-2}$

(c) $c^0 g^2 p^{-1}$ (d) $c^2 g^2 p^{-2}$

6. The dimensions of $\varepsilon_0 E^2/2$ (ε_0 = permittivity of free space, E = electric field) are **[IIT 2000]**

(a) MLT^{-1} (b) ML^2T^{-2}

(c) $ML^{-1}T^{-2}$ (d) ML^2T^{-1}

7. The dimensional formula for μ_0 is given by ; **[Haryana CET 2000]**

(a) M^0LT (b) $MLT^{-2}A^{-2}$

(c) $M^0L^2T^{-1}A^2$ (d) none of the above

8. If L and R denote inductance and resistance respectively, which of the following has the dimensions of frequency? **[MP CET 1999]**

(a) R/L (b) L/R

(c) $\sqrt{R/L}$ (d) $\sqrt{L/R}$

9. The ratio of one micron to one nanometre is **[MNR 1993]**

(a) 10^3 (b) 10^{-3}

(c) 10^{-6} (d) 10^{-9}

10. Which of the following is equal to joule-Ohm/second-volt ? **[Haryana CET 1998]**

(a) ampere (b) watt

(c) tesla (d) volt

11. The unit of permittivity of free space is **[Haryana CET 1998]**

(a) coulomb^2/newton-metre^2

(b) coulomb/(newton-metre)^2

(c) newton-metre^2/coulomb^2

(d) coulomb/newton-metre

12. The physical quantity which has the dimensional formula MT^{-3} is **[Pb CET 1998]**

(a) surface tension (b) solar constant

(c) density (d) compressibility

13. The dimensions of the quantities in one of the following pairs are the same. Identify the pair. **[IIT 1986]**

(a) torque and work

(b) angular momentum and work

(c) energy and Young's modulus

(d) light year and frequency

14. The SI unit of inductance, the henry can be **[IIT 1998]**

(a) weber/ampere

(b) volt-second /ampere

(c) joule/(ampere)^2

(d) all of above

15. A gas bubble from an explosion under water oscillates with a period T proportional to $p^a d^b E^c$ where p is the static pressure, d is the density of water and E is the total energy of explosion. The calculated values of a, b and c on the dimensional analysis will be **[Haryana CET 1994]**

(a) $a = -5/6, b = 2, c = 3$

(b) $a = b = c = 1$

(c) $a = 5, b = \frac{1}{2}, c = \frac{1}{3}$

(d) $a = -\frac{5}{6}, b = \frac{1}{2}, c = \frac{1}{3}$

16. In the equation $\left(P + \frac{a}{V^2}\right)(V - b) =$ constant (P = pressure , V = Volume), the units of a are **[MNR 1995]**

(a) dyne × cm^2 (b) dyne × cm^4

(c) dyne/cm^2 (d) dyne × cm^3

17. L, C and R represent the physical quantities inductance, capacitance and resistance respectively. Identify the quantity that represents frequency. **[IIT 1984]**

(a) $\frac{1}{RC}$ (b) $\frac{L}{R}$

(c) $\frac{C}{L}$ (d) $\frac{L}{C}$

18. Dimensions of $\frac{1}{\mu_0 \varepsilon_0}$, where symbols have their usual meanings are : **[AIEEE 2003]**

(a) $[L^{-1}T]$ (b) $[L^2T^2]$

(c) $[L^2T^{-2}]$ (d) $[LT^{-1}]$

19. Pressure depends on distance as, $P = \frac{\alpha}{\beta}$ exp. $\left(\frac{-\alpha z}{k\theta}\right)$, where α, β are constants, z is distance, k is Boltzmann's constant and θ is temperature. The dimensions of β are **[IIT 2004]**

(a) $M^0L^0T^0$ (b) $M^{-1}L^{-1}T^{-1}$

(c) $M^0L^2T^0$ (d) $M^{-1}L^{-1}T^2$

20. Which of the following set have different dimensions? **[IIT 2005]**

(a) Pressure, Young's modulus, stress

(b) Emf, potential difference, electric potential

(c) Heat, work done, energy

(d) Dipole moment, electric flux, electric field

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (d)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (c)
6. (c)	**7.** (b)	**8.** (a)	**9.** (a)	**10.** (d)
11. (a)	**12.** (b)	**13.** (a)	**14.** (d)	**15.** (d)
16. (b)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (d)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. $$F = G\frac{Mm}{r^2} \quad \text{or} \quad G = \frac{Fr^2}{mM}$$

$$\therefore \quad \text{Unit of } G = \frac{\text{Unit of force} \times \text{Unit of } r^2}{\text{Unit of } m \times \text{Unit of } M} = \frac{\text{N} \times \text{m}^2}{\text{kg} \times \text{kg}} = \text{Nm}^2\text{kg}^{-2}$$

2. The unit of moment of inertia is kg m^2 and, therefore, its dimensions are $\left[ML^2T^0\right]$. The dimensions of moment of force are $\left[ML^2T^{-2}\right]$. Therefore, this pair does not have the same dimensions.

3. $$X = 3CB^2 = \left(\frac{Q}{V}\right)\left(\frac{F}{Il}\right)^2$$

$$\therefore \quad \text{Dimensions of } X = \frac{[AT]}{\left[ML^2T^{-3}A^{-1}\right]} \times \left[\frac{MLT^{-2}}{AL}\right]^2 = [ML^{-2}]$$

4. $$X = \varepsilon_0 L\frac{\Delta V}{\Delta t} \quad \text{and} \quad \varepsilon_0 L = \text{Capacitance} = \frac{Q}{\Delta V}$$

$$\therefore \quad \left[\varepsilon_0 L\frac{\Delta V}{\Delta t}\right] = \left[\frac{Q}{\Delta V}\right]\left[\frac{\Delta V}{\Delta t}\right] = \left[\frac{Q}{\Delta t}\right] = [I]$$

5. Let $G = c^x g^y p^z$

$$\text{or} \quad \left[M^{-1}L^3T^{-2}\right] = \left[LT^{-1}\right]^x\left[LT^{-2}\right]^y\left[ML^{-1}T^{-2}\right]^z$$

$$= [M^z L^{x+y-z} T^{-x-2y-2z}]$$

$\therefore \quad z = -1 ; \quad x + y - z = 3 ; \quad -x - 2y - 2z = -2$

on solving, we have, $x = 0, \quad y = 2. \quad \therefore \; G = c^0 g^2 p^{-1}$

6. $$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \quad ; \quad E = \frac{F}{q}$$

$$\therefore \qquad \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\left(\frac{q_1 q_2}{4\pi F r^2}\right)\left(\frac{F^2}{q^2}\right)$$

$\therefore$ The unit of $\varepsilon_0 E^2/2$ is that of F/r^2.

$$\text{Unit of } \varepsilon_0 E^2/2 = \frac{\text{kg ms}^{-2}}{\text{m}^2} = \text{kg m}^{-1}\text{s}^{-2}$$

$\therefore$ Dimensions of $\varepsilon_0 E^2/2 = [ML^{-1}T^{-2}]$

7. The force (F) between two magnetic poles of strength m_1 and m_2 separated by a distance r is

$$F = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^2} \quad \text{or} \quad \mu_0 = \frac{4\pi F r^2}{m_1 m_2}$$

$$\therefore \qquad \mu_0 = \frac{\left(MLT^{-2}\right)\left(L^2\right)}{AL \times AL} = [MLT^{-2}A^{-2}]$$

8. L/R is the time constant of R–L series circuit. Therefore, R/L has the dimensions of frequency.

9. $$\frac{1\,\text{micron}}{1\,\text{nanometre}} = \frac{10^{-6}\,\text{m}}{10^{-9}\,\text{m}} = 10^3$$

10. $$\frac{J}{s} \times \frac{\text{ohm}}{\text{volt}} = \frac{J}{Q} \times \frac{Q}{s} \times \frac{\text{volt/A}}{\text{volt}} = \text{volt} \times \text{A} \times \frac{1}{\text{A}} = \text{volt}$$

11. $$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \qquad \therefore \; \varepsilon_0 = \frac{q_1 q_2}{4\pi F r^2}$$

$$\text{Unit of } \varepsilon_0 = \frac{\text{coulomb} \times \text{coulomb}}{\text{newton} \times (\text{metre})^2} = \text{coulomb}^2/\text{N–m}^2$$

12. $$\text{Solar constant} = \frac{\text{Energy}}{\text{Area} \times \text{Time}} = \frac{ML^2T^{-2}}{L^2T} = [MT^{-3}]$$

13. Torque has the same dimensions $\left[ML^2T^{-2}\right]$ as that of work.

14. $$L = \frac{N\phi}{I} = \frac{\text{Weber}}{\text{ampere}}$$

$$e = L\frac{dI}{dt} \quad \therefore \quad L = \frac{edt}{I} = \text{volt-second/ampere}$$

$$\text{Now } I = \frac{V}{X_L} = \frac{V}{2\pi f L} \qquad \therefore \quad L = \frac{V}{2\pi I f} = \frac{J/Q}{2\pi I \times 1/T} = \frac{J}{2\pi I^2} = \text{joule/(ampere)}^2$$

15. $$T \propto p^a d^b E^c$$

or $$\left[M^0L^0T\right] = k\left[ML^{-1}T^{-2}\right]^a\left[ML^{-3}\right]^b\left[ML^2T^{-2}\right]^c$$

$$= k\left[M^{a+b+c}\right]\left[L^{-a-3b+2c}\right]\left[T^{-2a-2c}\right]$$

Equating the powers of M, L and T, we get,

$$a + b + c = 0 \; ; \quad -a - 3b + 2c = 0 \; ; \quad -2a - 2c = 1$$

On solving, $a = -5/6$, $b = 1/2$, $c = 1/3$

16. The units of $\frac{a}{V^2}$ must be that of pressure.

$$\therefore \quad \frac{a}{V^2} = P \quad \text{or} \quad a = PV^2 = \frac{\text{dyne}}{\text{cm}^2} \times (\text{cm}^3)^2 = \text{dyne} \times \text{cm}^4$$

17. $R = \frac{V}{I} = \frac{\text{Energy}/Q}{I} \quad \therefore \quad [R] = \frac{[M L^2 T^{-2}]}{[AT][A]} = [M L^2 T^{-3} A^{-2}]$

Now $$C = \frac{Q}{V} = \frac{Q}{\text{Energy}/Q} = \frac{Q^2}{\text{Energy}}$$

$$\therefore \quad [C] = \frac{[A^2 T^2]}{[M L^2 T^{-2}]} = [M^{-1} L^{-2} T^4 A^2]$$

$$\therefore \quad \frac{1}{RC} = \frac{1}{[M L^2 T^{-3} A^{-2}][M^{-1} L^{-2} T^4 A^2]} = [T^{-1}]$$

Hence the quantity $1/RC$ represents frequency.

18. The velocity of light is given by ;

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \text{or} \quad v^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\therefore \quad \left[\frac{1}{\mu_0 \varepsilon_0}\right] = [v^2] = [LT^{-1}]^2 = [L^2 T^{-2}]$$

19. $[k] = \left[\frac{\text{joule}}{\text{kelvin}}\right] = [ML^2 T^{-2} \theta^{-1}]$

An exponential is a constant with zero dimensions *i.e.*, $M^0 L^0 T^0$.

$$\therefore \quad \left[\frac{\alpha z}{k\theta}\right] = [M^0 L^0 T^0]$$

or $$[\alpha] = \left[\frac{k\theta}{z}\right] = \left[\frac{ML^2 T^{-2} \theta^{-1} \times \theta}{L}\right] = [MLT^{-2}]$$

Pressure, $[P] = [M L^{-1} T^{-2}]$

$$\therefore \quad [\beta] = \frac{\alpha}{P} \exp\left(-\frac{\alpha z}{k\theta}\right)$$

or $$[\beta] = \left[\frac{\alpha}{P}\right] \times [M^0 L^0 T^0] = \frac{[MLT^{-2}]}{[ML^{-1} T^{-2}]} = [L^2]$$

$$\therefore \quad [\beta] = [M^0 L^2 T^0]$$

20. **Dipole moment**, $p = q \times 2a \quad \therefore [p] = [AT \times L] = [M^0 LTA]$

Electric field, $E = F/q \quad \therefore [E] = \frac{[MLT^{-2}]}{[AT]} = [MLT^{-3} A^{-1}]$

Electric flux, $\phi = EA \cos\theta \quad \therefore [\phi] = [MLT^{-3} A^{-1}] \times [L^2] = [ML^3 T^{-3} A^{-1}]$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **A gas bubble from an explosion under water oscillates with a period T proportional to $P^a d^b E^c$ where P is the static pressure, d is the density of water and E is the total energy of the explosion. Find the values of a, b and c.** **[I.I.T.]**

Hint. $T \propto P^a d^b E^c$ or $T = k P^a d^b E^c$ ... (*i*)

where k is a dimensionless constant and a, b and c are the numbers we want to find. The dimensions of T, P, d and E are $[M^0 L^0 T]$, $[ML^{-1}T^{-2}]$, $[M L^{-3}]$ and $[M L^2 T^{-2}]$ respectively. Putting the dimensions of various physical quantities in eq. (*i*), we have,

$$[M^0 L^0 T] = [M L^{-1} T^{-2}]^a [M L^{-3}]^b [M L^2 T^{-2}]^c$$

or $$[M^0 L^0 T] = [M^{a+b+c} L^{-a-3b+2c} T^{-2a-2c}]$$

Equating the powers of M, L and T on both sides, we have,

$$a + b + c = 0 \; ; \quad -a - 3b + 2c = 0 \; ; \quad -2a - 2c = 1$$

Solving these equations, we get, $a = -\frac{5}{6}$; $b = \frac{1}{2}$; $c = \frac{1}{3}$

2. **If the escape velocity depends upon (*i*) acceleration due to gravity of the planet (*ii*) the radius of the planet, then establish relationship between them by dimensional analysis.**

Hint. $V_e \propto g^a R^b$ ... (*i*)

The dimensions of V_e, g and R are $[M^0 LT^{-1}]$, $[M^0 LT^{-2}]$ and $[M^0 LT^0]$ respectively. Putting the dimensions of various physical quantities in eq. (*i*), we have,

$$[M^0 LT^{-1}] = [M^0 LT^{-2}]^a [M^0 LT^0]^b \text{ or } [LT^{-1}] = [L^{a+b} T^{-2a}]$$

Equating the dimensions of L and T on both sides, we have,

$$a + b = 1 \; ; \quad -2a = -1 \; \therefore \; a = \frac{1}{2} \; ; \quad b = \frac{1}{2}$$

Putting the values of a and b in eq. (*i*), we have, $V_e \propto \sqrt{gR}$

3. **If velocity, time and force were chosen the basic quantities, find the dimensions of mass.**

Hint. $$\text{Force} = \text{Mass} \times \text{Acceleration} = \text{Mass} \times \frac{\text{Velocity}}{\text{Time}}$$

$$\therefore \quad \text{Mass} = \frac{\text{Force} \times \text{Time}}{\text{Velocity}} \text{ or } [\text{Mass}] = FTV^{-1}$$

4. **Find the dimensions of (*i*) capacitance (*ii*) resistance.**

Hint. (*i*) Capacitance, $C = \dfrac{\text{Charge}}{\text{Potential difference}}$

But Potential difference $= \dfrac{\text{Work done}}{\text{Charge}}$ $\therefore$ $C = \dfrac{(\text{Charge})^2}{\text{Work done}}$

or $$[C] = \left[\frac{Q^2}{M L^2 T^{-2}}\right] = [M^{-1} L^{-2} T^2 Q^2]$$

(*ii*) Resistance, $R = \dfrac{\text{Voltage}}{\text{Current}} = \dfrac{V}{I}$; Voltage, $V = \dfrac{\text{Work done}}{\text{Charge}}$

$\therefore$ $R = \dfrac{\text{Work done}}{\text{Charge} \times \text{Current}}$

$\therefore$ $[R] = \dfrac{[M L^2 T^{-2}]}{[Q][I]} = \dfrac{[ML^2T^{-2}]}{[IT][I]} = [M\ L^2 I^{-2} T^{-3}]$

5. **In the formula $x = 3yz^2$, x and z have the dimensions of capacitance and magnetic induction respectively. Find the dimensions of y in MKSQ system.** **[I.I.T.]**

Hint. MKSQ stands for Metre, Kilogram, Second and Charge.

$$x = 3yz^2$$

or $$y = \frac{x}{3z^2}$$

Now Capacitance, $x = \dfrac{(\text{Charge})^2}{\text{Work done}}$

$\therefore$ $[x] = \dfrac{[Q]^2}{[M L^2 T^{-2}]} = [M^{-1} L^{-2} T^2 Q^2]$

Magnetic induction, $z = \dfrac{\text{Force}}{\text{Current} \times \text{Length}}$

$\therefore$ $[z] = \dfrac{[M LT^{-2}]}{[Q/T][L]} = [M T^{-1} Q^{-1}]$

$\therefore$ $[y] = \dfrac{[M^{-1} L^{-2} T^2 Q^2]}{[M T^{-1} Q^{-1}]^2} = [M^{-3} L^{-2} T^4 Q^4]$

6. **Check the correctness of the following equation :**

$$h = \frac{2T\cos\theta}{rdg}$$

where θ is the angle of contact, d is the density of the liquid, r is the radius of the tube, g is acceleration due to gravity, h is the height of the liquid and T is surface tension.

Hint.
$$h = \frac{2T\cos\theta}{rdg} \quad \ldots (i)$$

The dimensions of h, r, d and g are $[L]$, $[L]$, $[ML^{-3}]$ and $[LT^{-2}]$ respectively.

Surface tension, $T = \dfrac{\text{Force}}{\text{Length}}$ $\therefore$ $[T] = \dfrac{[M LT^{-2}]}{[L]} = [M T^{-2}]$

Note that 2 and $\cos\theta$ are dimensionsless.

Putting the dimensions of the various physical quantities in eq. (*i*), we have,

$$[L] = \frac{[MT^{-2}]}{[L][ML^{-3}][LT^{-2}]} = [L]$$

We find that dimensions of L.H.S. are the same as that of R.H.S. Hence, the equation is correct.

7. **If force [F], length [L], current [I] and time [T] are taken as the basic quantities, find the dimensions of ε_0.**

Hint. $$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

or $$\varepsilon_0 = \frac{1}{4\pi} \times \frac{q_1 q_2}{F r^2}$$

$$\therefore \quad [\varepsilon_0] = \frac{[IT][IT]}{[F][L^2]} = \frac{[I^2T^2]}{[FL^2]} = [F^{-1}L^{-2}T^2I^2]$$

8. **Derive by the method of dimensions, an expression for the energy E of a body executing S.H.M., assuming that this energy depends upon the mass m, the frequency f and amplitude r of the vibrating body.**

Hint. $$E \propto m^a f^b r^c \quad \text{or} \quad E = k m^a f^b r^c \qquad \ldots (i)$$

where k is a dimensionless constant and a, b and c are the numbers we want to find.

The dimensions of E, m, f and r are [ML^2T^{-2}], [M], [T^{-1}] and [L] respectively. Putting the dimensions of the various physical quantities in eq. (i), we have,

$$[ML^2T^{-2}] = [M]^a[T^{-1}]^b[L]^c \quad \text{or} \quad [ML^2T^{-2}] = [M^aL^cT^{-b}]$$

Equating the powers of M, L and T on both sides, we have,

$$a = 1\,;\ c = 2\,;\ -b = -2 \quad \text{or} \quad b = 2$$

Putting the values of a, b and c in eq. (i), we get, $E = kmf^2r^2$

9. **Young's modulus of steel is 19×10^{10} N/m^2. Express it in dynes/ cm^2. Here dyne is the C.G.S. unit of force.**

Hint. The SI unit of Young's modulus Y is N/m^2. Therefore, dimensions of Y are :

$$[Y] = \frac{[F]}{[L^2]} = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$$

Comparing with $M^aL^bT^c$, $a = 1$; $b = -1$; $c = -2$

SI units	C.G.S. units
$n_1 = 19 \times 10^{10}$ N/m^2	$n_2 = ?$
$M_1 = 1$ kg	$M_2 = 1$ g
$L_1 = 1$ m	$L_2 = 1$ cm
$T_1 = 1$ s	$T_2 = 1$ s

Now $$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c = 19\times10^{10}\left[\frac{1\text{kg}}{1\text{g}}\right]^1\left[\frac{1\text{m}}{1\text{cm}}\right]^{-1}\left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

$$= 19\times10^{10}\left[\frac{1000\text{g}}{1\text{g}}\right]^1\left[\frac{100\text{ cm}}{1\text{cm}}\right]^{-1}[1]^{-2}$$

$$= 19 \times 10^{10}\left[1000 \times \frac{1}{100} \times 1\right]$$

$$= 19 \times 10^{10}\,[10] = 19 \times 10^{11}\ \text{dynes/cm}^2$$

$\therefore$ $19 \times 10^{10}\ \text{N/m}^2 = 19 \times 10^{11}\ \text{dynes/cm}^2$

10. Obtain the dimensional formula of torque τ and pressure P from their defining equations. Hence, obtain the dimensional formula of viscosity η and rigidity modulus (η) using the equations :

(*i*) $Q = \frac{V}{t} = \frac{\pi P r^4}{8\eta l}$ (*ii*) $\tau = \frac{\pi \eta r^4 \theta}{2l}$

Hint. Torque, τ = Force × Distance $\therefore$ $[\tau] = [M\,LT^{-2}][L] = [ML^2T^{-2}]$

$$\text{Pressure, } P = \frac{\text{Force}}{\text{Area}} \quad \therefore\ [P] = \frac{[M\,LT^{-2}]}{[L^2]} = [M\,L^{-1}T^{-2}]$$

(*i*) $Q = \frac{V}{t} = \frac{\pi P r^4}{8\eta l}$ or Viscosity, $\eta = \frac{(\pi P r^4)t}{8Vl}$

$$\therefore\ [\eta] = \frac{[M\,L^{-1}T^{-2}][L^4][T]}{[L^3][L]} = [M\,L^{-1}T^{-1}]$$

(*ii*) $\tau = \frac{\pi \eta r^4 \theta}{2l}$ or Rigidity modulus, $\eta = \frac{\tau \times 2l}{\pi r^4 \theta}$

$$\therefore\ [\eta] = \frac{[M\,L^2T^{-2}][L]}{[L^4]} = [M\,L^{-1}T^{-2}]$$

11. Check the correctness of the following equations : (*i*) $F = 6\pi\eta r v$ (*ii*) $v = \sqrt{\frac{2GR}{M}}$

(*iii*) $\frac{V}{t} = \frac{\pi P r^4}{8\eta l}$, where symbols have their usual meanings.

Hint. (*i*) $F = 6\pi\eta r v$

Dimensions of force F, viscosity η, radius r and velocity v are $[M\,LT^{-2}]$, $[M\,L^{-1}T^{-1}]$, $[L]$ and $[LT^{-1}]$ respectively. Note that 6π is a dimensionless constant. Putting the dimensions of various physical quantities in the above equation, we have,

$$[M\,LT^{-2}] = [M\,L^{-1}T^{-1}][L][LT^{-1}]$$

or $$[M\,LT^{-2}] = [M\,LT^{-2}]$$

$\therefore$ Dimensions of L.H.S. = Dimensions of R.H.S.

Therefore, the given equation is **correct.**

(*ii*) $v = \sqrt{\frac{2GR}{M}}$

The dimensions of velocity v, gravitational constant G, radius R and mass M are $[LT^{-1}]$, $[M^{-1}L^3T^{-2}]$, $[L]$ and $[M]$ respectively. Note that 2, being a number, has no dimensions.

Dimensions of L.H.S. = $[LT^{-1}]$

$$\text{Dimensions of R.H.S.} = \sqrt{\frac{\left[M^{-1}L^3T^{-2}\right][L]}{[M]}} = \sqrt{\left[M^{-2}L^4T^{-2}\right]} = \left[M^{-1}L^2T^{-1}\right]$$

$\therefore$ Dimensions of L.H.S. $\neq$ Dimensions of R.H.S. Therefore, the equation is not correct.

(*iii*) $$\frac{V}{t} = \frac{\pi P r^4}{8\eta l}$$

The dimensions of volume V, time t, pressure P, viscosity η, radius r and length l are $[L^3]$, $[T]$, $[M L^{-1}T^{-2}]$, $[M L^{-1}T^{-1}]$, $[L]$ and $[L]$ respectively. Note that π and 8 have no dimensions.

$\therefore$ $$\text{Dimensions of L.H.S.} = \frac{\left[L^3\right]}{[T]} = \left[L^3T^{-1}\right]$$

$$\text{Dimensions of R.H.S.} = \frac{\left[M L^{-1}T^{-2}\right]\left[L^4\right]}{\left[M L^{-1}T^{-1}\right][L]} = \left[L^3T^{-1}\right]$$

$\therefore$ Dimensions of L.H.S. = Dimensions of R.H.S.

Hence, the equation is **correct.**

12. **Prove that the dimensions of the quantity $\sqrt{1/\mu_0 \varepsilon_0}$ are the same as that of velocity, where the symbols have their usual meanings.**

Hint. The dimensions of absolute permeability of free space μ_0 and absolute permittivity of free space ε_0 are $M LT^{-2}I^{-2}$ and $M^{-1}L^{-3}T^4I^2$.

$$\therefore \quad \left[\frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right] = \frac{1}{\sqrt{\left[M LT^{-2}I^{-2}\right]\left[M^{-1}L^{-3}T^4I^2\right]}} = \frac{1}{\sqrt{\left[L^{-2}T^2\right]}} = \left[\frac{1}{L^{-1}T}\right] = \left[LT^{-1}\right]$$

Now the dimensions of velocity are $[LT^{-1}]$. Therefore, the quantity $\sqrt{1/\mu_0 \varepsilon_0}$ has the dimensions of velocity.

13. **Obtain an expression for the acceleration of a particle in uniform circular motion by the method of dimensions.**

Hint. The acceleration a of a particle undergoing uniform circular motion depends upon (*i*) the radius r of the path (*ii*) the constant speed v of the particle and (*iii*) mass m of the particle. Therefore, the formula for acceleration a can be written as :

$$a \propto r^a v^b m^c \quad \text{or} \quad a = k r^a v^b m^c \qquad ...(i)$$

where k is a dimensionless constant.

The dimensions of acceleration a, radius r, velocity v and mass m are $[LT^{-2}]$, $[L]$, $[LT^{-1}]$ and $[M]$ respectively. Putting the dimensions of the various physical quantities in eq. (*i*), we have,

$$\left[LT^{-2}\right] = [L]^a \left[LT^{-1}\right]^b [M]^c$$

or $$\left[M^0LT^{-2}\right] = \left[M^c L^{a+b}T^{-b}\right]$$

Equating the powers of M, L and T on both sides, we have,

$$c = 0 \; ; -b = -2 \text{ or } b = 2 \; ; \; a + b = 1 \text{ or } a = -1$$

Putting the values of a, b and c in eq. (i), we have, $a = k\, r^{-1} v^2 m^0$

or $$a = k\frac{v^2}{r}$$

14. The velocity v of water waves may depend on their wavelength λ, the density of water ρ and the acceleration due to gravity g. Find the relation between these quantities by the method of dimensions. **[N.C.E.R.T.]**

Hint. As per the conditions of the problem,

$$v \propto \lambda^a \rho^b g^c \quad \text{or} \quad v = k\lambda^a \rho^b g^c \qquad ...(i)$$

where k is a dimensionless constant.

The dimensions of velocity v, wavelength λ, density ρ and acceleration due to gravity g are $[LT^{-1}]$, $[L]$, $[ML^{-3}]$ and $[LT^{-2}]$ respectively. Putting the dimensions of various physical quantities in eq. (i), we have,

$$\left[LT^{-1}\right] = [L]^a \left[ML^{-3}\right]^b \left[LT^{-2}\right]^c$$

or $$\left[M^0 LT^{-1}\right] = \left[M^b L^{a-3b+c} T^{-2c}\right]$$

Equating the powers of M, L and T on both sides, we have,

$$b = 0\ ;\ a - 3b + c = 1\ ;\ -2c = -1$$

On solving, we get, $a = 1/2$; $b = 0$ and $c = 1/2$.

Putting the values of a, b and c in eq. (i), we have,

$$v = k\, \lambda^{1/2} \rho^0 g^{1/2} = k\, \lambda^{1/2} g^{1/2}$$

or $$v^2 \propto \lambda g$$

15. The position x of a particle depends upon time t according to the equation $x = a\,t + b\,t^2$. Determine the dimensions and the units of a and b. What are the physical quantities denoted by them ? **[UPB]**

Hint. $$x = a\,t + b\,t^2$$

In this equation, the dimensions of the term x on L.H.S. are $[L]$. Therefore, the dimensions of at as well as $b\,t^2$ on R.H.S. must also be $[L]$.

$\therefore$ Dimensions of $a\,t = [L]$ or Dimensions of $a = \dfrac{[L]}{[T]} = \left[LT^{-1}\right]$

Therefore, the SI unit of a is $\mathrm{ms^{-1}}$ and it represents the physical quantity speed

Dimensions of $b\,t^2 = [L]$ or Dimensions of $b = \dfrac{[L]}{\left[T^2\right]} = \left[LT^{-2}\right]$

Therefore, the SI unit of b is $\mathrm{ms^{-2}}$ and it represents the physical quantity acceleration

16. The force F is given in terms of time t and displacement x by the equation :

$$F = A\cos Bx + C\sin Dt$$

What are the dimensions of D/B ?

Hint. Bx and Dt are angles and hence dimensionless.

i.e., $$[Bx] = [Dt] \quad \text{or} \quad \left[\frac{D}{B}\right] = \frac{x}{t} = \frac{[L]}{[T]} = \left[LT^{-1}\right]$$

17. The velocity v of a particle depends upon time t according to the relation :

$$v = at + \frac{b}{t+c}$$

What are the dimensions of a, b and c ?

Hint. $$v = at + \frac{b}{t+c}$$

In this equation, the dimensions of the term v on L.H.S. are $[LT^{-1}]$. Therefore, the dimensions of both the terms at and $b/(t + c)$ must also be $[LT^{-1}]$. According to the theory of dimensions, the dimensions of c must be that of time because only like quantities can be added *i.e.*,

$$[c] = [T]\text{; Dimensions of } a\,t = [LT^{-1}]$$

$$\therefore \text{ Dimensions of } a = \frac{[LT^{-1}]}{[T]} = [LT^{-2}]$$

Note that physical quantity a represents acceleration.

$$\text{Dimensions of } \frac{b}{t+c} = [LT^{-1}]$$

or $$\text{Dimensions of } b = [LT^{-1}][T+T] = [LT^{-1}][T] = [L]$$

18. The density of a material in C.G.S. units is 8 g/cm^3. In a system of units in which the unit of length is 5 cm and the unit of mass is 20 g, what is the density of the material?

[AIIMS]

Hint. The dimensions of density are :

$$[\text{Density}] = [M\,L^{-3}]$$

Comparing with $M^a L^b T^c$, we have, $a = 1$, $b = -3$, $c = 0$.

C.G.S. units	Units of new system
$n_1 = 8$ g/cm^3	$n_2 = ?$
$M_1 = 1$ g	$M_2 = 20$ g
$L_1 = 1$ cm	$L_2 = 5$ cm

$$\text{Now } n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b = 8\left[\frac{1g}{20g}\right]^1\left[\frac{1\text{cm}}{5\text{cm}}\right]^{-3} = 8\left[\frac{1}{20}\right]^1\left[\frac{1}{5}\right]^{-3}$$

$$= 8 \times \frac{1}{20} \times 5^3 = \mathbf{50\ g/cm^3}$$

19. The number of particles n crossing a unit area perpendicular to X-axis in a unit time is given by ;

$$n = -D\frac{(n_2 - n_1)}{x_2 - x_1}$$

where n_1 and n_2 are the number of particles per unit volume for the value of x meant to x_1 and x_2. What are the dimensions of the diffusion constant D ?

Hint. $$D = -\frac{n(x_2 - x_1)}{n_2 - n_1} \qquad ...(i)$$

$$n = \frac{\text{number of particles}}{\text{area} \times \text{time}} \quad \therefore \quad [n] = \frac{1}{[L^2][T]} = [L^{-2}T^{-1}]$$

Now $n_2 - n_1 = \dfrac{\text{number of particles}}{\text{volume}} \quad \therefore \quad [n_2 - n_1] = \dfrac{1}{[L^3]} = [L^{-3}]$

Now $x_2 - x_1 = \text{distance} \quad \therefore \quad [x_2 - x_1] = [L]$

Putting the dimensions of n, $n_2 - n_1$ and $x_2 - x_1$ in eq. (i), we get,

$$[D] = \frac{[L^{-2}T^{-1}][L]}{[L^{-3}]} = [L^2T^{-1}]$$

20. **If the fundamental quantities are velocity (v), mass (M) and time (T), what will be the dimensions of η in the equation $V = \pi Pr^4/8\eta l$?** **[CBSE 2004]**

Hint. $$V = \frac{\pi P r^4}{8\eta l} \quad \therefore \quad \eta = \frac{\pi P r^4}{8 V l}$$

Here V stands for the rate of flow (*i.e.*, volume per unit time).

$$\therefore \quad [\eta] = \frac{[P][r^4]}{[V][L]} = \frac{[ML^{-1}T^{-2}]\,[L^4]}{[L^3T^{-1}][L]} = [ML^{-1}T^{-1}]$$

Now distance = velocity × time

$$\therefore \quad [L] = [v\ T]$$

$$\therefore \quad [\eta] = [M\ v^{-1}\ T^{-1}\ T^{-1}] = [M\ v^{-1}\ T^{-2}]$$

measure the length of the same rod with a vernier calliper, the recorded length may be 3.242 cm. Now the measurement is expressed to four significant figures – three certain, one estimated. Obviously, the number of significant figures obtained in the above measurements depend upon the least count of the measuring instrument.

5.4. RULES FOR SIGNIFICANT FIGURES

When you work a physics problem, you must use the right number of significant figures to express the result. Zeros may or may not be significant depending on whether they are known values or are just being used to locate the decimal place. The following are the rules for determining the number of significant figures:

(*i*) *All nonzero digits are significant e.g.* 128.25 g contains five significant figures.

(*ii*) *All zeros between two nonzero digits are significant e.g.* 107.004m contains six significant figures.

(*iii*) Unless stated otherwise, *all zeros to the left of an* **understood** *decimal point but to the right of a nonzero digit are not significant e.g.* 208,000 m contains three significant figures.

(*iv*) *All zeros to the left of an* **expressed** *decimal point and to the right of a nonzero digit are significant e.g.* 202.000 contains six significant figures.

(*v*) *All zeros to the right of a decimal point but to the left of a nonzero digit are* **not** *significant e.g.* 0.000248kg contains three significant figures.

(*vi*) *All zeros to the right of a decimal point and to the right of a nonzero digit are significant e.g.* 0.06020 cm and 30.00 cm each contains four significant figures.

The table below gives the number of significant figures in different measurements.

S.No.	Measurement	Number of significant figures	Remarks
1.	4.2 cm	2	—
2.	3.56 m s^{-1}	3	—
3.	4.002 kg	4	Both zeros significant
4.	0.00826 kg	3	Zeros merely locate decimal point
5.	*8.26 × 10^{-3} kg	3	Same quantity as previous example
6.	6200 m	2, 3 or 4	Ambiguous. Cannot tell
7.	6.2 × 10^3 m	2	Above confusion removed
8.	6.200 × 10^3 m	4	Above confusion removed

Note. Suppose the length of a rod is exactly 3 cm. Three different persons measure the length of the rod and record the observation as 3.0 cm, 3.00 cm and 3.000 cm respectively. Although these

* It is a common practice in physics to write numbers in "powers of tens" or "exponential notation" such as 36,900 as 3.69 × 10^4 or 0.0021 as 2.1 × 10^{-3}. The obvious advantage of exponential notation is that it allows the number of significant figures to be clearly expressed. For example, it is not clear whether 36,900 has three, four or five significant figures. If the number is known to an accuracy of three significant figures, we write it as 3.69 × 10^4 but if it is known to four significant figures, we write it as 3.690 × 10^4.

observations are identical from the mathematical point of view, they are different from scientific point of view. The first measurement has 2 significant figures; second has three significant figures and the third has four significant figures. Obviously, the measurement has been made with devices having different least count. Needless to say, the last measurement is the most accurate from scientific point of view.

5.5. SIGNIFICANT FIGURES IN CALCULATIONS

Any physics experiment involves a series of measurements and each of these measurements is made upto a certain degree of accuracy. The number of significant figures which you can quote in any measurement will depend upon the least count of the measuring device. For example, if you quote three different lengths as 3.76 cm, 48.855 cm and 0.2 cm, it is clear that the measuring devices have different least count. In doing calculation work on measurements, you may have to perform such operations as addition, subtraction, multiplication or division. While making calculations, it is important to remember that *calculations cannot increase either the precision or the number of significant figures of the measured quantities.* In other words, the precision of the answer can only be as great as the *least* precise term in calculations. Therefore, while doing calculations, the following rules may be followed :

(*i*) *In addition or subtraction, the number of decimal places in the result should be equal to the smallest number of decimal places of any term in the sum or subtraction.*

(*ii*) *In multiplication or division, the number of significant figures in the result should be equal to the number of significant figures of the least precise term in multiplication or division.*

Illustration. Suppose you make three different measurements of lengths, using instruments of various precision and get 3.76 cm, 46.855 cm and 0.2 cm. You are required to find the sum of these values. The sum = 3.76 + 46.855 + 0.2 = 50.815 cm. The question is, what is the correct answer? The least precise quantity (*i.e.* 0.2 cm) is known only to one decimal place. Therefore, the answer should be expressed to one decimal place. The *correct* answer is 50.8 cm.

To calculate the area of a rectangle 11.3 cm by 6.8 cm, the result of multiplication would be 76.84 cm^2. Again the question is, what is the correct answer? The least precise quantity (*i.e.* 6.8 cm) in the multiplication is known only to two significant figures. Therefore, the answer should be expressed to two significant figures. The correct answer is 76 cm^2 or if rounded off it will be 77 cm^2.

5.6. ROUNDING OFF

Sometimes, we want to *round off* the answer to the correct number of significant figures. For this purpose, the following rules should be followed :

(*i*) *If the first digit beyond the last significant figure is 5 or greater, the last significant figure should be increased by 1. All other figures beyond the last significant figure are dropped.*

(*ii*) *If the digit beyond the last significant figure is less than 5, the last significant figure remains unchanged.*

Let us illustrate the above rules with numerical examples. Thus 10.05 rounds off to 10.1 but 10.04 rounds off to 10.0. Most pocket calculators that round off automatically use this rule.

Example 5.2. What is the number of significant figures in the following expressions?

(*i*) 173.2 m (*ii*) 205 cm (*iii*) 4000 km (*iv*) 0.025 g

5

Error Analysis

INTRODUCTION

Physics is an experimental science. Every physics experiment involves a series of measurements. For example, the calculation of speed requires the measurement of a time and a distance. The accuracy of the final result in an experiment depends upon how accurately the physical quantities involved are measured. Measurement can never be made with absolute precision; physical quantities obtained from experimental results always have some error. For example, a distance measured with a metre rod is usually accurate only to the nearest millimetre while a vernier calliper can measure distances upto 0.1 mm. There are two basic types of errors that may appear in a measurement *viz.* (*i*) errors due to known causes and (*ii*) errors due to unknown causes. The first type of errors can be eliminated by suitable methods because we know their causes. The second type of errors cannot be eliminated but can be reduced by making a number of readings of a given physical quantity and then taking the arithmetic mean. In this chapter, we shall discuss the various aspects of errors in measurements.

5.1. ACCURACY OF MEASURING INSTRUMENT

We use measuring instruments to measure various physical quantities (*e.g.* length, mass, speed *etc.*). The accuracy of measurement depends upon the least count of the measuring instrument.

The **least count** *of a measuring instrument is the smallest measurement that can be made accurately with it.*

For example, a metre rod usually has numbered centimetre divisions subdivided into millimetres. Therefore, the least distance that can be measured accurately with a metre rod is 1 mm. We say that the least count of a metre rod is 1mm.

(*i*) The least count of a metre rod is 1 mm or 0.1cm.

(*ii*) The least count of ordinary vernier calliper is 0.1mm or 0.01cm.

(*iii*) The least count of ordinary screw gauge is 0.01mm or 0.001cm.

The smaller the least count of a measuring instrument, the greater is the accuracy in measurement. For example, the length of a given rod measured with a metre rod may give the result as 3.1 cm. However, the length of the same rod measured with a vernier calliper may give the result as 3.12 cm. Clearly, the second measurement is more accurate as compared to the first measurement.

Example 5.1. Three observers record the length of a rod as 2.2cm, 2.24cm and 2.242cm. What do you infer from these observations?

Solution. The statement shows that the length of the rod has been measured with instruments of different least count. The first measurement (*i.e.* 2.2cm) is made with an instrument having

a least count of 1 mm (or 0.1cm) ; the second observation is made with an instrument having least count of 0.1 mm (or 0.01cm) and the third observation is made with an instrument having least count of 0.01mm (or 0.001cm). Note that last observation is the most accurate.

5.2. LIMIT OF PRECISION OF MEASURING INSTRUMENT

Every measuring instrument has an inherent limit of precision or accuracy. This limit is determined by the least count of the instrument. For example, a distance measured with an ordinary metre rod is correct upto 1 mm or 0.1 cm. No matter how carefully we measure, we can never obtain a result more precise than the limit of our measuring device. A general guideline is that a given measuring instrument has a limit of precision equal to one-half of the smallest division of measurement built into the device. When you make a measurement, you read the measured quantity to the nearest mark on the device. The "true value" of this measurement thus lies within a range of one-half of the smallest division of the device above or below the indicated mark.

The **limit of precision** *of a measuring instrument is* $\pm\frac{1}{2}$ *of the smallest division of measurement the device is able to display.*

For example, a metre rod marked in millimetres (mm) has a limiting precision of ± 0.5 mm or ± 0.05 cm. A vernier calliper that can read to nearest 0.1 mm has a limiting precision of ± 0.005 cm. Thus if the length of a rod as measured by a metre rod is 23.2 cm, the measurement can be written as 23.2 ± 0.05 cm. This means that "true value" of length most likely lies between 23.25 cm and 23.15 cm. *Sometimes limiting precision is taken equal to the least count of the instrument.*

5.3. SIGNIFICANT FIGURES

Suppose you measure the length of a table with a metre rod and find that it lies between 52 cm and 53 cm. Looking carefully at the metre rod, you see that the end of the table lies between 52.3 cm and 52.4 cm marks. You directly read 52.3 cm from the metre rod but cannot tell about further distance because there is no marking between 3 mm and 4 mm marks. You *estimate that this distance is 0.7 mm or 0.07 cm so that you record the length of table as 52.37 cm. Note that you are certain about the digits 5, 2 and 3 but you are doubtful about the last digit (*i.e.*7). We say that number of significant figures in the measurement is 4.

The number of digits in a measurement about which we are certain plus one additional digit which is uncertain are known as **significant figures**.

Thus in the above example, we expressed the length of the table as 52.37 cm. This measurement is expressed to four significant figures – three certain, one estimated. It may be noted that in recording a measurement or calculation, *one and only one doubtful digit is retained.*

The greater the number of significant figures obtained in a measurement, the more accurate is the measurement. Since measuring instruments have a limit of precision, there is a limit on the number of significant figures that can be recorded in a measurement. Let us illustrate this point. Suppose you measure the length of a rod with a metre scale and find it to be 3.24 cm. The measurement is expressed to three significant figures – two certain, one estimated. If you

* It is difficult for the observer to estimate between the smallest divisions of the measuring device. The estimation would vary from observer to observer.

Solution. (*i*) 173.2 m contain *four* significant figures.

(*ii*) 205 cm contain *three* significant figures.

(*iii*) 4000 km contain *one* significant figure. Here zeros merely locate the decimal point.

(*iv*) 0.025 g contains *two* significant figures.

Example 5.3. **Perform the indicated operations to correct significant figures :**
(*i*) 3.2 + 14.235 (*ii*) 27.482 – 27.4 (*iii*) 3.437 × 12 (*iv*) 105 ÷ 5.5

Solution. (*i*) 3.2 + 14.235 = 17.435

The least precise term (*i.e.* 3.2) is known only to one decimal place. Therefore, the answer should be expressed to one decimal place. The correct answer is **17.4.**

(*ii*) 27.482 – 27.4 = 0.082

The answer is to be written to first decimal place after rounding off. Therefore, the correct answer is **0.1.**

(*iii*) 3.437 × 12 = 41.244

The result is to be given upto two significant figures because the least precise term (12) has two significant figures. Therefore, the correct answer is **41.**

(*iv*) 105 ÷ 5.5 = 19.09

The result is to be given upto two significant figures. Therefore, the correct answer is **19.**

Example 5.4. Calculate the volume of a cylinder with a diameter of 9.9 cm and a height of 13.5 cm.

Solution. Volume of cylinder, V = area of base × height

$$= \frac{\pi}{4} d^2 \times h = \frac{\pi}{4} \times (9.9)^2 \times 13.5$$

$$= 1039.19 \text{ cm}^3 = 1.03919 \times 10^3 \text{ cm}^3$$

Since the least precise quantity is the diameter and is known to two significant figures, the answer must be expressed to two significant figures. Therefore, $V = \mathbf{1.0 \times 10^3 \text{ cm}^3}$

Example 5.5. 5.74 g of the substance occupies 1.2 cc. Find the density of the substance to correct significant figures.

Solution. Since the least precise quantity is the volume (1.2 cc) and is known to two significant figures, the answer must have two significant figures.

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{5.74}{1.2} = 4.783 \text{ g/cc} = \mathbf{4.8 \text{ g/cc}}$$

Note that final result of two significant figures has been obtained after rounding off.

Example 5.6. **Perform the following operation to correct significant figures:**

$$\mathbf{6.8 \times 10^4 \text{ s} + 3.1 \times 10^5 \text{ s}}$$

Solution. To add quantities of different exponents, we must first express them with same exponent.

$$6.8 \times 10^4 + 3.1 \times 10^5 = 6.8 \times 10^4 + 31 \times 10^4 = 37.8 \times 10^4 \text{ s}$$

Since the least precise term is known to one decimal place, the final result must be expressed to one decimal place. After rounding off, the answer is $\mathbf{3.8 \times 10^5 \text{ s}}$.

Example 5.7. **What is the volume of a box whose sides are 31.3 cm, 28 cm and 51.85 cm?**

Solution. Volume of box, V = length × breadth × height

$$= 31.3 \times 28 \times 51.85 = 45{,}441.34 \text{ cm}^3$$

Since the least precise quantity (28 cm) is known to two significant figures, the answer must be expressed to two significant figures. Therefore, $V = 45000 \text{ cm}^3 = 4.5 \times 10^4 \text{ cm}^3$.

Example 5.8. **Each side of a cube is measured to be 7.203 m. What is the total surface area and the volume of the cube to appropriate significant figures?**

Solution. Since each side of the cube is known to four significant figures, the answers must be expressed to four significant figures.

$$\text{Area of cube} = 6 \times (\text{side})^2 = 6 \times (7.203)^2 = 311.299254 \text{ m}^2 = 311.3 \text{ m}^2$$

$$\text{Volume of cube} = (\text{side})^3 = (7.203)^3 = 373.71475 \text{ m}^3 = 373.7 \text{ m}^3$$

PROBLEMS FOR PRACTICE

1. Solve the following to correct significant figures:
   (*i*) 6.32 × 7.4695 (*ii*) 9.75 + 18.1395 (*iii*) 25 ÷ 5.00 **[(*i*) 47.2 (*ii*) 27.89 (*iii*) 5.0]**
2. Divide 68.73 by 22.9 and give the answer in correct significant figures. **[3.00]**
3. The length, breadth and thickness of a rectangular metal sheet are 4.234 m, 1.005 m and 2.01 cm respectively. Give the area and volume of the sheet to correct significant figures. **[4.255 m²; 0.0855 m³]**
4. A tank is 25.00 m long, 10.00 m wide and filled with water to a depth of 2.00 m. What is the mass of water in kg to correct significant figures? **[5.00 × 10⁵ kg]**
5. Convert each of the following to scientific notation:
   (*i*) 175,000,000 (*ii*) 0.000047 (*iii*) 4500
   [(*i*) 1.75 × 10⁸ (*ii*) 4.7 × 10⁻⁵ (*iii*) 4.5 × 10³]
6. Add 16.24, 3.8 and 27.965. Give the answer by keeping the significant figures in view. **[48.0]**
7. How many significant figures are there in each of the following expressions ?
   (*i*) 6.032 m (*ii*) 0.007 kg (*iii*) 0.0006032 m **[(*i*) 4 (*ii*) 1 (*iii*) 4]**
8. The mass of a box measured by a grocer's balance is 2.3kg. Two gold pieces of masses 20.15g and 20.17g are added to the box. What is (*i*) the total mass of the box (*ii*) the difference in the masses of the pieces to correct significant figures?
   [(*i*) 2.3kg (*ii*) 0.02g]

5.7. ERRORS IN MEASUREMENT

Today accurate measurement is an important part of physics. But no measurement is absolutely precise; there is an uncertainty associated with every measurement.

The uncertainty in a measurement is called **error.**

When giving the result of a measurement, it is a good practice to state the estimated uncertainty or error. For example, if the measured length of a table is 23.2 cm and the uncertainty or error is about 0.1 cm, then length will be recorded as 23.2 ± 0.1 cm. The ± 0.1 cm ("plus or minus 0.1 cm") represents the error in the measurement. The true length of the table most likely lies between 23.1 cm and 23.3 cm. Error in measurement may arise due to several sources.

Types of errors. The errors in measurement can be divided into the following three types :

1. Errors due to known causes — Systematic errors
2. Errors due to unknown causes — Random errors
3. Gross errors

1. Systematic errors. *Those errors that appear in a measurement due to known causes are called systematic errors.* Some important causes of such errors are :

(*i*) Incorrect design or calibration of the instrument.

(*ii*) Limitations of the method used for measurement.

(*iii*) Incorrect-reading or interpretation of the instrument.

(*iv*) Lack of accuracy of the formula being used.

The systematic errors cause the measurement to be consistently higher or consistently lower than the true value. In other words, systematic error is of the same size and sign. Repeating the measurement a number of times will have no effect on this type of error and it may not even be suspected until the final result is calculated. Systematic errors can be reduced

(*a*) by careful design of instrument and calibration.

(*b*) by using improved methods of measurement.

2. Random errors. *Those errors which appear in a measurement due to unknown causes are called random errors.* These arise due to a variety of factors which cannot be taken into account. For example, the reading of a sensitive physical balance may be affected on account of settling of dust particles on the pans. These errors have no set pattern and take place in a random manner. The size of these errors depends on how well the experimenter can *use* the apparatus. The better the experimenter you are, the smaller will be the random error that you will introduce into an experiment.

It is not possible to completely eliminate random errors. However, these errors can be reduced by making a number of readings of a given quantity and taking the average. This *arithmetic mean* may be taken as the best estimate of the *true value* of the quantity.

3. Gross errors. *These errors are due to sheer carelessness of the experimenter.* These may result from :

(*i*) Recording the observations wrongly.

(*ii*) Not caring for sources of error and precautions.

(*iii*) Reading the instruments incorrectly.

(*iv*) Entering wrong values of observations in calculations.

No correction can be applied for such errors. These errors can be minimised if the experimenter is careful and mentally alert.

5.8. COMMONLY USED TERMS IN ERROR ANALYSIS

(*i*) **True value.** We cannot determine the correct or true or absolute value of a physical quantity. It is because no measurement is absolutely precise; there is an uncertainty associated with every measurement. In practice, we take a number of readings of the quantity to be measured and take the arithmetic mean of the measured values. This arithmetic mean is accepted as the true value of the quantity.

The **true value** *of a physical quantity is the arithmetic mean of a large number of readings of that quantity.*

If a_1, a_2, a_3 ... a_n are the n different readings of a physical quantity in an experiment, then true value of that quantity is

$$\text{True value, } \bar{a} = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$$

or $$\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$$

The bar (–) over a is the standard symbol for arithmetic mean. Note that $\bar{a}$ is accepted as the true or absolute value of the physical quantity.

(ii) **Absolute error.** *The difference in the magnitudes of true value and the measured value of a physical quantity is called* **absolute** *(i.e. actual)* **error.**

If $a_1, a_2, a_3 ..., a_n$ are n different readings of a physical quantity in an experiment, then true value ($\bar{a}$) of the quantity is

$$\bar{a} = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$$

The absolute errors in the various measured values are :

$$\Delta a_1 = \bar{a} - a_1 ; \ \Delta a_2 = \bar{a} - a_2 \ ..... \ \Delta a_n = \bar{a} - a_n$$

(iii) **Mean absolute error.** *The arithmetic mean of all absolute errors in the measured values is called* **mean absolute error.**

$$\overline{\Delta a} = \frac{|\Delta a_1| + |\Delta a_2| + |\Delta a_3| + ... + |\Delta a_n|}{n}$$

Note that $\overline{\Delta a}$ is also called the *final absolute error.* If the mean value of a physical quantity is $\bar{a}$ and $\overline{\Delta a}$ is the mean absolute error, then the quantity is recorded as :

$$a = \bar{a} \pm \overline{\Delta a}$$

This means that if we make a single measurement, then the result of measurement will lie between $\bar{a} - \overline{\Delta a}$ and $\bar{a} + \overline{\Delta a}$.

(iv) **Relative error and percentage error.** *The ratio of mean absolute error to the mean value (or true value) of the quantity being measured is called* **relative error** or **fractional error.**

$$\text{Relative error} = \frac{\overline{\Delta a}}{\bar{a}}$$

If we multiply the relative error by 100, we get the percentage error.

$$\therefore \quad \text{Percentage error} = \frac{\overline{\Delta a}}{\bar{a}} \times 100$$

Example 5.9. The period of oscillation of a simple pendulum in an experiment is recorded as 2.63 s, 2.56 s, 2.42 s, 2.71 s and 2.80 s respectively. Find *(i)* mean time period *(ii)* absolute error in each observation and percentage error.

Solution. *(i)* Mean time period of the pendulum is given by ;

$$\bar{T} = \frac{2.63 + 2.56 + 2.42 + 2.71 + 2.80}{5} = \frac{13.12}{5} = 2.62 \text{ s}$$

Note that answer is given to correct significant figures. Further, this mean value is accepted as the true value of time period.

(ii) The absolute error in each observation is

$2.62 - 2.63 = -0.01$ s ; $2.62 - 2.56 = 0.06$ s ; $2.62 - 2.42 = 0.20$ s ; $2.62 - 2.71 = -0.09$ s ;

$2.62 - 2.80 = -0.18$ s

Mean absolute error, $\overline{\Delta T} = \frac{\Sigma|\Delta T|}{5} = \frac{0.01+0.06+0.2+0.09+0.18}{5} = 0.11$ s

$\therefore$ Percentage error $= \frac{\overline{\Delta T}}{\overline{T}} \times 100 = \frac{0.11}{2.62} \times 100 = 4.2\%$

In this experiment, the result should be written as: $T = 2.62 \pm 0.11$ s

Example 5.10. **What is the percentage error in the measurement 8.9 ± 0.2 cm?**

Solution. Percentage error $= \frac{0.2}{8.9} \times 100 = 2.2\%$

5.9. PROPAGATION OF ERRORS

Any physics experiment involves a series of measurements and each measurement is associated with a certain amount of error. In order to calculate the final result, we may have to perform mathematical operations like addition, subtraction, multiplication, division *etc.* on these measurements. The final error not only depends upon the errors in individual measurements but also on the mathematical operations performed on these measurements. We shall now calculate the *maximum permissible error in different mathematical operations.

(*i*) Error in sum. Suppose the result Q is given as the sum of two observed quantities a and b *i.e.*

$$Q = a + b$$

If Δa and Δb are the absolute errors in the measurements of quantities a and b, the measurements should be recorded as $a \pm \Delta a$ and $b \pm \Delta b$. If ΔQ is the absolute error in the sum, then,

$$Q \pm \Delta Q = (a \pm \Delta a) + (b \pm \Delta b)$$

or $$Q \pm \Delta Q = (a + b) \pm (\Delta a + \Delta b)$$

or $$\pm \Delta Q = \pm (\Delta a + \Delta b)$$

$\therefore$ Maximum error in $Q = \Delta Q = +(\Delta a + \Delta b) = -(\Delta a + \Delta b)$

$\therefore$ **Maximum absolute error in Q = Maximum absolute error in a + Maximum absolute error in b**

Therefore, when two physical quantities are added, the maximum absolute error in the result is equal to the sum of the absolute errors in the physical quantities.

Illustration. Suppose the two lengths a and b are recorded as:

$a = 25.4$ cm $\pm$ 0.1 cm and $b = 16.5$ cm $\pm$ 0.1 cm

We are required to find the sum $Q = a + b$.

(a) Q has its average value when both a and b have their average values.

$\therefore$ Average value of $Q = 25.4 + 16.5 = 41.9$ cm

(b) Q has its maximum value when both a and b have the largest values.

$\therefore$ Maximum value of $Q = (25.4 + 0.1) + (16.5 + 0.1) = 42.1$ cm

$\therefore$ Magnitude of maximum error in $Q = 42.1 - 41.9 = 0.2$ cm

Note that error in Q is 0.2 cm (= 0.1 cm + 0.1 cm) which is simply the sum of the errors in a and b.

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* We generally define error as the maximum amount by which a physical quantity is expected to differ from its true value. Therefore, we indicate maximum error in the result.

(*c*) Q has its minimum value when both a and b have their minimum values.

$\therefore$ Minimum value of $Q = (25.4 - 0.1) + (16.5 - 0.1) = 41.7$ cm

$\therefore$ Magnitude of maximum error in $Q = 41.9 - 41.7 = 0.2$ cm

Again the error in Q is 0.2 cm *i.e.* the sum of errors in a and b.

The answer should be written as $Q = 41.9 \pm 0.2$ cm.

$$\% \text{ error } = \frac{0.2}{41.9} \times 100 = 0.48\%$$

(*ii*) **Error in difference.** Let $Q = a - b$

If Δa and Δb are the absolute errors in the measurements of quantities a and b, the measurements should be recorded as $a \pm \Delta a$ and $b \pm \Delta b$. If ΔQ is the absolute error in the difference, then,

$$Q \pm \Delta Q = (a \pm \Delta a) - (b \pm \Delta b)$$

or $$Q \pm \Delta Q = (a - b) \pm \Delta a \mp \Delta b$$

or $$\pm \Delta Q = \pm \Delta a \mp \Delta b$$

$\therefore$ Maximum error in $Q = \Delta Q = +(\Delta a + \Delta b) = -(\Delta a + \Delta b)$

$\therefore$ **Maximum absolute error in Q = Maximum absolute error in a + Maximum absolute error in b**

Therefore, when two physical quantities are subtracted, the maximum absolute error in the result is equal to the sum of errors in the physical quantities.

Illustration. Suppose the two lengths a and b are recorded as :

$$a = 25.4 \text{ cm} \pm 0.1 \text{ cm}; \; b = 16.5 \text{ cm} \pm 0.1 \text{ cm}$$

We are required to find the difference $Q = a - b$.

Average value of $Q = 25.4 - 16.5 = 8.9$ cm

Maximum value of $Q^* = (25.4 + 0.1) - (16.5 - 0.1) = 9.1$ cm

$\therefore$ Magnitude of maximum error in $Q = 9.1 - 8.9 = 0.2$ cm

Note that error in Q is 0.2 cm (0.1 cm + 0.1 cm) which is simply the sum of errors in a and b.

Minimum value of $Q^{**} = (25.4 - 0.1) - (16.5 + 0.1) = 8.7$ cm

Magnitude of maximum error in $Q = 8.9 - 8.7 = 0.2$ cm

Again the error in Q is 0.2 cm *i.e.* the sum of errors in a and b.

The answer should be written as $Q = 8.9 \pm 0.2$ cm.

$$\% \text{ error } = \frac{0.2}{8.9} \times 100 = 2.2\%$$

(*iii*) **Error in Product.** Let $Q = ab$

If Δa and Δb are the absolute errors in the measurement of quantities a and b, then measurements should be recorded as $a \pm \Delta a$ and $b \pm \Delta b$. If ΔQ is the absolute error in the result, then,

$$Q \pm \Delta Q = (a \pm \Delta a)(b \pm \Delta b)$$

---

* The value of Q will be maximum when a has largest value and b has smallest value.

** Q has its minimum value when b has its largest value and a has the smallest value.

or $$Q \pm \Delta Q = ab \pm a\,\Delta b \pm b\,\Delta a \pm \Delta a \Delta b$$

or $$\pm \Delta Q = \pm b\,\Delta a \pm a\,\Delta b \pm \Delta a\,\Delta b$$

Dividing both sides by Q (= ab), we get,

$$\frac{\pm \Delta Q}{Q} = \pm\frac{\Delta a}{a} \pm \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \times \frac{\Delta b}{b}$$

The terms $\frac{\Delta Q}{Q}$, $\frac{\Delta a}{a}$ and $\frac{\Delta b}{b}$ are the relative or fractional errors in Q, a and b respectively.

We can ignore the term $\frac{\Delta a}{a} \times \frac{\Delta b}{b}$ since it is the product of two small quantities and is therefore small *in comparison* with other terms.

$\therefore$ $$\pm\frac{\Delta Q}{Q} = \pm\frac{\Delta a}{a} \pm \frac{\Delta b}{b}$$

$\therefore$ Maximum fractional error in $Q = \frac{\Delta Q}{Q} = +\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right) = -\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$

$\therefore$ **Maximum fractional error in Q = Maximum fractional error in a + Maximum fractional error in b**

Therefore, when two physical quantities are multiplied, the maximum relative error in the result is equal to the sum of relative errors in the physical quantities.

Now $$\frac{\Delta Q}{Q} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

or $$\frac{\Delta Q}{Q} \times 100 = \frac{\Delta a}{a} \times 100 + \frac{\Delta b}{b} \times 100$$

$\therefore$ **Max. percentage error in Q = Max. percentage error in a + Max. percentage error in b**

Therefore, when two physical quantities are multiplied, the maximum percentage error in the result is equal to the sum of percentage errors in the physical quantities.

Illustration. Consider the relation $F = m\,a$. Here F is the force acting on a body of mass m and produces acceleration a in the body. Suppose mass and acceleration are recorded as:

$$m = 3.5 \text{ kg} \pm 0.1 \text{ kg}; \; a = 10 \text{ m s}^{-2} \pm 1 \text{ ms}^{-2}$$

It is desired to find the fractional error and percentage error in the measurement of F.

$$F = m\,a \;\therefore\; \frac{\Delta F}{F} = \frac{\Delta m}{m} + \frac{\Delta a}{a}$$

or $$\frac{\Delta F}{F} = \frac{0.1}{3.5} + \frac{1}{10} = 0.03 + 0.1 = 0.13$$

Therefore, fractional error in F is 0.13 and percentage error = 0.13 × 100 = 13%. We can also achieve this result as:

$$\% \text{ error in } F = \% \text{ error in } m + \% \text{ error in } a$$

$$= 0.03 \times 100 + 0.1 \times 100 = 3\% + 10\% = 13\%$$

Note. $F = m\,a = 3.5 \times 10 = 35$ N and $\frac{\Delta F}{F} = 0.13 \;\therefore \Delta F = 0.13 \times F = 0.13 \times 35 = 4.6$ N.

The answer should be given as $F = 35 \pm 4.6$ N ; % error in $F = \frac{4.6}{35} \times 100 = 13\%$

(*iv*) **Error in division.** Let $Q = \frac{a}{b}$

If Δa and Δb are the absolute errors in the measurements of quantities a and b, the measurement should be recorded as $a \pm \Delta a$ and $b \pm \Delta b$. If ΔQ is the absolute error in the result, then,

$$Q \pm \Delta Q = \frac{(a \pm \Delta a)}{(b \pm \Delta b)} = \frac{a\left(1 \pm \frac{\Delta a}{a}\right)}{b\left(1 \pm \frac{\Delta b}{b}\right)} = \frac{a}{b}\left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)^{-1}$$

or $$Q \pm \Delta Q = \frac{a}{b}\left(1 \pm \frac{\Delta a}{a}\right)\left(1 \mp \frac{\Delta b}{b}\right)^{*}$$

or $$Q \pm \Delta Q = \frac{a}{b}\left(1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b} \mp \frac{\Delta a \Delta b}{ab}\right)$$

We can ignore the term $\Delta a \Delta b/ab$ since it is small in comparison with other terms.

$\therefore$ $$Q \pm \Delta Q = \frac{a}{b}\left(1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}\right)$$

or $$Q \pm \Delta Q = Q\left(1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}\right) \qquad \left(\because Q = \frac{a}{b}\right)$$

or $$\frac{Q \pm \Delta Q}{Q} = 1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}$$

or $$1 \pm \frac{\Delta Q}{Q} = 1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}$$

or $$\pm \frac{\Delta Q}{Q} = \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}$$

$\therefore$ Maximum fractional error in $Q = \frac{\Delta Q}{Q} = +\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right) = -\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$

$\therefore$ **Maximum fractional error in Q = Maximum fractional error in a + Maximum fractional error in b**

Therefore, when two physical quantities are divided, the maximum relative error in the result is equal to the sum of relative errors in the physical quantities.

Now $$\frac{\Delta Q}{Q} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

or $$\frac{\Delta Q}{Q} \times 100 = \frac{\Delta a}{a} \times 100 + \frac{\Delta b}{b} \times 100$$

$\therefore$ **Max. percentage error in Q = Max. percentage error in a + Max. percentage error in b**

Therefore, when two physical quantities are divided, the maximum percentage error in the result is equal to the sum of percentage errors in the physical quantities.

Illustration. Consider the relation $\rho = m/V$. Here ρ is the density of an object of mass m and having volume V. Suppose mass and volume are recorded as:

$$m = 3.5 \text{ kg} \pm 0.1 \text{ kg} ; V = 10 \text{ m}^3 \pm 1 \text{ m}^3$$

* Other terms being small and, therefore, neglected.

It is desired to find the fractional error and percentage error in the measurement of ρ.

$$\rho = \frac{m}{V} \qquad \therefore \frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta V}{V} = \frac{0.1}{3.5} + \frac{1}{10} = 0.03 + 0.1 = 0.13$$

Therefore, fractional error in ρ is 0.13 and percentage error = 0.13 × 100 = 13%.

Alternatively; % error in ρ = % error in m + % error in V

= 0.03 × 100 + 0.1 × 100 = 3% + 10% = 13%

(*v*) **Error in power of a quantity.** Let $Q = a^2$

If Δa is the absolute error in the measurement of the quantity a, then the measurement should be recorded as $a \pm \Delta a$. If ΔQ is the absolute error in the result, then,

$$Q \pm \Delta Q = (a \pm \Delta a)^2 = a^2 + (\Delta a)^2 + 2a\ \Delta a$$

We can ignore the term $(\Delta a)^2$, since it is the product of two small quantities and is therefore small in comparison with other terms.

$$\therefore \qquad Q \pm \Delta Q = a^2 \pm 2a\ \Delta a$$

or $$\pm \Delta Q = \pm 2a\ \Delta a$$

Dividing both sides by $Q = a^2$, we have,

$$\pm\frac{\Delta Q}{Q} = \pm\frac{2a\Delta a}{a^2} = \pm 2\frac{\Delta a}{a}$$

$\therefore$ Maximum fractional error in $Q = \frac{\Delta Q}{Q} = +2\times\frac{\Delta a}{a} = -2\times\frac{\Delta a}{a}$

$\therefore$ Maximum fractional error in Q = 2 × fractional error in a

In general, if $Q = a^n$, then, $\pm\frac{\Delta Q}{Q} = \pm n\frac{\Delta a}{a}$

i.e. the fractional error (or percentage error) is n times the fractional error (or percentage error) in a. Note that pure numbers have no errors.

(*vi*) **General case.** Let $Q = \frac{a^l b^m}{c^n}$

It can be proved that : $\frac{\Delta Q}{Q} = l\times\frac{\Delta a}{a} + m\times\frac{\Delta b}{b} + n\frac{\Delta c}{c}$

i.e. maximum fractional error in $Q = l \times$ fractional error in $a + m \times$ fractional error in $b + n \times$ fractional error in c

Example 5.11. The force acting on an object of mass m, travelling at velocity v in a circle of radius r is given by ;

$$F = \frac{mv^2}{r}$$

The measurements are recorded as m = 3.5 kg ± 0.1kg ; v = 20 ms^{-1} ± 1ms^{-1} ; r = 12.5 m ± 0.5 m.

Find the maximum possible (*i*) fractional error (*ii*) percentage error in the measurement of force. How will you record the reading ?

Solution.

(*i*) $$F = \frac{mv^2}{r} \qquad \therefore \frac{\Delta F}{F} = \frac{\Delta m}{m} + 2\times\frac{\Delta v}{v} + \frac{\Delta r}{r}$$

PROBLEMS FOR PRACTICE

1. The error in measuring mass is 6% and volume is 1%. What will be the error in the measurement of density? **[7%]**

2. The period of oscillation of a simple pendulum is given by :

$$T = 2\pi\sqrt{\frac{l}{g}}$$

100 vibrations are taken and the total time measured is 200 seconds. The least count of stop watch is 0.1 s. The length of the pendulum is 1 m. The least count of scale is 1 mm. What is the maximum error in the measurement of g ? **[0.2%]**

3. Calculate the maximum possible error in Y in the following relation:

$$Y = \frac{4MgL}{\pi D^2 l}$$

where mass M = 1000 g; original length L = 200.00 cm; diameter D = 0.075 cm; extension l = 0.325 cm. **[3.03%]**

Hint. Mass is constant and does not play any part in error determination. Note that significant figures indicate the least count of the measuring device. Thus length is measured with a metre scale whose least count is 0.1 cm. The diameter and extension are measured with screw gauge whose least count is 0.001 cm.

4. A physical quantity P is given by : $P = ab^2c^3d^4$

The percentage error in measurement of a, b, c and d are 0.5% each. What is the percentage error in P? **[5%]**

5. The length, breadth and thickness of a block of wood are recorded as:

l = (15.12 ± 0.01) cm ; b = (10.15 ± 0.01) cm ; h = (5.29 ± 0.01) cm

Find the percentage error in the volume of the block. **[0.36%]**

CONCEPTUAL QUESTIONS

Q.1. The period of oscillation of a simple pendulum is given by ;

$$T = 2\pi\sqrt{\frac{l}{g}}$$

In finding the value of g, which quantity should be measured most accurately and why?

Ans. $$T = 2\pi\sqrt{\frac{l}{g}}$$

or $$T^2 = \frac{4\pi^2 l}{g} \quad \text{or} \quad g = 4\pi^2\frac{l}{T^2}$$

$$\therefore \text{ Fractional error in } g = \frac{\Delta g}{g} = \frac{\Delta l}{l} + \frac{2\Delta T}{T}$$

Note that any error in T is doubled up. Therefore, time should be measured most accurately.

Q.2. The error in the measurement of the radius of a sphere is 0.4 %. What is the permissible error in the measurement of surface area?

Ans. Fractional error in radius $= \frac{0.4}{100} = 0.004$

Surface area of sphere, $A = 4\pi r^2$ $\therefore \frac{\Delta A}{A} = 2\frac{\Delta r}{r} = 2\times 0.004 = 0.008$

$\therefore$ % error in $A = \frac{\Delta A}{A}\times 100 = 0.008\times 100 = 0.8\%$

Note that %age error in A is twice the error in r.

Q.3. Three lengths are given as 3.7cm, 48.78cm and 6.71cm. What do you infer from these readings ?

Ans. It is clear from these readings that measurements have been made using instruments of different least count.

Q.4. The diameter of a steel rod is given as (56. 47 ± 0.02) mm. What does it mean?

Ans. It means that true value of diameter is unlikely to be less than 56.45 mm or greater than 56.49 mm.

Q.5. A bag contains 24 identical balls. A person makes measurement on one of them and determines its volume to be 1.75 cm³. What will be the volume of all balls ?

Ans. Total volume of balls = 1.75 × 24 = 42.0 cm³

Note that the answer has been expressed to three significant figures. It is because the number of balls is exact and there is no error in counting them.

VERY SHORT ANSWER QUESTIONS

Q.1. Define significant figures.

Ans. The number of digits in a measurement about which we are certain *plus* one additional digit which is uncertain are known as significant figures.

Q.2. Which of the following length measurement is most accurate and why?
(*i*) 2.0cm (*ii*) 2.00cm (*iii*) 2.000cm.

Ans. The measurement 2.000 cm is most accurate because it has four significant figures.

Q.3. What is the difference between 4.0 and 4.000?

Ans. The measurement 4.0 has two significant figures while 4.000 has four significant figures. In other words, measurement 4.0 is made accurate upto first decimal place while the measurement 4.000 is made accurate upto third decimal place.

Q.4. In a number without decimal, what is the significance of zeros on the right of non-zero digits?

Ans. All such zeros are not significant. For example, measurement 625000 has only three significant figures.

Q.5. The mass of a body as measured by two students is given as 9.2kg and 9.23kg. Which is more accurate and why?

Ans. The measurement of 9.23kg is more accurate as it has been made upto second place of decimal.

Q.6. Round off to 3 significant figures (*i*) 15.87m (*ii*) 0.04134.

Ans. (*i*) 15.9m (*ii*) 0.0413.

Q.7. Round off to four significant figures (*i*) 34.879 (*ii*) 1.0074.

Ans. (*i*) 34.88 (*ii*) 1.007.

Q.8. Which quantity in a given formula should be measured most accurately?

Ans. The quantity in the formula which has maximum power (n) should be measured most accurately. It is because any error in the measurement of this quantity is multiplied n times in the final result.

Q.9. What is the percentage error in the volume of a sphere when error in measuring its radius is 2%?

Ans. $V = \frac{4}{3}\pi r^3$ $\therefore \frac{\Delta V}{V} = 3\times\frac{\Delta r}{r}$ = 3(±2%) = ±6%.

Q.10. A person takes 10 observations in an experiment. If he repeats the same experiment by taking 50 observations, how is the probable error affected?

Ans. Since the number of observations is made 5 times, the probable error is reduced 5 times.

Q.11. If $\theta_1 = 25.4 \pm 0.1°C$ and $\theta_2 = 52.7 \pm 0.1°C$, find $\theta_2 - \theta_1$.

Ans. $\theta_2 - \theta_1 = (52.7 \pm 0.1) - (25.4 \pm 0.1) = 27.3 \pm 0.2°C$.

Q.12. A substance of mass 5.74 g occupies a volume of 1.2 cm^3. Find its density with due regard to significant figures.

Ans. Density $= \dfrac{\text{Mass}}{\text{Volume}} = \dfrac{5.74\,g}{1.2\,cm^3} = 4.78 g/cm^3 = 4.8 g/cm^3$ (rounded off to 1 decimal place).

Q.13. Two resistances $R_1 = 100 \pm 3$ ohm and $R_2 = 200 \pm 4$ ohm are connected in series. Find equivalent resistance with error limits.

Ans. Equivalent resistance $= R_1 + R_2 = (100 \pm 3) + (200 \pm 4) = (300 \pm 7)$ ohm.

Q.14. If $m_1 = 1.2$ kg and $m_2 = 5.42$ g, find $m_1 + m_2$ with due regard to significant figures.

Ans. $m_1 = 1.2 kg$; $m_2 = 5.42 g = 0.00542 kg$

$\therefore \quad m_1 + m_2 = 1.2 + 0.00542 = 1.20542$ kg

The mass m_1 has the least number of decimal places (*i.e.*, 1). Therefore, rounding off the above result, $m_1 + m_2 = 1.2$ kg.

Q.15. If $l_1 = 25.2 \pm 0.1$cm and $l_2 = 16.8 \pm 0.1$cm, find $l_1 + l_2$ with the limits of error.

Ans. $l_1 + l_2 = (25.2 \pm 0.1) + (16.8 \pm 0.1) = 42.0 \pm 0.2$cm.

SHORT ANSWER QUESTIONS

Q.1. A potential difference of $V = (100 \pm 2)$ volt applied to a resistance R gives a current $I = (10 \pm 0.5)$ A. Find percentage error in R if $R = V/I$.

Ans. $\dfrac{\Delta R}{R} \times 100 = \left(\dfrac{\Delta V}{V} + \dfrac{\Delta I}{I}\right) \times 100 = \pm\left(\dfrac{2}{100} + \dfrac{0.5}{10}\right) \times 100 = \pm 7\%$.

Q.2. A physical quantity X is given by ;

$$X = \frac{A^4 B^{1/3}}{CD^{3/2}}$$

Find the fractional error in X.

Ans. $\dfrac{\Delta X}{X} = 4\dfrac{\Delta A}{A} + \dfrac{1}{3}\dfrac{\Delta B}{B} + \dfrac{\Delta C}{C} + \dfrac{3}{2}\dfrac{\Delta D}{D}$.

Q.3. A physical quantity X is given by ;

$$X = \frac{a^2 b^3}{c\sqrt{d}}$$

If the percentage errors of measurements in a, b, c and d are 4%, 2%, 3% and 1% respectively, find the percentage error in X.

Ans. $$X = \frac{a^2 b^3}{c\sqrt{d}}$$

$$\% \text{ error in } X = 2 \times \% \text{ error in } a + 3 \times \% \text{ error in } b + \% \text{ error in } c + \frac{1}{2} \times \% \text{ error in } d$$

$$= 2 \times 4\% + 3 \times 2\% + 3\% + \frac{1}{2} \times 1\% = 17.5\%.$$

Q.4. The diameter of a circle is 1.06 m. Calculate its area with due regard to significant figures.

Ans. $$\text{Area} = \frac{\pi D^2}{4} = \frac{\pi \times (1.06)^2}{4} = 0.88202\ \text{m}^2$$

Since the diameter has three significant figures, the result must have three significant figures. Rounding off the result to three significant figures, area = 0.882m^2.

Q.5. The original length of a wire is (153.7 ± 0.6)cm. It is stretched to (155.3 ± 0.2)cm. Calculate the elongation in wire with error limits.

Ans. Elongation = 155.3 – 153.7 = 1.6 cm ; $\Delta l = \pm(\Delta l_1 + \Delta l_2) = \pm(0.6 + 0.2) = \pm 0.8\text{cm}$.

$\therefore$ Elongation = (1.6 ± 0.8) cm

Q.6. What is the percentage error in the density of a cube when errors in measuring its mass and length are ±2% and ±1% respectively?

Ans. Density, $\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{L^3}$ $\quad\therefore\quad \frac{\Delta\rho}{\rho} = \frac{\Delta M}{M} + \frac{3\Delta L}{L}$

or $$\frac{\Delta\rho}{\rho} = \pm(2\% + 3 \times 1\%) = \pm 5\%$$

Q.7. If all measurements in an experiment are taken upto same number of significant figures, then which measurement is responsible for maximum error?

Ans. The quantity in the formula which has maximum power is responsible for maximum error. If all the quantities in the formula have the same powers, then the quantity which is least in magnitude is responsible for maximum error.

Q.8. Subtract with due regard to significant figures: $3.9 \times 10^5 - 2.5 \times 10^4$.

Ans. $3.9 \times 10^5 - 2.5 \times 10^4 = 390000 - 25000 = 365000 = 3.65 \times 10^5$

Since 3.9×10^5 and 2.5×10^4 have two significant figures, the answer should also have two significant figures. Rounding off to two significant figures, the answer is 3.7×10^5.

Q.9. What do you understand by absolute error?

Ans. The difference in the magnitudes of true value and the measured value of a physical quantity is called absolute (*i.e.,* actual) error.

Q.10. Solve with due regard to significant figures: $\frac{0.9996 \times 3.52}{1.758}$.

Ans. $\frac{0.9996 \times 3.52}{1.758} = 2.00147$.

Since 3.52 has least number of significant figures (*i.e.,* 3), the answer must have three significant figures. Rounding off to three significant figures, the answer is 2.00.

LONG ANSWER QUESTIONS

Q.1. Give the rules for counting significant figures. **[Refer to Art. 5.4]**

Q.2. Discuss briefly the various types of errors in measurement. **[Refer to Art. 5.7]**

Q.3. Discuss how errors propagate in sum, difference, product and division of quantities. **[Refer to Art. 5.9]**

Q.4. Explain systematic errors and random errors. **[Refer to Art. 5.7]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The **limit of precision** of a measuring device is ±1/2 of the least count of the device. We generally take limiting precision equal to the least count of the device.
2. **Significant Figures.** In order to count significant figures in a quantity, observe the following rule:

   The number of significant figures in a number is the number of digits other than leading or trailing zero/zeros. Trailing zeros do count if they are after the decimal point. Thus 0.2457 has four significant figures as does 345600. Similarly 3.14 has three significant figures and 0.003800 has four significant figures.

   (*i*) The digit in a number that is farthest to the right is called **least significant digit.** For example, the least significant figure in 3.456 is 6 and in 0.003800 is the rightmost zero.

   (*ii*) When numbers are multiplied or divided, the result has the same number of significant figures as are in the least precise quantity given. For example :

   $$\frac{0.456\times7.8}{9.0123}=0.39$$

   (*iii*) In addition or subtraction, the number of decimal places in the result should be equal to the smallest number of decimal places of any term in the sum or subtraction.
3. **Communication of errors.** We generally define error as the maximum amount by which a physical quantity is expected to differ from its true value.

S.No.	Mathematical Operation	Error
(*i*)	Addition : $x = a + b$	$\Delta x = \pm(\Delta a + \Delta b)$
(*ii*)	Subtraction : $x = a - b$	$\Delta x = \pm(\Delta a + \Delta b)$
(*iii*)	Division : $x = \frac{a}{b}$	$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$
(*iv*)	Multiplication : $x = ab$	$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$
(*v*)	Exponential power : $x = a^n$	$\frac{\Delta x}{x} = \pm n\frac{\Delta a}{a}$

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The percentage errors in the measurement of mass and speed are 2% and 3% respectively. The error in kinetic energy obtained by measuring mass and speed will be **[AFMC - Pune 2001]**

   (*a*) 12% (*b*) 10%

   (*c*) 8% (*d*) 5%

2. The length and breadth of a metal sheet are 3.124 m and 3.002 m respectively. The area of this sheet upto four correct significant figures is **[AIIMS 2001]**

   (*a*) 9.37 m^2 (*b*) 9.378 m^2

   (*c*) 9.3782 m^2 (*d*) 9.378248 m^2

3. The number of significant figures in 5418000 are **[Pb PMT 1999]**

(*a*) 4 (*b*) 5

(*c*) 6 (*d*) 7

4. The significant figures in the number 6.0023 are **[Pb PMT 1998]**

(*a*) 1 (*b*) 4

(*c*) 5 (*d*) 2

5. If the length of the pendulum is (0.8 ± 0.01) m and time period is (2.5 ± 0.12) sec, then percentage error in acceleration due to gravity will be **[AFMC - Pune 1997]**

(*a*) 10.85% (*b*) 12.75%

(*c*) 15.85% (*d*) 21.12%

6. Two resistances are expressed as $R_1 = (4 \pm 0.5)\,\Omega$ and $R_2 = (12 \pm 0.5)\,\Omega$. What is the maximum absolute error when they are connected in series? **[AFMC - Pune 1995]**

(*a*) 5 Ω (*b*) 1 Ω

(*c*) 10 Ω (*d*) 15 Ω

7. A force F is applied on to a square plate of side L. If percentage error in determining L is 2% and that in F is 4%, what is the permissible error in pressure ? **[CPMT 1993]**

(*a*) 2% (*b*) 4%

(*c*) 6% (*d*) 8%

8. The density of a cube is measured by measuring its mass and length. If maximum errors in the measurement of mass and length are 4% and 3% respectively, the maximum error in the measurement of density would be **[CBSE PMT 1996]**

(*a*) 9% (*b*) 13%

(*c*) 12% (*d*) 7%

9. An experimenter measures quantities a, b, c and then X is calculated from : **[CPMT 1989]**

$$X = \frac{ab^2}{c^3}$$

If the percentage errors in a, b, c are ± 1%, ± 3%, ± 2% respectively, the percentage error in X can be

(*a*) ± 13% (*b*) ± 7%

(*c*) ±4% (*d*) ± 1%

10. If error in measurement of radius of sphere is 1%, what will be the error in measurement of volume ? **[AFMC 2005]**

(*a*) 1% (*b*) 1/3%

(*c*) 3% (*d*) 10%

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*c*)	**2.** (*b*)	**3.** (*a*)	**4.** (*c*)	**5.** (*a*)
6. (*b*)	**7.** (*d*)	**8.** (*b*)	**9.** (*a*)	**10.** (*c*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. $\frac{\Delta m}{m} = 2\% = 0.02$; $\frac{\Delta v}{v} = 3\% = 0.03$

Kinetic energy, K.E. $= \frac{1}{2}mv^2 \therefore \frac{\Delta K.E.}{K.E.} = \frac{\Delta m}{m} + 2\frac{\Delta v}{v} = 0.02 + 2\times 0.03 = 0.08 = 8\%$

2. Length, l = 3.124 m; Breadth, b = 3.002 m; Area = lb = 3.124 × 3.002 = 9.378248 m^2

The area upto four significant figures is **9.378 m^2**.

3. Since there is no decimal point, the trailing zeros are not significant. Therefore, the number of significant figures is **4**.

4. Since the trailing zeros are after the decimal point, they are to be counted. Therefore, the number of significant figures is **5**.

5. $l = 0.8$ m ; $\Delta l = 0.01$m ; $T = 2.5$ sec ; $\Delta T = 0.12$ sec

Now $g = 4\pi^2 \dfrac{l}{T^2}$

or $\dfrac{\Delta g}{g} = \dfrac{\Delta l}{l} + \dfrac{2\Delta T}{T} = \dfrac{0.01}{0.8} + 2\times\dfrac{0.12}{2.5} = 0.1085 = \mathbf{10.85\%}$

6. $\Delta R_1 = 0.5\Omega$; $\Delta R_2 = 0.5\Omega$

When resistances are connected in series, $R = R_1 + R_2$

$\therefore$ $\Delta R = \Delta R_1 + \Delta R_2 = 0.5 + 0.5 = \mathbf{1\Omega}$

7. $\dfrac{\Delta L}{L} = 2\% = 0.02$; $\dfrac{\Delta F}{F} = 4\% = 0.04$

Pressure, $P = \dfrac{\text{Force}}{\text{Area}} = \dfrac{F}{L^2}$ $\therefore$ $\dfrac{\Delta P}{P} = \dfrac{\Delta F}{F} + \dfrac{2\Delta l}{L}$

$\therefore$ $\dfrac{\Delta P}{P} = 0.04 + 2\times 0.02 = 0.08 = \mathbf{8\%}$

8. $\dfrac{\Delta l}{l} = 3\% = 0.03$; $\dfrac{\Delta m}{m} = 4\% = 0.04$

Density, $d = \dfrac{\text{mass}}{\text{volume}} = \dfrac{m}{l^3}$ $\therefore$ $\dfrac{\Delta d}{d} = \dfrac{\Delta m}{m} + 3\dfrac{\Delta l}{l} = 0.04 + 3\times 0.03 = 0.13 = \mathbf{13\%}$

9. $\dfrac{\Delta a}{a} = \pm 1\% = \pm\ 0.01$; $\dfrac{\Delta b}{b} = \pm 3\% = \pm 0.03$; $\dfrac{\Delta c}{c} = \pm 2\% = \pm 0.02$

$X = \dfrac{ab^2}{c^3}$ $\therefore$ $\dfrac{\Delta X}{X} = \dfrac{\Delta a}{a} + 2\dfrac{\Delta b}{b} + 3\dfrac{\Delta c}{c}$

or $\dfrac{\Delta X}{X} = (\pm 0.01) + (2\times \pm 0.03) + (3\times \pm 0.02)$

$= \pm 0.01 + (\pm 0.06) + (\pm 0.06) = \pm 0.13 = \pm\mathbf{13\%}$

10. Volume of sphere, $V = \dfrac{4}{3}\pi r^3$ $\therefore$ Error in volume, $\dfrac{\Delta V}{V} = 3\left(\dfrac{\Delta r}{r}\right)$

$\therefore$ % error in volume = 3 × percentage error in radius

$= 3 \times 1\% = \mathbf{3\%}$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A simple pendulum completes 20 oscillations in 25 s. The time is measured with a stop watch of least count 0.1s. The error in the measurement of time is **[CET 1999]**

(*a*) 1.6% (*b*) 1.2%

(*c*) 0.8% (*d*) 0.4%

2. A student measured the diameter of a wire using a screw gauge with least count of 0.001 cm and listed the measurement. The correct measurement is **[SCRA 1998]**

(*a*) 5.3 cm (*b*) 5.32 cm

(*c*) 5.320 cm (*d*) 5.3200 cm

3. The external and internal diameter of a hollow cylinder are measured to be (4.23 ± 0.01) cm and (3.89 ± 0.01) cm. The thickness of the wall of the cylinder is **[MP PET 1993]**

(*a*) (0.34 ± 0.02) cm (*b*) (0.17 ± 0.02) cm

(*c*) (0.17 ± 0.01)cm (*d*) (0.34 ± 0.01) cm

4. If a current of 2.33A is passed through a resistance of 10.485 Ω, the potential is 24.43005 V. Its value in proper significant figures would be **[Karnataka CET 1994]**

(a) 24.43 V (b) 24.4 V

(c) 24.430 V (d) 24.43005 V

5. When a current of (2.5 ± 0.5)A flows through a wire, it develops a potential difference of (20 ± 1)V. The resistance of the wire is **[CET 1994]**

(a) (8 ± 2) Ω (b) (8 ± 1.5) Ω

(c) (8 ± 0.5) Ω (d) (8 ± 3) Ω

6. The relative density of a metal may be found by hanging a block of the metal from a spring balance and noting that in air, the balance reads (5.00 ± 0.05)N while in water it reads (4.00 ± 0.05) N. The relative density of metal would be quoted as **[MP PET 1995]**

(a) 5.00 ± 0.05 (b) 5.00 ± 0.10

(c) 5.00 ± 6% (d) 5.00 ± 11%

7. A quantity is measured by $X = M^aL^bT^c$. The %age error in the measurement of M, L and T are $\alpha\%$, $\beta\%$ and $\gamma\%$ respectively. The %age error in X would be **[CEET 1995]**

(a) $(\alpha a + \beta b + \gamma c)\%$

(b) $(\alpha a - \beta b + \gamma c)\%$

(c) $(\alpha a - \beta b - \gamma c)\%$

(d) none of above

8. A wire has a length $l = 6 \pm 0.06$ cm and radius $r = 0.5 \pm 0.005$ cm and mass $m = 0.3 \pm 0.003$ gm. Maximum percentage error in density is **[IIT 2004]**

(a) 4 (b) 2

(c) 1 (d) 6.8

9. A student performs an experiment for determination of $g\left(=\frac{4\pi^2 l}{T^2}\right)$. The error in length l is Δl and in time T is ΔT and n is the number of times the reading is taken. The measurement of g is most accurate for **[IIT 2006]**

	Δl	ΔT	n
(a)	5 mm	0.2 sec	10
(b)	5 mm	0.2 sec	20
(c)	5 mm	0.1 sec	10
(d)	1 mm	0.1 sec	50

10. A student performs an experiment to determine the Young's modulus of a wire, exactly 2 m long, by Searle's method. In a particular reading, the student measures the extension in the length of the wire to be 0.8 mm with an uncertainty of ± 0.05 mm at a load of exactly 1.0 kg. The student also measures the diameter of the wire to be 0.4 mm with an uncertainty of ±0.01 mm. Take $g = 9.8$ m/s^2 (exact). The Young's modulus obtained from the reading is **[IIT 2007]**

(a) $(2.0 \pm 0.3) \times 10^{11}$ N/m^2

(b) $(2.0 \pm 0.2) \times 10^{11}$ N/m^2

(c) $(2.0 \pm 0.1) \times 10^{11}$ N/m^2

(d) $(2.0 \pm 0.05) \times 10^{11}$ N/m^2

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (d)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (a)
6. (d)	**7.** (a)	**8.** (a)	**9.** (d)	**10.** (b)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Error in measuring 25 sec = 0.1s $\therefore$ % error $= \frac{0.1}{25} \times 100 = 0.4\%$

2. The measurement is correct upto 3 decimal places. Therefore, it must be 5.320 cm.

3. $$x = x_1 - x_2 = 4.23 - 3.89 = 0.34 \text{ cm}$$
$$\Delta x = \pm(0.01 + 0.01) = \pm 0.02 \text{cm}$$
$\therefore$ Thickness of wall of cylinder $= x \pm \Delta x = (0.34 \pm 0.02)$ cm

4. The least precise term is 2.33 A (three significant figures). Therefore, the answer should be upto three significant figures. Consequently, the proper answer is **24.4 V.**

5. $R = \frac{V}{I} = \frac{20}{2.5} = 8\ \Omega$ or $\frac{\Delta R}{R} = \frac{\Delta V}{V} + \frac{\Delta I}{I} = \frac{1}{20} + \frac{0.5}{2.5} = \frac{1}{4}$

$\therefore \quad \Delta R = R \times \frac{1}{4} = 8 \times \frac{1}{4} = 2\Omega$ or $R = (8 \pm 2)\ \Omega$

6. Relative density, $d = \frac{\text{Wt. in air}}{\text{Loss of wt. in water}} = \frac{5}{5-4} = 5$

$$\frac{\Delta d}{d} = \frac{0.05}{5.00} + \frac{0.05+0.05}{1.00} = 0.11 = 11\%$$

$\therefore$ Relative density = **5.00 ± 11%**

7. $X = M^a L^b T^c$

$$\therefore \quad \frac{\Delta X}{X} \times 100 = \left(a\frac{\Delta M}{M} + b\frac{\Delta L}{L} + c\frac{\Delta T}{T} \right) \times 100 = (a\alpha + b\beta + c\gamma)\%$$

8. Density, $\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{m}{\pi r^2 l}$ $\quad \therefore \frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{2\Delta r}{r} + \frac{\Delta l}{l}$

or $\frac{\Delta\rho}{\rho} = \frac{0.003}{0.3} + \frac{2\times0.005}{0.5} + \frac{0.06}{6} = 0.04$

$\therefore$ % error in density = 100 × 0.04 = **4%**

9. $g = \frac{4\pi^2 l}{T^2}$ $\quad \therefore \frac{\Delta g}{g} = \frac{\Delta l}{l} + 2\frac{\Delta T}{T}$

It is clear from the given table that Δl is minimum in option (*d*), ΔT is minimum in options (*c*) and (*d*) and n is maximum in option (*d*). Therefore, error will be minimum in option (*d*). Consequently, g is most accurate for option (*d*).

10. Young's modulus, $Y = \frac{\text{Stress}}{\text{Strain}} = \frac{F/(\pi d^2/4)}{\Delta L/L}$

or $$Y = \frac{4FL}{\pi d^2 \Delta L} = \frac{4\times(1\times9.8)\times2}{\pi\times(0.4\times10^{-3})^2\times0.8\times10^{-3}} = 2 \times 10^{11}\ \text{N/m}^2$$

Now $$\frac{\Delta Y}{Y} = \frac{2\Delta d}{d} + \frac{\Delta(\Delta L)}{\Delta L} = \left(\frac{2\times0.01}{0.4}\right) + \left(\frac{0.05}{0.8}\right)$$

$\therefore$ $$\Delta Y = \left[\frac{2\times0.01}{0.4} + \frac{0.05}{0.8}\right] \times 2\times10^{11} = 0.2 \times 10^{11}\ \text{N/m}^2$$

Therefore, Young's modulus obtained from the reading is $(2.0 \pm 0.2) \times 10^{11}\ \text{N/m}^2$.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **A jeweller puts a diamond in a box of mass 1.5 kg. If the mass of the diamond is 3.57g, find the total mass of the box and diamond with due regard to significant figures.**

Hint. Mass of box = 1.5 kg; Mass of diamond = 3.57 g = 0.00357 kg
$\therefore$ Total mass = 1.5 + 0.00357 = 1.50357 kg

The least precise term (*i.e.*, 1.5) is known only to one decimal place. Therefore, the answer should be expressed to one decimal place. Therefore, total mass = **1.5 kg.**

2. **The initial temperature of a body is (15 ± 0.5)°C and the final temperature is (17±0.3)°C. What is the rise of temperature of the body?**

Hint. Rise in temperature = Final temperature–Initial temperature

$$= (17 \pm 0.3) - (15 \pm 0.5) = (17 - 15) \pm (0.3 + 0.5) = (2 \pm 0.8)^\circ C$$

Note that if two physical quantities are to be added or subtracted from each other, the errors in the two quantities are added together for both addition and subtraction.

3. **The length, breadth and thickness of a block of wood were measured with the help of a metre stick .The results after calculating the errors are as follows :**

$l = (15.12 \pm 0.01)$ cm ; $b = (10.15 \pm 0.01)$ cm ; $h = (5.28 \pm 0.01)$ cm

Find the percentage error in the volume of the block.

Hint. The volume of the block is given by ; $V = lb\,h$

Taking log of both sides, we have, $\log V = \log l + \log b + \log h$

Differentiating, we get, $\frac{\Delta V}{V} = \frac{\Delta l}{l} + \frac{\Delta b}{b} + \frac{\Delta h}{h}$

∴ Percentage error in the volume of the block

$$= \left(\frac{\Delta l}{l} + \frac{\Delta b}{b} + \frac{\Delta h}{h}\right) \times 100 = \left(\frac{0.01}{15.12} + \frac{0.01}{10.15} + \frac{0.01}{5.28}\right) \times 100$$

$$= (0.07 + 0.10 + 0.19) = \mathbf{0.36\ \%}$$

4. **The length and breadth of a rectangular field are measured as :**

length, $l = (250 \pm 5)$ m ; breadth, $b = (150 \pm 4)$ m

What is the area of the field?

Hint. Area of the field is given by ; $A = lb$

Taking log of both sides, we have, $\log A = \log l + \log b$

Differentiating, we get, $\frac{\Delta A}{A} = \frac{\Delta l}{l} + \frac{\Delta b}{b}$

Here $A = 250 \times 150 = 37500\ m^2$; $\Delta l = 5$ m; $\Delta b = 4$ m; $l = 250$m; $b = 150$m

$$\therefore \quad \frac{\Delta A}{A} = \frac{5}{250} + \frac{4}{150} = 0.02 + 0.0266 = 0.0466$$

or $\quad \Delta A = A \times 0.0466 = 37500 \times 0.0466 = 1747.5\ m^2$

$$\therefore \quad \text{Area} = (A \pm \Delta A) = (37500 \pm 1747.5)\ m^2 = (3.75 \pm 0.17) \times 10^4\ m^2$$

5. **The radius of a thin wire is 0.24 mm. Find the area of cross-section taking into account the significant figures.**

Hint. Since the radius of the wire has two significant figures, the answer should also have two significant figures.

Area, $A = \pi r^2 = \pi (0.24)^2 = 0.180864\ mm^2 = \mathbf{0.18\ mm^2}$

6. **In an experiment with simple pendulum, a time period measured was 40 s for 20 vibrations when the length of the pendulum was taken as 100 cm.**

(*i*) If the least count of the stop watch is 0.1 s and that of the metre scale is 0.1 cm, calculate the maximum permissible error in the measurement of g.

(*ii*) If the actual value of g at Delhi is 9.7915 ms^{-2}, calculate the percentage error.

Hint. (*i*) The time-period of a simple pendulum is given by ;

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad T^2 = 4\pi^2 \times \frac{l}{g} \quad \therefore\ g = \frac{4\pi^2 l}{T^2}$$

Taking log of both sides, $\log g = \log 4 + 2\log \pi + \log l - 2\log T$

Differentiating above equation, $\frac{\Delta g}{g} = \frac{\Delta l}{l} - 2\frac{\Delta T}{T}$

Since errors are always added, $\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2\frac{\Delta T}{T}$

$\therefore$ Max. permissible error in $g = \frac{\Delta l}{l} + 2\frac{\Delta T}{T} = \frac{0.1}{100} + 2\times\frac{0.1}{40} = 0.001 + 0.005 = 0.006 = 0.60\%$

(ii) $g = \frac{4\pi^2 l}{T^2}$. Here l=100 cm = 1m ; T = 40/20 = 2s

$\therefore\ g = \frac{4\pi^2 \times 1}{(2)^2} = 9.8596\ \text{ms}^{-2}$ $\therefore$ % error $= \frac{9.8596 - 9.7915}{9.7915} \times 100 = 0.6955\%$

7. **A stone weighs (10.0 ± 0.1) kg in air. The weight of the stone in water is (5.0 ± 0.1) kg. Find the maximum percentage error in the measurement of specific gravity.**

Hint. Weight of stone in air = (10.0 ± 0.1) kg

Loss of weight in water = (10.0 ± 0.1) –(5.0 ± 0.1) = (5 ± 0.2) kg

Note that when two quantities are subtracted (or added), the absolute errors are added up.

Now, Specific gravity $= \frac{\text{Weight in air}}{\text{Loss of weight in water}} = \frac{(10.0 \pm 0.1)\text{kg}}{(5 \pm 0.2)\text{kg}}$

$\therefore$ Maximum percentage error in specific gravity

$$= \frac{0.1}{10.0} \times 100 + \frac{0.2}{5.0} \times 100 = 1\% + 4\% = 5\%$$

8. **The density ρ of a piece of metal of mass m and volume V is given by the formula ρ = m/V . If m= (375.32 ± 0.01) g and V = (136.41 ± 0.01) cm^3, find the percentage error in ρ .**

Hint. $\rho = \frac{m}{V}$ $\therefore$ $\frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta V}{V} = \frac{0.01}{375.32} + \frac{0.01}{136.41}$

$\therefore$ $\frac{\Delta\rho}{\rho}$ = 0.0000266 + 0.0000733 = 0.0000999

$\therefore$ Percentage error in density $= \frac{\Delta\rho}{\rho} \times 100 = 0.0000999 \times 100 = 0.00999\% = 0.01\%$

9. **In the formula $g = 4\pi^2 l / T^2$, l has 2% uncertainty and T 5% uncertainty. What is the maximum uncertainty in the value of g ?**

Hint. $g = \frac{4\pi^2 l}{T^2}$ $\therefore$ $\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2\frac{\Delta T}{T}$

$\therefore$ Max. percentage uncertainty in $g = \frac{\Delta l}{l} \times 100 + 2\frac{\Delta T}{T} \times 100 = 2\% + 2(5\%) = 2\% + 10\% = 12\%$

10. **The heat generated in a circuit depends upon the current, resistance and time for which current flows. If the errors in measuring the above are 2%, 1% and 1% respectively, find the maximum error in measuring heat.**

Hint. Heat generated in the circuit is given by ; $H = I^2 R\ t$

$$\therefore \quad \frac{\Delta H}{H} = 2\frac{\Delta I}{I} + \frac{\Delta R}{R} + \frac{\Delta t}{t}$$

$\therefore$ Maximum percentage error in measuring heat

$$= 2\frac{\Delta I}{I}\times 100 + \frac{\Delta R}{R}\times 100 + \frac{\Delta t}{t}\times 100 \quad = 2\,(2\%) + 1\% + 1\% = \mathbf{6\%}$$

11. In an experiment to determine the value of Young's modulus of elasticity of steel, a wire of length 325 cm (measured by a metre scale of least count 0.1 cm) is loaded by a mass of 2 kg and it is found that it stretches by 0.227 cm (measured by a micrometer having least count 0.001cm). The diameter of the wire as measured by a screw gauge (least count= 0.001cm) is found to be 0.043cm. Calculate the maximum permissible error.

Hint. Young's modulus of elasticity is given by ;

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{l/L} = \frac{FL}{Al} = \frac{Mg\,L}{\pi r^2 l} = \frac{4Mg\,L}{\pi D^2 l} \qquad \left(\because r = \frac{D}{2}\right)$$

or $\quad Y = \dfrac{4Mg\,L}{\pi D^2 l} \quad \therefore \quad \dfrac{\Delta Y}{Y} = \dfrac{\Delta L}{L} + 2\dfrac{\Delta D}{D} + \dfrac{\Delta l}{l}$

$\therefore$ Maximum permissible error in Y

$$= \left(\frac{\Delta L}{L} + 2\frac{\Delta D}{D} + \frac{\Delta l}{l}\right)\times 100 = \left(\frac{0.1}{325} + 2\times\frac{0.001}{0.043} + \frac{0.001}{0.227}\right)\times 100$$

$$= (0.031 + 4.651 + 0.441) = \mathbf{5.123\%}$$

12. A screw gauge having 100 equal divisions and a pitch of length 1mm is used to measure the diameter of a wire of length 5.6cm. The main scale reading is 1mm and 47^{th} circular division coincides with the main scale. Find the curved surface area of wire in cm^2 to appropriate significant figures (use π = 22/7).

[IIT 2004]

Hint. Least count of screw gauge = $\dfrac{\text{Pitch}}{N} = \dfrac{1\,\text{mm}}{100}$ = 0.01mm

$\therefore$ Diameter of wire, d = 1 + (47 × 0.01) = 1.47mm

Area of curved surface, $S = \pi\, d\, l = \dfrac{22}{7} \times 1.47 \times 56\ \text{mm}^2 = 2.58724\text{cm}^2$

The required curved surface area to appropriate significant figures is **2.6 cm^2**.

13. The side of a cube is measured by a vernier calliper (10 divisions of a vernier scale coincide with 9 divisions of main scale, where 1 division of main scale is 1 mm). The main scale reads 10 mm and first division of vernier scale coincides with the main scale. Mass of the cube is 2.736 g. Find the density of cube in appropriate significant figures. **[IIT 2005]**

Hint. Least count of vernier calliper = 1 M.S.D. − 1 V.S.D.

$$= 1\ \text{mm} - \frac{9}{10}\text{mm} = \frac{1}{10}\text{mm} = 0.1\text{mm}$$

$\therefore$ Side of cube, l = 10 mm + 1 × 0.1mm = 10.1mm = 1.01cm

$\therefore$ $$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{2.736}{(1.01)^3} = \mathbf{2.66\ gcm^{-3}}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 1 : PHYSICAL WORLD

Q.1. Some of the most profound statements on the nature of science have come from Albert Einstein, one of the greatest scientists of all times. What do you think did Einstein mean when he said: "The most incomprehensible thing about the world is that it is comprehensible?"

Ans. There are a very large number of phenomena taking place in nature so that a layman may think that this world is incomprehensible *i.e.,* it cannot be understood. But all these phenomena can be explained by a few concepts and scientific laws. Therefore, this world which seems to be incomprehensible is actually comprehensible.

Q.2. Every great physical theory starts as a hearsay and ends as a dogma. Give some examples from the history of science of the validity of this incisive remark.

Ans. This statement is true. Thomson opined that matter consists of atoms and each atom has uniformly distributed positive charge with electrons embedded in it. Rutherford suggested that an atom has a positively charged nucleus with electrons revolving around it in different circular paths. Later on, several modifications were made to these suggestions and today we have a modern theory of matter.

In other words, every theory starts with a hearsay—a guess and ends with a dogma—a principle.

Q.3. 'Politics is the art of possible'. Similarly, 'Science is the art of the soluble'. Explain this beautiful aphorism on the nature and practice of science.

Ans. This aphorism (truth) applies to science also. Just as politicians have a solution to every political problem through political wisdom, the scientists have a solution to every scientific problem. The scientists have concepts and scientific laws that can readily be applied to solve any scientific phenomenon.

Q.4. Though India now has a large base in science and technology, which is fast expanding, it is still a long way from realising its potential of becoming a world leader in science. Name some important factors, which in your view have hindered the advancement of science in India.

Ans. Some of the important factors which have hindered the advancement of science in India are :

(*i*) The infrastructure for science education in schools and colleges is quite inadequate.

(*ii*) Due to poverty, education is not a priority for a poor man.

(*iii*) Sufficient funds are not available for science and technology.

(*iv*) Due to lack of education, a common man in India does not know the importance of science and technology.

(*v*) Indian society is full of superstitions and people are reluctant to adopt new scientific trends.

(*vi*) There is very little research and development in science and technology in India.

Q.5. No physicist has ever "seen" an electron. Yet, all physicists believe in the existence of electrons. An intelligent but superstitious man advances his analogy to argue that 'ghosts' exist even though no one has 'seen' one. How will you refute this argument?

Ans. It is true that no physicist has ever seen an electron. However, there are indirect experimental evidences which show the existence of electrons. But in case of ghosts, there is no direct or indirect evidence to show the existence of ghosts.

Q.6. The shells of crabs found around a particular coastal location in Japan seem mostly to resemble the legendary face of a Samurai. Given below are two explanations of this observed fact. Which of these strikes you as a scientific explanation?

(*a*) **A tragic sea accident several centuries ago drowned a young Samurai. As a tribute to his bravery, nature through its inscrutable ways immortalised his face by imprinting it on the crab shells in that area.**

(*b*) **After the sea tragedy, fishermen in that area, in a gesture of honour to their dead hero, let free any crab shell caught by them which accidentally had a shape resembling the face of a Samurai. Consequently, the particular shape of the crab shell survived longer and therefore in course of time the shape was genetically propagated. This is an example of evolution by artificial selection.**

[Note: This interesting illustration taken from Carl Sagan's 'The Cosmos' highlights the fact that often strange and inexplicable facts which on the first sight appear 'supernatural' actually turn out to have simple scientific explanations. Try to think out other examples of this kind.]

Ans. That the shells of crabs resemble the face of a Samurai, option (*b*) offers scientific explanation.

Q.7. **The industrial revolution in England and Western Europe more than two centuries ago was triggered by some key scientific and technological advances. What were these advances?**

Ans. The industrial revolution in England and Western Europe in 1750 A.D. was triggered by the following scientific and technological advances :

(*i*) Development of *steam engine* by James Watt.

(*ii*) Development of *blast furnace* which converts low grade iron into steel.

(*iii*) Development of *cotton gin* which separates the seeds from cotton much faster than used to be done manually.

(*iv*) Development of *power loom.*

Q.8. **It is often said that the world is witnessing now a second industrial revolution, which will transform the society as radically as did the first. List some key contemporary areas of science and technology, which are responsible for this revolution.**

Ans. The following are some areas of science and technology which are going to bring second industrial revolution:

(*i*) Development of superconducting materials at room temperature. This will lead to transmit electrical energy without any loss of energy.

(*ii*) Progress in biology specially in biotechnology and genetic engineering may totally change the face of the society in the next 25 to 30 years.

(*iii*) Advancement in the field of information technology including electronic media, internet, superfast computers and satellite communication.

(*iv*) Development of robots so that tasks which involve risk to human lives may be performed safely and efficiently.

(*v*) Advancement in space science including use of satellite for exploring conditions for life on other planets, geological survey *etc.*

Q.9. **Write in about 1000 words a fiction piece based on your speculation on the science and technology of the twenty-second century.**

Ans. The pace at which biotechnology and genetic engineering are progressing, the times are not far off when the man will conquer diseases and ageing. It will be possible to produce humans, animals and plants with specific characteristics.

Today, we go from one place to another by using such means as cars, aeroplanes *etc.* The distances covered are in the range of hundreds or thousands of kilometres. But in the twenty-second century, the man may undertake a journey to a distant star hundreds of light years away from earth without worrying about fuel requirement.

Today, the world is facing fuel problems. We are spending crores of rupees to solve this problem. In the twenty-second century, we will not have to bother about fuel requirements. The sun is a huge source of energy and we can use solar energy to meet fuel requirements.

Q.10. Attempt to formulate your 'moral' views on the practice of science. Imagine yourself stumbling upon a discovery, which has great academic interest but is certain to have nothing but dangerous consequences for the human society. How, if at all, will you resolve your dilemma?

Ans. If a scientific discovery is of great academic interest but dangerous to human society, it must be made public. For example, the discovery of nuclear fission is of great academic and practical interest because it can help in electric power generation. However, it can also lead to develop a nuclear bomb—a device of mass destruction. We should educate the society to use this discovery for the welfare of the humanity and not for its destruction.

Q.11. Science, like any knowledge, can be put to good or bad use, depending on the user. Given below are some of the applications of science. Formulate your views on whether the particular application is good, bad or something that cannot be so clearly categorised: (a) Mass vaccination against small pox to curb and finally eradicate this disease from the population. (This has already been successfully done in India.) (b) Television for eradication of illiteracy and for mass communication of news and ideas. (c) Prenatal sex determination. (d) Computers for increase in work efficiency. (e) Putting artificial satellites into orbits around the earth. (f) Development of nuclear weapons. (g) Development of new and powerful techniques of chemical and biological warfare. (h) Purification of water for drinking. (i) Plastic surgery. (j) Cloning.

Ans. (*a*) It is good. (*b*) It is good. (*c*) It is bad. (*d*) It is good. (*e*) It is good. (*f*) It is bad. (*g*) It is bad. (*h*) It is good. (*i*) It is good. (*j*) Something which cannot be clearly categorised.

Q.12. India has had a long and unbroken tradition of great scholarship—in mathematics, astronomy, linguistics, logic and ethics. Yet, in parallel with this, several superstitious and obscurantistic attitudes and practices flourished in our society and unfortunately continue even today—among many educated people too. How will you use your knowledge of science to develop strategies to counter these attitudes?

Ans. The superstitions and several other illogical practices in India are so deeply rooted that there is no short cut way to eradicate them. However, the following methods can be used to tackle this problem:

(*i*) We should educate a common man about the benefits that science has given to society.

(*ii*) The electronic media (*e.g.,* radio, television, *etc.*) and print media (*e.g.,* newspapers) can play vital role for this purpose.

(*iii*) Teachers should play a lead role in this regard.

Q.13. Though the law gives women equal status in India, many people hold unscientific views on a woman's innate nature, capacity and intelligence, and in practice give them a secondary status and role. Demolish this view using scientific arguments, and by quoting examples of great women in science and other spheres; and persuade yourself and others that, given equal opportunity, women are on par with men.

Ans. This difference is man-made. The gender factor is in no way involved in the development of human mind. Therefore, given equal opportunities (*e.g.,* diet, environment, education, care, *etc.*), women are at par with men. Recent developments show that women even excel men in many fields of science, technology, politics *etc.* We can cite many examples of successful women in science and other fields: (*i*) Madam Curie won Nobel prize in physics in 1903 (*ii*) Mrs. Indra Gandhi, Mrs. Margaret Thatcher and many other women proved excellent politicians. Therefore, we conclude that scientifically women are not inferior to men.

Q.14. "It is more important to have beauty in the equations of physics than to have them agree with experiments." The great British physicist P.A.M. Dirac held this view. Criticise this statement. Look out for some equations and results in this book which strike you as beautiful.

Ans. This statement of British physicist is partially true. Some equations of physics (*e.g.,* $F = ma$, $E = mc^2$ *etc.*) are simple and can be readily understood and applied in many situations. Some equations involving field of modern physics are neither simple nor easy to understand. Many times we

perform experiments to get empirical relations to solve a complex physics problem. Therefore, it depends on the situation where to use equations and where to perform experiments.

Q.15. Though the statement quoted above may be disputed, most physicists do have a feeling that the great laws of physics are at once simple and beautiful. Some of the notable physicists, besides Dirac, who have articulated this feeling are: Einstein, Bohr, Heisenberg, Chandrasekhar and Feynman. You are urged to make special efforts to get access to the general books and writings by these and other great masters of physics. (See the Bibliography at the end of N.C.E.R.T. textbook). Their writings are truly inspiring!

Ans. It is true that great laws of physics are at once so simple and beautiful that anybody can grasp them. However, general books on physics make an interesting reading. Therefore, students are advised to read general books on physics. This will help them to enjoy this subject.

Q.16. Textbooks on science may give you a wrong impression that studying science is dry and all too serious and that scientists are absent-minded introverts who never laugh or grin. This image of science and scientists is patently false. Scientists, like any other group of humans, have their share of humorists, and many have led their lives with a great sense of fun and adventure, even as they seriously pursued their scientific work. Two great physicists of this genre are Gamow and Feynman. You will enjoy reading their books listed in the N.C.E.R.T. book bibliography.

Ans. It is true that scientists, like any other group of humans, are lively, jovial and fun loving people. If you study the life history of great scientists like Newton and Einstein, you will find that they were very funny people. However, they pursued their work very seriously.

NCERT CHAPTER 2 : UNITS AND DIMENSIONS

Q.1. Fill in the blanks.

(*a*) The volume of a cube of side 1cm is equal to ..... m^3.

(*b*) The surface area of a solid cylinder of radius 2.0cm and height 10.0cm is equal to ..... $(mm)^2$.

(*c*) A vehicle moving with a speed of 18kmh^{-1} covers ..... m in 1s.

(*d*) The relative density of lead is 11.3. Its density is ..... gcm^{-3} or ..... kgm^{-3}.

Ans. (*a*) The length of cube side, l = 1cm = 10^{-2}m. Therefore, volume of cube = $l^3 = (10^{-2})^3$ = **$10^{-6}m^3$**.

(*b*) Here, r = 2cm = 20mm and h = 10cm = 100mm.

$\therefore$ Surface area of solid cylinder = $(2\pi r)h = (2\pi \times 20) \times 100$ = **$1.26 \times 10^4 mm^2$**

(*c*) Speed of vehicle = 18kmh^{-1} = $\frac{18\times10^3}{60\times60}$ = 5ms^{-1}

Therefore, distance covered in 1s is **5 m**.

(*d*) Relative density = 11.3 $\therefore$ Density = **11.3 g/cc** = $\frac{11.3\times10^{-3}\,kg}{10^{-6}m^3}$ = **11.3×10^3 kgm^{-3}**

Q.2. Fill in the blanks by suitable conversion of units.

(*a*) 1kgm^{2}s^{-2} = ... gcm^{2}s^{-2} (*b*) 1m = ... ly (*c*) 3.0ms^{-2} = ... kmh^{-2}

(*d*) $G = 6.67 \times 10^{-11}$ Nm^2(kg)$^{-2}$ = ... (cm)^{3}s^{-2}g^{-1}.

Ans. (*a*) 1 kgm^{2}s^{-2} = (1 $\times$ 10^3g) $\times$ (10^2cm)^2 $\times$ s^{-2} = **10^7gcm^{2}s^{-2}**

(*b*) We know that 1 light year = 9.46×10^{15}m.

$\therefore$ 1 m = $\frac{1}{9.46\times10^{15}}$ light year = **1.057×10^{-16} light year**

(*c*) $3\ ms^{-2} = (3 \times 10^{-3} km)\left(\frac{1}{60 \times 60} h\right)^{-2} = \mathbf{3.888 \times 10^4 km\ h^{-2}}$

(*d*) $G = 6.67 \times 10^{-11}\ N\ m^2\ kg^{-2} = 6.67 \times 10^{-11}\ (kg\ ms^{-2}) m^2 kg^{-2}$

$= 6.67 \times 10^{-11}\ m^3\ s^{-2}\ kg^{-1} = 6.67 \times 10^{-11}\ (100\ cm)^3 s^{-2}\ (1000\ g)^{-1}$

$= \mathbf{6.67 \times 10^{-8}\ cm^3 s^{-2} g^{-1}}$

Q.3. A calorie is a unit of heat energy and it is equal to about 4.2 J where 1 J = 1 kgm^2s^{-2}. Suppose we employ a system of units in which the unit of mass equals α kg, the unit of length equals β m and the unit of time γs. Show that a calorie has a magnitude of 4.2 $\alpha^{-1}\beta^{-2}\gamma^2$ in terms of new units.

Ans. 1 calorie = 4.2 J = 4.2 kg m^2s^{-2} ..(*i*)

New unit of mass = α kg

$\therefore$ 1 kg = $\frac{1}{\alpha}$ new unit of mass = α^{-1} new unit of mass

Likewise 1 m = β^{-1} new unit of length and 1s = γ^{-1} new unit of time.

Putting these values in eq. (*i*), we have, 1 calorie = **4.2 $\alpha^{-1}\beta^{-2}\gamma^2$ new units.**

Q.4. Explain this statement clearly:

"To call a dimensionless quantity 'large' or 'small' is meaningless without specifying a standard for comparison." In view of this, reframe the following statements wherever necessary: (a) atoms are very small objects. (b) a jet plane moves with great speed. (c) The mass of jupiter is very large. (d) The air inside this room contains a large number of molecules. (e) A proton is much more massive than an electron. (f) The speed of sound is much smaller than the speed of light.

Ans. The statement is correct. For example, angle is dimensionless. Now $\angle\theta = 70°$ is greater than $\angle\theta = 30°$ but smaller than $\angle\theta = 90°$. (*a*) The size of an atom is much smaller than that of the tip of a pin. (*b*) A jet plane moves faster than a train. (*c*) The mass of jupiter is very large as compared to that of the earth. (*d*) The air in a room contains a very large number of molecules as compared to that in a small balloon. (*e*) The statement is correct. (*f*) The statement is correct.

Q.5. A new unit of length is chosen such that the speed of light in vacuum is unity. What is the distance between the sun and the earth in terms of the new unit, if light takes 8 min and 20 s to cover this distance?

Ans. It is given that velocity of light in vacuum, c = 1 new unit of length s^{-1}. The time taken by sunlight to reach earth, t = 8 min. 20s = 500s.

$\therefore$ Distance S between sun and earth is

$S = ct$ = 1 new unit of length s^{-1} × 500 s = **500 new units of length**

Q.6. Which of the following is the most precise device for measuring length?

(*a*) a vernier calliper with 20 divisions on sliding scale, coinciding with 19 main scale divisions, (*b*) a screw gauge of pitch 1 mm and 100 divisions on the circular scale, (c) an optical instrument that can measure length to within a wavelength of light.

Ans. The most precise device is that which has the minimum value of least count.

(*a*) Least count of vernier calliper = 1 MSD − 1 VSD

$= 1 - \frac{19}{20} = \frac{1}{20}$ mm = 0.005 cm

(*b*) Least count of screw gauge = $\frac{\text{Pitch of screw}}{\text{Total no. of divisions on circular scale}}$

$$= \frac{1}{100} \text{ mm} = 0.001\text{cm}$$

(*c*) Wavelength of light, $\lambda = 10^{-5}$ cm = 0.00001cm

It is clear that the most precise measurement will be with the optical instrument.

Q.7. **A student measures the thickness of a human hair by looking at it through a microscope of magnification 100. He makes 20 observations and finds that the average width of the hair in the field of view of the microscope is 3.5 mm. What is the estimate on the thickness of hair?**

Ans. $$\text{Magnification} = \frac{\text{Observed width}}{\text{True width}} \text{ or } 100 = \frac{3.5\text{ mm}}{\text{True width}}$$

$$\therefore \quad \text{True width (thickness)} = \frac{3.5}{100} = 0.035\text{mm}$$

Q.8. **Answer the following :**

(*a*) **You are given a thread and a metre scale. How will you estimate the diameter of the thread?**

(*b*) **A screw guage has a pitch of 1.0mm and 200 divisions on the circular scale. Do you think it is possible to increase the accuracy of screw guage arbitrarily by increasing the number of divisions on the circular scale?**

(*c*) **The mean diameter of a thin brass rod is to be measured by vernier callipers. Why is a set of 100 measurements of the diameter expected to yield a more reliable estimate than a set of 5 measurements only?**

Ans. (*a*) The diameter of a thread is so small that it cannot be measured directly with a metre scale. We can use an indirect method. Take the metre scale and wrap the thread around it such that the turns of the thread are tight and each turn touches the immediate neighbouring turns on either side. Measure the length (l) of the wound thread and count the number of turns (n).

$$\therefore \quad \text{Diameter of thread} = \frac{l}{n}$$

(*b*) The accuracy of the instrument depends upon its least count (L.C.). The smaller the L.C., the more accurate is the instrument.

$$\text{L.C.} = \frac{\text{Pitch of screw}}{\text{No. of div. on circular scale}}$$

Theoretically, we can decrease L.C. (and hence increase accuracy) by increasing the number of divisions on the circular scale. But there is a limit to it. Therefore, the accuracy cannot be increased infinitely because there is a limit of resolution of human eye.

(*c*) When we make measurements, there is a possibility of positive as well as negative random errors (*i.e.*, errors due to unknown causes) in the measurement. Therefore, when the number of observations is large, random errors are likely to cancel and the result may be more reliable.

Q.9. **The photograph of a house occupies an area of 1.75cm^2 on a 35mm slide. The slide is projected on to a screen and the area of the house on the screen is 1.55m^2. What is the linear magnification of the projector screen arrangement?**

Ans. $$\text{Areal magnification} = \frac{\text{Size of image}}{\text{Size of object}} = \frac{1.55\text{ m}^2}{1.75\times10^{-4}\text{ m}^2} = 8857.14$$

$$\therefore \quad \text{Linear magnification} = \sqrt{\text{Areal magnification}} = \sqrt{8857.14} = 94.1$$

Q.10. State the number of significant figures in the following:

(*a*) $0.007 m^3$ (*b*) $2.64 \times 10^{24} kg$ (*c*) $0.2370 gcm^{-3}$

(*d*) 6.320J (*e*) $6.032 Nm^{-2}$ (*f*) $0.0006032 m^2$

Ans. (*a*) one (*b*) three (*c*) four (*d*) four (*e*) four (*f*) four.

Q.11. The length, breadth and thickness of a metal sheet are 4.234m, 1.005m and 2.01cm respectively. Give the area and volume of the sheet to correct significant figures.

Ans. Length, l = 4.234 m ; Breadth, b = 1.005 m ; Thickness, t = 2.01 cm = 2.01×10^{-2} m

Area of sheet = $2(l \times b + b \times t + t \times l) = 2(4.234 \times 1.005 + 1.005 \times 0.0201 + 0.0201 \times 4.234)$

$= 2(4.36047) = 8.72094 \text{ m}^2$

Since the least precise term has three significant figures, the answer must be expressed to three significant figures.

After rounding off, area of sheet = **$8.72 m^2$**.

Volume = $4.234 \times 1.005 \times 2.01 \times 10^{-2} = 0.085528917 m^3$

Since the least precise term (*i.e.*, 2.01cm) is given to three significant figures, the answer must be expressed to three significant figures. After rounding off, volume of sheet = **$0.0855 m^3$**.

Q.12. The mass of a box measured by a grocer's balance is 2.300kg. Two gold pieces of masses 20.15g and 20.17g are added to the box. What is (*a*) the total mass of the box (*b*) the difference in the masses of the pieces to correct significant figures?

Ans. Mass of box, m = 2.300kg ; Mass of one gold piece, m_1 = 20.15g = 0.02015kg ; Mass of other gold piece, m_2 = 20.17 g = 0.02017 kg

(*a*) $\therefore$ Total mass = $m + m_1 + m_2 = 2.300 + 0.02015 + 0.02017 = 2.34032$kg

As the mass of the box has the least number of decimal places, therefore, after rounding off, total mass = **2.3kg**.

(*b*) Mass difference in pieces = $m_2 - m_1$ = 20.17g − 20.15g = 0.02g

As the mass of each gold piece is measured upto two decimal places, the answer must be expressed upto two decimal places. Therefore, difference in the masses of the two gold pieces is **0.02g.**

Q.13. A physical quantity P is related to four observables a, b, c and d as follows:

$$P = a^3 b^2 / (\sqrt{c}\, d)$$

The percentage errors of measurement in a, b, c and d are 1%, 3%, 4% and 2% respectively. What is the percentage error in P? If the value of P calculated using the above relation turns out to be 3.763, to what value should you round off the result?

Ans. $$P = \frac{a^3 b^2}{\sqrt{c}\, d} \quad \therefore \quad \frac{\Delta P}{P} = \frac{3\Delta a}{a} + \frac{2\Delta b}{b} + \frac{1}{2}\frac{\Delta c}{c} + \frac{\Delta d}{d}$$

or $$\frac{\Delta P}{P} \times 100 = \frac{3\Delta a}{a} \times 100 + 2\frac{\Delta b}{b} \times 100 + \frac{1}{2}\frac{\Delta c}{c} \times 100 + \frac{\Delta d}{d} \times 100$$

Now $\frac{\Delta a}{a} \times 100 = 1\%$; $\frac{\Delta b}{b} \times 100 = 3\%$; $\frac{\Delta c}{c} \times 100 = 4\%$; $\frac{\Delta d}{d} \times 100 = 2\%$

$\therefore$ % error in $P = 3 \times 1\% + 2 \times 3\% + \frac{1}{2} \times 4\% + 2\% = 3\% + 6\% + 2\% + 2\% = 13\%$

As the result (= 13% or 0.13) has two significant figures, therefore, if P turns out to be 3.763, the result would be rounded off to **3.8**.

Q.14. A book with many printing errors contains four different formulas for the displacement y of a particle undergoing a certain periodic motion:

(a) $y = a \sin 2\pi t/T$ (b) $y = a \sin vt$

(c) $y = (a/T) \sin t/a$ (d) $y = (a/\sqrt{2})(\sin 2\pi t/T + \cos 2\pi t/T)$

(a = maximum displacement of the particle, v = speed of the particle, T = time period of motion). Rule out the wrong formulas on dimensional grounds.

Ans. The argument of a trigonometric function (*i.e.*, angle) is always dimensionless. The wrong formulas are those whose arguments are not dimensionless.

(a) $\frac{2\pi t}{T} = \frac{T}{T} = [M^0L^0T^0]$ (b) $vt = (LT^{-1})(T) = [M^0L^1T^0]$

(c) $\frac{t}{a} = \frac{T}{L} = [M^0L^{-1}T^1]$ (d) $\frac{2\pi t}{T} = \frac{T}{T} = [M^0L^0T^0]$

Since in the second and third formulas the arguments are not dimensionless, the formulas (*b*) and (*c*) are wrong.

Q.15. A famous relation in physics relates 'moving mass' m to the 'rest mass' m_0 of a particle in terms of its speed v and the speed of light c. (This relation first arose as a consequence of special theory of relativity due to Albert Einstein.) A boy recalls the relation almost correctly but forgets where to put the constant c. He writes : $m = \frac{m_0}{(1-v^2)^{1/2}}$. Guess where to put the missing c.

Ans. The dimensions of L.H.S. of the expression are ML^0T^0. The dimensions of R.H.S. must be the same *i.e.*, ML^0T^0 (*i.e.*, dimensions of mass). This is only possible if we have v^2/c^2 in the above relation. Therefore, the correct formula is

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

Q.16. The unit of length convenient on the atomic scale is known as an angstrom and is denoted by Å : 1 Å = 10^{-10} m. The size of a hydrogen atom is about 0.5 Å. What is the total atomic volume in m^3 of a mole of hydrogen atoms?

Ans. Volume of one hydrogen atom $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(0.5\times10^{-10})^3 = 5.24\times10^{-31}\,m^3$

One mole of hydrogen contains 6.02×10^{23} atoms.

$\therefore$ Atomic volume of one mole of hydrogen atoms

= Volume of one atom $\times N_A$

= $5.24 \times 10^{-31} \times 6.02 \times 10^{23} = 3.154 \times 10^{-7} m^3$

Q.17. One mole of an ideal gas at standard temperature and pressure occupies 22.4 litres (molar volume). What is the ratio of molar volume to the atomic volume of a mole of hydrogen? (Take the size of hydrogen molecule to be about 1 Å). Why is this ratio so large?

Ans. Volume of one mole of ideal gas = 22.4 litres = 22.4×10^{-3} m^3. The radius of hydrogen atom, R = 1/2 Å = 0.5 Å = 0.5×10^{-10}m. If N_A is Avogadro's number, then,

$$\text{Atomic volume} = \frac{4}{3}\pi R^3 N_A = \frac{4}{3}\pi(0.5\times10^{-10})^3\times6.023\times10^{23} = 3.154 \times 10^{-7} m^3$$

$$\therefore \quad \frac{\text{Molar volume}}{\text{Atomic volume}} = \frac{22.4\times10^{-3}}{3.154\times10^{-7}} = 7.1 \times 10^4$$

This large ratio is due to the fact that the intermolecular separation in a gas is much larger than the size of hydrogen molecule.

Q.18. Explain this common observation clearly. If you look out of the window of a fast moving train, the nearby trees, houses etc. seem to move rapidly in a direction opposite to the train's motion, but the distant objects (hill tops, the moon, the stars etc.) seem to be stationary.

Ans. A person sitting in a moving train finds the objects shifting in the opposite direction due to relative velocity. This relative shift depends upon the distance of the object from the person. If this distance is small, the objects appear to be shifting fast in the opposite direction. When we look at a distant object, the relative shift is negligible so that it appears to move in the direction of the train due to parallax.

Q.19. The principle of 'parallax' is used in the determination of distances of very distant stars. The baseline *AB* is the line joining the earth's two locations six months apart in its orbit around the sun. That is, the baseline is about the diameter of the earth's orbit $\simeq 3 \times 10^{11}$m. However, even the nearest stars are so distant that with such a long baseline, they show parallax only of the order of 1″ (second) of arc or so. A parsec is a convenient unit of length on the astronomical scale. It is the distance of an object that will show a parallax of 1″ (second) of arc from opposite ends of a baseline equal to the distance from the earth to the sun. How much is a parsec in terms of metres?

Ans. Length of baseline, l = 1 A.U. = 1.5×10^{11}m ; r = 1 parsec

$$\text{Parallax angle, } \theta = 1'' = \frac{1'}{60} = \frac{1^0}{60\times 60} = \frac{\pi}{180}\times\frac{1}{60\times 60}\text{ rad} = 4.85 \times 10^{-6}\text{rad}$$

From parallax method, we have,

$$r = \frac{l}{\theta} = \frac{1.5\times 10^{11}}{4.85\times 10^{-6}} = 3.1 \times 10^{16}\text{ m} \therefore 1\text{ parsec} = 3.1 \times 10^{16}\text{m}$$

Q.20. The nearest star to our solar system is 4.29 light years away. How much is this distance in terms of parsec? How much parallax would this star show when viewed from two locations of the earth six months apart in its orbit around the sun?

Ans. x = 4.29 ly = $4.29 \times 9.46 \times 10^{15}$m = $\frac{4.29\times 9.46\times 10^{15}}{3.08\times 10^{16}}$ parsec = 1.32parsec

$$\text{Parallax angle, } \theta = \frac{l}{r} = \frac{2\text{ A.U.}}{x} = \frac{2\times 1.496\times 10^{11}}{4.29\times 9.46\times 10^{15}}\text{ rad} = 1.52\text{sec}$$

Q.21. Precise measurements of physical quantities are a need of science. For example, to ascertain the speed of an aircraft, one must have an accurate method to find its positions at closely separated instants of time. This was the actual motivation behind the discovery of radar in World War II. Think of different examples in modern science where precise measurements of length, time, mass etc. are needed. Also, wherever you can, give a quantitative idea of the precision needed.

Ans. Precise measurements of physical quantities like mass, length, time etc. are required in physics or any other science to obtain accurate results. A few examples are given by way of illustration.

(*i*) For measuring the distance of moon from the earth by Laser beam, the time taken should be measured very accurately. In some Laser applications, time interval involved is of the order of 10^{-9}s.

(*ii*) Precise measurement of time (of the order of 10^{-9}s) and mass (of the order of 10^{-9}kg) are required for nuclear reactions.

(*iii*) For measuring the distance of a star by parallax method, parallax of the star should be measured accurate upto 1″ of arc.

Q.22. Just as precise measurements are necessary in science, it is equally important to be able to make rough estimates of quantities using rudimentary ideas and common observations. Think of ways by which you can estimate the following (where an estimate is difficult to obtain, try to get an upper bound on the quantity): (*a*) the total mass of rain-bearing clouds over India during the Monsoon, (*b*) the mass of an elephant, (*c*) the wind speed during a storm, (*d*) the number of strands of hair on your head, (*e*) the number of air molecules in your classroom.

Ans. (*a*) During monsoon in India, the average rainfall is 100cm *i.e.*, h = 100cm = 1 m. The area of the country is about $A = 3.3 \times 10^6$ km^2 = 3.3×10^{12}m^2.

$\therefore$ Volume of rain water, $V = A \times h = 3.3 \times 10^{12} \times 1 = 3.3 \times 10^{12}$m^3

The density of water is $\rho = 10^3$ kg m^{-3} so that the total mass of rain-bearing clouds over India = $V \times \rho = 3.3 \times 10^{12} \times 10^3 = \mathbf{3.3 \times 10^{15}}$**kg**.

(*b*) To estimate the mass of an elephant, consider a boat of known area A in water. Mark a point on the boat upto which it is inside the water. Now move the elephant into the boat. The boat goes deeper into water. Again mark a point upto which the boat is inside the water. Let h be the distance between the two marks.

$\therefore$ Volume of water displaced by elephant = Ah

According to Archimede's principle,

Mass of elephant = Mass of water displaced = $Ah\rho$

Here ρ (= 10^3kg m^{-3}) is the density of water.

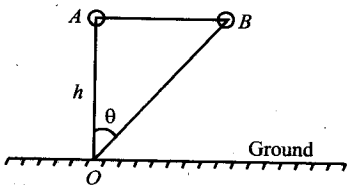


Fig. 2.01

(*c*) The wind speed during storm can be easily estimated by using a gas filled balloon. Fig. 2.01 illustrates the basic idea. When no wind is blowing, the balloon is at point A at a vertical height h from the ground. Suppose the wind blows to the right and the balloon moves to point B in one second. Then $AB = x = h\theta$. Since AB (= x) is the distance moved by the balloon in one second, it is the wind speed (= AB m/s).

(*d*) Suppose the area of hair-bearing head is A and d is the thickness (*i.e.*, diameter) of the hair. Assuming hair distribution to be uniform and neglecting the interspaces between the hair,

$$\text{No. of strands of hair} = \frac{\text{Area of hair-bearing head}}{X-\text{section area of hair}} = \frac{A}{\pi d^2/4} = \frac{4A}{\pi d^2}$$

The area A of the head can be measured by a scale and the diameter of hair can be determined by a fine micrometer. The number of hair is of the order of 10^6.

(*e*) Suppose the volume of the room is V m^3. We know that one mole of air occupies a volume of 22.4 litres (= 22.4×10^{-3} m^3) at N.T.P. and contains molecules equal to Avogadro's number (= 6.023×10^{23}).

$\therefore$ No. of air molecules in volume $V = \dfrac{6.023 \times 10^{23}}{22.4 \times 10^{-3}} \times V$

Q.23. The sun is hot plasma (ionised matter) with its inner core at a temperature exceeding 10^7K, and its outer surface at a temperature of about 6000K. At these high temperatures, no substance remains in a solid or liquid phase. In what range do you expect the mass density of the sun to be, in the range of densities of solids and liquids or gases? Check if your guess is correct from the following data; mass of the sun = 2.0×10^{30} kg, radius of the sun = 7.0×10^8 m.

Ans. Mass of sun, $M = 2 \times 10^{30}$kg ; Radius of sun, $R = 7 \times 10^{8}$m

$$\text{Volume of sun, } V = \frac{4}{3}\pi R^3 = \frac{4}{3}\times\pi(7\times10^8)^3 = 1.437\times10^{27}\ \text{m}^3$$

$$\therefore \quad \text{Density of sun} = \frac{M}{V} = \frac{2\times10^{30}}{1.437\times10^{27}} = 1.39 \times 10^3\ \text{kgm}^{-3} = 1390\text{kgm}^{-3}$$

The density of sun is in the range of the densities of solids and liquids (not gases). The high density of sun is owing to the inward gravitational attraction on outer layers, due to the inner layers of the sun.

Q.24. When the planet jupiter is at a distance of 824.7 million kilometres from the earth, its angular diameter is measured to be 35.72″ of arc. Calculate the diameter of jupiter.

Ans. Distance of jupiter, D = 824.7 million km = 824.7×10^6 km

$$\theta = 35.72'' = \frac{35.72}{60\times60}\times\frac{\pi}{180}\ \text{rad}$$

$$\therefore \quad \text{Diameter of jupiter, } d = D\theta = (824.7 \times 10^6) \times \frac{35.72}{60\times60}\times\frac{\pi}{180}\text{km} = 1.428 \times 10^5\text{km}$$

Q.25. A man walking briskly in rain with speed v must slant his umbrella forward making an angle (with the vertical). A student derives the following relation between θ and v : $\tan\theta = v$ and checks that the relation has a correct limit: as $v \to 0$, $\theta \to 0$ as expected (We are assuming there is no strong wind and that the rain falls vertically for a stationary man). Do you think this relation can be correct? If not, guess the correct relation.

Ans. The relation $\tan\theta = v$ has correct limits as $v \to 0$, $\theta \to 0$. However, this relation is not dimensionally correct. It is because R.H.S. ($M^0L^0T^0$) has no dimensions while the L.H.S. (M^0LT^{-1}) has the dimensions of speed. As we shall see in chapter 11, the correct relation is

$$\tan\theta = \frac{v^2}{rg}.$$

Q.26. It is claimed that two cesium clocks, if allowed to run for 100 years, free from any disturbance, may differ by only about 0.02 s. What does this imply for the accuracy of the standard cesium clock in measuring a time-interval of 1s?

Ans. The error (*i.e.*, difference shown by two clocks) in 100 years = 0.02 s.

$$\therefore \quad \text{Error in 1 second} = \frac{0.02\text{s}}{100\ \text{years}} = \frac{0.02\text{s}}{100\times365.25\times24\times60\times60\text{s}}$$

$$= 6.34 \times 10^{-12}\ \text{s} \simeq 10^{-11}\text{s}$$

Therefore, accuracy of atomic clock is 1 in 10^{11}.

Q.27. Estimate the average mass density of a sodium atom assuming its size to be about 2.5 Å. (Use the known values of Avogadro's number and the atomic mass of sodium). Compare it with the density of sodium in its crystalline phase: 970 kg m^{-3}. Are the two densities of the same order of magnitude? If so, why?

Ans. Radius of sodium atom, R = 2.5/2Å = 1.25Å = 1.25×10^{-10}m

$\therefore$ Volume of 1 sodium atom, $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(1.25\times10^{-10})^3 = 8.18 \times 10^{-30}\text{m}^3$

Now one mole (*i.e.*, 23g or 23×10^{-3} kg) of sodium contains 6.023×10^{23} atoms (= Avogadro's number).

$\therefore$ Mass of 1 sodium atom, $m = \frac{23\times10^{-3}}{6.023\times10^{23}} = 3.82 \times 10^{-26}$ kg

$\therefore$ Density of sodium $= \frac{m}{V} = \frac{3.82\times10^{-26}}{8.18\times10^{-30}} = 4.67 \times 10^{3}$ kgm^{-3}

The density of sodium in crystalline state is 970 kgm^{-3} *i.e.*, the two densities are not of the same order of magnitude. The difference is due to interatomic spacing in the crystalline state.

Q.28. The unit of length convenient on the nuclear scale is a fermi: $1f = 10^{-15}$ m. Nuclear sizes obey roughly the following empirical relation: $r = r_0 A^{1/3}$ where r is the radius of the nucleus, A its mass number and r_0 is a constant equal to about 1.2 f. Show that the rule implies that nuclear mass density is nearly constant for different nuclei. Estimate the mass density of sodium nucleus. Compare it with the average mass density of a sodium obtained in exercise 27.

Ans. Let m be the average mass of a nucleon (proton or neutron). It is given that nucleus contains A nucleons.

$\therefore$ Mass of nucleus $= mA$; Radius of nucleus, $r = r_0 A^{1/3}$

$\therefore$ Density of nucleus, $\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{mA}{(4/3)\pi r^3} = \frac{3\,mA}{4\pi\,(r_0\,A^{1/3})^3} = \frac{3m}{4\pi\,r_0^3}$

Since m and r_0 are constant, therefore, nuclear density is the same for different nuclei. Its value can be found as under :

$$m = 1.66 \times 10^{-27} \text{ kg and } r_0 = 1.2f = 1.2 \times 10^{-15}\text{m}$$

$\therefore$ Nuclear density, $\rho = \frac{3\,m}{4\pi\,r_0^3} = \frac{3\times1.66\times10^{-27}}{4\pi\times(1.2\times10^{-15})^3} = 2.29 \times 10^{17}$ kgm^{-3}

Therefore, density of sodium nucleus (or of any nucleus) is 2.29×10^{17} kgm^{-3}.

$$\frac{\text{Density of sodium nucleus}}{\text{Density of sodium atom}} = \frac{2.29\times10^{17}}{4.67\times10^{3}} = 4.9 \times 10^{13}$$

Q.29. A LASER is a source of very intense, monochromatic and unidirectional beam of light. These properties of a laser light can be exploited to measure long distances. The distance of the moon from the earth has been already determined very precisely using a laser as a source of light. A laser light beamed at the moon takes 2.56s to return after reflection at the moon's surface. How much is the radius of the lunar orbit around the earth?

Ans. The speed of laser light is 3×10^8 ms^{-1}. If S is the distance of the moon from earth, then total distance travelled by laser light in going and coming back is $2S$.

$\therefore$ $2S = c \times t$

or $S = \frac{c\times t}{2} = \frac{3\times10^8\times2.56}{2} = 3.84 \times 10^8$m

Q.30. A SONAR (Sound navigation and ranging) uses ultrasonic waves to detect and locate objects under water. In a submarine equipped with a SONAR, the time delay between generation of a probe wave and the reception of its echo after reflection from an enemy submarine is found to be 77.0s. What is the distance of the enemy submarine? [Speed of sound in water = 1450ms^{-1}]

Ans. Suppose the distance of the enemy submarine is S from the observation point. Then total distance travelled by sound in going and coming back is $2S$.

$\therefore$ $2S = v \times t$

or $S = \frac{v\times t}{2} = \frac{1450\times77}{2} = 55825$m

Q.31. The farthest objects in our universe discovered by modern astronomers are so distant that light emitted by them takes billions of years to reach the earth. These objects (known as quasars) have many puzzling features, which have not yet been satisfactorily explained.

What is the distance in km of a quasar from which light takes 3.0 billion years to reach us?

Ans. Time taken by light to reach earth from quasar, t = 3.0 billion years = 3×10^9 years = $3 \times 10^9 \times 365 \times 24 \times 60 \times 60$s. Now velocity of light, $c = 3 \times 10^8$ ms^{-1}.

$\therefore$ Distance of quasar = $ct = (3 \times 10^8) \times (3 \times 10^9 \times 365 \times 24 \times 60 \times 60) = 2.84 \times 10^{22}$ km

Q.32. It is well known fact that during a total solar eclipse, the disk of the moon almost completely covers the disk of the sun. From the following data (See Fig. 2.02), determine the approximate diameter of the moon :

$ME = 3.84 \times 10^8$m ; $SE = 1.496 \times 10^{11}$m ;

$AB = 1.39 \times 10^9$m.

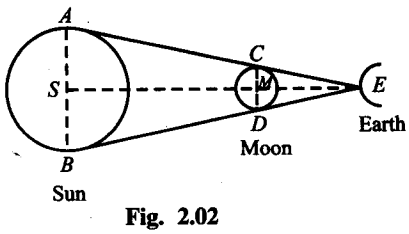


Fig. 2.02

Ans. Fig. 2.02 shows the conditions when total solar eclipse takes place.

As $\Delta s\, ABE$ and CDE are similar, therefore,

$$\frac{AB}{CD} = \frac{SE}{ME}$$

$$\therefore \quad CD = AB \times \frac{ME}{SE} = \frac{1.39 \times 10^9 \times 3.84 \times 10^8}{1.496 \times 10^{11}} = 3.568 \times 10^6 \text{m}$$

Q.33. A great physicist of this century (P.A.M. Dirac) loved playing with numerical values of fundamental constants of nature. This led him to an interesting observation. Dirac found that from the basic constants of atomic physics (c, e, mass of electron, mass of proton) and the gravitational constant G, he could arrive at a number with the dimensions of time. Further, it was a very large number, its magnitude being close to the present estimate on the age of the universe (≃ 15 billion years). From the table of fundamental constants in this book, try to see if you too can construct this number (or any other interesting number you can think of). If its coincidence with the age of the universe were significant, what would this imply for the constancy of fundamental constants?

Ans. The basic constants of atomic physics are: Speed of light c ; charge on electron e ; mass of electron m_e; mass of proton m_p; universal gravitational constant G.

Using these basic constants of atomic physics, we can arrive at a quantity which has the dimensions of time. One such quantity is

$$t = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \times \frac{e^4}{m_e^2\, m_p\, c^3\, G}$$

Here, $\dfrac{1}{4\pi\varepsilon_0} = 9 \times 10^9$; $e = 1.6 \times 10^{-19}$ C; $m_e = 9.1 \times 10^{-31}$ kg;

$m_p = 1.67 \times 10^{-27}$ kg; $c = 3 \times 10^8$ ms^{-1}; $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}

$$\therefore \quad t = (9\times10^9)^2 \times \frac{(1.6\times10^{-19})^4}{(9.1\times10^{-31})^2 \times 1.67\times10^{-27} \times (3\times10^8)^3 \times 6.67\times10^{-11}}$$

$$= 2.13 \times 10^{16} \text{ s}$$

This time is of the age of universe. This calculation shows that the estimate based on constants of atomic physics is fairly close.

Unit II

KINEMATICS

(Description of Motion in One Dimension)

- Motion in a Straight Line
- Uniformly Accelerated Linear Motion
- N.C.E.R.T. Textbook Exercises

6

Motion in a Straight Line

INTRODUCTION

We commonly observe that bodies are either in motion or at rest. A body is said to be in motion if it changes its position w.r.t. its surroundings with the passage of time. A car speeding along a highway, a train running on a railway track, a man walking on the road — all these are the examples of objects in motion. A body is said to be at rest if it does not change its position *w.r.t.* its surroundings with the passage of time. A book placed on the table remains on the table and we say that the book is at rest. Although the study of objects at rest is important, a large part of our everyday experience concerns the things that move. For this reason, the study of motion is one of the basic studies in physics. In this chapter, we shall discuss the motion in a straight line *i.e.,* rectilinear motion.

6.1. MECHANICS

The branch of physics which deals with the study of forces and motion and their relationship is called **mechanics.**

In mechanics, we seek to understand and explain the motion of material objects and also the conditions under which objects remain at rest. The reader may wonder why such a study is important. By learning a few basic principles of mechanics, we can describe and explain the motions of stars and planets, build bridges, fly aeroplanes and put satellites into orbits. Further, many concepts of mechanics play an important role in the study of other branches of physics.

The study of mechanics is divided into three parts *viz.* (*i*) statics (*ii*) kinematics and (*iii*) dynamics.

(*i*) **Statics.** *The branch of mechanics which deals with objects at rest is called statics.* Statics is of central importance in the design of bridges, buildings and other structures.

(*ii*) **Kinematics.** *The branch of mechanics which deals with the motion of objects only without considering the cause of motion is called kinematics.* In other words, kinematics does not consider the size, shape, mass *etc.* of the body; it is restricted to properties of the motion. Note that the word kinematics is derived from the Greek word kinema meaning "*motion*".

(*iii*) **Dynamics.** *The branch of mechanics which deals with the cause of motion is called dynamics.* In fact, in dynamics, we study the effect of forces in causing motion. Therefore, mass of the object must be considered. Note that the word dynamics is derived from Greek word dynamis meaning "*power*".

6.2. REST AND MOTION

(*i*) **Rest.** *An object is said to be at rest if it does not change its position w.r.t. its surroundings with the passage of time.* For example, a book placed on the table remains on the

table *i.e.*, it does not change its position with time *w.r.t.* its surroundings. We say that the book is at rest.

(*ii*) **Motion.** *An object is said to be in motion if it changes its position w.r.t. its surroundings with the passage of time.* For example, a speeding car is in motion because as the time passes, the car changes its position *w.r.t.* its surroundings *e.g.*, electric poles, trees, buildings *etc.*

Rest and motion are relative. Rest and motion are relative terms. For example, a book on the table is at rest *w.r.t.* table and other objects in the room. But all these objects are sharing the motion of the earth. If an observer is located on the moon, he will observe that the book and other objects in the room are moving. Thus the book is at rest if viewed from the room but is moving if viewed from the moon. This means that when we discuss the motion of something, we describe its motion *w.r.t.* something else. When we say that a car moves at 30 km/hr, we mean *w.r.t.* the surface of the earth. Similarly, when we say that a train is moving at 100 km/hr, we mean *w.r.t.* the track or surface of the earth. *Unless stated otherwise, when we discuss motion, we mean with respect to the surface of the earth.*

6.3. TYPES OF MOTION

In general, the motion of a body can be of three types *viz.*

(*i*) Translatory motion

(*ii*) Rotational motion

(*iii*) Oscillatory motion

(*i*) **Translatory motion.** *A body is said to have transalational motion if each particle of the body has the same displacement in the same time interval.* In other words, in this type of motion, every particle of the body covers a definite distance in definite times in linear paths. The motion of the body of a car moving on a straight road is an example of translational motion.

If a single particle or a point mass moves in a straight line, it is called *rectilinear motion.*

(*ii*) **Rotational motion.** *A body is said to have rotational motion or rotatory motion if each particle of the body (except those on the axis of rotation) travels in a circle.* The *axis of rotation* is a straight line that consists of the centres of the circular motion of the particles. The motion of a door being opened or closed is an example of rotational motion. The line along the hinges is the axis of rotation.

If a single particle or a point mass moves in a circle, it is called *circular motion.*

(*iii*) **Oscillatory motion.** *A body is said to have an oscillatory motion if it moves to and fro repeatedly about a fixed point called mean position e.g.*, swinging of a simple pendulum. If the amplitude of oscillatory motion is extremely small, the motion is called *vibratory motion.*

6.4. CONCEPT OF POINT OBJECT OR PARTICLE

We normally define a **particle** *as an isolated point mass having no size and incapable of rotation.* Everyday objects, of course, are not particles since they do take up space. Can we treat a real object as a particle? The answer is that it depends upon the circumstances. A few examples given below illustrate that a real object can be treated as a particle.

(*i*) In describing the motion of a car that is going a hundred kilometres, the length of the car is very small as compared to the distance involved. For all practical purposes, we can consider the motion of car as the motion of a particle.

(*ii*) The bob of a swinging pendulum can be considered as a particle since its size is very small as compared to the length of the string.

(*iii*) A spacecraft travelling towards the moon can be considered as a particle. It is because the length of the spacecraft can be ignored as compared to the distance involved.

Therefore, a particle is not necessary a very small object as commonly imagined. Rather we take particle to be simply an object that is small enough for its size to be unimportant for the scale of our observation. *Therefore, we are free to consider an object or a body as a particle to simplify the problems.*

6.5. LOCATION OF POINT IN SPACE

The space we live in is three dimensional *i.e.*, we can specify the position of any point in space with three numbers. To do so, we choose a convenient point *O* as the origin. Through *O*, we draw three mutually perpendicular axes *OX* , *OY* and *OZ* as shown in Fig. 6.1. This is known as rectangular (cartesian) coordinate system. The three numbers – the distances along *OX* axis, *OY* axis and *OZ* axis – will specify the position of the point *P* in space. We say that coordinates of point *P* are *x, y* and *z*. If all the three coordinates *x, y* and *z* of an object remain unchanged with time, we say the object is at rest *w.r.t.* the coordinate system. If any one or more of the object's coordinates change with time, we say that the object is moving *w.r.t.* the coordinate system.

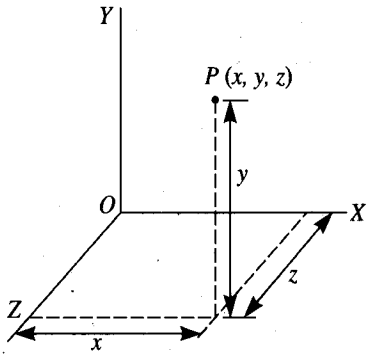


Fig. 6.1

6.6. FRAME OF REFERENCE

When we say that a car is moving at 80 km/hr, what does it mean? It usually means *w.r.t.* the surface of the earth. But earth is moving at 30 km/hr relative to the sun so that car's speed relative to sun is different from 80 km/hr. Statements about motion are meaningful only when we have an answer to the question "motion relative to what"?

The place from which motion is observed and measured is called **frame of reference.**

In order to measure motion (*i.e.*, to locate the position of an object), we need a coordinate system. This coordinate system is attached to some specified body, usually earth's surface. *Unless stated otherwise, frame of reference means coordinate system attached to the earth.* We add clock to the frame of reference to measure time.

6.7. MOTION IN THREE, TWO AND ONE DIMENSIONS

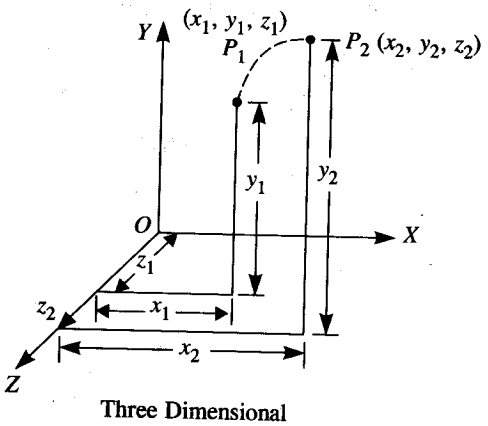
In order to describe the motion of an object, we consider the earth as our frame of reference *i.e.*, the coordinate system is attached to the surface of the earth. We also add clock to our frame of reference to measure time.

(*i*) **Three dimensional motion.** The motion of an object in space is generally three dimensional motion *i.e.*, all the three coordinates (*x, y* and *z*) change with time.

The motion of an object is said to be **three dimensional** *if all the three coordinates* (*x, y, z*) *of the object change with time.*

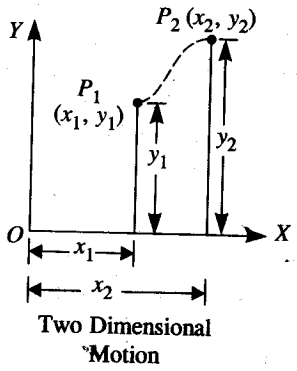
Fig. 6.2 illustrates the three dimensional motion of an object. As the object moves from P_1 to P_2, all the three coordinates change *i.e.*, from (x_1, y_1, z_1) to (x_2, y_2, z_2). The examples of three dimensional motion are (*a*) a bird flying in space (*b*) an aeroplane in motion in space *etc.*

(*ii*) **Two dimensional motion.** The motion of an object in a plane is two dimensional motion *i.e.*, only two of the three coordinates (say x and y) change with time.



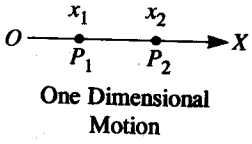
Three Dimensional Motion

Fig. 6.2



Two Dimensional Motion

Fig. 6.3



One Dimensional Motion

Fig. 6.4

The motion of an object is said to be **two dimensional** *if two of the three coordinates of the object change with time.*

Fig. 6.3 illustrates the two dimensional motion of an object. Note that motion of the object is restricted to XY plane. Therefore, as the object moves in this plane, only x and y coordinates change. Thus as the object moves from P_1 to P_2, the coordinates (x_1, y_1) change to (x_2, y_2). The examples of two dimensional motion are (*a*) projectile motion *i.e.*, the motion of a body thrown in air (*b*) motion of a body in a circle with constant speed *etc.*

(*iii*) **One dimensional motion.** The motion of an object along a straight line is one dimensional motion *i.e.*, only one of the three coordinates (say x) changes with time.

The motion of an object is said to be **one dimensional** *if only one of the three coordinates of the object changes with time.*

Fig. 6.4 illustrates the one dimensional motion of an object. Note that motion of the object is along OX axis. As the object moves from P_1 to P_2 along OX axis, the x-coordinate changes from x_1 to x_2. This is the simplest type of motion. Examples of one dimensional motion are (*a*) motion of a train along a straight railway track (*b*) the vertical fall of a stone *etc.*

6.8. SCALAR AND VECTOR QUANTITIES

The physical quantities (*e.g.* mass, length, velocity, force *etc.*) in physics can be divided into two classes *viz.* (*i*) scalar quantities (*ii*) vector quantities.

(*i*) **Scalar quantity.** *A physical quantity which has magnitude only but no direction is called a* **scalar quantity** *or a* **scalar**.

Examples of scalar quantities are mass, length, energy, temperature, speed *etc.* The magnitude or size of a scalar indicates its numerical value and unit of measure *e.g.* mass of 10 kg ; time of 2 hours and so on. The scalars can be added, subtracted, multiplied and divided by using ordinary laws of algebra.

(*ii*) **Vector quantity.** *A physical quantity which has magnitude as well as direction is called a* **vector quantity** *or a* **vector**.

Examples of vector quantities are force, velocity, acceleration, momentum and so on. The magnitude of a vector is a scalar that indicates only how large or small the vector is. Often

the direction of a vector is as important as its magnitude. For the *addition, subtraction or multiplication of vectors, their directions must be considered. Therefore, rules of ordinary algebra cannot be used to carry out these operations. Instead, we use vector algebra for this purpose.

A vector is represented graphically by a straight line with an arrowhead as shown in Fig. 6.5. The length of the line represents the magnitude of the vector (on suitable scale) and arrowhead indicates its direction. When we write a vector such as in Fig. 6.5 in hand, it is represented as $\vec{F}$. The magnitude of the vector (called modulus of vector) is represented as $|\vec{F}|$ or F (lightface). However, we print its symbol in boldface type as $\boldsymbol{F}$.

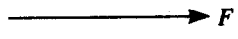


Fig. 6.5

6.9. MOTION IN A STRAIGHT LINE

We can describe the motion of an object in two ways : *with mathematical equations* and *with graphs.* In order to study the motion graphically, we must be able to measure the position of the object and time. Since motion along a straight line (*i.e.,* one dimensional motion) is the simplest, we shall use it to describe these concepts.

(*i*) **Position measurement.** In order to locate the position of an object, we need a frame of reference. A convenient way is to fix the coordinate system to earth's surface as shown in Fig. 6.6. Here O is the origin and serves as a reference point for all measurements. *An object moving along X-axis or Y-axis undergoes one dimensional motion.* The positive and negative directions along the axes shown are the usual convention. According to this convention, the measurements to the right of O are taken as positive while the measurements to the left of O are taken negative. Similarly, measurements above O are positive while those below O are negative. It may be noted that the choice of origin is arbitrary. However, once the choice of origin is made in a problem, it should remain unchanged. The SI unit of distance measurement is metre.

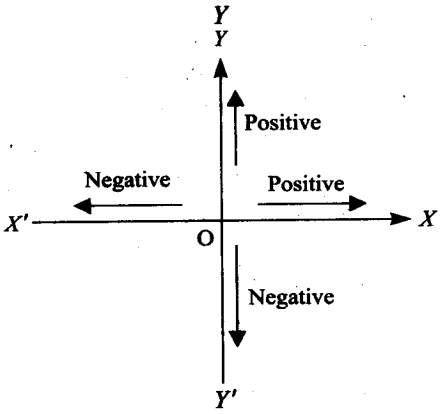


Fig. 6.6

(*ii*) **Time measurement** We can add a clock to our reference frame to measure time. In order to describe the motion of an object, it is necessary to choose some instant of time as the origin. This instant is assigned zero value of time *i.e.*, $t = 0$. It may be noted that $t = 0$ is not necessarily the beginning of the motion. *In fact $t = 0$ is the instant at which we start the study of motion of the object.* The usual unit of time is second but sometimes it is more convenient to use minute or hour as the unit of time.

Illustration. Consider the one dimensional motion of an object along OX axis as shown in Fig. 6.7. Here O is the origin for position measurement. Suppose we start the observation of motion of the object when it is at the origin O. Therefore, the object is at O, at $t = 0$. Let at any

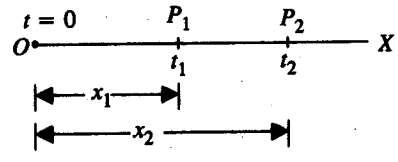


Fig. 6.7

* Division of vector by a vector is not a valid operation in vector algebra. It is because the division of a vector by a direction is not possible.

time t_1, the object be at point P_1 and covers a distance (= displacement) x_1. We say that the position coordinate of the object is x_1. Similarly, at time t_2, the position coordinate of the object is x_2. Note that when we say that position of an object is x_1, it means that object is on the x-axis and its distance (or displacement) from the origin O is x_1.

Note. In one dimensional motion, magnitude of displacement is equal to distance travelled.

6.10. DESCRIPTION OF MOTION

To describe the motion of an object, we need the concepts of *displacement, velocity* and *acceleration.* These are vector quantities *i.e.*, they possess magnitude as well as direction. For the motion of an object in two or three dimensions, the detailed knowledge of vectors is essential. However, for one dimensional motion, we need only one axis (*e.g.*, x-axis or y-axis) to indicate the position of the object. For such a restricted motion, there are only two *possible directions *viz.* positive and negative. Therefore, vector symbols are not needed for one dimensional motion. The sign (+ or –) indicates the direction of motion along the axis. In this entire unit, we shall study the concepts of displacement, velocity, acceleration and related topics considering one dimensional motion.

6.11. DISTANCE AND DISPLACEMENT

(*i*) Distance. *The total length of the path travelled by an object during a given time is called the distance travelled or covered by the object in that time.* It is a scalar quantity and is measured in metres in SI units. The value of distance cannot be negative.

(*ii*) Displacement. *The shortest distance from the initial position to the final position of an object is called displacement of the object.* It is a vector quantity *i.e.*, it has magnitude as well as direction. The displacement is measured in metres in SI units and its value can be zero, positive or negative.

Fig. 6.8 shows the concept of displacement. The path of the object may be *ACB*, *ADB etc.* but the displacement is *AB i.e.*, the shortest distance between the initial position A and the final position B of the object. The magnitude of the displacement is the length of the straight line *AB* and its direction is from the initial position (A) to the final position (B). The displacement of an object between two points does not tell exactly how the object has actually moved between these points. Since displacement is a vector quantity, its magnitude as well as direction must be specified.

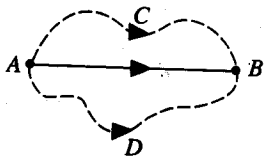


Fig. 6.8

Consider the one dimensional motion of an object along OX axis as shown in Fig. 6.9. Let the time be measured from the instant the object is at the origin O. Suppose at time t_1, the object is at point P_1 with coordinate x_1 and at time t_2, it is at P_2 where its coordinate is x_2. The change in the position of the object (*i.e.*, $x_2 - x_1$) is the *displacement* of the object during time interval (t_2–t_1).

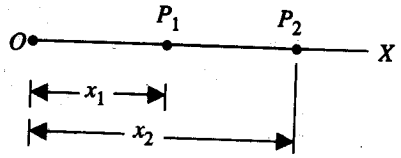


Fig. 6.9

$$\text{Displacement, } \Delta x = x_2 - x_1$$

It is a usual practice to use Greek letter Δ to indicate the change in a quantity. Thus, the change in x is written as **Δx .

---

* There are two directions from the origin—left and right or up and down.

** The notation Δx (read 'delta x') stands for a single quantity ; it is not a product of Δ and x.

6.12. SPEED

The speed of a moving object is defined as the total distance covered divided by the time taken to cover the distance i.e.,

$$\text{Speed} = \frac{\text{Total distance covered}}{\text{Time taken}}$$

The SI unit of speed is ms^{-1} but sometimes the speed of an object is expressed in km/hr.

(*i*) The dimensional formula of speed is $[M^0 LT^{-1}]$.

(*ii*) Speed is a scalar quantity *i.e.*, it has magnitude only but no direction.

(*iii*) The speed of an object can be positive or zero but not negative. It is because distance cannot be negative.

1. Uniform speed. *An object is said to be moving with a uniform speed (i.e., constant speed) if it covers equal distances in equal intervals of time, however small these time intervals may be.*

   Thus if a car moves with a uniform speed of 60 km/hr for 2 hours, the distance covered by the car in 2 hours = 60 × 2 = 120km.

2. Variable speed. *An object is said to be moving with a variable speed if it covers equal distances in unequal intervals of time, however small these time intervals may be.*

3. Average speed. When an object moves with a variable speed, we generally describe its motion in terms of average speed.

   The average speed of an object is the total distance covered divided by the time taken to cover the distance i.e.,

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Time taken}}$$

   For example, if a car covers a distance of 200 km in 4 hours, then average speed of the car = 200/4 = 50 km/hr. Note that average speed tells us nothing about the different speeds and variations that may have taken place during the journey.

4. Instantaneous speed. When an object is moving with a variable speed, it has different speeds at different instants of time.

   The speed of an object at a given instant of time is called its instantaneous speed.

The speedometer of a car indicates the instantaneous speed. When we say that the speed of a car at some particular instant is 60 km/hr, we are specifying its instantaneous speed. Note that instantaneous speed is quite different from the average speed.

Example 6.1. **An object travels different distances with different speeds in the same direction. Find the expression for the average speed.**

Solution. Suppose an object travels distances S_1, S_2, S_3 etc. with speeds v_1, v_2, v_3 etc. in the same direction respectively.

$$\text{Total distance travelled} = S_1 + S_2 + S_3$$

$$\text{Total time taken} = \frac{S_1}{v_1} + \frac{S_2}{v_2} + \frac{S_3}{v_3}$$

$$\therefore \quad \text{Average speed} = \frac{S_1 + S_2 + S_3}{\frac{S_1}{v_1} + \frac{S_2}{v_2} + \frac{S_3}{v_3}}$$

Special case. If the object travels two equal distances ($S_1 = S_2 = S$) with different speeds v_1 and v_2, then,

$$\text{Average speed} = \frac{S+S}{\frac{S}{v_1}+\frac{S}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$$

Example 6.2. An object travels with different speeds during different time intervals in the same direction. Find the expression for the average speed.

Solution. Suppose an object travels with speeds v_1, v_2, v_3 etc. during time intervals t_1, t_2, t_3 etc. in the same direction.

$$\text{Total distance travelled} = v_1 t_1 + v_2 t_2 + v_3 t_3$$

$$\text{Total time taken} = t_1 + t_2 + t_3$$

$$\therefore \quad \text{Average speed} = \frac{v_1 t_1 + v_2 t_2 + v_3 t_3}{t_1 + t_2 + t_3}$$

Special case. If $t_1 = t_2 = t_3 = t$, then,

$$\text{Average speed} = \frac{v_1 + v_2 + v_3}{3}$$

6.13. VELOCITY

Velocity of an object is defined as the displacement divided by the time taken for the displacement i.e.,

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time taken}}$$

The SI unit of velocity is ms^{-1}. Velocity is a vector quantity *i.e.,* it has magnitude as well as direction. For example, if a body has a displacement of 50 m in 5 s towards north, then its velocity = 50/5 = 10 ms^{-1} towards north.

(*i*) The dimensional formula of velocity is $[M^0 LT^{-1}]$.

(*ii*) The direction of velocity is the same as that of the displacement.

(*iii*) Since the displacement can be positive, zero or negative, the velocity of an object may be positive, zero or negative.

1. Uniform velocity. *An object is said to be moving with a uniform velocity (i.e., constant velocity) if it undergoes equal displacements in equal intervals of time, however small these time intervals may be.* Uniform velocity means that magnitude as well as direction of the velocity remains constant.

2. Variable velocity. *A body is said to be moving with a variable velocity (i.e., non-uniform velocity) if there is a change in its magnitude or direction or both.*

A body moving with a constant speed in a circle has a variable velocity. Although speed (magnitude of velocity) is constant, the direction of velocity of the body is changing from instant to instant.

3. Average velocity. When an object moves with a variable velocity, we describe its motion in terms of average velocity.

The average velocity of an object is defined as the displacement divided by the time taken for the displacement i.e.,

$$\text{Average velocity, } \bar{v} = \frac{\text{Displacement}}{\text{Time taken}}$$

The bar above the v indicates average velocity.

Consider one dimensional motion of an object along OX axis as shown in Fig. 6.10. Let at $t = 0$, the object be at the origin O. Suppose at time t_1, the object is at point P_1, with coordinate x_1 and at time t_2, it is at point P_2 where its coordinate is x_2. Therefore, during time interval $(t_2 - t_1)$:

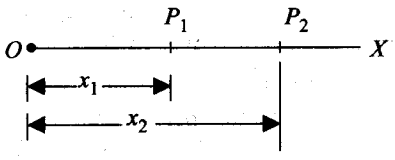


Fig. 6.10

$$\text{Displacement, } \Delta x = x_2 - x_1$$

$$\text{Time interval, } \Delta t = t_2 - t_1$$

$$\therefore \quad \text{Average velocity, } \bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

Note that displacement and velocity may be positive or negative depending upon the direction of motion of the object. A positive value indicates that the motion is to the right of origin O; a negative value indicates that motion is to the left (*i.e.*, towards the decreasing value of x).

4. Instantaneous velocity. *The velocity of an object at any given instant of time is called its instantaneous velocity.*

Consider an object moving with a uniform velocity of 5 ms^{-1}. It means that the object is moving with a speed (magnitude of velocity) of 5 ms^{-1} in a straight line. The velocity of the object at any instant (*i.e.*, instantaneous velocity) is 5 ms^{-1} and, therefore, its average velocity is also 5 ms^{-1}.

Hence for uniform motion, the average velocity of an object is equal to its instantaneous velocity.

Note. When we are dealing with one dimensional motion (*i.e.*, motion in a straight line), we need not use vector notation for displacement and velocity. Positive and negative signs suffice.

6.14. SPEED VERSUS VELOCITY

Loosely speaking, we often use the words speed and velocity interchangeably. However, the two physical quantities are different. This becomes apparent from the following discussion :

(*i*) Speed is a scalar quantity and has magnitude only. However, velocity is a vector quantity and has magnitude as well as direction.

(*ii*) Speed indicates how fast something is moving whereas velocity tells us how fast and in which direction it is moving.

(*iii*) The distinction between speed and velocity can be made more clear if we refer to Fig. 6.11. Here a car is moving round a circular track at a constant speed of 40 km/hr. At every point on the track, the speed is the same *i.e.*, 40 km/hr. But at every point (such as A, B and C) the velocity is different. Thus the magnitude of velocity at points A, B and C is the same but the direction of velocity is different. Therefore, we note that velocities at all points (*e.g.*, A, B and C) are different, even though the magnitudes of the velocities are same.

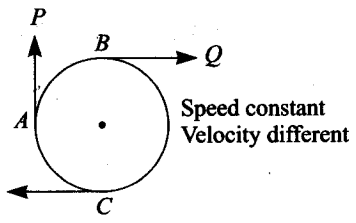


Fig. 6.11

Example 6.3. **A particle is moving along X-axis. At time $t_1 = 2$ s, its position is $x_1 = 3$ m and at $t_2 = 7$ s, its position is $x_2 = 18$ m. What is the average velocity of the particle?**

Solution. $\text{Displacement, } \Delta x = x_2 - x_1 = 18 - 3 = 15\text{m}$

$$\text{Time interval, } \Delta t = t_2 - t_1 = 7 - 2 = 5 \text{ s}$$

$$\therefore \quad \text{Average velocity, } \bar{v} = \frac{\Delta x}{\Delta t} = \frac{15}{5} = 3 \text{ ms}^{-1}$$

Note that the displacement and average velocity are positive, indicating that the particle moved to the right *i.e.*, towards increasing values of x.

Example 6.4. A train is travelling due east at an average speed of 120 km/hr. How far does this train travel in 6 s?

Solution. Since the time interval is given in seconds, we shall convert the magnitude of velocity in metres per second.

$$120 \text{ km/hr} = \frac{120\,\text{km}}{1\,\text{hr}} = \frac{120\times10^3\,\text{m}}{1\times3600\,\text{s}} = 33.3\,\text{ms}^{-1}$$

$$\text{Average velocity, } \bar{v} = \frac{\Delta x}{\Delta t}$$

$$\therefore \quad \text{Displacement, } \Delta x = \bar{v}\,\Delta t$$

$$\text{Here, } \bar{v} = 33.3\,\text{ms}^{-1};\ \Delta t = 6\,\text{s}$$

$$\therefore \quad \Delta x = 33.3 \times 6 = 200\,\text{m}$$

Example 6.5. A runner runs 100 m in 10 s, then turns around and jogs 50 m back toward the starting point in 30 s. Find his (*i*) average speed (*ii*) average velocity.

Solution. (*i*) Total distance travelled = 100 + 50 = 150 m

Total time taken = 10 + 30 = 40 s

$$\therefore \quad \text{Average speed} = \frac{150}{40} = 3.75\ \text{ms}^{-1}$$

(*ii*) Total displacement, $\Delta x = 100 - 50 = 50\text{m}$

Time interval, $\Delta t = 10 + 30 = 40\text{s}$

$$\therefore \quad \text{Average velocity, } \bar{v} = \frac{\Delta x}{\Delta t} = \frac{50}{40} = 1.25\ \text{ms}^{-1}$$

Example 6.6. A particle moves along a circle of radius R as shown in Fig. 6.12. It starts from A and moves in anticlockwise direction. Calculate the distance travelled and displacement (*i*) from A to B (*ii*) form A to C and (*iii*) from A to D.

Solution. (*i*) Distance travelled from A to $B = \frac{2\pi R}{4} = \frac{\pi R}{2}$

$$\text{Displacement, } |AB| = \sqrt{OA^2 + OB^2}$$

$$= \sqrt{R^2 + R^2} = \sqrt{2}\,R$$

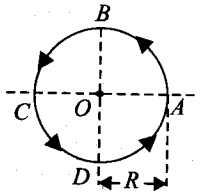


Fig. 6.12

(*ii*) Distance travelled from A to $C = \frac{2\pi R}{2} = \pi R$

Displacement, $|AC| = 2R$

(*iii*) Distance travelled from A to $D = 2\pi R \times \frac{3}{4} = 1.5\pi R$

Displacement, $|AD| = \sqrt{R^2 + R^2} = \sqrt{2}\,R$

Example 6.7. A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km/h. Finding the market closed, he instantly turns and walks back home with a speed of 7.5 km/h. What is the magnitude of average velocity and average speed of the man over the interval of time (*i*) 0 – 30 minutes (*ii*) 0 – 50 minutes (*iii*) 0 – 40 minutes?

Solution. (*i*) **Time interval 0 – 30 minutes.** While going to the market, the man walks at a speed of 5 km/h.

$\therefore$ Displacement in 30 min = 5/2 = 2.5 km

$$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Time}} = \frac{2.5\,\text{km}}{0.5\,\text{h}} = 5\ \text{km/h}$$

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{2.5\,\text{km}}{0.5\,\text{h}} = 5\ \text{km/h}$$

(*ii*) **Time interval 0 – 50 minutes.** While going to the market the man walks at a speed of 5 km/h. Since the market is 2.5 km from his home, he takes 30 minutes to reach the market. While coming back from the market, he walks at a speed of 7.5 km/h so that time taken to reach home = 2.5/7.5 = 1/3 hour = (1/3) × 60 = 20 minutes. This means that in 50 minutes the man walks back home. Therefore, net displacement = 0.

$\therefore$ $$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Time}} = \frac{0}{50\,\text{min}} = 0$$

Now in 50 min, the total distance travelled = 2.5 + 2.5 = 5 km and total time taken = 50/60 hr = 5/6 hr.

$\therefore$ $$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{5}{5/6} = 6\ \text{km/h}$$

(*iii*) **Time interval 0 – 40 minutes.** In 30 minutes, the man covers a distance of 2.5 km in going from home to the market. While returning home, he covers a distance of 1.25 km in 10 minutes ($\because$ return speed is 7.5 km/h).

$\therefore$ Net displacement during time interval 0 – 40 min

= 2.5 – 1.25 = 1.25 km

$$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Total time}} = \frac{1.25}{40/60} = 1.875\ \text{km/h}$$

Total distance travelled in 40 min = 2.5 + 1.25= 3.75 km

$\therefore$ $$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time}} = \frac{3.75}{40/60} = 5.625\ \text{km/h}$$

PROBLEMS FOR PRACTICE

1. A particle is moving along *X*-axis. At time t_1 = 3s, its position is x_1 = 6m and at t_2 = 8 s, its position is x_2 = 21 m. What is its average velocity ? **[3 ms^{-1}]**
2. A particle is moving along *X*-axis. At time t_1 = 4s, its position is x_1 = 40 m and at t_2 = 6 s, its position is x_2 = 20 m. What is its average velocity ? **[–10 ms^{-1}]**
3. A train travels a distance with a speed of 30 km/hr and returns with a speed of 50 km/hr. Find the average speed of the train. **[37.5 km/h]**
4. On a 60 km track, a train travels the first 30 km with a uniform (*i.e.*, constant) speed of 30 km/h. How fast must the train travel the next 30 km so as to average 40 km/h for the entire trip? **[60 km/h]**
5. A stopwatch is used to time the motion of a car. At t_1 = 14 s, the car is at x = 60 m. At t_2 = 17 s, the car is at x = 15 m. What is the average velocity of the car ? **[–15 ms^{-1}]**

Solution. (*i*) **Time interval 0 – 30 minutes.** While going to the market, the man walks at a speed of 5 km/h.

$\therefore$ Displacement in 30 min = 5/2 = 2.5 km

$$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Time}} = \frac{2.5\,\text{km}}{0.5\,\text{h}} = 5\ \text{km/h}$$

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{2.5\,\text{km}}{0.5\,\text{h}} = 5\ \text{km/h}$$

(*ii*) **Time interval 0 – 50 minutes.** While going to the market the man walks at a speed of 5 km/h. Since the market is 2.5 km from his home, he takes 30 minutes to reach the market. While coming back from the market, he walks at a speed of 7.5 km/h so that time taken to reach home = 2.5/7.5 = 1/3 hour = (1/3) × 60 = 20 minutes. This means that in 50 minutes the man walks back home. Therefore, net displacement = 0.

$\therefore$ $$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Time}} = \frac{0}{50\,\text{min}} = 0$$

Now in 50 min, the total distance travelled = 2.5 + 2.5 = 5 km and total time taken = 50/60 hr = 5/6 hr.

$\therefore$ $$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{5}{5/6} = 6\ \text{km/h}$$

(*iii*) **Time interval 0 – 40 minutes.** In 30 minutes, the man covers a distance of 2.5 km in going from home to the market. While returning home, he covers a distance of 1.25 km in 10 minutes ($\because$ return speed is 7.5 km/h).

$\therefore$ Net displacement during time interval 0 – 40 min

= 2.5 – 1.25 = 1.25 km

$$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Total time}} = \frac{1.25}{40/60} = 1.875\ \text{km/h}$$

Total distance travelled in 40 min = 2.5 + 1.25= 3.75 km

$\therefore$ $$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time}} = \frac{3.75}{40/60} = 5.625\ \text{km/h}$$

PROBLEMS FOR PRACTICE

1. A particle is moving along X-axis. At time t_1 = 3s, its position is x_1 = 6m and at t_2 = 8 s, its position is x_2 = 21 m. What is its average velocity ? **[3 ms^{-1}]**
2. A particle is moving along X-axis. At time t_1 = 4s, its position is x_1 = 40 m and at t_2 = 6 s, its position is x_2 = 20 m. What is its average velocity ? **[–10 ms^{-1}]**
3. A train travels a distance with a speed of 30 km/hr and returns with a speed of 50 km/hr. Find the average speed of the train. **[37.5 km/h]**
4. On a 60 km track, a train travels the first 30 km with a uniform (*i.e.*, constant) speed of 30 km/h. How fast must the train travel the next 30 km so as to average 40 km/h for the entire trip? **[60 km/h]**
5. A stopwatch is used to time the motion of a car. At t_1 = 14 s, the car is at x = 60 m. At t_2 = 17 s, the car is at x = 15 m. What is the average velocity of the car ? **[–15 ms^{-1}]**

6.15. UNIFORM MOTION IN A STRAIGHT LINE

Uniform motion means that the object moves with a uniform velocity *i.e.*, the *magnitude* and *direction* of the velocity remain the same. Clearly, such a motion can take place *only* if the object moves in a straight line.

An object is said to be in uniform motion if it undergoes equal displacements (in the same direction) in equal intervals of time, however small these time intervals may be.

If an object is moving with a uniform velocity v, then displacement of the object in time t will be $= vt$. This is the key point in uniform motion.

Formulas for Uniform Motion. Consider an object moving with uniform velocity v along the straight line OX as shown in Fig. 6.13. Here O is the origin. Let the time be measured from the instant when the object is at position A *i.e.*, at $t = 0$, $OA = x_0$. Let the object be at positions B and C after times t_1 and t_2 respectively. Then, $OA = x_0$; $OB = x_1$; $OC = x_2$; $AB = vt_1$; $AC = vt_2$.

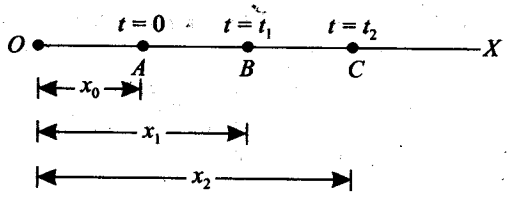


Fig. 6.13

Now $OB = OA + AB$

or $x_1 = x_0 + vt_1$ ...(*i*)

Also $OC = OA + AC$

or $x_2 = x_0 + vt_2$ ...(*ii*)

From eqs. (*i*) and (*ii*), $x_2 - x_1 = v(t_2 - t_1)$ ...(*iii*)

or $x_2 = x_1 + v(t_2 - t_1)$ ...(*iv*)

Eqs. (*i*) to (*iv*) give the formulas for uniform motion. Here eq. (*i*) gives the position of the object at $t = t_1$ while eqs. (*ii*) and (*iv*) give the position of the object at $t = t_2$. However, eq. (*iii*) gives the displacement of the object during the time interval $(t_2 - t_1)$. If $x_2 - x_1 = S$, then displacement during the time $t = (t_2 - t_1)$ is

$$S = vt$$

Note. If the time is measured from the instant the object is at the origin O, then at $t = 0$, $x_0 = 0$. Therefore, the above equations for uniform motion (put $x_0 = 0$) become :

$$x_1 = vt_1 \; ; \; x_2 = vt_2 \; ; \; x_2 - x_1 = v(t_2 - t_1) \; ; \; x_2 = x_1 + v(t_2 - t_1)$$

6.16. GRAPHICAL REPRESENTATION OF UNIFORM MOTION

The graphical representation of uniform motion is very helpful because it provides more physical insight than the mathematical relations. We shall study the following graphs :

1. Displacement-time ($x - t$) graph
2. Velocity-time ($v - t$) graph

While interpreting the graphs for uniform motion, the following points may be kept in mind :

(*i*) The displacement (x) and velocity (v) will be taken along the Y-axis and time (t) along the X-axis.

(*ii*) Positive velocity (or displacement) means that the motion of the object is to the right of the origin O. Negative velocity (or displacement) means that the motion is to the left of

the origin *O*. *Therefore, if $x - t$ or $v - t$ graph is above X-axis (i.e., time axis), the velocity (v) is positive and if it is below X-axis, velocity is negative.*

(*iii*) If at $t = 0$, the initial displacement of the object from the origin is zero (*i.e.*, $x_0 = 0$), the $x - t$ graph will start from the origin of the graph. If at $t = 0$, the initial displacement is x_0, the $x - t$ graph will start at x_0.

(*iv*) Zero velocity of the object means the object is at rest.

6.17. DISPLACEMENT-TIME ($x - t$) GRAPH FOR UNIFORM MOTION

We have seen that in uniform motion, the position x of the object at any time t is related to the uniform velocity v as :

$$x = vt \quad \text{...starting location } O$$

$$x = x_0 + vt \quad \text{...starting location } x_0$$

Both these equations reveal that x is a linear function of t. *Therefore, $x - t$ graph for uniform motion is a straight line.*

(*i*) **Slope of $x - t$ graph.** Fig. 6.14 shows the $x - t$ graph for uniform motion. The graph OA represents the condition when the starting location of the object is O while graph $O'B$ represents the condition when the same object has a starting location x_0. Since the two graphs are parallel ($x \propto t$), they have the same slope.

$$\text{Slope of graph} = \tan\theta = \frac{x}{t} = v$$

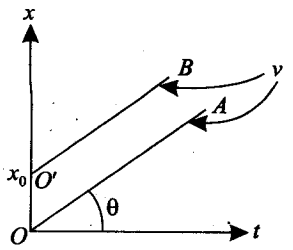


Fig. 6.14

Hence we arrive at a very important conclusion that slope of $x - t$ graph (true for any type of motion) of an object gives the velocity (v) of the object. In the present case, the slope of graph ($= x/t$) gives the uniform velocity v of the object.

(*ii*) **When the object is rest.** Fig. 6.15 shows the $x - t$ graph for an object at rest for two conditions *viz.* for starting locations at O and x_0. Note that displacement does not change with time *i.e.*, velocity is zero. This fact (zero velocity) is also revealed by the graphs being parallel to the X-axis (time axis) *i.e.*, $\tan\theta = \tan 0° = 0$.

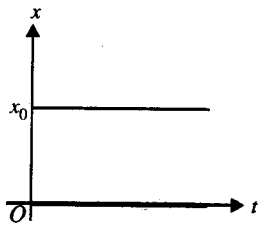


Fig. 6.15

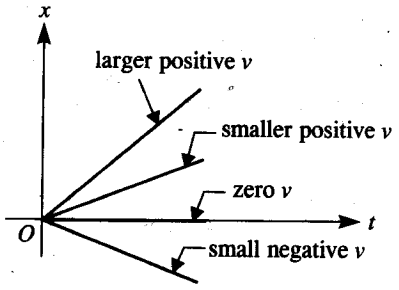


Fig. 6.16

(*iii*) **When object moves with uniform velocity.** Fig. 6.14 (Refer back) shows the situation when two objects have the same uniform velocity v but have different starting locations. Fig. 6.16 shows the $x - t$ graph for different velocities for the same starting location O. Note that if $x - t$ graph is above X-axis (time axis), the velocity is positive. However, if the $x - t$ graph is below X-axis, the velocity is negative. Further, greater the slope ($= \tan\theta$) of the $x - t$ graph, the greater is the velocity and vice-versa.

6.18. VELOCITY-TIME ($v - t$) GRAPH FOR UNIFORM MOTION

$$x = vt \quad \therefore \frac{dx}{dt} = \frac{d}{dt}(vt) = v = \text{constant}$$

$$x = x_0 + vt \quad \therefore \frac{dx}{dt} = \frac{d}{dt}(x_0 + vt) = v = \text{constant}$$

Therefore, the velocity of the object is constant at all times so that velocity-time graph is a straight line parallel to time axis as shown in Fig. 6.17. Note that $v - t$ graph (solid line) above O indicates positive velocity *i.e,* the object is moving in the positive x-direction. The $v - t$ graph (dotted line) below O indicates negative velocity *i.e.*, the object is moving in the negative x-direction.

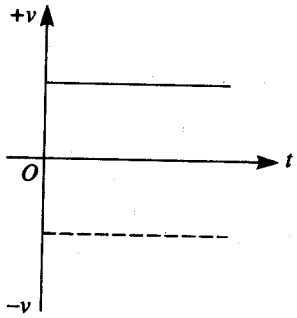


Fig. 6.17

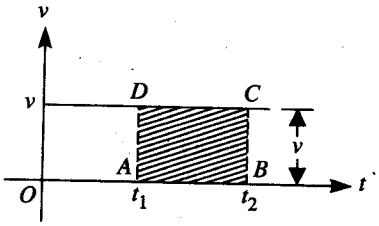


Fig. 6.18

Net displacement from $v - t$ graph. The velocity-time graph of a moving object can be used to calculate the *net* displacement of the object during a given time interval. Thus referring to Fig. 6.18, A and B are the two points on the $v - t$ graph corresponding to times t_1 and t_2. Since velocity of the object is the same (*i.e.*, v) at t_1 and t_2, the area under $v - t$ graph during the time interval t_1 to t_2 is equal to the area of the rectangle $ABCD$.

Area under $v - t$ graph = Area $ABCD = AB \times BC = (t_2 - t_1)v$

But $v(t_2 - t_1)$ is the displacement of the object during the time interval $(t_2 - t_1)$.

Hence the displacement of an object during the time interval $(t_2 - t_1)$ is equal to the area under velocity-time graph during that time interval.

This statement is true for any type of motion. Note that the area of the $v - t$ graph above X-axis (time axis) is positive and that below the X-axis is negative. The *net* displacement will be the algebraic sum of the areas under the $v - t$ graph.

Note. If we are to find distance travelled by the object from the $v - t$ graph, then the negative area of $v - t$ graph (*i.e.*, area of $v - t$ graph below X-axis) is also to be considered positive.

6.19. SUMMARY OF UNIFORM MOTION IN STRAIGHT LINE

The following points are worth noting about uniform motion in a straight line :

(*i*) The magnitude (v) and direction of the velocity remain the same throughout the journey.

(*ii*) The sign of v gives the direction of motion.

(*iii*) The magnitude of velocity gives the time rate of covering distance or displacement.

(*iv*) The displacement-time graph is a straight line. The slope of this graph gives the magnitude of the constant velocity.

(*v*) The position x of the object at any time t is given by $x = x_0 + v\,t$. Here x_0 is the position of the object at $t = 0$ (*i.e.*, its distance from origin at $t = 0$) and v is the constant velocity.

(*vi*) The velocity-time graph is a straight line parallel to the time axis.

(*vii*) The displacement during any time interval is equal to the area under the velocity-time graph during that time interval.

6.20. GRAPHICAL ANALYSIS OF VARIABLE VELOCITY

In uniform motion, the magnitude and direction of the velocity remain the same. However, in *non-uniform motion,* the velocity of the object is variable. In such a case, the motion of the object is described by its average velocity and instantaneous velocity. Let us discuss these graphically.

(*i*) **Geometrical interpretation of average velocity.** Consider one dimensional motion of an object along *OX* axis as shown in Fig. 6.19. Suppose the object is moving with a variable velocity. At time t_1, the object is at point *P* with coordinate x_1 and at time t_2, it is at *Q* with coordinate x_2. The graph between x and t is shown in Fig. 6.20. Note that curve in this figure does not represent the path of the object; it is the curve showing the positions of the object at various times. The path of the object is along a straight line as shown in Fig. 6.19. The cordinates of point *P* are (x_1, t_1) and that of *Q* are (x_2, t_2). These are represented on the graph. The displacement $\Delta x = x_2 - x_1$ and the time interval $\Delta t = t_2 - t_1$. A straight line drawn from point $P(x_1, t_1)$ to point $Q(x_2, t_2)$ forms the hypotenuse of the right angled triangle whose sides are Δx and Δt. The ratio $\Delta x/\Delta t$ is the slope of the line *PQ*. But $\Delta x/\Delta t$ is also the average velocity of the object during the time interval $\Delta t = t_2 - t_1$.

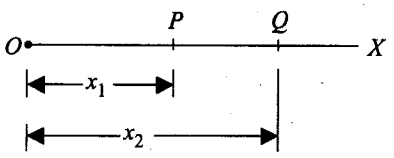


Fig. 6.19

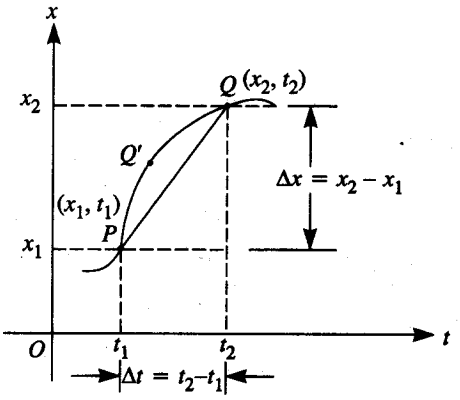


Fig. 6.20

$\therefore$ Average velocity $= \dfrac{\Delta x}{\Delta t}$ = Slope of line connecting points $P(x_1, t_1)$ and $Q(x_2, t_2)$

Thus we have the geometrical interpretation of the average velocity.

The average velocity of an object during the time interval $\Delta t = t_2 - t_1$ is equal to the slope of the straight line connecting two points $P(x_1, t_1)$ and $Q(x_2, t_2)$ on the $x - t$ graph.

Note that the average velocity depends upon the time interval choosen. For example in Fig. 6.20, if we choose a smaller time interval, the average velocity would be greater as indicated by the greater steepness of the line connecting *P* and *Q'*.

(*ii*) **Instantaneous velocity.** If you drive a car along a straight road for 100 km in 2 h, your average velocity is 50 km/h. However, it is unlikely that you were moving at 50 km/h at every instant. The velocity at any instant of time is called the instantaneous velocity. This is not a very precise definition. More precisely, the **instantaneous velocity** *at any moment is the average velocity over an indefinitely short time interval.* To make it clear, we shall seek the help of $x - t$ graph.

Fig. 6.21 shows the same $x - t$ graph as Fig. 6.20. Suppose we want to find the instantaneous velocity at point P *i.e.,* at time t_1. Consider the time interval $\Delta t = t_2 - t_1$. The average velocity during this time interval is slope of the line PQ. Now consider a time t' between t_1 and t_2 at which time the object is at x'. The slope of the line $P'Q'$ is greater than the slope of line PQ in this case. As the time interval Δt becomes smaller and smaller, the average velocity approaches the slope of tangent at point P (dotted line).

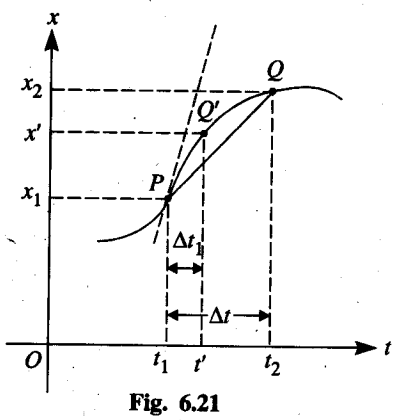


Fig. 6.21

We define the slope of this tangent to be the instantaneous velocity at the time t_1.

Thus instantaneous velocity is the limit of the ratio $\Delta x/\Delta t$ as Δt approaches zero *i.e.,*

$$\text{Instantaneous velocity, } v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The limit as $\Delta t \to 0$ is written in calculus notation as dx/dt and is called derivative of x with respect to t.

$$\therefore \quad v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

or
$$v = \frac{dx}{dt}$$

Thus, for a position versus time graph, the slope of the tangent line at a particular time is the object's velocity at that time.

This is the geometrical interpretation of instantaneous velocity. Since the slope may be positive (x increasing) or negative (x decreasing), the instantaneous velocity may be positive or negative in one dimensional motion.

Example 6.8. A car moves at a constant speed of 60 km/h for 1 km and at 40 km/h for next 1 km. What is the average speed of the car ?

Solution. $\text{Average speed} = \dfrac{\text{Total distance covered}}{\text{Total time taken}}$

$x(t_1) = 1$ km ; $v(t_1) = 60$ km/h ; $x(t_2) = 1$ km ; $v(t_2) = 40$ km/h

$$\therefore \quad t_1 = \frac{x(t_1)}{v(t_1)} = \frac{1}{60}\text{ h} \text{ and } t_2 = \frac{x(t_2)}{v(t_2)} = \frac{1}{40}\text{h}$$

$$\text{Total distance travelled} = x(t_1) + x(t_2) = 1 + 1 = 2 \text{ km}$$

$$\text{Total time taken} = t_1 + t_2 = \frac{1}{60} + \frac{1}{40} = \frac{1}{24}\text{ h}$$

$$\therefore \quad \text{Average speed} = \frac{2}{1/24} = 48 \text{ km/h}$$

Example 6.9. A car travels 1/3 of the distance on a straight road with a velocity of 10 km/h, next one-third with a velocity 20 km/h and the last one-third with a velocity of 60 km/h. What is the average velocity of the car for the entire journey ?

Solution. Let x be the total distance of the journey.

$x(t_1) = x/3$; $x(t_2) = x/3$; $x(t_3) = x/3$

$v(t_1) = 10$ km/h ; $v(t_2) = 20$ km/h ; $v(t_3) = 60$ km/h

(*vi*) The velocity-time graph is a straight line parallel to the time axis.

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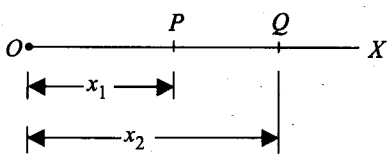

Fig. 6.19

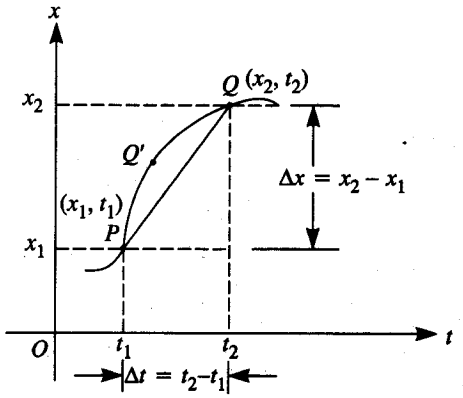

Fig. 6.20

$$\therefore \text{ Average velocity } = \frac{\Delta x}{\Delta t} = \text{Slope of line connecting points } P(x_1, t_1) \text{ and } Q(x_2, t_2)$$

Thus we have the geometrical interpretation of the average velocity.

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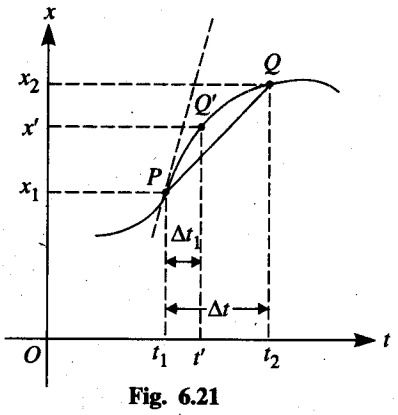

Fig. 6.21

We define the slope of this tangent to be the instantaneous velocity at the time t_1.

Thus instantaneous velocity is the limit of the ratio $\Delta x/\Delta t$ as Δt approaches zero *i.e.*,

$$\text{Instantaneous velocity, } v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The limit as $\Delta t \to 0$ is written in calculus notation as dx/dt and is called derivative of x with respect to t.

$$\therefore \quad v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

or
$$v = \frac{dx}{dt}$$

Thus, for a position versus time graph, the slope of the tangent line at a particular time is the object's velocity at that time.

This is the geometrical interpretation of instantaneous velocity. Since the slope may be positive (x increasing) or negative (x decreasing), the instantaneous velocity may be positive or negative in one dimensional motion.

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Solution. $\text{Average speed} = \dfrac{\text{Total distance covered}}{\text{Total time taken}}$

$x(t_1) = 1$ km ; $v(t_1) = 60$ km/h ; $x(t_2) = 1$ km ; $v(t_2) = 40$ km/h

$$\therefore \quad t_1 = \frac{x(t_1)}{v(t_1)} = \frac{1}{60} \text{ h} \quad \text{and} \quad t_2 = \frac{x(t_2)}{v(t_2)} = \frac{1}{40} \text{h}$$

$$\text{Total distance travelled} = x(t_1) + x(t_2) = 1 + 1 = 2 \text{ km}$$

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Solution. Let x be the total distance of the journey.

$x(t_1) = x/3$; $x(t_2) = x/3$; $x(t_3) = x/3$

$v(t_1) = 10$ km/h ; $v(t_2) = 20$ km/h ; $v(t_3) = 60$ km/h

$$\therefore \quad t_1 = \frac{x(t_1)}{v(t_1)} = \frac{x/3}{10} = \frac{x}{30} \text{ hours}$$

$$t_2 = \frac{x(t_2)}{v(t_2)} = \frac{x/3}{20} = \frac{x}{60} \text{ hours}$$

$$t_3 = \frac{x(t_3)}{v(t_3)} = \frac{x/3}{60} = \frac{x}{180} \text{ hours}$$

$$\text{Total time, } t = t_1 + t_2 + t_3 = \frac{x}{30} + \frac{x}{60} + \frac{x}{180} = \frac{10x}{180}$$

$$\therefore \quad \text{Average velocity} = \frac{x}{t} = \frac{x}{(10x/180)} = \mathbf{18\ km/h}$$

Example 6.10. A particle is moving along the *X*-axis as shown in Fig. 6.22. Its position as a function of time is given by :

$$x = A t^2 + B$$

Here $A = 2.1\ ms^{-2}$; $B = 2.8$ m

Determine (*i*) the displacement of the particle during the time interval from $t_1 = 3$ s to $t_2 = 5$ s (*ii*) the average velocity during this time interval and (*iii*) the magnitude of instantaneous velocity at $t = 5$ s.

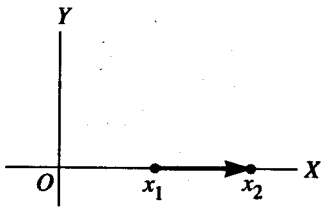


Fig. 6.22

Solution. (*i*) $x = A t^2 + B$

At $t_1 = 3$ s, the position of the particle is

$$x_1 = (2.1)\ (3)^2 + 2.8 = 21.7 \text{ m}$$

At $t_2 = 5$ s, the position of the particle is

$$x_2 = (2.1)\ (5)^2 + 2.8 = 55.3 \text{ m}$$

$$\text{Displacement, } \Delta x = x_2 - x_1 = 55.3 - 21.7 = \mathbf{33.6\ m}$$

(*ii*) Magnitude of average velocity is

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{33.6}{5-3} = \mathbf{16.8\ ms^{-1}}$$

(*iii*) $x = A t^2 + B$

$\therefore$ Instantaneous velocity v is given by :

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(At^2 + B\right) = 2At$$

$$\text{At } t = 5 \text{ s}, \quad v = 2 \times 2.1 \times 5 = \mathbf{21\ m\ s^{-1}}$$

Note that instantaneous velocity is quite different from the average velocity. Clearly, the particle is moving with a variable velocity.

Example 6.11. The position of a stone dropped from rest from a cliff is given by $x = 5 t^2$. What is the magnitude of the velocity of the stone at $t = 2$s ?

Solution. The stone is dropped from rest from the cliff. Therefore, it will fall vertically downward. Clearly, it is one dimensional motion along *Y*-axis.

$$x = 5 t^2$$

Instantaneous velocity is given by :

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(5t^2\right) = 10t$$

$$\text{At } t = 2 \text{ s} ; \quad v = 10 \times 2 = \mathbf{20\ ms^{-1}}$$

Example 6.12. The position of a ball rolling in a straight line is given by ;

$$x = 2 + 6.6\,t - 1.1\,t^2$$

What is its velocity at (*i*) $t = 2$ s and (*ii*) $t = 3$ s ?

Solution. $x = 2 + 6.6\,t - 1.1\,t^2$

The instantaneous velocity is given by ;

$$v = \frac{dx}{dt} = \frac{d}{dt}(2 + 6.6\,t - 1.1\,t^2) = 6.6 - 2.2\,t$$

(*i*) At $t = 2$ s, $v_1 = 6.6 - 2.2 \times 2 = 2.2$ ms^{-1}

(*ii*) At $t = 3$ s, $v_2 = 6.6 - 2.2 \times 3 = 0$ ms^{-1}

Obviously, the ball will come to rest in 3 seconds.

PROBLEMS FOR PRACTICE

1. A car travels 30 km at a uniform speed of 40 km/h and next 30 km at a uniform speed of 20 km/h. Find the average speed of the car. **[26.67 km/h]**
2. A person travelled a distance of 200 km between two cities by a car covering the first half distance of the journey at a constant speed of 40 km/h and the remaining half distance at a constant speed of v km/h. If the average speed during the entire journey is 48 km/h, find the value of v. **[60 km/h]**
3. The displacement at any time t of a particle moving with a uniform velocity along a straight line is $x = 3 + 5t$. What is the velocity of the particle? Displacement is measured in metres and time in seconds. **[5 m s^{-1}]**
4. Draw $x - t$ time graphs for the following cases : (*i*) $x = 3 - 5t$ (*ii*) $x = 2 + 3t$.
5. The position of a particle as a function of time is given by ;
   $x = 6t + 0.5\,t^3$
   What is its velocity at $t = 1$s? **[8.1 ms^{-1}]**
6. The position of a stone dropped from rest from a cliff is given by $x = 5\,t^2$. What is the velocity of the stone at $t = 3$ s? **[30 ms^{-1}]**
7. The position of a particle moving along X-axis is given by ;
   $x = At + 4\,Bt^3$
   What is its velocity at $t = 5$ s? **[120 B]**

6.21. RELATIVE VELOCITY IN ONE DIMENSION

Velocity is not an absolute quantity but is measured relative to other objects. Thus to measure the velocity of an object, we specify the frame of reference. Ordinarily, we consider the surface of earth as our frame of reference and specify the velocity of body with respect to it (*i.e.*, earth). Thus, when we say that a car is travelling at *60 km/h, we mean to say it is going 60 km/h relative to the road or earth's surface.

Suppose the two cars A and B are travelling on a straight level road in the same direction as shown in Fig. 6.23. Let the uniform velocities of A and B be v_A and v_B respectively. We want to find the velocity of car A with respect to car B. Note that v_A is the velocity of car A *w.r.t.* earth. But we want to find v_{AB} – now our frame of reference is car B. If we superpose a velocity $-v_B$ on the two cars, then car B comes to rest (*i.e.*, car B and earth become identical frames of reference), then the velocity of car A *w.r.t.* car B is $v_{AB} = v_A - v_B$.

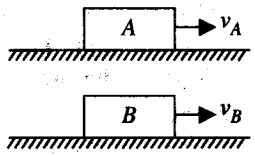


Fig. 6.23

* We generally omit mention of reference frame since we know that it is earth's surface.

Thus in order to find the relative velocity of body A w.r.t. body B in one dimensional motion, we add negative of velocity of B to both the bodies. This effectively brings the body B to rest The resultant velocity of body A is v_{AB} i.e., relative velocity of A w.r.t. B.

Suppose two cars A and B having velocities v_A and v_B are moving along a straight road in the same direction. In order to find the relative velocity of A *w.r.t.* B, we add velocity $-v_B$ to both the cars. This effectively brings the car B to rest and the resultant velocity of car A becomes $v_A - v_B$.

$\therefore$ Relative velocity of car A *w.r.t.* car B is

$$v_{AB} = v_A - v_B$$

The direction of the relative velocity will depend upon the direction of $(v_A - v_B)$.

Note.

$$v_{AB} = v_A - v_B$$

But v_A and v_B are the velocities of the cars *w.r.t.* earth and can be written as v_{AE} and v_{BE} respectively.

$\therefore$
$$v_{AB} = v_{AE} - v_{BE}$$

or
$$v_{AB} = v_{AE} + v_{EB} \quad ...(i) \ (\because v_{BE} = -v_{EB})$$

Eq. (i) is a very useful relation. By writing the equation in this form, we see that inner subscripts on the right-hand of eq. (i) are the same (the two E's) and the outer subscripts on the right of eq. (i) (the A and B) are the same as the subscripts of the term v_{AB} on the left-hand side of eq. (i).

6.22. MATHEMATICAL TREATMENT OF RELATIVE VELOCITY

We know that velocity of a body is the time rate of change of its position. Therefore, relative velocity of a body A *w.r.t.* body B can be defined as under :

The relative velocity of body A w.r.t. body B (v_{AB}) is the time rate at which body A changes its position w.r.t. body B.

Consider two bodies A and B moving along a straight line in the same direction with uniform velocities v_A and v_B respectively as shown in Fig. 6.24. Let x_{AO} and x_{BO} be their displacements from the origin O at $t = 0$. Clearly, $(x_{AO} - x_{BO})$ is the displacement between the bodies A and B at $t = 0$. If x_A and x_B are their position coordinates in time t, then,

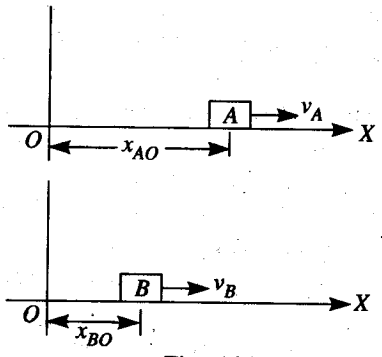


Fig. 6.24

$$x_A = x_{AO} + v_A t$$

and
$$x_B = x_{BO} + v_B t$$

$\therefore$
$$x_A - x_B = (x_{AO} - x_{BO}) + (v_A t - v_B t)$$

or
$$x_A - x_B = (x_{AO} - x_{BO}) + (v_A - v_B) t$$

Increase in displacements between the two bodies in time t

= Final displacement – Original displacement

$$= [(x_{AO} - x_{BO}) + (v_A - v_B) t] - [(x_{AO} - x_{BO})]$$

$$= (v_A - v_B) t$$

Therefore, relative velocity of A *w.r.t.* B is

$$v_{AB} = \frac{\text{displacement}}{\text{time}} = \frac{(v_A - v_B)t}{t} = v_A - v_B$$

It is left as an exercise for the reader to prove that $v_{BA} = v_B - v_A$.

6.23. POSITION-TIME GRAPH OF RELATIVE VELOCITY

We have seen above that the total displacement between the bodies A and B in time t is given by ;

$$x_A - x_B = (x_{AO} - x_{BO}) + (v_A - v_B)\, t$$

(*i*) When $v_A = v_B$. When $v_A = v_B$, then,

$$x_A - x_B = x_{AO} - x_{BO}$$

This means that distance between the two bodies will remain the same at all times. Since $^*v_A = v_B$, their position – time graphs are parallel to each other as shown in Fig. 6.25.

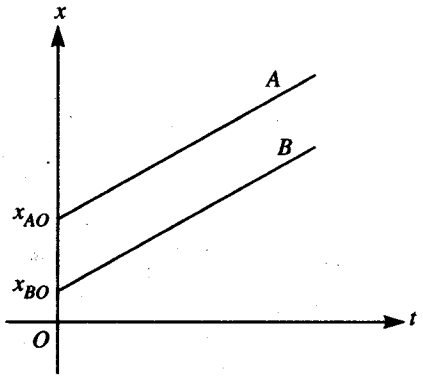


Fig. 6.25

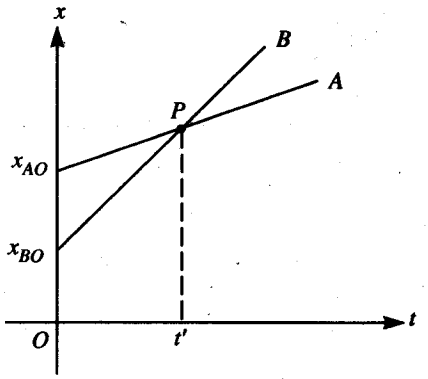


Fig. 6.26.

(*ii*) When $v_A \neq v_B$. In such a case, one of the position-time graphs will be steeper than the other. If $v_B > v_A$, then position-time graph of v_B will be steeper than that of body A as shown in Fig. 6.26. As a result, the two graphs will meet at point P which corresponds to time t'. At t', the two bodies meet *i.e.,* they have the same displacement from the origin. Before t', the body A is ahead of body B and after t', body B will be ahead of body A. On the other hand, if $v_A > v_B$, then $(v_A - v_B)$ will be positive. It means that relative separation between the two bodies will increase by an amount $(v_A - v_B)$ after each second. Therefore, the two graphs will never meet as shown in Fig. 6.27. In fact, the two graphs will open out gradually.

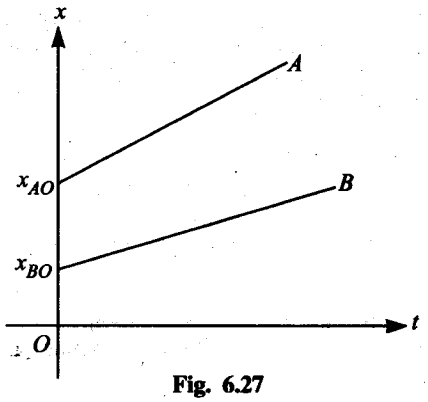


Fig. 6.27

Example 6.13. Two railway tracks are parallel to west-east direction. Along one track, train *A* moves with a speed of 54 km/h from west to east while along the second track, train *B* moves with a speed of 90 km/h from east to west. Find the (*i*) relative velocity of *B w.r.t. A* and (*ii*) relative velocity of ground *w.r.t. B*.

Solution. Considering the positive direction of x-axis from west to east, we have,

v_{AE} = Relative velocity of A *w.r.t.* earth = +54 km/h = +15 ms^{-1}

v_{BE} = Relative velocity of B *w.r.t.* earth = –90 km/h = –25 ms^{-1}

(*i*) Relative velocity of B *w.r.t.* A is given by ;

$$v_{BA} = v_{BE} + v_{EA} = v_{BE} - v_{AE} = (-25) - (+15) = \mathbf{-40\ ms^{-1}}$$

* This means their slopes are the same.

What is the meaning of negative sign? It means that train B appears to move from east to west with a speed of 40 ms^{-1}.

(*ii*) Relative velocity of earth *w.r.t.* B is

$$v_{EB} = v_{EE} + v_{EB} = v_{EE} - v_{BE} = 0 - (-25) = 25 \text{ ms}^{-1}$$

Positive sign shows that the ground or earth's surface will appear to move west to east with a speed of 25 ms^{-1}.

Example 6.14. Two trains, each of length 100 m, are running on parallel tracks. One overtakes the other in 20 seconds and one crosses the other in 10 seconds. Calculate the velocity of each train.

Solution. Let u and v be the velocities of the trains in ms^{-1}.

Relative velocity of overtaking $= u - v$

Relative velocity of crossing $= u + v$

Total distance $= 100 + 100 = 200$ m

$$\therefore \quad 20 = \frac{200}{u-v}$$

or $$u - v = 10 \qquad ...(i)$$

Also $$10 = \frac{200}{u+v}$$

or $$u + v = 20 \qquad ...(ii)$$

Solving eqs. (*i*) and (*ii*), we have, $u = 15$ ms^{-1} ; $v = 5$ ms^{-1}

Example 6.15. Two trains along the same straight rails are moving with constant velocity of 60 km/h and 30 km/h towards each other. If at time $t = 0$, the distance between them is 90 km, find the time when they collide.

Solution. Relative velocity of approach $= 60 - (-30) = 60 + 30 = 90$ km/h

Distance between trains $= 90$ km

$$\therefore \quad \text{Time when they will collide} = \frac{90}{90} = 1\text{h}$$

Example 6.16. Two cars A and B are at positions 100 m and 200 m from the origin at $t = 0$. They start simultaneously with velocities 10 ms^{-1} and 5 ms^{-1} respectively. Calculate the time and position at which they will overtake one another. Assume that they are moving in the same direction.

Solution. $x_{BO} = 200$ m ; $x_{AO} = 100$ m ; $v_A = 10$ ms^{-1} ; $v_B = 5$ ms^{-1}

$$x_B - x_A = (x_{BO} - x_{AO}) + (v_B - v_A)t$$

For the solution of the problem, $x_B - x_A = 0$.

$$\therefore \quad 0 = (200 - 100) + (5 - 10)t \quad \text{or} \quad 0 = 100 - 5t \quad \therefore \; t = 100/5 = 20 \text{ s}$$

Also $$x = 100 + 10 \times 20 = 100 + 200 = 300 \text{ m}$$

Example 6.17. The distance between two towns M and N is 400 km. Two cars A and B set off simultaneously from the towns M and N towards each other. The car A from M travels at a speed of $v_A = 60$ km/h and car B from N at a speed of $v_B = 40$ km/h. Find the time that will elapse before they meet and the point where they will meet.

Solution. $v_A = 60$ kmh^{-1} ; $v_B = -40$ kmh^{-1}

The negative sign is given because car B is moving in a direction opposite to that of the car A.

$\therefore$ Relative velocity of car A *w.r.t.* to car B is

$$v_{AB} = v_A - v_B = 60 - (-40) = 100 \text{ kmh}^{-1}$$

Distance between the towns M and N is x = 400 km.

$\therefore$ Time after which the two cars meet is

$$t = \frac{x}{v_{AB}} = \frac{400}{100} = 4 \text{ h}$$

The distance S from the town M when the two cars meet is

$$S = v_A \times t = 60 \times 4 = 240 \text{ km}$$

6.24. ACCELERATION

The acceleration of a moving object is defined as the change in velocity divided by the time taken for the change in velocity i.e.,

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time interval}}$$

The SI unit of acceleration is ms^{-2}. Acceleration is a vector quantity *i.e.,* it has magnitude as well as direction.

(*i*) The dimensional formula of acceleration is $[M^0 LT^{-2}]$.

(*ii*) The direction of acceleration is the same as that of the velocity.

(*iii*) The acceleration is positive if the velocity is increasing and is negative if the velocity is decreasing. The negative acceleration is also called **retardation** or **deceleration.**

1. Uniform acceleration. *An object is said to be moving with uniform acceleration (i.e., constant acceleration) if its velocity changes by equal amounts in equal intervals of time, however small these time intervals may be.*

   Note that uniform acceleration means that magnitude as well as direction of acceleration remains the same. For example, objects falling freely under gravity near earth's surface have uniform acceleration. In this case, the magnitude of acceleration in 9.8 ms^{-2} and is always directed vertically downward.

2. Variable acceleration. *An object is said to be moving with variable acceleration (or non-uniform acceleration) if its velocity changes in magnitude or direction or both.* The problems involving variable acceleration are solved either graphically or by the use of calculus.

3. Average acceleration. *The average acceleration of an object is defined as the change in velocity Δv (= $v_2 - v_1$) divided by the time interval Δt (= $t_2 - t_1$) for the change in velocity i.e.,*

   $$\text{Average acceleration, } \bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

   Suppose a body starts from rest and moving in a straight line attains a velocity of 20 ms^{-1} in 2 seconds. The change in velocity, Δv = 20 – 0 = 20 ms^{-1}.

   $\therefore$ $$\text{Average acceleration, } \bar{a} = \frac{\Delta v}{\Delta t} = \frac{20}{2} = 10 \text{ ms}^{-2}$$

Example 6.18. **A car travels in the positive x direction at 20 ms^{-1} and needs to pass a truck . The car's velocity is increased to 25 ms^{-1} in 2 s. Find the average acceleration of the car.**

Solution. Since the motion is in the positive x direction, both initial (v_1) and final (v_2) velocities are positive.

$$\text{Change in car's velocity, } \Delta v = v_2 - v_1 = 25 - 20 = 5 \text{ ms}^{-1}$$

$$\text{Time interval, } \Delta t = t_2 - t_1 = 2 \text{ s}$$

$\therefore$ $$\text{Average acceleration, } \bar{a} = \frac{\Delta v}{\Delta t} = \frac{5}{2} = 2.5 \text{ ms}^{-2}$$

What does average acceleration mean? In this case, it means that on the average, the velocity of the car increases by 2.5 m s^{-1} every second.

Example 6.19. **A car is moving in the positive x direction at 20 ms^{-1} and comes up behind a truck and is unable to pass. The car slows to 15 ms^{-1} in a time 2 s. Calculate car's average acceleration.**

Solution. Change in car's velocity, $\Delta v = v_2 - v_1 = 15 - 20 = -5\ ms^{-2}$

Note that the change in velocity is negative because the car's velocity in the positive x direction is decreased by 5 ms^{-1}.

Time interval, $\Delta t = t_2 - t_1 = 2s$

$\therefore$ Average acceleration, $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-5}{2} = -2.5\ ms^{-2}$

The average acceleration is negative because the car's velocity along the positive x direction is decreased. It, on the average, lost 2.5 ms^{-1} each second.

6.25. GRAPHICAL ANALYSIS OF VARIABLE ACCELERATION

In uniform acceleration, the magnitude and direction of the acceleration remain the same. However, in non-uniform acceleration, the acceleration of the object is variable. In such a case, the motion of the object is described by its average acceleration and instantaneous acceleration. Let us discuss these graphically.

(*i*) **Geometrical interpretation of average acceleration.** A graphical approach gives further insight into the relationship between average acceleration, velocity and time. Consider one dimensional motion of an object along the positive x direction as shown in Fig. 6.28. At time t_1, the object is at point P and has a velocity v_1 and at a later time t_2, the object is at Q and has a velocity v_2. The velocity-time graph of the object is shown in Fig. 6.29. Note that *curve in this figure does not represent the path of the object ; it is the curve showing the velocities of the object at various times. The path of the object is along a straight line as shown in Fig. 6.28.

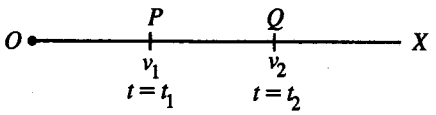


Fig. 6.28

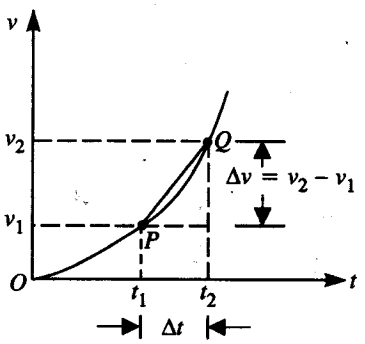


Fig. 6.29

The coordinates of point P are (v_1, t_1) and that of Q are (v_2, t_2). The change in velocity $\Delta v = v_2 - v_1$ and the time interval $\Delta t = t_2 - t_1$. The straight line PQ forms the hypotenuse of the right angled triangle whose sides are Δv and Δt. The ratio $\Delta v / \Delta t$ is the slope of the line PQ. But $\Delta v / \Delta t$ is also the average acceleration of the object during the time interval $\Delta t = t_2 - t_1$.

$\therefore$ Average acceleration, $\bar{a} = \frac{\Delta v}{\Delta t}$ = Slope of the line connecting points $P(v_1, t_1)$ and $Q(v_2, t_2)$

Thus we have the geometrical interpretation of the average acceleration.

* The graph is a curve (not a straight line) because the object is moving with a variable velocity.

The average acceleration of an object during the time interval $\Delta t = t_2 - t_1$ is equal to the slope of the straight line connecting two points $P(v_1, t_1)$ and $Q(v_2, t_2)$ on the $v - t$ graph.

The reader may note that the geometrical interpretation of the average acceleration is defined in a manner similar to the way we defined the average velocity. In case of average velocity, we used $x - t$ graph and in case of average acceleration, we have used $v - t$ graph.

(*ii*) **Instantaneous acceleration.** We can define instantaneous acceleration in a manner similar to the way we defined the instantaneous velocity. Fig. 6.30 shows a typical $v - t$ graph for the one dimensional motion of an object. The average acceleration of the object during the time interval $\Delta t = t_2 - t_1$ is the slope of the line QP. The instantaneous acceleration at any time t_1 is the slope of the tangent to *$v - t$ graph at that time. Thus at time t_1,

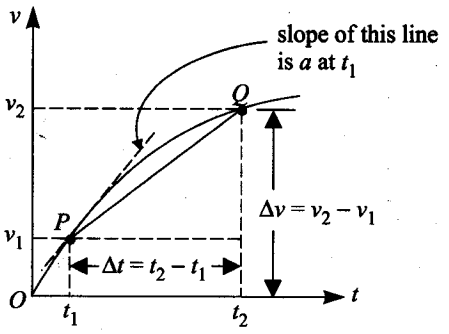


Fig. 6.30

$$\text{Instantaneous acceleration, } a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

$$\therefore \quad a = \frac{dv}{dt}$$

Like velocity, acceleration is a rate. The velocity of an object is the rate at which displacement changes with time, its acceleration is the rate at which its velocity changes with time. Therefore, acceleration can also be expressed as :

$$a = \frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \qquad \left(\because v = \frac{dx}{dt}\right)$$

$$\therefore \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Here d^2x/dt^2 is called the *second derivative* of x *w.r.t.* time. It is because we first take the derivate of x *w.r.t.* time (*i.e.*, dx/dt) and then we again take the derivative *w.r.t.* time [d/dt (dx/dt)] to get acceleration.

6.26. GRAPHICAL REPRESENTATION OF ACCELERATION

The acceleration of an object is very important in physics because it is related to the forces acting on the object. We shall see in chapter 10 that the acceleration is directly proportional to the net force acting on the object. In order to have more insight into this important physical quantity, we discuss its graphical interpretation.

(*i*) **When acceleration of the object is zero.**

$$a = \frac{dv}{dt} \quad \text{or} \quad 0 = \frac{dv}{dt}$$

This means that rate of change of velocity with time is zero. In other words, the velocity of the object is **constant. Therefore, velocity-time graph of an object moving with zero acceleration is a straight line parallel to the time axis as shown in Fig. 6.31. Note that graph corresponds to zero slope. This is expected because acceleration is the slope of $v - t$ graph.

---

* Note that as we go from t_1 to t_2, the velocity is increasing but acceleration is decreasing since slope of the curve is decreasing.

** If the acceleration of an object is zero, the object is either moving with constant velocity in a straight line or remains at rest. We are considering here the former case.

(*ii*) When acceleration of the object is constant. Consider the one dimensional motion of an object. Suppose at $t = 0$, the object is at the origin and starts with constant acceleration 'a'. Then,

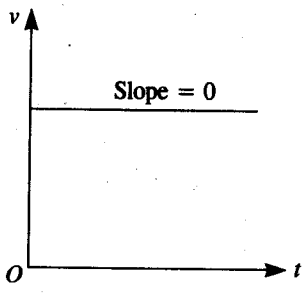


Fig. 6.31

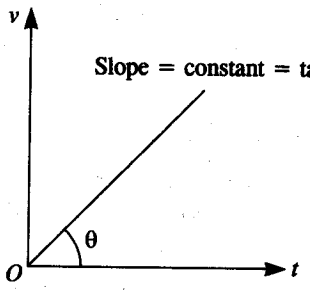


Fig. 6.32

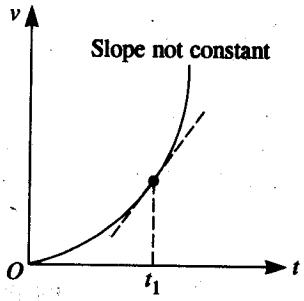


Fig. 6.33

$$a = \frac{dv}{dt} \quad \text{or} \quad dv = a\,dt$$

We integrate velocity from the initial velocity zero to the final value v corresponding to time t.

$$\int_0^v dv = a\int_0^t dt$$

$$\therefore \qquad v = at$$

Since a is constant, $v \propto t$. Therefore, the velocity–time graph of an object moving with constant acceleration is a straight line as shown in Fig. 6.32. Note that slope of line is constant and is equal to the acceleration of the object. Also note that slope of the line is everywhere the same, indicating that velocity is increasing at a constant rate.

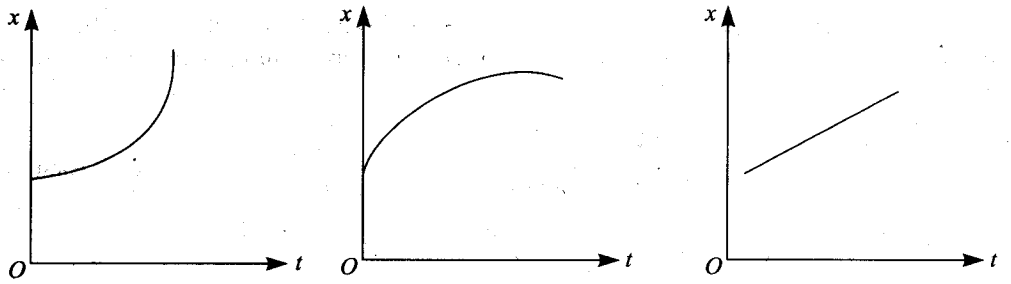
Note. The motion in a straight line with constant acceleration is a very important type of motion. It is called *uniformly accelerated linear motion.*

(*iii*) When acceleration of the object is variable. For zero acceleration or constant acceleration, the graphical analysis of acceleration is simple. We can easily find the acceleration of the object (= slope of graph) from its $v - t$ graph. But if the velocity of the object changes continuously with time (See Fig. 6.33), how can we determine the acceleration at a particular instant of time? In other words, the acceleration is variable. We have already seen that instantaneous acceleration given by ;

$$a = \frac{dv}{dt}$$

The instantaneous acceleration at any time t_1 is equal to the slope of the tangent (shown dotted in Fig. 6.33) to the $v - t$ graph at that time. In order to deal with variable acceleration, we use the techniques of calculus.

Note. Like velocity, acceleration can also be positive, negative or zero. Position–time graphs for motion with positive, negative and zero acceleration are shown in Figs. 6.34 (*i*), (*ii*) and (*iii*) respectively. Note that the graph curves upward for positive acceleration, downward for negative acceleration and is a straight line for zero acceleration.



Positive acceleration (*i*) Negative acceleration (*ii*) Zero acceleration (*iii*)

Fig. 6.34

Example 6.20. The position of stone dropped from a cliff is given by $x = 5\ t^2$ where x is in metres and t is in seconds. What is the acceleration of the stone at $t = 2$ s?

Solution. $x = 5\ t^2 \quad \therefore \quad \frac{dx}{dt} = 10\ t$ or $\frac{d^2x}{dt^2} = 10\ \text{ms}^{-2}$

But d^2x / dt^2 is acceleration of the stone. Therefore, $a = 10\ \text{m s}^{-2}$

Example 6.21. The position of a particle is given by $x = K\ t^3$ where K is a constant with the dimensions of m s^{-2}. Find (*i*) velocity (*ii*) acceleration of the particle as a function of time.

Solution. (*i*) $x = K\ t^3 \quad \therefore \quad v = \frac{dx}{dt} = \frac{d}{dt}\left(Kt^3\right) = 3Kt^2$ or $v = 3\ Kt^2$

Note that value of v varies as the square of time. Clearly, it is a case of variable velocity.

(*ii*) Now, $a = \frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}\left(3Kt^2\right) = 6\ Kt \quad \therefore \quad a = 6\ Kt$

Note that value of a varies directly with time. Clearly, the acceleration is not constant but varies with time.

Example 6.22. The position of a particle moving along a straight line is given by ;

$$x = 2 - 5t + 6t^2$$

Find the acceleration of the particle at $t = 2$ s.

Solution. $x = 2 - 5t + 6\ t^2$ or $\frac{dx}{dt} = \frac{d}{dt}(2 - 5t + 6\ t^2) = -5 + 12\ t$

$\therefore \quad a = \frac{d^2x}{dt^2} = \frac{d}{dt}\ (-5 + 12\ t) = 12$

Note that acceleration of the particle is constant and is independent of time.

CONCEPTUAL QUESTIONS

Q.1. What is translation motion?

Ans. Rigid objects that move without rotation are said to undergo translation motion. When we are interested only in translation motion and the object's size is not significant, we can treat the object as a particle.

Q.2. Which is generally our frame of reference for studying motion of objects?

Ans. Ordinarily our frame of reference for studying motion of objects is a coordinate system attached to the surface of earth. Unless stated otherwise, this is our frame of reference.

Q.3. An aeroplane is flying in the sky. Is its motion three dimensional?

Ans. Yes. It is because it may be changing its direction as well as the height from the ground during its flight.

Q.4. Which is the simplest type of motion?

Ans. The uniform motion in a straight line (one dimensional motion) is the simplest type of motion.

Q.5. Why is time period repeated twice in the unit of acceleration?

Ans. The velocity of a particle is the rate at which its displacement changes with time; its acceleration is the rate at which its velocity changes with time. Thus acceleration is the rate of a rate. So time is repeated twice.

Q.6. Fig. 6.35 shows $v - t$ graph for various situations. What does each graph indicate?

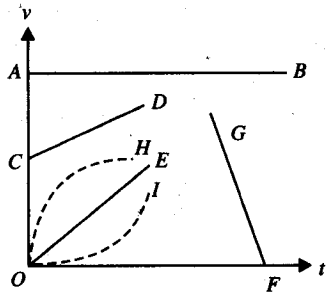


Fig. 6.35

Ans. Acceleration = Slope of $v - t$ graph

(*i*) Straight line *AB* indicates that the acceleration of the moving body is *zero*. Clearly, the body is moving with constant velocity.

(*ii*) Straight line *CD* indicates that the body has *constant positive acceleration* with initial velocity *OC*. In this case, the velocity of the body is increasing.

(*iii*) Straight line *OE* indicates that the body has positive constant acceleration with zero initial velocity.

(*iv*) Dotted curve *OI* shows the increasing acceleration. Here the slope of the graph increases with time.

(*v*) Dotted curve *OH* indicates decreasing acceleration. Here the slope of the graph decreases with time.

(*vi*) The straight line *FG* indicates that the body is moving with constant negative acceleration. Here the slope of the graph is negative. It means the velocity of the body is decreasing at a constant rate.

Q.7. Is the time variation of position as shown in Fig. 6.36 observed in nature?

Ans. The graph shown in Fig. 6.36 cannot be observed in nature. It is because with the increase of position, the time first increases and then decreases which is impossible.

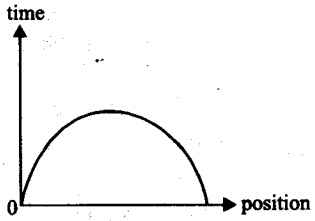


Fig. 6.36

VERY SHORT ANSWER QUESTIONS

Q.1. Are rest and motion absolute or relative terms?

Ans. Rest and motion are relative terms.

Q.2. Which speed is measured by the speedometer of a car?

Ans. Instantaneous speed *i.e.*, speed at a given instant of time.

Q.3. A body is moving with a uniform velocity. What does it mean?

Ans. It means that the body is moving with a constant speed in a straight line.

Q.4. Can a body have a constant velocity but a variable speed?

Ans. No. It is because speed is equal to the magnitude of velocity.

Q.5. Can a body have a constant speed but variable velocity?

Ans. Yes. When a body is rotated along a circular path with constant linear speed, the velocity is changing (because direction of motion is changing) but speed is constant.

Q.6. What does the slope of position-time graph of a body represent?

Ans. The slope of $x - t$ graph of a body represents its velocity.

Q.7. Under what condition will the distance and displacement of a moving object will have the same magnitude?

Ans. When the object moves in a straight line in the same direction.

Q.8. What is the nature of position-time graph for a stationary object?

Ans. The $x - t$ graph for a stationary object is a straight line parallel to time-axis.

Q.9. Can the speed of a body be negative?

Ans. No. It is because speed = distance/time and distance can never be negative.

Q.10. Can the slope of $x - t$ graph be negative?

Ans. Yes. It is so when the velocity of the body is negative.

Q.11. What is meant by a point object in physics?

Ans. An object is considered as a point object when its size is negligible compared to the scale of observation.

Q.12. What does slope of velocity-time graph represent?

Ans. The slope of velocity-time graph represents acceleration.

Q.13. How will you calculate distance travelled from velocity-time graph?

Ans. The area under $v - t$ graph for a given time interval gives the distance covered during that time interval.

Q.14. What will be the nature of $x - t$ graph for a uniform motion?

Ans. It will be a straight line inclined to the time-axis.

Q.15. Can an object be accelerated without speeding up or slowing down?

Ans. Yes, it can be so in case of uniform circular motion of an object.

Q.16. If the acceleration of a body is zero, what is the nature of motion?

Ans. If the acceleration of a body is zero, the body is either moving with constant velocity in a straight line or remains at rest.

Q.17. What is the basic difference between speed and velocity?

Ans. Speed indicates how fast the body is moving whereas velocity tells us how fast and in which direction it is moving.

Q.18. What does area under velocity-time graph represent?

Ans. The net area under velocity-time graph represents displacement for a given time interval.

Q.19. Can the displacement be greater than distance travelled by an object?

Ans. Displacement is the shortest distance between two positions. Therefore, displacement cannot be greater than the distance travelled.

Q.20. What will be the nature of velocity-time graph for a uniform motion?

Ans. It will be a straight line parallel to time-axis.

Q.21. When will the relative velocity of two moving objects be zero?

Ans. When the two objects are moving with same speed in the same direction (*i.e.*, equal velocities).

Q.22. How can a body be simultaneously at rest and in motion?

Ans. In relative motion, it is possible. A passenger in a train is at rest *w.r.t.* train but is in motion *w.r.t.* ground.

SHORT ANSWER QUESTIONS

Q.1. If the displacement of a body is zero, is the distance covered by it necessarily zero? Comment.

Ans. No. For example, when a body is thrown vertically upward, it comes back to the same point after some time. In this case, displacement of the body is zero but the distance travelled by the body is $2h$ where h is the maximum height to which the body rises. Similarly, when a body moving in a circle of radius r completes one revolution, the displacement of the body is zero but distance travelled by it is $2\pi r$.

Q.2. Explain that rest and motion are always relative.

Ans. Rest and motion are relative terms. For example, a book on the table is at rest *w.r.t.* table and other objects in the room. If an observer is located on the moon, he will observe that the book and other objects in the room are moving. Thus the book is at rest if viewed from the room but is moving if viewed from the moon.

Q.3. A body can have zero average velocity but not zero average speed. Comment.

Ans. The statement is true.

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}}; \quad \text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

If a body moving in a circular path of radius r completes one revolution in t seconds, then displacement is zero but distance travelled is $2\pi r$. Therefore, the average velocity of the body is zero while average speed is $2\pi r/t$.

Q.4. A car covers a distance S_1 with velocity v_1 and distance S_2 with velocity v_2. What is the average velocity of car?

Ans. $$\text{Average velocity} = \frac{S_1 + S_2}{(S_1/v_1) + (S_2/v_2)}$$ ... Refer to Example 6.1.

Q.5. During a given interval of time, can the average velocity of a body be greater than its average speed?

Ans. No. It is because during a given time interval, the total displacement can be equal to or smaller than the total distance travelled.

Q.6. A cyclist completes one round of circular track of radius R = 20 m in 20 seconds. What will be the displacement at the end of 20 s and 30 s?

Ans. At the end of 20 s, the cyclist completes one revolution and the displacement is zero. However, at the end of 30 s, the cyclist completes 1.5 revolutions and the displacement = diameter of circle = $2R = 2 \times 20 = 40$ m.

Q.7. In the above question, what is the average speed and average velocity?

Ans. After 20 s, average speed = $2\pi R/t = 2\pi \times 20/20 = 2\pi$ m/s and average velocity = 0 m/s.

After 30 s, average speed = $1.5 \times 2\pi \times 20/30 = 2\pi$ m/s and average velocity = $2\ R/t = 40/30 = 4/3 \text{ms}^{-1}$.

Q.8. The velocity-time graph of two bodies A and B make angles of 30° and 60° with the time-axis. What is the ratio of their accelerations?

Ans. $a_A/a_B = \tan 30°/\tan 60° = 1/3$.

Q.9. The displacement-time graph of two bodies A and B make angles of 45° and 30° with time-axis. What is the ratio of their velocities?

Ans. $v_A/v_B = \tan 45°/\tan 30° = \sqrt{3}$.

Q.10. A car travels half the distance with velocity v_1 and the second half with a velocity v_2. If the total distance is S, what is the average velocity of the car?

Ans. Average velocity $= \dfrac{2v_1v_2}{v_1+v_2}$ ... Refer to Example 6.1.

Q.11. Does velocity or acceleration decide the direction of motion of a body?

Ans. It is the velocity and not acceleration that decides the direction of motion of a body. Thus when a body is thrown upward, the direction of motion of the body and its velocity are in the upward direction. However, acceleration due to gravity (g) acts vertically downward. Therefore, direction of motion of the body is in the direction of velocity and not that of acceleration.

Q.12. If the distance (x) covered by a moving body is directly proportional to time (t), what conclusions you draw?

Ans. $x \propto t$ or $x = kt$ when k is constant of proportionality.

Now, $$v = \frac{dx}{dt} = \frac{d}{dt}(kt) = k = \text{constant}$$

Therefore, the body is moving with uniform velocity *i.e.*, with a constant speed in a straight line.

Q.13. If the displacement of a body is directly proportional to the square of time, what conclusions you draw?

Ans. The body is moving with uniform acceleration (Refer to Example 6.20).

Q.14. If the displacement of a body is proportional to t^3, what conclusions you draw?

Ans. Acceleration $\propto t$ (Refer to Example 6.21).

Q.15. A ship S_1 is sailing due east with a velocity of 30 km/h and another ship S_2 is sailing due south with the same velocity. What is the velocity of S_2 relative to S_1?

Ans. Impress a velocity of 30 km/h due west on both ships. This brings ship S_1 to rest. The velocity v of S_2 *w.r.t.* S_1 is the resultant of two velocities, each 30 km/h, one due west and the other due south.

$\therefore$ $$v = \sqrt{30^2+30^2} = 30\sqrt{2} \text{ km/h south-west}$$

Q.16. When two cars A and B move towards each other with constant velocities v_A and v_B, the distance between them decreases by 10 m/s. When they move in same direction with the same speed, the distance between them increases by 5 m/s. What are their velocities ?

Ans. $v_A + v_B = 10$ m/s and $v_A - v_B = 5$ m/s. Therefore, $v_A = 7.5$ m/s ; $v_B = 2.5$ m/s.

LONG ANSWER QUESTIONS

Q.1. Explain the terms : distance, displacement, average speed and average velocity. **[Refer to Arts. 6.11, 6.12 and 6.13]**

Q.2. Explain difference between speed and velocity. **[Refer to Art. 6.14]**

Q.3. What is the geometrical interpretation of average velocity? **[Refer to Art. 6.20]**

Q.4. Prove that displacement of an object in a given time interval is equal to the area under velocity-time graph in that time interval. **[Refer to Art. 6.18]**

Q.5. What is the importance of $x - t$ graph? **[Refer to Art. 6.17]**

Q.6. Explain the $x - t$ and $v - t$ graphs of uniform motion. **[Refer to Arts. 6.17 and 6.18]**

Q.7. Draw position-time graphs of two objects moving along a straight line when their relative velocity is (*i*) zero (*ii*) non-zero. **[Refer to Art. 6.23]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Distance is a scalar while displacement is a vector. Both have the same dimensions and units.
2. The displacement of a moving particle in a given time interval can be positive, negative or zero but distance travelled by it cannot be negative or zero.
3. An object or body can be considered as a particle if its size is small compared to the scale of observation.
4. In general, the magnitude of displacement of a moving body is not equal to the distance travelled. However, if the body moves in a straight line without changing direction, the magnitude of displacement is equal to the distance travelled.
5. The average velocity of a moving body in a given time interval can be positive, negative or zero but its average speed cannot be negative or zero.
6. For a given time interval, a moving body can have a single average velocity. However, it can have a number of average speeds depending on the path followed.
7. The average velocity of a body over a parituclar time interval is equal to the displacement Δx divided by time interval Δt.

   Average velocity, $\bar{v} = \dfrac{\Delta x}{\Delta t}$

   On a displacement-time graph, the slope of the chord over a given time interval gives the average velocity over that time interval.
8. The instantaneous velocity v (velocity at a given instant) is defined as the limit of the average velocity at a particular time as $\Delta t \to 0$.

   $$\therefore \quad v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

   On a displacement-time graph, the instantaneous velocity is equal to the slope of the tangent to the graph at that instant of time.
9. If a particle travels distances s_1, s_2 and s_3 with speeds v_1, v_2 and v_3 respectively, then,

   Total distance travelled, $s = s_1 + s_2 + s_3$

   $$\text{Total time taken, } t = \frac{s_1}{v_1} + \frac{s_2}{v_2} + \frac{s_3}{v_3}$$

   $$\therefore \quad \text{Average speed} = \frac{s}{t}$$
10. If a particle travels with speeds v_1, v_2 and v_3 during time intervals t_1, t_2 and t_3 respectively, then,

    Total distance travelled, $s = v_1t_1 + v_2t_2 + v_3t_3$

    Total time taken, $t = t_1 + t_2 + t_3$

    $$\therefore \quad \text{Average speed} = \frac{s}{t}$$
11. For a body at rest, displacement-time graph is a striaght line parallel to time axis.
12. Uniform velocity means magnitude of velocity as well as its direction remains the same.
13. The velocity of a body is positive if it is moving to the right of the origin and the velocity is negative if the body is moving to the left of the origin.

14. The displacement-time graph of a body moving with uniform velocity is a straight line inclined at an angle θ to the time axis. If $\theta < 90°$, the uniform velocity is positive and if $\theta > 90°$, the uniform velocity is negative.
15. The velocity-time graph of a body moving with uniform velocity is a straight line parallel to time axis.
16. Acceleration is the time rate of change of velocity. The average acceleration over a time interval Δt is equal to change of velocity during the time interval divided by Δt.

    Average acceleration, $\bar{a} = \frac{\Delta v}{\Delta t}$
17. The instantaneous acceleration a of a body is defined as the limit of the average acceleration at a particular time as $\Delta t \to 0$.

    $\therefore \quad a = \underset{\Delta t \to 0}{\text{Lim}} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$
18. If the velocity the time graph of a body is a striaght line parallel to time axis, it means the accelration of the body is zero. The body is, therefore, either at rest or moving with uniform velocity.
19. If the velocity-time graph of a body is a striaght line inclined at an angle θ to the time axis, the body is moving with uniform (constant in magnitude and direction) acceleration. If $\theta < 90°$, the acceleration is positive and if $\theta > 90°$, the acceleration is negative.
20. If the slope of velocity-time graph is changing, it means the body is moving with variable acceleration.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The displacement-time graph for two particles A and B are straight lines inclined at angles of 30° and 60° with time axis. The ratio of velocities of $V_A : V_B$ is **[CPMT 1990]**

   (a) 1 : 2 (b) 1 : $\sqrt{3}$

   (c) $\sqrt{3}$: 1 (d) 1 : 3
2. For a particle, displacement-time relation is $t = \sqrt{x} + 3$. Its displacement when its velocity is zero is **[CBSE PMT 1999]**

   (a) 2m (b) 4m

   (c) 0 (d) none of these
3. A car covers the first half of the distance between two places at 40 km/hr and the other half at 60 km/hr. The average speed of the car is **[CBSE PMT 1990]**

   (a) 40 km/h (b) 48 km/h

   (c) 50 km/h (d) 60 km/h
4. The area under acceleration-time graph represents **[MP PMT 1993]**

   (a) the displacement (b) velocity

   (c) change in velocity (d) distance travelled
5. A car moves along a straight line whose equation is given by $s = 12t + 3t^2 - 2t^3$ where s is in metres and t is in seconds. The velocity of the car at start will be . **[CBSE PMT 1998]**

   (a) 7 m/s (b) 9 m/s

   (c) 12 m/s (d) 16 m/s
6. The displacement-time graph of a moving particle is shown in Fig. 6.37. **[CBSE PMT 1994]**

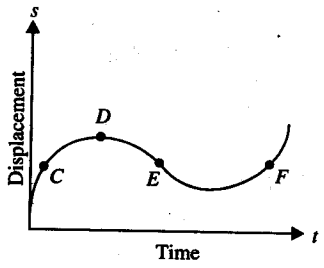


Fig. 6.37

   The instantaneous velocity of the particle is negative at the point

   (a) D (b) F

   (c) C (d) E
7. The position x of a particle varies with time t as $x = at^2 - bt^3$. The time t when the acceleration is equal to zero is **[CBSE PMT 1997]**

(a) $\frac{2a}{3b}$ (b) $\frac{a}{b}$

(c) $\frac{a}{3b}$ (d) zero

8. A particle moves along a straight line such that its displacement at any time t is given by: **[CBSE PMT 1994]**

$x = (t^3 - 6t^2 + 3t + 4)$ metres

The velocity when the acceleration is zero is

(a) –9 m/s (b) –15 m/s

(c) 3 m/s (d) 42 m/s

9. Which of the following curve in Fig. 6.38 does not represent motion in one dimension? **[CBSE PMT 1992]**

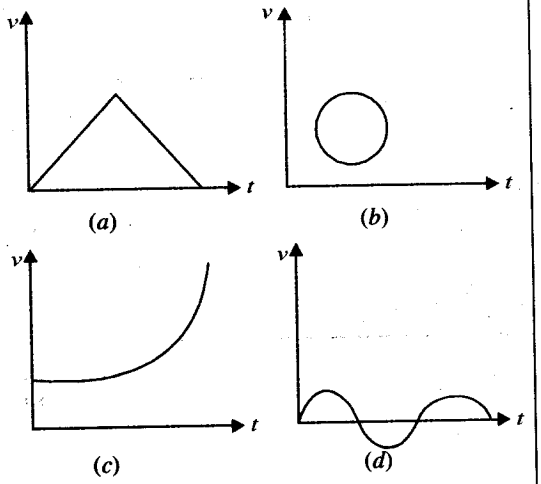


Fig. 6.38

10. In 1.0 s, a particle goes from point A to point B, moving in a semicircle of radius 1.0 m as shown in Fig. 6.39. The magnitude of average velocity is **[CPMT 2000]**

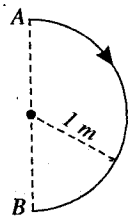


Fig. 6.39

(a) 3.14 m/s

(b) 2.0 m/s

(c) 1.0 m/s

(d) zero

11. One train A 120 m long is moving in a direction with a speed of 20 m/s. Another train B 130 m long is moving with a speed of 30 m/s in the opposite direction. The train B crosses the train A in a time **[CPMT 1996]**

(a) 5 s (b) 36 s

(c) 38 s (d) None of these

12. The displacement of a particle is given by ; $y = a + bt + ct^2 - dt^4$

The initial velocity and acceleration are respectively **[CPMT 1999]**

(a) b, $-4d$ (b) $-b$, $2c$

(c) b, $2c$ (d) $2c$, $-4d$

13. The displacement x of a particle varies with time t as $x = ae^{-\alpha t} + be^{\beta t}$, where a, b, α and β are positive constants. The velocity of the particle will **[CBSE 2005]**

(a) go on decreasing with time

(b) be independent of α and β

(c) drop to zero when $\alpha = \beta$

(d) go on increasing with time

14. Two boys are standing at the ends A and B of a ground, where $AB = a$. The boy at B starts running in a direction perpendicular to AB with velocity v_1. The boy at A starts running simultaneously with velocity v and catches the other boy in a time t, where t is : **[CBSE 2005]**

(a) $a/\sqrt{v^2 + v_1^2}$ (b) $\sqrt{a^2/(v^2 - v_1^2)}$

(c) $a/(v - v_1)$ (d) $a/(v + v_1)$

15. Which of the following velocity-time ($v - t$) graph shows a realistic situation for a body in motion (See Fig. 6.40)? **[AIIMS 2005]**

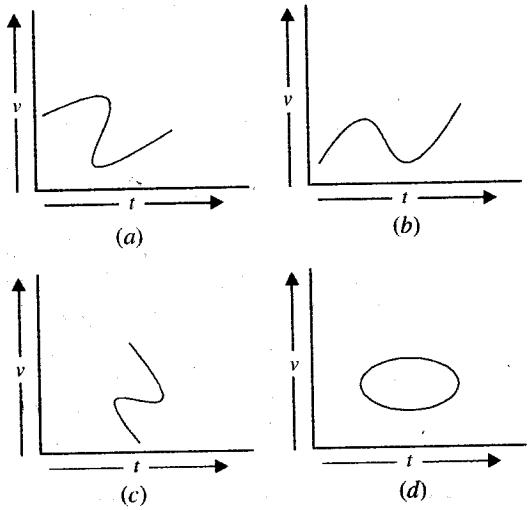


Fig. 6.40

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*d*)	**2.** (*c*)	**3.** (*b*)	**4.** (*c*)	**5.** (*c*)
6. (*d*)	**7.** (*c*)	**8.** (*a*)	**9.** (*b*)	**10.** (*b*)
11. (*a*)	**12.** (*c*)	**13.** (*d*)	**14.** (*b*)	**15.** (*b*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Velocity of particle is equal to the slope of displacement-time graph. (See Fig. 6.41).

$$\therefore \quad \frac{V_A}{V_B} = \frac{\tan 30^\circ}{\tan 60^\circ} = \frac{1}{3}$$

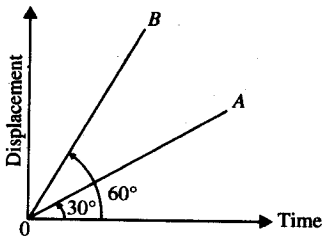


Fig. 6.41

2. $$t = \sqrt{x} + 3$$

Differentiating both sides *w.r.t.* t, we get,

$$1 = \frac{1}{2\sqrt{x}} \frac{dx}{dt}$$

Putting $dx/dt = 0$, we have $x = 0$.

3. Let the total distance be s.

$$\text{Total time taken} = \frac{s/2}{40} + \frac{s/2}{60} = \frac{s}{48} h$$

$$\therefore \quad \text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{s}{s/48} = 48 \text{ km/h}$$

4. Area under acceleration – time graph

= acceleration × time

= Change in velocity

5. The velocity at start means the velocity at $t = 0$.

$$s = 12t + 3t^2 - 2t^3$$

$$\therefore \quad \frac{ds}{dt} = 12 + 6t - 6t^2$$

$$\therefore \quad \left(\frac{ds}{dt}\right)_{t=0} = 12 + 6 \times 0 - 6 \times (0)^2 = 12 \text{ m/s}$$

6. Instantaneous velocity of a particle is the slope of the tangent to the displacement-time graph at a point. At point E, the slope is negative. Therefore, the instantaneous velocity of the particle is negative at point E.

7. $$x = at^2 - bt^3$$

or $$\frac{dx}{dt} = 2at - 3bt^2$$

or $$\frac{d^2x}{dt^2} = 2a - 6bt$$

Let at time t_1 the acceleration of the particle become zero. Then as per given conditions:

$$0 = 2a - 6bt_1$$

$$\therefore \quad t_1 = \frac{2a}{6b} = \frac{a}{3b}$$

8. $$x = t^3 - 6t^2 + 3t + 4$$

or $$\frac{dx}{dt} = 3t^2 - 12t + 3$$

or $$\frac{d^2x}{dt^2} = 6t - 12$$

When the acceleration (d^2x/dt^2) of the particle is zero, the time t is given by :

$$0 = 6t - 12 \quad \therefore \quad t = 2 \text{ sec.}$$

The velocity of the particle at $t = 2$ sec is

$$\left(\frac{dx}{dt}\right)_{t=2\text{sec}} = 3\times(2)^2 - 12\times 2 + 3 = -9\text{m/s}$$

9. In Fig. 6.38 (b), the particle has two velocities at the same time. This is not possible in one dimensional motion.

10. Displacement = 2 × radius = 2 × 1.0 = 2.0 m

$$\therefore \quad \text{Average velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{2\,\text{m}}{1\text{s}} = 2\text{m/s}$$

11. Total length of two trains, $x = 130 + 120 = 250$ m

Relative velocity of train B *w.r.t.* train A

$$= 30 - (-20) = 50 \text{ m/s}$$

$\therefore$ Time taken by train B to cross train A

$$= \frac{x}{50} = \frac{250}{50} = 5\text{s}$$

12. $$y = a + bt + ct^2 - dt^4$$

or $$\frac{dy}{dt} = b + 2ct - 4dt^3$$

or $$\frac{d^2y}{dt^2} = 2c - 12dt^2$$

$$\text{Initial velocity} = \left(\frac{dy}{dt}\right)_{t=0} = b + 2c\times 0 - 4d\times(0)^3 = b$$

$$\text{Initial acceleration} = \left(\frac{d^2y}{dt^2}\right)_{t=0} = 2c - 12d(0)^2 = 2c$$

13. $$x = ae^{-\alpha t} + be^{\beta t}$$

$\therefore$ $$v = \frac{dx}{dt} = \frac{d}{dt}\left(ae^{-\alpha t} + be^{\beta t}\right) = -a\alpha e^{-\alpha t} + b\beta e^{\beta t} = A + B$$

where $A = -a\alpha e^{-\alpha t}$ and $B = b\beta e^{\beta t}$

As time t increases, the term A approaches zero. However, as time t increases, the term B (positive) approaches infinity. As a result, velocity goes on increasing with time

14. Fig. 6.42 shows the conditions of the problem In time t, the distance covered by the boy starting at point A is $AC = vt$. In time t, the distance covered by the boy starting at point B is $BC = v_1 t$.

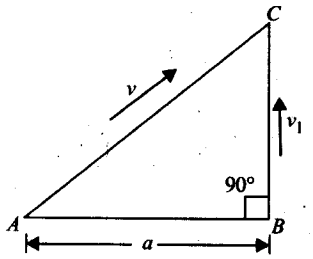


Fig. 6.42

Now $(AC)^2 = (AB)^2 + (BC)^2$

or $(vt)^2 = a^2 + (v_1 t)^2$

or $v^2t^2 = a^2 + v_1^2 t^2$

$\therefore \quad t^2(v^2 - v_1^2) = a^2 \quad$ or $\quad t = \sqrt{\dfrac{a^2}{\left(v^2 - v_1^2\right)}}$

15. We know that a body in motion has only one velocity at a particular time. However, this velocity can decrease or increase with time. In the given $v - t$ graphs, only graph (b) has only one velocity at a particular time. But graphs (a), (c) and (d) have two velocities at same times. Therefore, $v - t$ graph (b) shows a realistic situation for the body in motion.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A car travels for half of its time at 80 km/h and for the rest half time at 40 km/h. Total distance covered is 60 km. What is the average speed of the car? **[Rajasthan PET 1996]**
   (a) 60 km/h (b) 80 km/h
   (c) 120 km/h (d) 180 km/h
2. The displacement of a body is given to be proportional to the cube of time elapsed. The magnitude of the acceleration of the body is **[NCERT 1990]**
   (a) increasing with time
   (b) decreasing with time
   (c) constant but not zero
   (d) zero
3. The correct statement from the following is **[MP PET 1993]**
   (a) A body having zero velocity will not necessarily have zero acceleration.
   (b) A body having zero velocity will necessarily have zero acceleration.
   (c) A body having uniform speed can have only uniform acceleration.
   (d) A body having non-uniform velocity will have zero acceleration.
4. A particle moving in a straight line covers half the distance with a speed of 3m/s. The other half of the distance is covered in two equal time intervals with speed of 4.5 m/s and 7.5 m/s respectively. The average speed of the particle during the entire motion is **[IIT 1992]**
   (a) 4 m/s (b) 5 m/s
   (c) 5.5 m/s (d) 4.8 m/s
5. Which one of the following four statements is false? **[Manipal 1995]**
   (a) A body can have zero velocity and still be accelerated.
   (b) A body can have constant velocity and still have a varying speed.
   (c) A body can have a constant speed and still have a varying velocity.
   (d) The direction of a body can change when magnitude of its acceleration is constant.
6. Figs. 6.43 (i) and (ii) below show the displacement-time graph of two particles moving along x-axis. We can say that **[Haryana CEET 1996]**

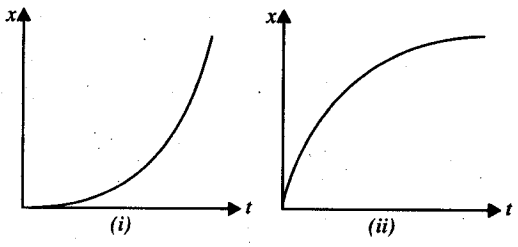


Fig. 6.43

   (a) both the particles are having a uniformly accelerated motion.
   (b) both the particles are having a uniformly retarded motion.
   (c) particle (i) is having a uniformly accelerated motion while the particle (ii) is having a uniformly retarded motion.

(*d*) particle *(i)* is having a uniformly retarded motion while the particle *(ii)* is having a uniformly accelerated motion.

7. The displacement of a particle starting from rest ($t = 0$) is given by $s = 6t^2 - t^3$. The time in seconds at which the particle will attain zero velocity is **[SCRA 1998]**
   (*a*) 2 (*b*) 4
   (*c*) 6 (*d*) 8

8. Which of the following in Fig. 6.44 cannot be speed-time graph ? **[Manipal 1999]**

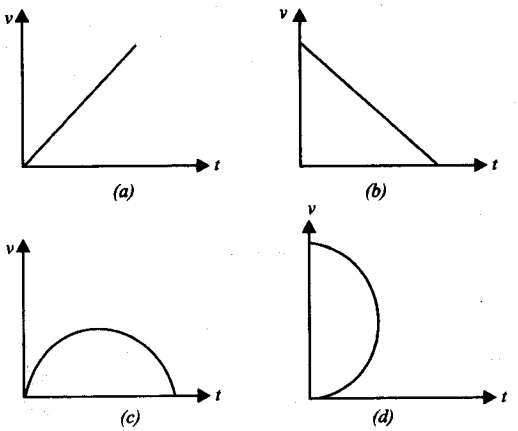


Fig. 6.44

9. An aeroplane is flying with a uniform speed of 100km/h along the circumference of a circle. The change in velocity in half the revolution will be (km/h) **[KCET 2000]**
   (*a*) 100 (*b*) 200
   (*c*) 300 (*d*) 150

10. The velocity of a body depends upon time according to the equation $v = 20 + 0.1t^2$. The body is undergoing **[MNR 1995]**
   (*a*) uniform acceleration
   (*b*) uniform retardation
   (*c*) non-uniform acceleration
   (*d*) zero acceleration

11. An athelete completes one round of a circular track of radius R in 40 seconds. What will be his displacement at the end of 2 min and 20 seconds? **[NCERT 1990]**
   (*a*) zero (*b*) $2R$
   (*c*) $2\pi R$ (*d*) $7\pi R$

12. The acceleration '*a*' in m/s^2 of a particle is given by $a = 3t^2 + 2t + 2$ where t is the time in seconds. If the particle starts out with a velocity $v = 2$ m/s at $t = 0$, then the velocity at the end of 2 seconds is **[SCRA 1994]**
   (*a*) 12 m/s (*b*) 18 m/s
   (*c*) 27 m/s (*d*) 36 m/s

13. A body starts from rest at time $t = 0$, the acceleration–time graph is shown in Fig. 6.45. The maximum velocity attained by the body will be : **[IIT 2004]**

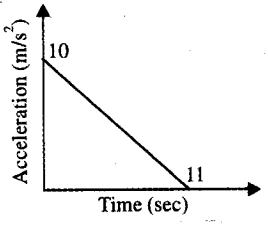


Fig. 6.45

   (*a*) 110 m/s (*b*) 55 m/s
   (*c*) 650 m/s (*d*) 550 m/s

14. The relation between time t and distance x is $t = ax^2 + bx$, where a and b are constants. The acceleration is **[AIEEE 2005]**
   (*a*) $-2abv^2$ (*b*) $2bv^3$
   (*c*) $-2av^3$ (*d*) $2av^2$

15. Which of the following option is correct for the object having a straight line motion represented by the following graph (See Fig. 6.46)? **[DCE 2004]**

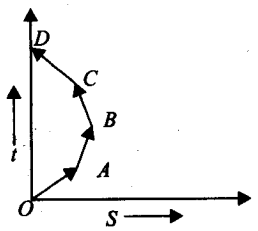


Fig. 6.46

   (*a*) average velocity is zero
   (*b*) velocity of the object increases uniformly
   (*c*) the object moves with constantly increasing velocity from O to A and then it moves with constant velocity
   (*d*) the graph shown is impossible

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*a*)	**2.** (*a*)	**3.** (*a*)	**4.** (*a*)	**5.** (*b*)
6. (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*b*)	**10.** (*c*)
11. (*b*)	**12.** (*b*)	**13.** (*b*)	**14.** (*c*)	**15.** (*a*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. $$\text{Average speed} = \frac{v_1 t_1 + v_2 t_2}{t_1 + t_2} = \frac{80 \times t/2 + 40 \times t/2}{t/2 + t/2} = 60 \text{ km/h}$$

2. $$x = kt^3 \quad \therefore \quad \frac{dx}{dt} = 3kt^2 \quad \text{or} \quad \frac{d^2x}{dt^2} = 6kt$$

   $\therefore$ Acceleration $\propto t$

3. When a body is projected vertically upward, at the highest point of motion, its velocity is zero. However, the body is acted upon by acceleration due to gravity (downward).

4. Let s metres be the total distance travelled by the particle. Suppose t_1 seconds is the time for first half journey and t_2 seconds two equal time intervals to cover the second half of journey.

   $\therefore$ $$t_1 = \frac{s/2}{3} = \frac{s}{6} \text{ seconds}; \quad s_1 = 4.5\, t_2 \quad \text{and} \quad s_2 = 7.5\, t_2$$

   Now $$s_1 + s_2 = \frac{s}{2} = 4.5t_2 + 7.5t_2 = 12t_2$$

   $\therefore$ $$t_2 = s/24$$

   $$\text{Total time} = t_1 + 2t_2 = \frac{s}{6} + 2 \times \frac{s}{24} = s/4$$

   $$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{s}{s/4} = 4 \text{ m/s}$$

5. The constant velocity of a body means that its magnitude (speed) as well as direction is unchanged throughout its motion.

6. The slope of $x - t$ graph at any time gives the velocity of the particle at that time. In Fig. 6.43 (*i*), the slope of $x - t$ graph increases and hence the body has accelerated motion. The reverse is the case for Fig. 6.43 (*ii*).

7. $$s = 6t^2 - t^3$$

   $$\text{Velocity} = \frac{ds}{dt} = \frac{d}{dt}(6t^2 - t^3) = 12t - 3t^2$$

   When velocity is zero, we have, $0 = 12t - 3t^2 \quad \therefore t = 4$ s

8. In Fig. 6.44 (*d*), there will be two speeds at one time. But a particle can have one speed at a time.

9. In half revolution, the velocity of the plane will be opposite to the initial velocity.

   $\therefore$ Change in velocity $= 100 - (-100) = 200$ km/h

10. $$\text{Acceleration, } a = \frac{dv}{dt} = \frac{d}{dt}(20 + 0.1t^2) = 0.2t$$

    Since $a \propto t$, the body is undergoing non-uniform acceleration.

11. Total time of motion $= 2 \times 60 + 20 = 140$ seconds

    Time/revolution = 40 seconds. In 120 seconds, the athelete will complete 3 revolutions and the displacement will be zero.

    $\therefore$ Displacement in 140 sec = Displacement in 20 sec.

    $$= R + R = 2R$$

12. $$a = \frac{dv}{dt} = 3t^2 + 2t + 2$$

    or $$dv = (3t^2 + 2t + 2)dt$$

or $$\int_u^v dv = \int_0^t (3t^2 + 2t + 2)dt$$

or $$v - u = t^3 + t^2 + 2t$$

or $$v = u + t^3 + t^2 + 2t$$

At the end of 2 seconds, the velocity is

$$(v)_{t = 2 \text{ sec}} = 2 + (2)^3 + (2)^2 + 2 \times 2 = 18 \text{ m/s}$$

13. The area under acceleration-time graph gives the change in velocity.

$$\therefore \quad v_{max} = \frac{1}{2} \times 10 \times 11 = 55 \text{ m/s}$$

Since the initial velocity is zero, the maximum velocity attained = 55 m/s

14. $$t = ax^2 + bx$$

Differentiating *w.r.t.* t, we have,

$$1 = 2a\,x\,\frac{dx}{dt} + b\frac{dx}{dt} = \frac{dx}{dt}(2ax + b)$$

$$\therefore \quad \frac{dx}{dt} = v = \frac{1}{2ax + b} = (2ax + b)^{-1} \qquad \ldots (i)$$

Again differentiating eq. (*i*) *w.r.t.* t, we have,

$$\frac{d^2x}{dt^2} = -1 \times (2ax + b)^{-2} \times 2a\,\frac{dx}{dt}$$

$$= \frac{-1}{(2ax + b)^2} \times \frac{2a}{(2ax + b)} \qquad \left(\because \frac{dx}{dt} = \frac{1}{(2ax + b)}\right)$$

$$= \frac{-2a}{(2ax + b)^3} = -2av^3 \qquad \left(\because v = \frac{1}{(2ax + b)}\right)$$

15. It is clear from the graph that the displacement of the object is zero. Therefore, average velocity of the object is zero.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **A person travelled a distance of 200 km between two cities by a car covering the first half distance of the journey at a constant speed of 40 km/h and the remaining half distance at a constant speed of v km/h. If the average speed of the person for the entire journey is 48 km/h, find v.**

Hint. $$t_1 = \frac{100}{40} = 2.5 \text{ hours} \; ; \; t_2 = \frac{100}{v} \text{ hours}$$

$$\text{Average speed} = \frac{\text{Total distance}}{t_1 + t_2}$$

or $$48 = \frac{200}{2.5 + (100/v)}$$

On solving, v = 60 km/h

2. **A train 100 m long is moving with a uniform velocity of 45 km/h. What is the time it will take to cross a bridge 1 km long ?**

Hint. Total distance to be covered

= Length of bridge + Length of train = 1 + 0.1 = 1.1 km

$\therefore$ Time taken by the train to cross the bridge is

$$t = \frac{\text{Total distance}}{\text{Velocity}} = \frac{1.1}{45}\text{hour} = \frac{1.1}{45}\times 3600\,\text{s} = 88\ \text{s}$$

3. **A motor ship covers a distance of 300 km between two localities on a river in 10 hours downstream and in 12 hours upstream. Find the velocity of the river and the velocity of the motor ship in still water, assuming that these velocities are constant.**

Hint. Suppose v_1 and v_2 are the velocities of the ship and water flow respectively in km/h. In case of downstream, the resultant velocity is $(v_1 + v_2)$ while in case of upstream, the resultant velocity is $(v_1 - v_2)$.

$$\text{In downstream, } t_1 = \frac{x}{v_1+v_2} \quad \text{or} \quad 10 = \frac{300}{v_1+v_2}$$

$$\therefore \quad v_1 + v_2 = 30 \qquad ...(i)$$

$$\text{In upstream, } t_2 = \frac{x}{v_1-v_2} \quad \text{or} \quad 12 = \frac{300}{v_1-v_2}$$

$$\therefore \quad v_1 - v_2 = 25 \qquad ...(ii)$$

On solving eqs. (i) and (ii), $v_1 = 27.5$ km/h ; $v_2 = 2.5$ km/h

4. **An electron starting from rest has a velocity v given by ;**

$$v = k\,t$$

where $k = 2\ \text{ms}^{-2}$ and t is the time in seconds. What will be the distance covered by the electron in first 3 seconds ? **[NCERT]**

Hint.

$$v = \frac{dx}{dt} = kt$$

Integrating, we get, $x = \frac{kt^2}{2}$

Here $k = 2\ \text{ms}^{-2}$; $t = 3\text{s}$

$$\therefore \quad x = \frac{2\times(3)^2}{2} = 9\ \text{m}$$

5. **Two cars started simultaneously towards each other from towns A and B which are 480 km apart. It took first car travelling from A to B 8 hours to cover the distance and the second car travelling from B to A took 12 hours. Determine when the cars meet after starting and at what distance from town A. Assume that both the cars travel at constant speed.**

Hint. Speed of car travelling from town A is

$$v_A = 480/8 = 60\ \text{km/h}$$

Speed of car travelling from town B is

$$v_B = 480/12 = 40\ \text{km/h}$$

Let the two cars meet after time t hours. In this time, the distance travelled from A is $60t$ and the distance travelled from B is $40t$. When the cars meet, the total distance travelled = 480 km.

$$\therefore \quad 60t + 40\,t = 480$$

or

$$t = \frac{480}{60+40} = 4.8\ \text{hours}$$

The distance from town A at the meeting point

$$= v_A \times t = 60 \times 4.8 = 288\ \text{km}$$

6. **Two cars are moving in the same direction with the same speed of 30 km/h. They are separated by a distance of 5 km. A truck moving in the opposite direction meets these two cars at an interval of 4 minutes. Find the speed of the truck.**

Hint. Suppose the truck is moving with a speed of v km/h as shown in Fig. 6.47. Apply a velocity of 30 km/h on all the vehicles in the direction of motion of the truck. The two cars come to rest with a fixed distance of 5 km while the velocity of truck becomes $(v + 30)$km/h. The time t hours taken by the truck to cover a distance of 5 km with a speed of $(v + 30)$km /h is given by ;

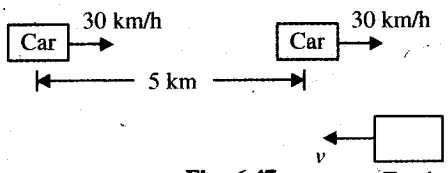


Fig. 6.47

$$t = \frac{\text{Distance}}{\text{Velocity}} = \frac{5}{v+30}$$

or $$\frac{1}{15} = \frac{5}{v+30} \qquad \left(\because t = 4\,\text{minutes} = \frac{1}{15}\,\text{hours}\right)$$

$$\therefore \qquad v = 45 \text{ km/h}$$

7. **A body is moving in a straight line along X-axis. Its distance x in metres from the origin is given by ; $x = 8\,t - 3\,t^2$**
The time t is in seconds. Find the average speed of the body in time interval $t = 0$ to $t = 1$ s.

Hint. $$x = 8\,t - 3\,t^2$$

$$\text{Velocity, } v = \frac{dx}{dt} = \frac{d}{dt}(8t - 3t^2) = 8 - 6t$$

$$\text{At } t = 0 \text{ s}, \quad v_0 = 8 - 6 \times 0 = 8 \text{ ms}^{-1}$$

$$\text{At } t = 1 \text{ s}, \quad v_1 = 8 - 6 \times 1 = 2 \text{ ms}^{-1}$$

$$\therefore \qquad \text{Average speed} = \frac{8+2}{2} = 5 \text{ ms}^{-1}$$

8. **Two trains are moving in opposite directions. Train A moves east with a speed of 72 km/h and train B moves west with a speed of 90 km/h. What is the (*i*) relative velocity of B *w.r.t.* A ? (*ii*) the relative velocity of ground w.r.t. B (*iii*) the velocity of a dog running on the roof of the train A against its motion (with a velocity of 54 km/h w.r.t. to train A) as observed by a man standing on the ground.**

Hint. If not mentioned, the speeds are understood to be *w.r.t.* earth or ground. Therefore, speeds of trains A and B are *w.r.t.* earth.

(*i*) $$\text{Speed of train } A, \; v_{AE} = 72 \text{ km/h} = \frac{72 \times 1000}{60 \times 60} = 20 \text{ ms}^{-1}$$

$$\text{Speed of train } B, \; v_{BE} = 90 \text{ km/h} = \frac{90 \times 1000}{60 \times 60} = 25 \text{ ms}^{-1}$$

Let us choose the positive direction from west to east.

$$\therefore \qquad v_{AE} = +20 \text{ ms}^{-1} \; ; \; v_{BE} = -25 \text{ ms}^{-1}$$

Relative velocity of B *w.r.t.* A is given by ;

$$v_{BA} = v_{BE} + v_{EA} = v_{BE} - v_{AE} = -25 - (+20)$$

$$= -45 \text{ ms}^{-1} \text{ or } 45 \text{ ms}^{-1} \text{ east to west}$$

(*ii*) The relative velocity of ground (*i.e.*, earth) *w.r.t. B* is

$$v_{EB} = v_{EE} + v_{EB} = v_{EE} - v_{BE} = 0 - (-25) = 25 \text{ ms}^{-1}$$

Thus, velocity is from west to east.

(*iii*) Velocity of dog *w.r.t.* to train *A* is

$$v_{DA} = 54\text{km/h} = \frac{54\times1000}{60\times60} = 15\text{ms}^{-1}$$

Since our positive direction is from west to east, $v_{DA} = -15 \text{ ms}^{-1}$.

Now $v_{DA} = v_{DE} + v_{EA} = v_{DE} - v_{AE}$

$\therefore$ $v_{DE} = v_{DA} + v_{AE} = -15 + 20 = 5 \text{ ms}^{-1}$

9. The motion of a particle along a straight line is described by the equation :

$$x = 6 + 4t^2 - t^4$$

where x is in metres (m) and t is the positive time in seconds (s).

(*i*) Find the position, velocity and acceleration at $t = 2$ s.

(*ii*) During what time interval is the velocity positive ?

(*iii*) During what time interval is x positive ?

(*iv*) What is the maximum possible velocity attained by the particle ?

Hint. (*i*) $x = 6 + 4t^2 - t^4$

$$\text{Velocity, } v = \frac{dx}{dt} = \frac{d}{dt}\left(6+4t^2-t^4\right) = 8t - 4t^3$$

$$\text{Acceleration, } a = \frac{dv}{dt} = \frac{d}{dt}\left(8t-4t^3\right) = 8 - 12t^2$$

At $t = 2$ s, $x = 6 + 4(2)^2 - (2)^4 = 6$ m

At $t = 2$ s, $v = 8 \times 2 - 4(2)^3 = -16 \text{ ms}^{-1}$

At $t = 2$ s, $a = 8 - 12(2)^2 = -40 \text{ ms}^{-2}$

(*ii*) The velocity v is positive if $8t - 4t^3 \geq 0$. That is if

$$4t(2 - t^2) > 0$$

i.e., $0 < t < \sqrt{2}$ or $0 < t < 1.41$ s

(*iii*) The position x is positive if $6 + 4t^2 - t^4 \geq 0$. That is, $t^2 \leq 2 \pm \sqrt{10}$ or $t^2 \leq 2 \pm 3.16$ s. Since t is positive, t lies between 0 and $\sqrt{5.16}$ or 0 and 2.27 s.

(*iv*) The velocity is maximum for t given by ;

$$\frac{dv}{dt} = 0 \quad \text{or} \quad 8 - 12t^2 = 0 \quad \text{or} \quad t = \sqrt{\frac{2}{3}} = 0.816\text{s}$$

$\therefore$ Maximum velocity, $v_{max} = (8t - 4t^3)_{t=0.816 \text{ s}}$

$$= [8 \times 0.816 - 4 \times (0.816)^3] = 4.355 \text{ ms}^{-1}$$

10. Two cars *A* and *B* are at positions 100 m and 200 m from the origin at $t = 0$. They start simultaneously with velocities 10 ms^{-1} and 5 ms^{-1} respectively. Calculate the time and position at which they will overtake one another. Assume that they are moving in the same direction.

Hint. $x_{BO} = 200$ m ; $x_{AO} = 100$ m ; $v_A = 10$ ms^{-1} ; $v_B = 5$ ms^{-1}

$$x_B - x_A = (x_{BO} - x_{AO}) + (v_B - v_A)t$$

For the solution of the problem, $x_B - x_A = 0$.

$\therefore$ $0 = (200 - 100) + (5 - 10)t$

or $0 = 100 - 5t$

$\therefore$ $t = 100/5 = 20$ s

$x = 100 + 10 \times 20 = 100 + 200 = 300$ m

11. The motion of a particle along X-axis is given by the equation $x = 20 + 4t^2$ where x is in metres and t in seconds. Find (*i*) the displacement of the particle in time interval $t_1 = 2$ s and $t_2 = 5$ s (*ii*) the average velocity in the same time interval (*iii*) the instantaneous velocity at t = 2 s.

Hint. $x = 20 + 4t^2$

(*i*) The position of the particle at $t_1 = 2$ s is $x_1 = 20 + 4(2)^2 = 36$ m

The position of the particle at $t_2 = 5$ s is $x_2 = 20 + 4(5)^2 = 120$ m

$\therefore$ Displacement of the particle during time interval 2 s to 5 s

$= x_2 - x_1 = 120 - 36 = 84$ m

(*ii*) Average velocity $= \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{84}{5-2} = \dfrac{84}{3} = 28$ ms^{-1}

(*iii*) $x = 20 + 4t^2$

Instantaneous velocity is given by ;

$$v = \frac{dx}{dt} = \frac{d}{dt}(20 + 4t^2) = 8t$$

At $t = 2$ s ; $v = 8 \times 2 = 16$ ms^{-1}

12. If velocity (m/s) varies with time as $v = 4t$, find the distance travelled by the body in the interval of 2 to 4 seconds. **[CBSE 2004]**

Hint. $v = 4t$ or $\dfrac{dx}{dt} = 4t$

$\therefore$ $\displaystyle\int_0^x dx = \int_2^4 4t\, dt$

or $\displaystyle x = 4\left[\frac{t^2}{2}\right]_2^4 = 2[4^2 - 2^2] = 24$ m

13. The position of a particle as a function of time is described as $y(t) = 5\cos(10t + 15°)$ in metres. What is the average velocity of the particle from $t = 0$ to $t = 3$ seconds?

Hint. $y(t) = 5\cos(10t + 15°)$

When $t = 0$ s ; $y(0) = 5\cos(10 \times 0 + 15°) = 5\cos 15° = 4.83$ m

When $t = 3$ s ; $y(3) = 5\cos(10 \times 3 + 15°) = 5\cos 45° = 3.53$ m

$\therefore$ Average velocity, $\bar{v} = \dfrac{\text{Total displacement}}{\text{Total time}} = \dfrac{4.83 + 3.53}{3} = 2.79$ m/s

7

Uniformly Accelerated Linear Motion

INTRODUCTION

When an object *moves with a constant acceleration in a straight line,* it is said to have **uniformly accelerated linear motion.** It is the simplest kind of accelerated motion. There are many practical situations in which the acceleration is uniform (*i.e.,* constant). For example, near the surface of earth, all objects fall vertically with a constant acceleration of gravity g (= 9.8 ms^{-2}). There are only *four mathematical equations* that describe the linear motion of an object with constant acceleration. In this chapter, we shall deal with mathematical and graphical aspects of uniformly accelerated linear motion.

7.1. EQUATIONS FOR UNIFORMLY ACCELERATED LINEAR MOTION

Consider an object moving with a uniform acceleration 'a' along a straight line OX with origin at O as shown in Fig. 7.1. Let at $t = 0$, the position of the object be x_0 (initial displacement from O) and its velocity v_0 (initial velocity). Let at $t = t$, the position of the object be x and its velocity v (final velocity). Clearly, in time t, the displacement of the object is $x - x_0$.

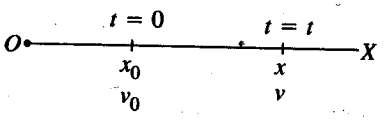


Fig. 7.1

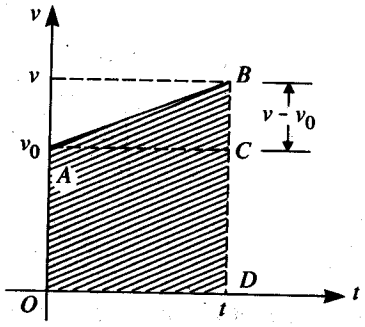


Fig. 7.2

(*i*) **Displacement – Average Velocity relation.** The velocity-time ($v - t$) graph for the motion in Fig. 7.1 is AB as shown in Fig. 7.2. The velocity increases uniformly from v_0 at $t = 0$ to v in time t (= OD). We know that displacement during any time interval is equal to the area under velocity-time graph during that time interval. In this case, the time interval is from 0 to t so that displacement is equal to the shaded area in Fig. 7.2.

$\therefore$ Displacement = Shaded area = Area of trapezium $OABD$

$$= \frac{AO + BD}{2} \times OD = \frac{v_0 + v}{2} \times t$$

$\therefore$ Average velocity, $\bar{v} = \dfrac{\text{Displacement}}{\text{Time interval}} = \dfrac{v_0 + v}{2} \times t \div t$

or $$\bar{v} = \frac{v_0 + v}{2}$$

Note that for the case of constant acceleration *and only for this case,* the average velocity is equal to one-half the sum of initial and final velocities.

Now Displacement = Average velocity × Time interval

or $$x - x_0 = \left(\frac{v_0 + v}{2}\right) t \qquad \ldots(i)$$

(*ii*) **Velocity–Time relation.** Since object's velocity has changed by an amount $v - v_0$ during time t, the constant acceleration 'a' during this time is

$$a = \frac{v - v_0}{t}$$

or $$v = v_0 + at \qquad \ldots(ii)$$

(*iii*) **Position–Time relation.** This can be found as under :

Average velocity, $\bar{v} = \dfrac{\text{Displacement}}{\text{Time}} = \dfrac{x - x_0}{t}$

or $$x - x_0 = \bar{v}\, t$$

or $$x - x_0 = \left(\frac{v_0 + v}{2}\right) t \qquad \left(\because \bar{v} = \frac{v_0 + v}{2}\right)$$

or $$x - x_0 = \left(\frac{v_0 + v_0 + at}{2}\right) t \qquad (\because v = v_0 + at)$$

or $$x - x_0 = v_0 t + \frac{1}{2} a t^2$$

or $$x = x_0 + v_0 t + \frac{1}{2} a t^2 \qquad \ldots(iii)$$

If $x - x_0 = S$, the displacement of the object in time t, then eq. (*iii*) can be written as :

$$S = v_0 t + \frac{1}{2} a t^2$$

(*iv*) **Velocity–Displacement relation.** This can be found as under :

$$x - x_0 = \left(\frac{v_0 + v}{2}\right) t \qquad \text{[From eq. (}i\text{)]}$$

or $$x - x_0 = \left(\frac{v_0 + v}{2}\right) \left(\frac{v - v_0}{a}\right) \qquad \left[\text{From eq. } (ii),\ t = \frac{v - v_0}{a}\right]$$

or $$x - x_0 = \frac{v^2 - v_0^2}{2a}$$

$\therefore$ $$v^2 - v_0^2 = 2a(x - x_0) \qquad \ldots (iv)$$

If $x - x_0 = S$, the displacement of the object in time t, then eq. (*iv*) can be written as :

$$v^2 - v_0^2 = 2\, aS$$

7.2. SUMMARY OF EQUATIONS FOR UNIFORM ACCELERATION

The *four equations* of motion for uniform acceleration are given in the table below. Each equation is also identified by the variable (x, v, a or t) that *does not* appear in it.

Equation	Variable not appearing
$v = v_0 + at$	x
$x - x_0 = v_0 t + \frac{1}{2} at^2$	v
$x - x_0 = \left(\frac{v_0 + v}{2}\right) t$	a
$v^2 - v_0^2 = 2a(x - x_0)$	t

(*i*) The *first two equations* respectively give the final velocity v and final displacement x in terms of their initial values v_0 and x_0, the acceleration 'a' and time interval t.

(*ii*) The *last two equations* are very important because they are usually sufficient to solve problems for constant acceleration.

7.3. DISPLACEMENT IN nth SECOND FOR UNIFORM ACCELERATION

Consider an object moving with a uniform acceleration 'a' along a straight line OX with origin at O as shown in Fig. 7.3. Let at $t = 0$, the position of the object be x_0 and its velocity v_0 (initial velocity). Let at $t = t$, the position of the object be x and its velocity v. Clearly, in time t, the displacement of the object is $x - x_0 = S$. The displacement of the object in time t is given by ;

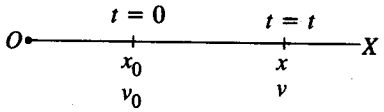


Fig. 7.3

$$S = v_0 t + \frac{1}{2} at^2$$

If S_n and S_{n-1} are the displacements of the object in n and $n - 1$ seconds, then displacement of the object in the nth second is

$$S_{nth} = S_n - S_{n-1}$$

$$= \left[v_0 n + \frac{1}{2} an^2\right] - \left[v_0 (n-1) + \frac{1}{2} a(n-1)^2\right]$$

$$= \left[v_0 n + \frac{1}{2} an^2\right] - \left[v_0 n - v_0 + \frac{1}{2} an^2 - an + \frac{a}{2}\right]$$

$$= v_0 (n - n + 1) + \frac{a}{2}\left(n^2 - n^2 + 2n - 1\right)$$

$$= v_0 + \frac{a}{2}(2n - 1)$$

$$\therefore \quad S_{nth} = v_0 + \frac{a}{2}(2n - 1) \qquad \ldots (i)$$

Eq. (*i*) gives the expression for the displacement in the nth second for uniform acceleration.

7.4. EQUATIONS OF UNIFORM ACCELERATION FROM v–t GRAPH

Fig. 7.4 shows the velocity-time graph for an object moving along a straight line with constant acceleration a. The velocity-time graph is a straight line because a (= slope of graph) is constant. Note that graph starts from v_0 because at $t = 0$, the initial velocity is assumed to be v_0.

(*i*) Velocity at any time *t*. From velocity-time graph, we can derive the relation for final velocity *v* of the object at any time *t*. Thus in Fig. 7.4, the final velocity *v* is the sum of two parts — initial velocity v_0 and the velocity change *at* during the time *t*. (Note that time interval = $t - 0 = t$).

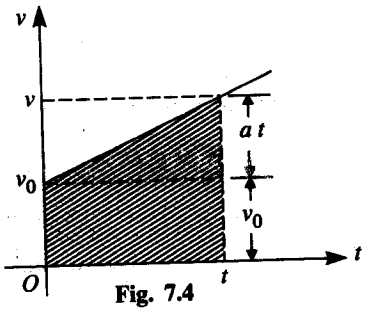


Fig. 7.4

$$\therefore \qquad v = v_0 + at$$

It is the same result that we derived earlier from algebraic considerations. Note that how clearly this relation is indicated in the $v - t$ graph.

(*ii*) Final position *x* at any time *t*. From the velocity-time graph in Fig. 7.4, we can derive the relation for the position *x* of the object as a function of time. We know that displacement *$x - x_0$ of the object during the time *t* is equal to the area of the graph for that time interval (*i.e.*, 0 to *t* in this case). The area under the graph for time interval 0 to *t* is the shaded area. It consists of a rectangle of width *t* and height v_0 plus right angled triangle of base *t* and height *a t*.

$$\therefore \qquad x - x_0 = \text{Shaded area}$$

$$\text{or} \qquad x - x_0 = v_0 t + \frac{1}{2} a t^2$$

$$\text{or} \qquad x = x_0 + v_0 t + \frac{1}{2} at^2$$

This is the same result which we derived earlier from algebraic considerations. Note the ease with which we have found the above relation from the $v - t$ graph. Indeed, graphical study of motion provides more insight than the mathematical equations.

(*iii*) $$v^2 - v_0^2 = 2aS$$

We shall derive the above relation graphically. Referring to Fig 7.4, we have,

$$x - x_0 = \text{Area of trapezium (shaded portion)}$$

$$= \frac{1}{2}(v + v_0)t \qquad \ldots(i)$$

$$\text{Acceleration, } a = \text{Slope of velocity-time graph}$$

$$= \frac{v - v_0}{t}$$

$$\text{or} \qquad t = \frac{v - v_0}{a}$$

Putting the value of *t* in eq. (*i*), we have,

$$x - x_0 = \frac{1}{2}(v + v_0) \times \frac{v - v_0}{a}$$

$$\therefore \qquad v^2 - v_0^2 = 2a\,(x - x_0)$$

If $x - x_0 = S$, the displacement of the particle in time *t*, then, $v^2 - v_0^2 = 2aS$.

(*iv*) Average velocity. Compare the velocity-time graphs in Fig. 7.5 (*i*) and in Fig. 7.5 (*ii*). The shaded area under $v - t$ graph is the same for both cases for the same time interval.

In Fig. 7.5 (*i*), the area is under the trapezoid and in Fig. 7.5 (*ii*), the same area is under the rectangle. Both areas are **equal to the change in displacement $x - x_0$. The horizontal line

---

* At $t = 0$, velocity is v_0 and the position of the object (*i.e.*, its displacement from the origin) is x_0.

** The velocity-time graphs are with identical areas under the graph. In Fig. 7.5 (*i*), the area is with constant acceleration and in Fig. 7.5 (*ii*), the same area is shown with constant velocity.

(solid line) in Fig. 7.5 (*ii*) has the value $(v_0 + v)/2$; it is in fact the average velocity $\bar{v}$.

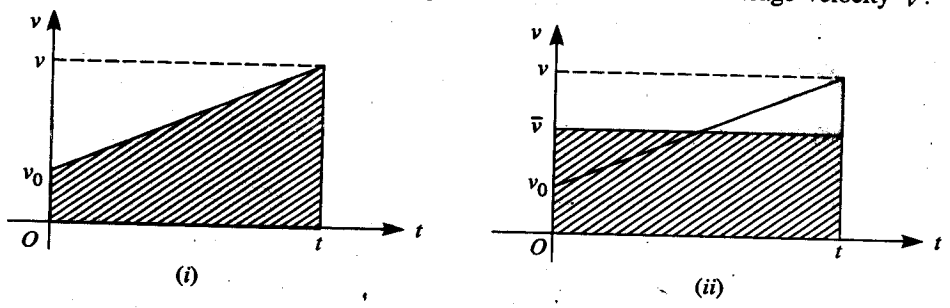


Fig. 7.5

$$\therefore \quad \text{Average velocity, } \bar{v} = \frac{v_0 + v}{2}$$

Referring to Fig. 7.5 (*ii*), the area under the $v - t$ graph for $0 - t$ time interval is equal to the width t of the rectangle multiplied by height $\bar{v}$ $[= (v_0 + v)/2]$.

$$\therefore \quad x - x_0 = \left(\frac{v_0 + v}{2}\right)t$$

7.5. POSITION – TIME GRAPH FOR CONSTANT +VE ACCELERATION

The position–time relation for uniformly accelerated motion in a straight line is given by ;

$$x = x_0 + v_0 t + \frac{1}{2} at^2 \qquad ...(i)$$

Eq. (*i*) shows that x is a quadratic function of time t. Therefore, the x –t graph is a ***parabola** (See Fig. 7.6). Note that slope of the tangent at $t = 0$ is equal to the initial velocity v_0 and slope of the tangent at time t is equal to the velocity v at time t. It is clear that the slope of the graph is continuously increasing (*i.e.*, velocity is continuously increasing). The measurements would show that the rate of increase of slope is constant.

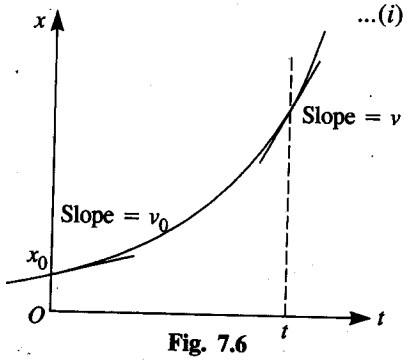


Fig. 7.6

Note. $x = x_0 + v_0 t + \frac{1}{2} at^2$

At first sight, it may appear that in this equation, acceleration is not constant. However, by differential calculus, it can be easily shown that acceleration is constant.

$$\frac{dx}{dt} = \frac{d}{dt}\left(x_0 + v_0 t + \frac{1}{2}at^2\right) = v_0 + at$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(v_0 + at) = a$$

Note that acceleration a is independent of time and, therefore, it is constant. The reader may note the important role of mathematics in physics. From the **$x - t$ graph, it is not clear that acceleration a is constant. But the use of differential calculus readily proves it.

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* If x is ploted against t, the curve is a parabola whose symmetry axis is parallel to the direction of the constant acceleration.

** However, from the v–t graph, it is clear that acceleration is constant in this case.

7.6 VERTICAL MOTION UNDER GRAVITY

One important example of linear motion with constant acceleration is the vertical motion (downward or upward) of an object under gravity. Objects that move under gravity alone are called freely falling bodies. *A freely falling body always has an acceleration of 9.8 ms^{-2} vertically downward.* Note that magnitude (9.8 ms^{-2}) as well as direction (vertically downward) of the acceleration is constant.

A freely falling body has a constant acceleration of 9.8 ms^{-2} acting vertically downward. It does not matter whether the body is rising or falling; its acceleration is still g (= 9.8 ms^{-2}) downward. When a body is rising, its velocity decreases at the rate of 9.8 ms^{-1} every second until it reaches at the top of its motion. At this instant, the velocity of the body is zero and it starts downward journey. As the body is falling, its velocity increases at the rate of 9.8 ms^{-1} every second until it hits the ground.

Equations of Motion. Since vertical motion under gravity alone is the linear motion with constant acceleration, all the equations of motion derived in Art. 7.2 are applicable to freely falling bodies. The only thing to be done is to change the nomenclature *i.e.*, displacement S is replaced by h (vertical displacement) and a by g. Therefore, equations of motion for a freely falling body are :

$$(i) \qquad h = \left(\frac{v_0 + v}{2}\right) t$$

$$(ii) \qquad v = v_0 + g\,t$$

$$(iii) \qquad h = v_0\,t + \frac{1}{2}\,g\,t^2$$

$$(iv) \qquad v^2 - v_0^2 = 2\,g\,h$$

$$(v) \qquad h_{nth} = v_0 + \frac{g}{2}(2n - 1)$$

Note. It may be noted that eq. (v) is not the basic equation for uniform acceleration but is only a special case of eq. (*iii*).

Sign convention. While solving problems regarding freely falling bodies, it is very important to adopt one sign convention for displacements and velocities. *It is a usual practice to choose h to be positive in the upward direction from the selected origin and negative in the downward direction from the origin.* Therefore, you must specify the origin of measurement in problems involving motion under gravity.

Example 7.1. A car starts from rest and accelerates uniformly for 10 s to a velocity of 36 km/h. It then runs at constant velocity and is finally brought to rest in 50 m with a constant retardation. The total distance covered by the car is 500 m. Find the value of acceleration, retardation and total time taken.

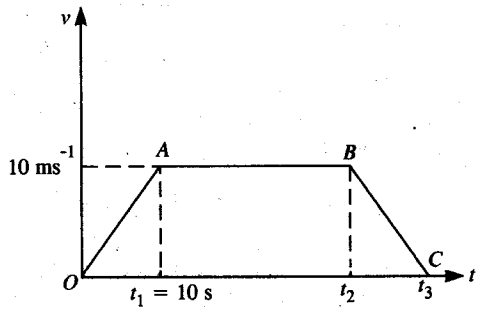


Fig. 7.7

Solution. Fig. 7.7 shows the velocity-time graph for the motion. The car uniformly accelerates for time t_1 (= 10 s) and attains a constant velocity v (= 10 ms^{-1}). In the time inverval t_1 to t_2, it moves with this constant velocity (*i.e.*,10 ms^{-1}). In time interval t_2 to t_3, it retards uniformly and comes to stop.

(*i*) Constant Acceleration part (*OA*)

$v_0 = 0$; v = 36 km/h = 10 ms^{-1} ; t_1 = 10 s

$\therefore$ Constant acceleration, $a = \dfrac{v - v_0}{t_1} = \dfrac{10-0}{10} = 1 \text{ ms}^{-2}$

Average velocity, $\bar{v} = \dfrac{v_0 + v}{2} = \dfrac{0+10}{2} = 5 \text{ ms}^{-1}$

Distance covered, $S_1 = \bar{v}\, t_1 = 5 \times 10 = 50 \text{ m}$

(ii) Constant Retardation part (*BC*)

$v_0 = 10 \text{ ms}^{-1}$; $v = 0$; $t_3 - t_2 = ?$; $S_3 = 50 \text{ m}$

Average velocity, $\bar{v} = \dfrac{v_0 + v}{2} = \dfrac{10+0}{2} = 5 \text{ ms}^{-1}$

Distance covered, $S_3 = \bar{v}\,(t_3 - t_2)$

$\therefore$ $t_3 - t_2 = \dfrac{S_3}{\bar{v}} = \dfrac{50}{5} = 10 \text{ s}$

Retardation, $a = \dfrac{v - v_0}{t_3 - t_2} = \dfrac{0-10}{10} = -1 \text{ ms}^{-2}$

(iii) Constant-velocity part (*AB*)

Distance covered, $S_2 = S - S_1 - S_3 = 500 - 50 - 50 = 400 \text{ m}$

$\therefore$ Time taken, $t_2 - t_1 = \dfrac{S_2}{\text{constant velocity}} = \dfrac{400}{10} = 40 \text{ s}$

$\therefore$ Total time for journey $= t_1 + (t_2 - t_1) + (t_3 - t_2)$

$= 10 + 40 + 10 = 60 \text{ s}$

Note. The statement of the problem suggests that magnitudes of acceleration and retardation must be same. For the same reason, times for acceleration and retardation must be same ; also the distances covered during acceleration and retardation periods must be the same.

Example 7.2. **Fig. 7.8 shows the speed-time graph of a car.**

(i) What is the acceleration of the car in first two hours, in the next two hours and in the last two hours?

(ii) What is the total distance travelled by the car?

(iii) What is the average speed of the car?

Solution. (*i*) Since the speed-time graph during acceleration (*AB*) period is a straight line, the car moves with constant acceleration during this time interval.

$\therefore$ Constant acceleration, a = Slope of line AB

$$= \frac{BD}{AD} = \frac{60}{4} = 15 \text{ km/hr}^2$$

The car retards uniformly (graph *BC*) from 4 hours to 8 hours. Therefore, retardation is constant and is given by :

$$a = \text{Slope of line } BC = -\frac{BD}{DC} = -\frac{60}{4} = -15 \text{ km/hr}^2$$

(*ii*) Total distance = Area of $\Delta ABC = \dfrac{1}{2} \times 60 \times 8 = 240 \text{ km}$

(*iii*) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}} = \dfrac{240}{8} = 30 \text{ km/h}$

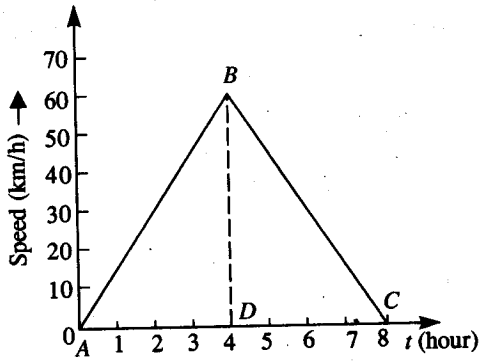


Fig. 7.8

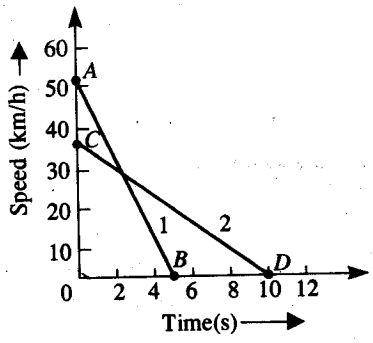


Fig. 7.9

Example 7.3. **A driver of a car travelling at 52 km/h applies the brakes and decelerates uniformly. The car stops in 5 seconds. Another driver going at 34 km/h applies his brakes slower and stops after 10 seconds. On the same graph paper, plot speed-time graphs for the two cars. Which of the two cars travelled more distance after the brakes were applied?**

Solution. Since both the cars decelerate uniformly, their speed-time graphs will be straight lines as shown in Fig. 7.9. We know that distance travelled by an object during any time interval is equal to the area under speed-time graph during that time interval.

Distance travelled by car 1 = Area OAB

$$= \frac{1}{2} \times *OA \times OB = \frac{1}{2} \times \frac{52 \times 1000}{60 \times 60} \times 5 = 36.1 \text{ m}$$

Distance travelled by car 2 = Area OCD

$$= \frac{1}{2} \times OC \times OD = \frac{1}{2} \times \frac{34 \times 1000}{60 \times 60} \times 10 = 47.2 \text{ m}$$

Therefore, car 2 travels more distance after the brakes were applied.

Example 7.4. **A body moving with uniform acceleration covers 24 m in the 4th second and 36 m in the 6th second. Calculate acceleration and initial velocity.**

Solution. The distance covered in the nth second is given by :

$$S_{nth} = v_0 + \frac{a}{2}(2n-1) \quad \therefore \quad 24 = v_0 + \frac{a}{2}(2 \times 4 - 1) \qquad ...(i)$$

Also $$36 = v_0 + \frac{a}{2}(6 \times 2 - 1) \qquad ...(ii)$$

Subtracting eq. (i) from eq. (ii), we get,

$$12 = 2a \quad \therefore \quad a = 12/2 = 6 \text{ ms}^{-2}$$

Putting the value of a in eq. (i), we get,

$$24 = v_0 + \frac{6}{2}(2 \times 4 - 1) \qquad \therefore \quad v_0 = 3 \text{ ms}^{-1}$$

$* OA = 52\text{km/h} = \frac{52 \times 1000}{60 \times 60} \text{ms}^{-1}$

Example 7.5. A body is moving with uniform acceleration. Its velocity after 5 seconds is 25 ms^{-1} and after 8 seconds, it is 34 ms^{-1}. Calculate the distance it will travel in 12th second.

Solution. $v = v_0 + a\,t$

For the first case : $v = 25\ ms^{-1}\ ;\ t = 5\ s$

$\therefore \quad 25 = v_0 + 5a \quad \ldots (i)$

For the second case : $v = 34\ ms^{-1}\ ;\ t = 8\ s$

$\therefore \quad 34 = v_0 + 8a \quad \ldots (ii)$

Solving eqs. (*i*) and (*ii*), we get, $v_0 = 10\ ms^{-1}\ ;\ a = 3\ ms^{-2}$.

The distance travelled in the 12th second is given by ;

$$S_{12th} = v_0 + \frac{a}{2}(2n - 1) = 10 + \frac{3}{2}(2 \times 12 - 1) = 44.5\ m$$

Example 7.6. A tennis ball is dropped on to the floor from a height of 10 m. It rebounds to a height of 2.5 m. If the ball is in contact with the floor for 0.01 second, what is the average acceleration during the contact ? Take $g = 10\ ms^{-2}$.

Solution. Let v_1 be the velocity of the ball just before striking the floor and v_2 be the velocity of the ball just after striking the floor. Since the two velocities are acting in the opposite directions, the change in velocity $\Delta v = v_1 - (-v_2) = v_1 + v_2$ and time for the change $\Delta t = 0.01$ second.

$\therefore$ Average acceleration, $\bar{a} = \dfrac{v_1 + v_2}{\Delta t}$

Now $v_1 = \sqrt{2gh_1} = \sqrt{2 \times 10 \times 10} = \sqrt{200}\ ms^{-1}$; $v_2 = \sqrt{2gh_2} = \sqrt{2 \times 10 \times 2.5} = \sqrt{50}\ ms^{-1}$

$\therefore \quad \bar{a} = \dfrac{\sqrt{200} + \sqrt{50}}{0.01} = 2121\ ms^{-2}$

Example 7.7. A car starts from rest and accelerates unifomiily at a rate of 2 ms^{-2} for 20 seconds. It then maintains a constant velocity for 10 seconds. The brakes are then applied and the car is uniformly retarded and comes to rest in 5 seconds. Draw the velocity-time graph for the motion and find (*i*) the maximum velocity (*ii*) the retardation in the last 5 seconds (*iii*) total distance travelled and (*iv*) average velocity.

Solution. Fig. 7.10 shows the velocity-time graph for the motion of the car. The graph is *OABC*.

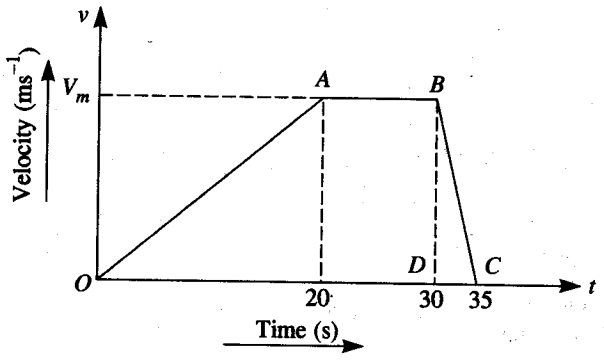


Fig. 7.10

(*i*) $a = 2\ \text{ms}^{-2}$; $v_0 = 0$; $t = 20$ s ; $v_m = ?$

$v_m = v_0 + at = 0 + 2 \times 20 = 40\ \text{ms}^{-1}$

(*ii*) Retardation = Slope of line $BC = -\dfrac{BD}{DC} = -\dfrac{40}{5} = -8\ \text{ms}^{-2}$

Alternatively; Retardation = $\dfrac{v - v_m}{t} = \dfrac{0 - 40}{5} = -8\ \text{ms}^{-2}$

(*iii*) Total distance travelled, S = Area of trapezium $OABC$

$$= \frac{1}{2}(AB + OC) \times BD$$

$$= \frac{1}{2}(10 + 35) \times 40 = 900\ \text{m}$$

(*iv*) Average velocity = $\dfrac{S}{t} = \dfrac{900}{35} = 25.71\ \text{ms}^{-1}$

Example 7.8. **A ball is thrown vertically upward with a velocity of 29.4 ms^{-1}. Describe the motion of the ball.**

Solution. Let us take origin at the ground. Then upward displacement and velocity will be positive and downward displacement and velocity will be negative. As g is always directed downward, $g = -9.8\ \text{ms}^{-2}$.

Upward Journey. Since the ball is under constant acceleration (= –9.8 m s^{-2}), its velocity will decrease by 9.8 ms^{-1} every second.

Velocity of ball after 1s = 29.4 – 9.8 = 19.6 ms^{-1}

Velocity of ball after 2s = 19.6 – 9.8 = 9.8 ms^{-1}

Velocity of ball after 3s = 9.8 – 9.8 = 0

Thus the ball will reach the highest point in 3 s. At the highest point, the velocity of the ball is zero. Now the ball starts *downward journey.

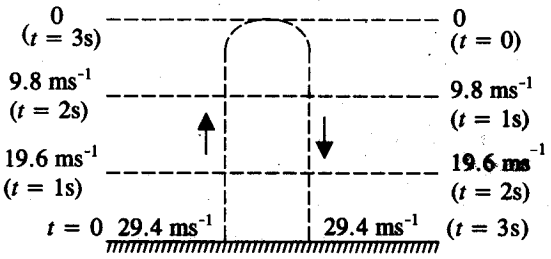


Fig. 7.11

Downward Journey. Note that during the downward journey of the ball, g is again –9.8 ms^{-2}. But displacement and velocity are negative.

Velocity after 1 s = 0 – 9.8= –9.8 ms^{-1}

Velocity after 2 s = $-(v_0 + g\,t) = -(9.8 + 9.8) = -19.6\ \text{ms}^{-1}$

Velocity after 3 s = – (19.6 + 9.8) = –29.4 ms^{-1}

Thus the ball reaches the ground in 3 s and strikes it with a velocity of 29.4 m s^{-1}.

Conclusions. Three important conclusions can be drawn about a freely falling body.

(*i*) The time taken for upward journey is equal to the time taken for the downward journey.

(*ii*) The magnitude of velocity (*i.e.* speed) with which the ball was thrown upward, it hits the ground with the same speed.

* In fact, the downward journey of the ball will be along the same path along which it performed upward journey. But in Fig.7.11, the two paths are shown separate for explaining the behaviour of a freely falling body.

(*iii*) The magnitude of velocity of the ball at any point in its upward and downward journey is the same.

The above behaviour of a freely falling body is due to the fact that value of g is constant.

Example 7.9. **A person throws a ball vertically upward with an initial velocity of 15 ms^{-1}. Calculate (*i*) how high it goes and (*ii*) how long the ball is in air before it comes to his hand.**

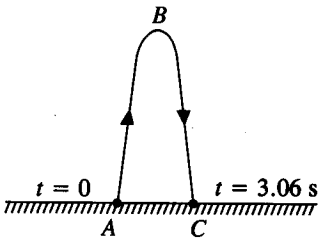


Fig. 7.12

Solution. We select earth as the origin so that $g = -9.8$ ms^{-2}.

(*i*) At the highest point, velocity is zero.

Now $v^2 - v_0^2 = 2\,g\,h$

Here $v = 0\ ;\ v_0 = +15$ ms^{-1} ; $g = -9.8$ ms^{-2}

$\therefore$ $(0)^2 - (+15)^2 = 2 \times (-9.8) \times h$

$\therefore$ Maximum height, $h = \dfrac{-(+15)^2}{2\times(-9.8)} = 11.5$ m

(*ii*) $h = v_0\,t + \dfrac{1}{2}gt^2$

Here *$h = 0\ ;\ v_0 = +15$ ms^{-1}; $g = -9.8$ m s^{-2}

$\therefore$ $0 = 15t - \dfrac{1}{2}\times 9.8\,t^2$ or $0 = 15\,t - 4.9\,t^2$

$\therefore$ $0 = t\,(15 - 4.9\,t)$

$\therefore$ $t = 0$ and $15 - 4.9\,t = 0$ or $t = \dfrac{15}{4.9} = 3.06$ s

Note that there are two solutions for t ; 0 s and 3.06 s. The first solution corresponds to initial point A (See Fig. 7.12) and second solution corresponds to the return–point C. Therefore, the ball is in the air for 3.06 s.

Example 7.10. **An object is thrown vertically upward from the ground at 30 ms^{-1}.**

(*i*) What is the displacement after 4 s? (*ii*) What is the velocity after 4 s?

(*iii*) What is the maximum height it attains ? (*iv*) What is the time of flight ?

Solution. We select earth as the origin. Then $g = -9.8$ ms^{-2}. For upward journey, $v_0 = +30$ ms^{-1} ; $g = -9.8$ ms^{-2}.

(*i*) $h = v_0\,t + \dfrac{1}{2}g\,t^2$

Here $v_0 = +30$ ms^{-1} ; $g = -9.8$ ms^{-2} ; $t = 4$ s ; $h = ?$

$\therefore$ $h = 30 \times 4 + \dfrac{1}{2}(-9.8) \times (4)^2 = 42$ m

(*ii*) $v = v_0 + g\,t$

Here $v_0 = +30$ ms^{-1} ; $g = -9.8$ ms^{-2} ; $t = 4$ s

$\therefore$ $v = 30 + (-9.8) \times 4 = -9$ ms^{-1}

Thus four seconds after the object is thrown, it is 42 m above the ground and is travelling downward (negative sign shows downward motion) with an instantaneous speed of 9 ms^{-1}.

* The net displacement for entire journey is zero.

(*iii*) At the highest point, the velocity of the object is zero.

Now $v^2 - v_0^2 = 2\,g\,h$

Here $v = 0$; $v_0 = +30\ \text{ms}^{-1}$; $g = -9.8\ \text{ms}^{-2}$

$\therefore \quad (0)^2 - (30)^2 = 2\,(-9.8)\,h$

or $$h = \frac{0^2 - (30)^2}{2(-9.8)} = 46\ \text{m}$$

(*iv*) The displacement h is zero when the object strikes the ground. We want to find the corresponding time of flight.

$$h = v_0\,t + \frac{1}{2}g\,t^2$$

Here $h = 0$; $v_0 = +30\ \text{ms}^{-1}$; $g = -9.8\ \text{ms}^{-2}$

$$\therefore \quad 0 = 30\,t - \frac{1}{2} \times 9.8\,t^2$$

or $$0 = 30\,t - 4.9\,t^2$$

or $$0 = t\,(30 - 4.9\,t)$$

$$\therefore \quad t = 0 \text{ and } 0 = 30 - 4.9\,t$$

or $$t = \frac{30}{4.9} = 6.1\text{s}$$

Therefore, time of flight is 6.1s. Note that $t = 0$ corresponds to the point of throwing the object upward. Fig. 7.13 shows the graphical solution of the problem.

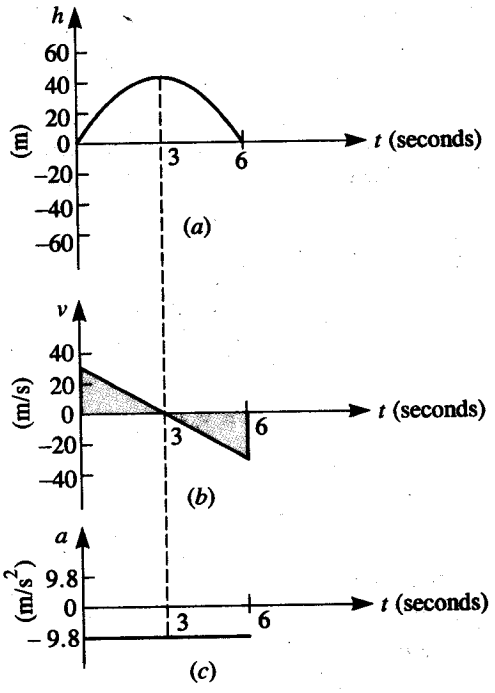


Fig. 7.13

(*a*) The h–t graph of the motion is a parabola. This is not surprising because the position-time graph for a linear motion with constant acceleration is a parabola. Note that the displacement is zero at 6.1s. This is not only indicated on h–t graph but also by the net area on v–t graph. The net area of v–t graph is zero so that net displacement is zero.

(*b*) The $v - t$ graph of the motion is a straight line. It is because the acceleration is constant during the entire motion of the object. Note that as the object moves upward, its velocity decreases uniformly (9.8 ms^{-1} every second). When the object reaches the highest point of motion, its velocity is zero. From now onwards, the object starts its downward journey.

(*c*) The $a - t$ graph of motion is a straight line parallel to the time axis. This implies that acceleration is constant (= $-9.8\ \text{ms}^{-2}$) throughout the journey of the object. Since origin is selected at the ground and g always acts downward, the value of g is negative.

Example 7.11. **Water drops fall at regular intervals from the nozzle of a shower on to the floor 3.24 m below. The first drop strikes the floor at the instant the fourth one begins to fall from the nozzle. Find the positions of the drops when a drop just strikes the floor.**

Solution. Fig. 7.14 shows the positions of the four drops *A, B, C* and *D*. When drop *A* strikes the floor, the drop *D* just beigns to fall from the nozzle. The positions of the second and third drops

are mid-way between them. Let t be the time interval between two successive drops. Referring to Fig. 7.14,

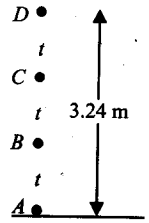


Fig. 7.14

$$CD = \frac{1}{2}gt^2 \; ; \; BD = \frac{1}{2}g\,(2t)^2 = 2gt^2 \; ; \; AD = \frac{1}{2}\,g\,(3t)^2 = 4.5\,gt^2$$

It is given that $AD = 3.24$ m.

$\therefore \quad 4.5\,gt^2 = 3.24$

or $\quad gt^2 = 3.24/4.5 = 0.72$

$\therefore \quad CD = \frac{1}{2}gt^2 = \frac{1}{2} \times 0.72 = 0.36$ m

$BD = 2gt^2 = 2 \times 0.72 = 1.44$ m ; $\quad AD = 3.24$ m

Example 7.12. **A parachutist bails out from an aeroplane and after dropping through a distance of 40 m opens the parachute and decelerates at 2 ms^{-2}. If he reaches the ground with a speed of 2 ms^{-1}, how long he is in the air? At what height did he bail out from the plane?**

Solution. Time for first 40 m journey.

Now $\quad v_0 = 0 \; ; \; h = -40$ m $\; ; \; g = -9.8$ m/s^2 ; $v = ?$

$$(-v)^2 - v_0^2 = 2gh \quad \text{or} \quad v^2 - (0)^2 = 2 \times (-9.8) \times (-40)$$

$\therefore \quad v = \sqrt{2 \times 9.8 \times 40} = 28$ m/s

Now $\quad v = v_0 + gt_1 \quad \therefore \; t_1 = \frac{v - v_0}{g} = \frac{-28 - 0}{-9.8} = 2.9$ s

Time for second journey.

$v_0 = -28$ m/s ; $v = -2$ m/s ; $a = +2$ m/s^2

$v = v_0 + at_2 \quad$ or $\quad -2 = -28 + 2t_2 \quad \therefore \; t_2 = 13$s

Total time t for which the parachute is in air is

$t = t_1 + t_2 = 2.9 + 13 = 15.9$ s

Height from which parachute has fallen during second journey.

$v^2 - v_0^2 = 2ah$ or $(-2)^2 - (-28)^2 = 2 \times (+2) \times -h \quad \therefore \quad h = 195$ m

$\therefore$ Height at which the parachutist bailed out $= 40 + 195 = 235$ m

Example 7.13. **A balloon is ascending at the rate of 12 ms^{-1}. When it is at a height of 65 m from the ground, a packet is dropped from it. After how much time and with what velocity does the packet reach the ground? Take $g = 10$ ms^{-2}.**

Solution. $h = v_0 t + \frac{1}{2}gt^2$

Taking downward motion as positive ; $v_0 = -12$ ms^{-1} ; $g = +10$ ms^{-2}

$\therefore \quad 65 = -12 \times t + \frac{1}{2} \times 10 \times t^2$

or $\quad 5t^2 - 12t - 65 = 0 \quad$ or $\quad (5t + 13)\,(t - 5) = 0$

$\therefore \quad t = -13/5$ sec ; $t = 5$ sec

As the time cannot be negative ; $t = 5$ sec.

$\therefore \quad v = v_0 + gt = -12 + 10 \times 5 = 38$ ms^{-1}

Example 7.14. A **bottle dropped from a balloon reaches the ground in 20 s. Determine the height of the balloon if (*i*) it was at rest in the air (*ii*) it was ascending with a speed of 50 ms^{-1}, when the bottle was dropped.**

Solution. (*i*) $h = v_0 t + \frac{1}{2} gt^2$

Taking upward motion as positive, $g = -9.8\ \text{ms}^{-2}$

$\therefore \quad h = 0 \times t - \frac{1}{2} \times 9.8 \times (20)^2 = -1960\ \text{m} \qquad \therefore |h| = 1960\ \text{m}$

(*ii*) Taking upward motion as positive, $g = -9.8\ \text{ms}^{-2}$

$\therefore \quad h = v_0 t + \frac{1}{2} gt^2 = 50 \times 20 + \frac{1}{2} \times (-9.8) \times (20)^2 = -960\ \text{m}$

$\therefore \quad |h| = 960\ \text{m}$

Example 7.15. **A stone is dropped from the top of a tower 400 m high and at the same time another stone is projected upward vertically from the ground with a velocity of 100 ms^{-1}. Find when and where the two will meet.**

Solution. Suppose the two stones meet after t seconds when the stone from the top of the tower has covered a distance of h. Then the distance of the meeting point from the ground is $(400 - h)$ as shown in Fig. 7.15. Take the downward motion positive.

For downward motion, $g = +9.8\ \text{ms}^{-2}; v_0 = 0.$

$\therefore \quad h = v_0 t + \frac{1}{2} gt^2 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2$

or $\quad h = 4.9\ t^2 \qquad \ldots (i)$

For upward motion, $g = -9.8\ \text{ms}^{-2}$; $v_0 = 100\ \text{ms}^{-1}$.

$\therefore \quad 400 - h = v_0 t + \frac{1}{2} gt^2 = 100 \times t + \frac{1}{2}(-9.8)\ t^2$

or $\quad 400 - h = 100\ t - 4.9\ t^2$

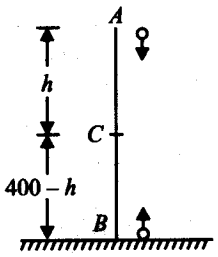


Fig. 7.15 ... (*ii*)

Adding eqs. (*i*) and (*ii*), we have,

$$400 = 100\ t \quad \therefore \quad t = 400/100 = 4\ \text{s}$$

Putting $t = 4$ s in eq. (*i*), $h = 4.9 \times (4)^2 = 78.4$ m.

Therefore, the two stones meet 78.4 m below the top of tower after 4 s.

Example 7.16. **A body is dropped from rest at a height of 150 m and simultaneously another body is dropped from rest from a point 100 m above the ground. What is the difference between their heights after they have fallen for 3 seconds? Take g = 10 ms^{-2}.**

Solution. Since initial velocity in the two cases is zero, $h = \frac{1}{2} gt^2$.

Downward distance covered by *first body* in 3 seconds

$$= \frac{1}{2} \times 10 \times (3)^2 = 45\ \text{m}$$

$\therefore$ Height of first body above the ground = 150 – 45 = 105 m

Downward distance covered by *second body* in 3 seconds

$$= \frac{1}{2} \times 10 \times (3)^2 = 45\ \text{m}$$

$\therefore$ Height of second body above the ground = 100 – 45 = 55 m

$\therefore$ Difference in height between the two after 3 seconds = 105 – 55 = 50 m

PROBLEMS FOR PRACTICE

1. A train starting from a station is uniformly gaining speed until after 5 minutes it acquires the maximum uniform speed of 80 km/h. Find the average velocity and the distance travelled during variable state of motion. **[40 km/h ; 3.33 km]**
2. A car travels with a uniform velocity of 72 km/h. The driver applies the brakes and the car comes to rest with a uniform retardation in 10 seconds. Find (*i*) the retardation (*ii*) velocity of the car after 3 seconds and (*iii*) the distance car travels after the brakes are applied. **[(*i*) –2 m s^{-2} (*ii*) 14 m s^{-1} (*iii*) 100 m]**
3. A car travels at a uniform velocity of 20 ms^{-1} for 5 seconds. The brakes are then applied and the car comes to rest with uniform retardation in further 8 seconds. Determine (*i*) the distance travelled in first 5 seconds (*ii*) retardation in the last 8 seconds (*iii*) distance travelled by the car after the brakes are applied and (*iv*) total distance travelled. **[(*i*) 100 m (*ii*) –2.5 m s^{-2} (*iii*) 80 m (*iv*) 180 m]**
4. A motorcycle starts from rest and maintains a constant acceleration of 3 ms^{-2} for 6 seconds and then a constant acceleration of 2 ms^{-2} for another 6 seconds. Find the acquired velocity and total distance travelled. **[30 ms^{-1}; 216 m]**
5. A body has a velocity of 5 ms^{-1} and a uniform acceleration of 4 ms^{-2}. Find the distance travelled in the 11th second of its motion. **[47 m]**
6. A body starts from rest and slides down an inclined plane (assumed frictionless) of length 15 m in 15 s. Find its acceleration and its speed at the bottom of the plane. **[0.113 ms^{-2}; 1.995 ms^{-1}]**
7. A body travels 200 cm in the first two seconds and 220 cm in the next four seconds. What will be the velocity at the end of seventh second from the start ? **[10 cm s^{-1}]**
8. An electron travelling with a speed 75 × 10^{13} ms^{-1} passes through an electric field with an acceleration of 10^{12} ms^{-2}. How long will it take for the electron to double its speed ? **[5 × 10^{-9} s]**
9. A boy drops a stone into a well. If the stone takes 2.5 s to hit the water, find (*i*) depth of the well (*ii*) the speed with which the stone hits the water. **[(*i*) 30.63 m (*ii*) 24.5 ms^{-1}]**
10. A boy standing near the edge of a cliff 125 m above a river throws a stone downward with a speed of 10 ms^{-1}. Find (*i*) with what speed will the stone hit water and (*ii*) how long will it take to descend ? **[(*i*) 50.5 m s^{-1} (*ii*) 4.13 s]**
11. A ball is thrown vertically upward from the ground with a velocity of 39.2 ms^{-1}. Calculate (*i*) the maximum height to which the ball rises and (*ii*) the time taken by the ball to reach the highest point. **[(*i*) 78.4 m (*ii*) 4 s]**
12. A stone is dropped from the top of a building 200 m high and at the same time another stone is projected vertically upward from the ground with a velocity of 50 ms^{-1}. Find where and when the two stones will meet. **[after 4 s at a height of 121.6 m from ground]**
13. A ball thrown vertically upward reaches a height of 80 m. Calculate (*i*) the time to reach the highest point (*ii*) the speed of the ball upon arrival on the ground. **[(*i*) 4.04 s (*ii*) 39.59 m s^{-1}]**

7.7. EQUATIONS FOR UNIFORMLY ACCELERATED LINEAR MOTION (CALCULUS METHOD)

In the earlier sections, we derived the equations for uniformly accelerated motion in one dimension by making use of algebra or graphical analysis. Now we shall derive these equations directly from definitions with the help of integral calculus.

(*i*) Velocity-Time relation. The instantaneous acceleration of a moving object is given by ;

$$a = \frac{dv}{dt}$$

or $$dv = a\,dt$$

When $t = 0$; $v = v_0$ and when $t = t$, $v = v$.

We integrate velocity from the initial value v_0 to the final value v, the corresponding limits on the time being 0 and t.

$$\therefore \quad \int_{v_0}^{v} dv = \int_{0}^{t} a\,dt$$

Since acceleration a is constant, it can be taken out of the integral sign.

$$\therefore \quad \int_{v_0}^{v} dv = a\int_{0}^{t} dt$$

or $$|v|_{v_0}^{v} = a\,|t|_0^t$$

or $$v - v_0 = at$$

$$\therefore \quad v = v_0 + a\,t \qquad ...(i)$$

(*ii*) Position-Time relation. $v = v_0 + a\,t$

or $$\frac{dx}{dt} = v_0 + at$$

or $$dx = v_0\,dt + at\,dt$$

When $t = 0$; $x = x_0$ and when $t = t$; $x = x$.

Integrating the above equation for dx through proper limits, we get,

$$\int_{x_0}^{x} dx = \int_{0}^{t} v_0 dt + \int_{0}^{t} at\,dt$$

Since v_0 (= initial velocity) and acceleration a are constant, they can be taken out of the integral sign.

$$\therefore \quad \int_{x_0}^{x} dx = v_0\int_{0}^{t} dt + a\int_{0}^{t} t\,dt$$

or $$|x|_{x_0}^{x} = v_0\,|t|_0^t + a\left|\frac{t^2}{2}\right|_0^t$$

$$\therefore \quad x - x_0 = v_0 t + \frac{1}{2}at^2$$

or $$x = x_0 + v_0\,t + \frac{1}{2}a\,t^2 \qquad ...(ii)$$

If $x - x_0 = S$, the displacement of the object in time t, then eq. (*ii*) can be written as :

$$S = v_0 t + \frac{1}{2}\,a\,t^2$$

(*iii*) Velocity-Displacement relation.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$

or $$a = \frac{dv}{dx}v \qquad \left(\because \frac{dx}{dt} = v\right)$$

or $$a\,dx = v\,dv$$

When $x = x_0$, $v = v_0$ and when $x = x$, $v = v$.

$\therefore$ Integrating above equation with proper limits, we have,

$$\int_{x_0}^{x} a\,dx = \int_{v_0}^{v} v\,dv$$

Since acceleration a is constant, it can be taken out of the integral sign.

$$\therefore \quad a\int_{x_0}^{x} dx = \int_{v_o}^{v} v\,dv \quad \text{or} \quad a\,|x|_{x_0}^{x} = \left|\frac{v^2}{2}\right|_{v_0}^{v}$$

or $$a(x - x_0) = \frac{v^2}{2} - \frac{v_0^2}{2} \quad \therefore \quad v^2 - v_0^2 = 2a(x - x_0) \qquad ...(iii)$$

If $x - x_0 = S$, the displacement of the object in time t, then eq. (*iii*) can be written as :

$$v^2 - v_0^2 = 2\,a\,S$$

Note. We can verify the correctness of these equations of uniformly accelerated linear motion by differential calculus.

$$x = x_0 + v_0 t + \frac{1}{2}a\,t^2$$

(*i*) $$v = \frac{dx}{dt} = \frac{d}{dt}\left(x_0 + v_0 t + \frac{1}{2}at^2\right) = v_0 + at$$

$\therefore$ $$v = v_0 + a\,t$$

(*ii*) $$a = \frac{dv}{dt} = \frac{d}{dt}(v_0 + at) = a = \text{constant}$$

The reader may note the important role of mathematics in physics. We derived these equations directly from definitions with the help of integral calculus and then proved their correctness by differential calculus.

CONCEPTUAL QUESTIONS

Q.1. **When is an object said to be accelerating?**

Ans. Whenever the velocity (magnitude or direction or both) of an object changes, we say that the object is accelerating. However, an object moving at a constant speed in a straight line is not accelerating because its velocity is not changing *i.e.*, magnitude as well as direction of velocity is the same.

Q.2. **What do you mean by uniformly accelerated linear motion?**

Ans. Uniformly accelerated linear motion means that the magnitude of the acceleration is constant and the motion is in a straight line.

Q.3. **Give one most common example of uniformly accelerated linear motion.**

Ans. One of the most common and practically important examples of uniformly accelerated linear motion is the vertical movement of an object in the gravitational field of the

earth. The effect of gravitational attraction is to impart all unsupported bodies an equal downward acceleration of 9.8 ms^{-2} (neglecting the retarding effect of air).

Q.4. We say that acceleration due to gravity is constant only if we ignore air resistance. For what situations can air resistance be ignored?

Ans. The answer to this question depends upon several factors :

(*i*) If you do not require greater accuracy, you can ignore air resistance.

(*ii*) If the body is moving through air at a moderate speed, air resistance can be ignored. It is because the faster an object moves, the more air resistance affects its acceleration.

(*iii*) If the body is heavy and/or streamlined, you can ignore air resistance. It is because shape and surface area of an object greatly affect air resistance. Air resistance is much more important for a light object, such as a ping-pong ball, than for a heavier object of the same size, such as a golf ball.

Q.5. Is the value of g positive or negative?

Ans. The acceleration due to gravity always acts vertically downward whether the body is rising or falling. The value of g will be negative or positive depending upon whether we choose upward direction positive or negative. If we choose upward direction positive, g, being directed downward, is $-9.8\ ms^{-2}$. If we choose downward direction positive, then g is $+9.8\ ms^{-2}$.

Q.6. If a body moves with constant acceleration, what is the average velocity?

Ans. If a body moves with constant acceleration, the velocity-time graph is a straight line: the slope of graph = constant acceleration. Therefore, the average velocity lies midway between the initial velocity (v_0) and the final velocity (v) *i.e.*,

$$\bar{v} = \frac{v_0 + v}{2}$$

Note that it is true *only* for constant acceleration.

Q.7. What is a freely falling body?

Ans. When an object is under the action of gravity alone, it is called a freely falling body. The presence of other forces, in particular air resistance, may alter dramatically the statement that all objects fall with the same constant acceleration g (= 9.8 m s^{-2}).

Q.8. The position of a stone dropped from rest from a cliff is given by $x = 4.9\ t^2$ where x is in metres and t is in seconds. What is the acceleration of the stone?

Ans. $$x = 4.9\ t^2 \quad \text{or} \quad \frac{dx}{dt} = 2(4.9)t = 9.8t$$

or $$\frac{d^2x}{dt^2} = 9.8\ ms^{-2} \quad \text{or} \quad a = 9.8\ ms^{-2}$$

We arrive at an important conclusion that if the displacement of a body varies as the square of elapsed time, the acceleration of the body is constant.

Q.9. In one dimensional motion, can a particle have zero speed at an instant but non-zero acceleration at that instant?

Ans. The answer is yes. For example, when a body starts falling freely, its speed is zero but acceleration is $9.8\ ms^{-2}$.

Q.10. In the position-time ($x - t$) graph shown in Fig. 7.16, what is the acceleration?

Ans. The $x - t$ graph is a straight line. Therefore, slope is constant. This means that velocity of the body is constant. Therefore, acceleration is zero.

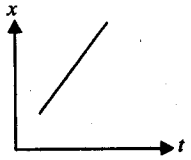


Fig. 7.16

Q.11. The distance travelled by a body is found to be directly proportional to time. Is the body moving with uniform velocity?

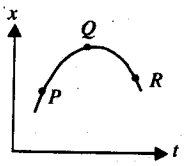


Fig. 7.17

Ans. $s \propto t$ or $s = kt$ $\therefore$ $v = \dfrac{ds}{dt} = k$ (constant)

Therefore, the body is moving with uniform velocity.

Q.12. Fig. 7.17 shows $x - t$ graph for a moving body. At what point velocity is zero?

Ans. The velocity at any point on the graph is the slope of the tangent line at that point. The slope of the tangent line at Q is zero. Therefore, velocity is zero at point Q.

VERY SHORT ANSWER QUESTIONS

Q.1. A ball is thrown straight up. What is its velocity and acceleration at the top?

Ans. Velocity at top = 0 ; Acceleration at top = 9.8 ms^{-2} downwards.

Q.2. Can a body have zero velocity and still be accelerating?

Ans. Yes. In case of a body projected vertically upwards, the velocity at the highest point is zero but acceleration is g (= 9.8 ms^{-2}) downwards.

Q.3. A body is thrown vertically upwards with a velocity of 30 m/s. What is its velocity when it hits the ground?

Ans. 30 m/s because the velocity with which the body hits the ground is equal to the velocity with which it is thrown up. The reason for this is that acceleration due to gravity g(= 9.8 ms^{-2}) remains constant during upward or downward journey.

Q.4. Two balls of different masses (one lighter and other heavier) are thrown vertically upward with the same initial speed. Which one will rise to greater height?

Ans. Both the balls will rise to the same height. It is because in the formula $v^2 - v_0^2 = 2gh$, no mass term is involved.

Q.5. What does the area under acceleration-time graph for any time interval represent?

Ans. Area under $a - t$ graph = $\dfrac{1}{2}at = \dfrac{1}{2} \times \dfrac{v - v_0}{t} \times t = \dfrac{v - v_0}{2}$.

Therefore, area under $a - t$ graph during time initerval t represents change in velocity during that time interval.

Q.6. If the acceleration of a particle is constant in magnitude but not in direction, what type of path does the particle follow?

Ans. It will follow a non-linear path.

Q.7. A ball is thrown vertically upwards and takes 5 s to reach the highest point. What is the time for downward journey?

Ans. 5 s because the value of g (= 9.8 ms^{-2} downwards) is constant for upward and downward journey.

Q.8. A man standing on the edge of a cliff throws a stone straight up with initial velocity u and then throws another stone straight down with same initial speed u and from the same position. Find the ratio of the speeds, the two stones would have attained when they hit ground at the base of the cliff.

Ans. When the stone is thrown up with speed u, it will return to the thrower with speed u. Since both stones are falling under gravity with the same initial speed u, they will hit the ground with the same speed. Hence the ratio of the speeds of the two stones will be 1 : 1.

Q.9. A stone is thrown vertically upward from the ground at 30 ms^{-1}. What is the time of flight?

Ans. t = 6.1s (Refer to Example 7.10).

Q.10. The initial and final velocities of a uniformly accelerated body are 20 ms^{-1} and 60 ms^{-1}. What is the average velocity?

Ans. Average velocity = $\dfrac{\text{Initial vel.} + \text{Final vel.}}{2} = \dfrac{20 + 60}{2} = 40\ ms^{-1}$

Q.11. How will you find the distance covered by a uniformly accelerated body from its $v - t$ graph?

Ans. The distance covered by a body in a given interval of time is equal to the *total area* (positive and negative) under v–t graph during that time interval. Note that negative area (due to negative velocity) is also taken as positive.

Note. The displacement of a body during a given time interval is the *net area* (with due regard to sign to area) under v–t graph during that interval.

Q.12. A car starting from rest has an aceleration of 5 ms^{-2}. How fast will it be after 10 s?

Ans. $v = v_0 + at = 0 + 5 \times 10 = 50\ ms^{-1}$

Q.13. A body is thrown up vertically with velocity u. How high will it rise?

Ans. $v^2 - u^2 = 2gh$ or $0^2 - u^2 = 2(-g)h \quad \therefore\ h = u^2/2g.$

Q.14. Velocity and acceleration are interlinked but is it possible that one is zero while other is not?

Ans. Yes it is possible. A body thrown vertically upward has zero velocity at the highest point but it has acceleration g (= 9.8 ms^{-2} downwards). On the other hand, acceleration is zero for motion with uniform velocity.

Q.15. A body is thrown vertically upward from the surface of earth. What is the direction of acceleration of the body?

Ans. The acceleration is g (= 9.8 ms^{-2}) acting vertically downward whether the body is going upward or coming downward.

Q.16. What is the unique characteristic of uniformly accelerated motion?

Ans. In case of uniformly accelerated motion (*and only for this case*), the average velocity is equal to one-half of sum of initial and final velocities.

Q.17. What is the ratio of the distances travelled by a body falling freely from rest during first, second and third second of its fall?

Ans. $S_{nth} = v_0 + \frac{a}{2}(2n-1) = 0 + \frac{g}{2}(2n-1) = \frac{g}{2}(2n-1)$

$$\therefore\ S_1 : S_2 : S_3 = \frac{g}{2}(2\times1-1) : \frac{g}{2}(2\times2-1) : \frac{g}{2}(2\times3-1) = 1 : 3 : 5$$

Q.18. Can you cite an example where a body subjected to uniform acceleration does not move in a straight line?

Ans. A body subjected to uniform acceleration in one dimensional motion moves in a straight line. However, the projectile is subjected to uniform acceleration and has a parabolic path (motion in a plane).

Q.19. What is the acceleration of a body when its $v - t$ graph is (*i*) perpendicular to time-axis (*ii*) parallel to time-axis?

Ans. (*i*) Inifinity (*ii*) zero.

Q.20. A stone is thrown vertically upward with an initial velocity of 14 ms^{-1}. What is the maximum height reached? (g = 9.8 ms^{-2})

Ans. $h = v_0^2/2g = 14^2/2 \times 9.8 = 10$ m.

SHORT ANSWER QUESTIONS

Q.1. What are the three main characteristics of a freely falling body?

Ans. A body that moves under gravity alone is called a freely falling body.

(*i*) Time taken for upward journey is equal to that for downward journey.

(*ii*) The speed with which body is thrown upward, it hits the ground with the same speed.

(*iii*) The speed of the body at any point in its upward and downward journey is the same.

Q.2. A ball is thrown vertically upwards. Draw its height-time and velocity-time graph.

Ans. The $h - t$ graph for the motion is a parabola and is shown in Fig. 7.18 (*i*). The $v - t$ graph for the motion is a straight line [See Fig. 7.18 (*ii*)] because acceleration is constant during the entire motion of the body.

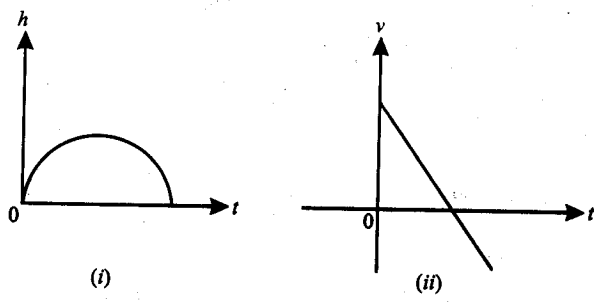


Fig. 7.18

Q.3. A ball is thrown vertically upward from the ground at 30 ms^{-1}. What is the time of flight?

Ans. The displacement $h = 0$ when the ball strikes the ground.

Now $h = v_0 t + \frac{1}{2} g t^2$ or $0 = 30 \times t - \frac{1}{2} \times 9.8\, t^2$ $\therefore$ $t = 6.1$ s

Q.4. Two balls of different masses (one lighter and other heavier) are thrown vertically upwards with the same speed. Which one will pass through the point of projection in their downward journey with greater speed?

Ans. Suppose v_0 is the velocity of projection of the ball and v is the velocity of the ball while passing downward through the point of projection. Since displacement $h = 0$,

$\therefore$ $v^2 - v_0^2 = 2gh$ or $v^2 - v_0^2 = 2g \times 0$ $\therefore$ $v = v_0$

Hence both the balls will have the same speed while passing through the point of projection in their downward journey. Alternatively, there is no mass term in the equation of motion.

Q.5. For a particle in one dimensional motion having uniform acceleration, the instantaneous speed is always equal to the magnitude of instantaneous velocity. Why?

Ans. In one dimensional motion having uniform acceleration, the direction of velocity does not change; only the magnitude of velocity changes (*i.e.* speed). Therefore, the instantaneous speed is always equal to the magnitude of instantaneous velocity of the particle in one dimensional (*i.e.*, in a straight line) motion.

Q.6. What will be the distance moved by a freely falling body from rest in the 5th second of its journey?

Ans. $S_{nth} = v_0 + \frac{a}{2}(2n-1)$. Here $v_0 = 0$; $a = 9.8 ms^{-2}$ and $n = 5$.

$\therefore$ $S_5 = 0 + \frac{9.8}{2}(2 \times 5 - 1) = 4.9\ (9) = 44.1$ m

Q.7. What is the magnitude and direction of acceleration due to gravity?

Ans. A freely falling body always has an acceleration of 9.8 ms^{-2} downward. It does not matter whether the body is rising or falling ; its acceleration is still g (= 9.8 ms^{-2}) downward.

Q.8. The position of a particle moving along a straight line is given by : $x = 2 - 5t + 6t^2$. What is the acceleration of the particle at $t = 2$s ?

Ans. $a = 12$ (Refer to Example 6.22).

Q.9. The acceleration due to gravity is 9.8 ms^{-2}. What does it mean?

Ans. It means that when the motion of a body is under gravity *alone*, its speed decreases at the rate of 9.8 ms^{-1} every second during upward motion. During downward journey, the speed of the body increases at the rate of 9.8 ms^{-1} every second until it hits the ground.

Q.10. A body starts from rest and has a uniform acceleration a. What will be the shape of its (*i*) $x - t$ graph (*ii*) $v - t$ graph (*iii*) $a - t$ graph?

Ans. (*i*) Parabola (*ii*) It will be a straight line passing through the origin (*iii*) It will be a straight line parallel to the time-axis.

Q.11. **A body is dropped from a height h. In how much time will it reach the ground?**

Ans. $h = v_0 t + \frac{1}{2}at^2$. Here $v_0 = 0$; $a = g$.

$\therefore h = 0 \times t + \frac{1}{2}gt^2$ or $t = \sqrt{\frac{2h}{g}}$

Q.12. **The velocity of a particle in ms^{-1} moving along a straight line is given by : $v = 10 + 3t^2$. What is instantaneous acceleration at t = 4s ?**

Ans. $a = \frac{dv}{dt} = \frac{d}{dt}(10 + 3t^2) = 6t$. Therefore, at t = 4s ; $a = 6 \times 4 = 24$ ms^{-2}.

Q.13. **Why does time occur twice in the unit of acceleration?**

Ans. Acceleration = $\frac{\text{Change in velocity}}{\text{Time}} = \frac{\text{Displacement/Time}}{\text{Time}} = \frac{\text{Displacement}}{(\text{Time})^2}$

Q.14. **A food packet is released from a helicopter which is rising steadily at 2 ms^{-1}. What is the velocity of the packet after 2 seconds? (Take g = 10 ms^{-2}).**

Ans. $v = v_0 + gt$. Taking upward direction positive, we have, $v_0 = 2$ ms^{-1}; $g = -10$ ms^{-2}. Therefore, $v = 2 - 10 \times 2 = -18$ ms^{-1}. Hence the packet is falling downward with a velocity of 18 ms^{-1} after 2 s.

Q.15. **Can you round a curve with zero acceleration?**

Ans. No. It is because when you round a curve, the direction of motion is changing from instant to instant. This is possible only by a force and hence by an acceleration.

Q.16. **Is the rate of change of acceleration with time important to mechanics? Comment.**

Ans. The rate of change of acceleration with time is not important to mechanics. It is because the laws of motion involve acceleration only and to solve problems relating to motion, we do not require rate of change of acceleration with time.

Q.17. **A car travelling at 36 km/h takes a U turn in 5 seconds. What is the acceleration of the car?**

Ans. Change in velocity in 5 seconds, $\Delta v = 36 - (-36) = 72$ km/h = 20 ms^{-1}

$\therefore$ Acceleration of car, $a = \Delta v/(\Delta t) = 20/5 = 4$ ms^{-2}.

LONG ANSWER QUESTIONS

Q.1. Explain the terms (*i*) uniform acceleration (*ii*) variable acceleration and (*iii*) average acceleration. **[Refer to Art. 7.1]**

Q.2. Show that in case of uniform acceleration, the average velocity is equal to one-half the sum of initial and final velocities. **[Refer to Art. 7.2]**

Q.3. For uniformly linear accelerated motion, draw and explain (*i*) position-time graph and (*ii*) velocity-time graph. **[Refer to Arts. 7.5 and 7.4]**

Q.4. Derive graphically the following equations for uniformly accelerated linear motion :

(*i*) $v = v_0 + at$ (*ii*) $S = v_0 t + \frac{1}{2}at^2$ (*iii*) $v^2 - v_0^2 = 2aS$ **[Refer to Art. 7.4]**

Q.5. Deduce the above equations of motion by calculus method. **[Refer to Art. 7.7]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. One dimensional motion with uniform acceleration means that magnitude as well as direction of acceleration remains the same.
2. The *equations of kinematics* for a particle moving along x-axis (one dimensional motion) with uniform acceleration 'a' (constant in magnitude and direction) are :

$(i)\ x - x_0 = \left(\frac{v_0 + v}{2}\right)t$

$(ii)\ v = v_0 + at$

$(iii)\ x = x_0 + v_0 t + \frac{1}{2}at^2$

$(iv)\ v^2 - v_0^2 = 2a(x - x_0)$

v_0 = initial velocity of particle
v = final velocity of particle in time t
$\bar{v}$ = average velocity of particle
x_0 = initial displacement of particle from origin *i.e.* at $t = 0$
x = final displacement of the particle from origin in time t
a = constant acceleration

3. The displacement of a particle in the *nth* second during uniformly accelerated motion in one dimension is

$$S_{nth} = v_0 + \frac{a}{2}(2n-1)$$

4. Objects that move under gravity alone are called *freely falling bodies. A freely falling body always has an acceleration of 9.8 m/s^2 downward.*
5. The motion of a freely falling body (neglecting air resistance) is equivalent to motion in one dimension with constant acceleration. Therefore, kinematic equations for constant acceleration can be applied.
6. While solving problems regarding freely falling bodies, it is very important to adopt one sign convention for displacement. We shall take the vertical displacement h positive upward. Since positive h is upward, the acceleration is negative and given by $a = -g$. *The negative sign simply indicates that the acceleration is downward.* It does not imply that the velocity of particle is increasing or decreasing. With these substitutions, we get the following equations for freely falling bodies :

$(i)\ h = \frac{(v_0 + v)}{2}t$ where $h = x - x_0$ $(ii)\ v = v_0 - g\,t$ $(iii)\ h = v_0 t - \frac{1}{2}gt^2$

$(iv)\ v^2 - v_0^2 = -2gh$ $(v)\ h_{nth} = v_0 - \frac{g}{2}(2n-1)$

You should note that *negative sign for acceleration is already included in these expressions.*

7. For upward journey, v_0, v and h are positive and equations containing 'g' term become:

$(i)\ v = v_0 - gt$ $(ii)\ h = v_0 t - \frac{1}{2}gt^2$ $(iii)\ v^2 - v_0^2 = -2gh$ $(iv)\ h_{nth} = v_0 - \frac{g}{2}(2n-1)$

For downward journey v_0, v and h are negative and the equations containing 'g' term become:

$$-v = -v_0 - g\,t \quad \text{or} \quad v = v_o + g\,t \quad \ldots (i)$$

$$-h = -v_0\,t - \frac{1}{2}gt^2 \quad \text{or} \quad h = v_o t + \frac{1}{2}g\,t^2 \quad \ldots (ii)$$

$$(-v)^2 - (-v_0)^2 = -2g(-h) \quad \text{or} \quad v^2 - v_0^2 = 2gh \quad \ldots (iii)$$

$$h_{nth} = v_0 + \frac{g}{2}(2n-1) \quad \ldots (iv)$$

8. Consider the case of a particle thrown vertically upward from the origin with velocity v_o. In this case, v_o is positive, $x_0 = 0$ (earth surface). Fig. 7.19 (*i*) shows displacement-time graph and Fig. 7.19 (*ii*) shows velocity-time graph for this motion. The following points are worth noting :

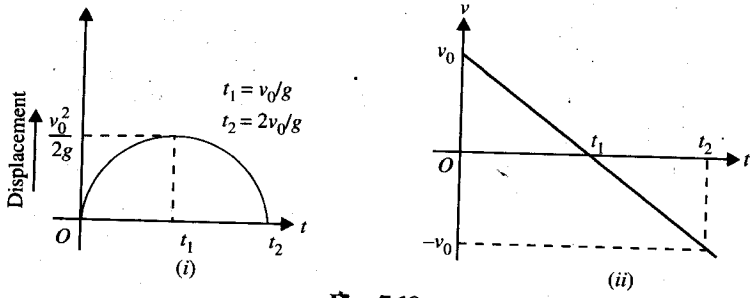


Fig. 7.19

(*i*) The velocity is initially positive but decreases with time and becomes zero at the peak of the path. This will occur at time t_1 given by;

$$0 = v_0 - gt_1 \quad \text{or} \quad t_1 = v_0/g \quad \text{... time for max. vertical displacement}$$

At this time, the vertical displacement is maximum and is given by ;

$$(0)^2 - (v_0)^2 = -2gh_{max} \therefore h_{max} = v_0^2/2g \quad \text{.... max. displacement}$$

(*ii*) At time $t_2 = 2t_1 = 2v_0/g$, the displacement is again zero *i.e.* the particle has returned to the starting point.

$$\therefore \quad t_2 = 2v_0/g \quad \text{... total journey time}$$

(*iii*) At time t_2, the velocity $v = -v_0$ as proved below :

$$v = v_0 - gt_2 = v_0 - \frac{g \times 2v_0}{g} = -v_0$$

Note that there is symmetry in motion. In other words, both the displacement and the magnitude of the velocity repeat themselves in the time interval $t = 0$ to $t = 2v_0/g$.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A body freely falling from rest has a velocity v after it falls through a height h. The distance it has to fall further for its velocity to become double is **[Pb PMT 2001]**
   (*a*) 4 h (*b*) 6 h
   (*c*) 8 h (*d*) 10 h

2. The velocity of a bullet is reduced from 200 m/s to 100 m/s while travelling through a wooden block of thickness 10 cm. The retardation, assuming it to be uniform, will be **[AIIMS 2001]**
   (*a*) 10×10^4 m/s^2 (*b*) 1.2×10^4 m/s^2
   (*c*) 13.5×10^4 m/s^2 (*d*) 15×10^4 m/s^2

3. A car moving with a velocity of 40 km/hr can be stopped by applying brakes in 2 m. If the same car is moving with a speed of 80 km / hr, what is the minimum stopping distance? **[AFMC - Pune 2000]**
   (*a*) 16 m (*b*) 12 m
   (*c*) 8 m (*d*) 4 m

4. A particle is thrown vertically upward. Its velocity at half of the height is 10 m/s, then maximum height attained by it is (g = 10 m/s^2) **[CBSE PMT 2001]**
   (*a*) 8 m (*b*) 20 m
   (*c*) 10 m (*d*) 16 m

5. A body A starts from rest with an acceleration a_1. After 2 seconds, another body B starts from rest with an acceleration a_2. If they travel equal distances in the fifth second after the start of A, then ratio of $a_1 : a_2$ is equal to **[AIIMS 2001]**
   (*a*) 5 : 9 (*b*) 5 : 7
   (*c*) 9 : 5 (*d*) 9 : 7

6. A stone is thrown up with an initial speed of 4.9 m/s from a bridge in vertically upward

direction. It falls down in water after 2 sec. The height of the bridge is

[AFMC - Pune 1999]

(*a*) 4.9 m (*b*) 9.8 m

(*c*) 19.8 m (*d*) 24.7 m

7. If a freely falling body in the last second travels a distance equal to the distance travelled in the first three seconds, the time of its travel is

[Pb PMT 1998]

(*a*) 3 sec (*b*) 4 sec

(*c*) 5 sec (*d*) 6 sec

8. A body dropped from a height h strikes the ground with a velocity of 3 m/s. Another body of same mass is dropped from the same height h with an initial velocity of 4 m/s. Final velocity of second mass with which it strikes the ground will be **[AFMC - Pune 1998]**

(*a*) 3 m/s (*b*) 4 m/s

(*c*) 8 m/s (*d*) 5 m/s

9. The acceleration of a particle is increasing linearly with time t as bt. The particle starts from origin with an initial velocity v_0. The distance travelled by the particle in time t will be

[CBSE PMT 1995]

(*a*) $v_0t+\frac{1}{6}bt^3$ (*b*) $v_0t+\frac{1}{3}bt^3$

(*c*) $v_0t+\frac{1}{3}bt^2$ (*d*) $v_0t+\frac{1}{2}bt^2$

10. If the water falls from a dam into a turbine wheel 19.6 m below, then the velocity of water at the turbine is (Take g = 9.8 m/s^2)

[AIIMS 1998]

(*a*) 9.8 m/s (*b*) 19.6 m/s

(*c*) 39.2 m/s (*d*) 98 m/s

11. A car is moving along a straight road with a uniform acceleration. It passes through two points P and Q separated by some distance with velocities 30 km/h and 40 km/h. The velocity of car midway between P and Q is

[CBSE PMT 1993]

(*a*) 33.3 km/h (*b*) $20\sqrt{3}$ km/h

(*c*) $25\sqrt{2}$ km/h (*d*) 35 km/h

12. If a ball is thrown vertically upwards with speed u, the distance covered during the last t seconds of its ascent is **[CBSE 2003]**

(*a*) $ut-\frac{1}{2}gt^2$ (*b*) $(u+gt)t$

(*c*) ut (*d*) $\frac{1}{2}gt^2$

13. A man throws balls with the same speed vertically upwards one after the other at an interval of 2 seconds. What should be the speed of the throw so that more than two balls are in the sky at any time? (Given g = 9.8 ms^{-2})

[CBSE 2003]

(*a*) Any speed less than 19.6 m/s

(*b*) Only with speed 19.6 m/s

(*c*) More than 19.6 m/s

(*d*) Atleast 9.8 m/s

14. A body moves for a total of nine seconds starting from rest with uniform acceleration and then with uniform retardation, which is twice the value of acceleration and then stops. The duration of uniform acceleration is :

[Manipal 2003]

(*a*) 3 s (*b*) 4.5 s

(*c*) 5 s (*d*) 6 s

15. A body initially at rest is moving with uniform acceleration a. Its velocity after n seconds is v. The displacement of the body in last 2 s is :

[Manipal 2005]

(*a*) $\frac{2v(n-1)}{n}$ (*b*) $\frac{v(n-1)}{n}$

(*c*) $\frac{v(n+1)}{n}$ (*d*) $\frac{2v(n+1)}{n}$

16. When a ball is thrown up vertically, with velocity v_0, it reaches a maximum height of h. If one wishes to triple the maximum height, then the ball should be thrown with a velocity of **[AIIMS 2005]**

(*a*) $3v_0$ (*b*) $9v_0$

(*c*) $\sqrt{3}\,v_0$ (*d*) $\frac{3}{2}v_0$

17. An earthquake generates both transverse (S) and longitudinal (P) sound waves in the earth. The speed of S waves is about 4.5 km/s and that of P waves is about 8.0 km/s. A seismograph records P and S waves from the earthquake. First P wave arrives 4.0 min. before the first S wave. Epicentre of the earthquake is located at a distance of about

[AIIMS 2003]

(*a*) 25 km (*b*) 250 km

(*c*) 2500 km (*d*) 5000 km

18. A ball is thrown vertically upwards. Which of the following plot represents the velocity-time

graph of the ball during its flight, if the air resistance is not ignored? **[AIIMS 2003]**

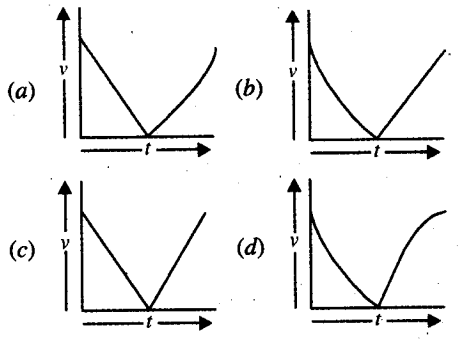


Fig. 7.20

19. A car accelerates from rest at constant rate for first 10 s and covers a distance x. It covers a distance y in next 10 s at the same acceleration. Which of the following is true? **[AFMC 2005]**

(a) $x = 3y$ (b) $y = 3x$

(c) $x = y$ (d) $y = 2x$

20. A coin is dropped in a lift. It takes time t_1 to reach the floor when lift is stationary. It takes time t_2 when lift is moving up with constant acceleration. Then : **[AFMC 2005]**

(a) $t_1 > t_2$ (b) $t_2 > t_1$

(c) $t_1 = t_2$ (d) $t_1 >> t_2$

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (a)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (a)
6. (b)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (b)
11. (b)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (a)
16. (c)	**17.** (c)	**18.** (d)	**19.** (d)	**20.** (a)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. $v^2 = v_0^2 + 2gh$ or $v^2 = 0 + 2gh$ $\therefore v^2 \propto h$

$\therefore \left(\frac{v_1}{v_2}\right)^2 = \frac{h_1}{h_2}$ or $\left(\frac{v}{2v}\right)^2 = \frac{h}{h_2}$ $\therefore h_2 = 4h$

2. Initial velocity, v_0 = 200 m/s ; Final velocity, v = 100 m/s

Distance travelled, S = 10 cm = 0.1 m

Now $v^2 - v_0^2 = 2aS$ $\therefore a = \frac{v^2 - v_0^2}{2S} = \frac{(100)^2 - (200)^2}{2 \times 0.1} = -15 \times 10^4$ m/s^2

The minus sign is due to retardation.

3. $v^2 = v_0^2 + 2as$ or $0 = v_0^2 + 2as$ $\therefore v_0^2 \propto s$

$\therefore \left(\frac{v_0}{v_0}\right)^2 = \frac{s_1}{s_2}$ or $\left(\frac{40}{80}\right)^2 = \frac{2}{s_2}$ $\therefore s_2 = 8$m

4. Suppose v_0 is the initial velocity. Let after travelling a distance s, the velocity of the particle be 10 m/s. At maximum height (*i.e* $2s$), $v = 0$.

$\therefore$ $0^2 = v_0^2 - 2g(2s)$ or $v_0^2 = 4gs = 4 \times 10\ s = 40\ s$

Also $(10)^2 = v_0^2 - 2 \times 10s$ or $100 = 40s - 20s$ $(\because v_0^2 = 40\ s)$

$\therefore$ $s = 5$ m

$\therefore$ Maximum height = $2s = 2 \times 5 = 10$ m

5. Distance travelled in nth second is

$$S_{nth} = v_0 + \frac{a}{2}(2n-1)$$

As per problem, distance covered by body A in the fifth second after start is equal to the distance covered by body B in the third second (5 – 2 = 3 sec) after start.

$$\therefore \quad 0+\frac{a_1}{2}(2\times5-1) = 0+\frac{a_2}{2}(2\times3-1)$$

or $$\frac{9a_1}{2} = \frac{5a_2}{2} \quad \text{or} \quad a_1 : a_2 = 5 : 9$$

6. The time t_1 taken by the stone to reach the highest point from the bridge is given by ;

$$v = v_0 - gt_1 \quad \text{or} \quad t_1 = \frac{v-v_0}{-g} = \frac{0-4.9}{-9.8} = 0.5 \text{ sec}$$

Therefore, total time of journey from the bridge and back to bridge = 0.5 + 0.5 = 1 sec. After reaching the bridge, the velocity is 4.9 m/s *i.e.* v_0 = 4.9 m/s, time to reach water, t = 2 – 1 = 1 sec.

$$\therefore \quad h = v_0t + \frac{1}{2}gt^2 = 4.9\times1 + \frac{1}{2}\times(9.8)\times(1)^2 = \mathbf{9.8\ m}$$

7. Distance travelled in first 3 seconds

$$= v_0t + \frac{1}{2}gt^2 = 0\times t + \frac{1}{2}\times9.8\times(3)^2 = 44.1 \text{ m}$$

Let t_1 be the total time of travel. Then,

44.1 = distance travelled in the last second

or $$44.1 = v_0 + \frac{1}{2}g(2t_1 - 1)$$

$$44.1 = 0 + \frac{1}{2}\times9.8(2t_1 - 1)$$

or $$t_1 = 5 \text{ sec}$$

8. For the first body, v_{01} = 0 ; v_1 = 3 m/s

Now $$v_1^2 - v_{01}^2 = 2gh$$

or $$(3)^2 - (0)^2 = 2 \times 9.8 \times h$$

$$\therefore \quad h = 0.46 \text{ m}$$

For the second body, v_{02} = 4 m/s ; v_2 = ?

Now $$v_2^2 - v_{02}^2 = 2gh$$

or $$v_2^2 - (4)^2 = 2\times9.8\times0.46$$

or $$v_2 = 5 \text{ m/s}$$

9. Acceleration, $a = bt$ or $\frac{dv}{dt} = bt$ or $v = \frac{bt^2}{2} + c$

At t = 0, $v = v_0$.

$$\therefore \quad v_0 = \frac{b\times(0)^2}{2} + c$$

$$\therefore \quad v = \frac{bt^2}{2} + v_0 \quad \text{or} \quad \frac{dS}{dt} = \frac{bt^2}{2} + v_0 \quad \text{or} \quad S = v_0t + \frac{bt^3}{6}$$

10. $$v_0 = 0 \ ; \ h = 19.6 \text{ m} \ ; \ v = ?$$

$$v^2 - v_0^2 = 2gh \text{ or } v^2 - (0)^2 = 2\times9.8\times19.6 \quad \therefore v = \mathbf{19.6\ m/s}$$

11. Since the acceleration is uniform, the velocity v of the car midway between P and Q is

$$v = \frac{2v_1v_2}{v_1+v_2} = \frac{2\times30\times40}{30+40} = 34.3 \text{ km/h} = 20\sqrt{3} \text{ km/h}$$

12. Let the body take T seconds to reach maximum height.

$\therefore$ $v = u - gT$ or $0 = u - gT$ $\therefore$ $T = u/g$

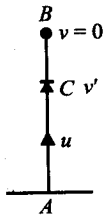


Fig. 7.21

Velocity attained by the body in $(T - t)$ sec. is

$$v' = u - g(T - t) = u - gT + gt$$

$$= u - g \times \frac{u}{g} + gt = gt$$

$\therefore$ Distance travelled in the last t sec. is

$$CB = v't - \frac{1}{2}gt^2 = gt^2 - \frac{1}{2}gt^2 = \frac{1}{2}gt^2$$

13. The time t taken by the ball to reach maximum height is

$$v = v_0 - gt \quad \text{or } 0 = v_0 - gt$$

$\therefore$ $v_0 = gt$

If t = 2 s, then $v_0 = 9.8 \times 2 = 19.6$ m/s. If the man throws the ball with a velocity of 19.6 m/s, then after 2 s, it attains a height of 19.6 m. When he throws the second ball, the first ball is at the top (= 19.6 m). When he throws the third ball, the first ball comes to the ground and the second ball will be at the top (= 19.6 m). Therefore, at any time, only two balls will be in air. If he wants to keep more than 2 balls in air, he should throw the ball with a speed greater than 19.6 m/s.

14. Total time, t = 9 s ; Acceleration = a ; Retardation = $-2a$

Let t_1 and t_2 be the time for acceleration and retardation respectively ($\therefore$ $t_1 + t_2 = 9$ s).

Now $v = v_0 + at_1$ or $v = 0 + at_1$, $\therefore$ $t_1 = v/a$

Also $0 = v - 2at_2$ or $t_2 = v/2a$

Now $t = t_1 + t_2$ or $9 = \frac{v}{a} + \frac{v}{2a}$ $\therefore$ $\frac{v}{a} = 6$

$\therefore$ Duration of acceleration, $t_1 = \frac{v}{a} = 6$ s

15. Displacement in last 2 seconds $= \frac{1}{2}an^2 - \frac{1}{2}a(n-2)^2 = \frac{1}{2}a\,[n^2 - (n-2)^2]$

$$= \frac{1}{2}a\,(4n-4) = 2a\,(n-1)$$

$$\text{Acceleration, } a = \frac{v-v_0}{t} = \frac{v-0}{n} = \frac{v}{n}$$

$\therefore$ Displacement in last 2s $= 2a(n-1) = 2 \times \frac{v}{n}(n-1) = \frac{2v(n-1)}{n}$

16. $v^2 - v_0^2 = -2gh$ or $(0)^2 - (v_0)^2 = -2\,gh$ $\therefore$ $v_0 \propto \sqrt{h}$

For the first case, $v_0 = v_0$; $h = h$

For the second case, $v_0' = ?$; $h' = 3h$

$\therefore$ $\frac{v_0'}{v_0} = \sqrt{\frac{h'}{h}} = \sqrt{\frac{3h}{h}} = \sqrt{3}$ $\therefore$ $v_0' = \sqrt{3}\,v_0$

17. Speed of S waves ; v_s = 4.5 km/s ; Speed of P waves, v_p = 8 km/s

Difference of time between two waves, Δt = 4 min. = 240 seconds

The relation for the location of epicentre of earthquake (s) is

$$\Delta t = \frac{s}{v_s} - \frac{s}{v_p} \quad \text{or} \quad 240 = \frac{s}{4.5} - \frac{s}{8} \qquad \therefore s \simeq 2500 \text{ km}$$

18. The velocity of the ball decreases during upward motion and becomes zero at the highest point. During downward motion, the velocity of the ball increases. Since there is air resistance, the velocity–time graph during upward as well as downward motion will not be linear. Remember that air resistance varies as the square of velocity. Therefore, velocity–time graph of the ball during its flight will be represented by graph (d).

19. Distance covered in first 10 seconds is

$$x = v_0 t + \frac{1}{2}at^2 = 0 \times 10 + \frac{1}{2}a \times (10)^2 = 50a$$

Velocity of car after 10 seconds is

$$v = v_0 + at = 0 + a \times 10 = 10a$$

Distance covered in the next 10 seconds is

$$y = vt = 10a \times 10 = 100a = 2x \qquad (\because x = 50a)$$

$$\therefore \quad y = 2x$$

20. In stationary lift, the time taken by the coin to reach the floor is given by ;

$$h = \frac{1}{2}gt^2 \quad \text{or} \quad t = \sqrt{\frac{2h}{g}} \qquad (\because v_0 = 0)$$

In the upward moving lift with constant acceleration a, $g' = g + a$.

$$\therefore \quad t_2 = \sqrt{\frac{2h}{g+a}} \qquad \therefore t_2 < t_1$$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A body projected vertically upwards with a velocity u returns to the starting point in 4 seconds. If g = 10 m/s^2, the value of u is

[Karnataka CET 1999]

(a) 5 m/s (b) 10 m/s

(c) 15 m/s (d) 20 m/s

2. The ratios of the distances covered by a freely falling particle, starting from rest in the first, second, third ...nth seconds of its motion

(a) form the series corresponding to the squares of the first n natural numbers

(b) form an A.P.

(c) form a series corresponding to the difference of the squares of the successive natural numbers

(d) both choices (b) and (c) are correct

[Haryana CET 2001]

3. A particle is thrown vertically upwards. The graph between its speed v and time t is given by (neglecting air resistance) the figure

[Roorkee 1998]

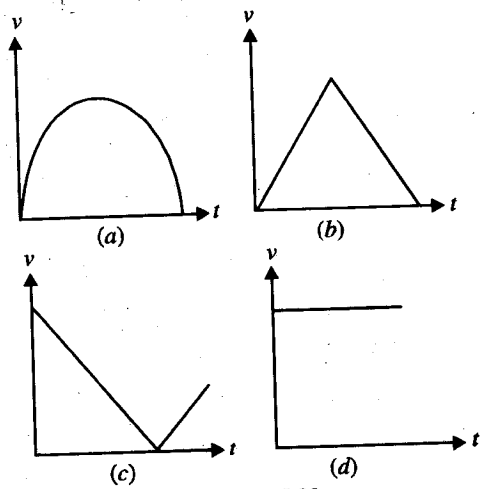


Fig. 7.22

4. A man in balloon rises vertically with an acceleration of 4.9 m/s^2 releases a ball 2 seconds after the balloon rises from the ground. The greatest height above the ground reached by the ball (g = 9.8 m/s^2) is **[Haryana CET 2000]**

(a) 9.8 m (b) 24.5 m

(c) 14.7 m (d) 19.6 m

5. A body falling from rest describes distances S_1, S_2 and S_3 in the first, second and third seconds of its fall. The ratio of $S_1 : S_2 : S_3$ is **[Karnataka CET 1997]**

(a) 1 : 2 : 3 (b) 1 : 3 : 5

(c) 1 : 1 : 1 (d) 1 : 4 : 9

6. A body released from a great height falls freely toward earth. Another body is released from the same height exactly one second later. The separation between both the bodies two seconds after the release of the second body is **[Karnataka CET 1997]**

(a) 4.9 m (b) 9.8 m

(c) 19.6 m (d) 24.5 m

7. A ball is dropped on to the floor from a height of 10 m. It rebounds to a height of 2.5 m. If the ball is in contact with the floor for 0.01 second, what is the average acceleration during contact? **[BHU 1997]**

(a) 1400 m/s^2 (b) 2100 m/s^2

(c) 700 m/s^2 (d) 2800 m/s^2

8. A ball P is dropped vertically and another ball Q is thrown horizontally from the same height and at the same time. If air resistance is neglected, then, **[BHU 1994]**

(a) ball P reaches the ground first

(b) ball Q reaches the ground first

(c) both the balls reach the ground at the same time

(d) respective masses of the two balls will decide the time

9. A stone falls from a balloon descending at a uniform rate of 12 m/s. The displacement of the stone from the point of release after 10 seconds is **[Haryana CET 1995]**

(a) 725 m (b) 610 m

(c) 510 m (d) 490 m

10. A body is projected vertically up with a velocity $\vec{v}$ and after some time it returns to the point from which it was projected. The average velocity and average speed of the body for the total flight are

(a) $\vec{v}$ and 0 (b) 0 and $v/2$

(c) 0 and 0 (d) 0 and $\vec{v}$

11. A particle moving with a uniform acceleration travels 24 m and 64 m in the first two consecutive intervals of 4 sec each. Its initial velocity is **[MP PET 1995]**

(a) 1 m/sec (b) 10 m/sec

(c) 5 m/sec (d) 2 m/sec

12. The velocity-time graph of a body moving in a straight line is shown in Fig. 7.23. The displacement and distance travelled by the body in 6 seconds are respectively. **[MP PET 1994]**

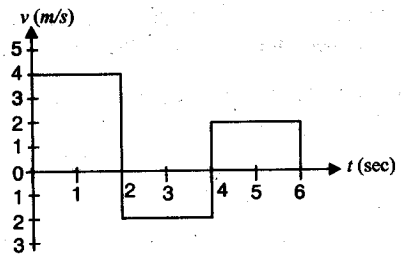


Fig. 7.23

(a) 8 m, 16 m (b) 16 m, 8 m

(c) 16 m, 16 m (d) 8 m, 8 m

13. Two bodies of different masses are dropped from heights of 16 m and 25 m respectively. The ratio of time taken by them to reach the ground is **[Pb CET 1991]**

(a) 25/16 (b) 5/4

(c) 4/5 (d) 16/25

14. A wooden block of mass 10 g is dropped from the top of a cliff 100 m high. Simultaneously a bullet of mass 10 g is fired from the foot of the cliff upward with a velocity of 100 m/s. The bullet and the wooden block will meet each other after a time (in second) **[EAMCET 1991]**

(a) 7 (b) 10

(c) 0.5 (d) 1

15. For the velocity-time graph shown in Fig. 7.24, the distance covered by the body in the last two seconds of its motion is what fraction of

the total distance covered in all the seven seconds ? **[MPPET 1998]**

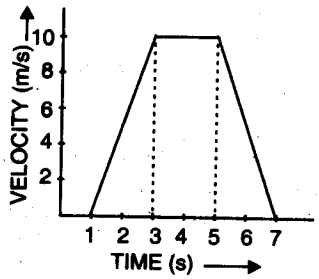


Fig. 7.24

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{4}$

(*c*) $\frac{1}{3}$ (*d*) $\frac{2}{3}$

16. A car, starting from rest, accelerates at the rate f through a distance S, then continues at constant speed for time t and then decelerates at the rate $\frac{f}{2}$ to come to rest. If the total distance travelled is $15S$, then : **[AIEEE 2005]**

(*a*) $S = ft$ (*b*) $S = \frac{1}{6}ft^2$

(*c*) $S = \frac{1}{72}ft^2$ (*d*) $S = \frac{1}{4}ft^2$

17. Three balls A, B and C are thrown from the top of a tower with the same speed. A is thrown straight up, B is thrown straight down and C is thrown horizontally. They hit the ground with speeds v_A, v_B and v_C respectively. Then: **[DCE 2006]**

(*a*) $v_A = v_B = v_C$ (*b*) $v_B > v_C > v_A$

(*c*) $v_A = v_B > v_C$ (*d*) $v_A > v_B = v_C$

18. A ball is released from the top of a tower of height h metres. It takes T seconds to reach the ground. What is the position of the ball in $T/3$ seconds? **[AIEEE 2004]**

(*a*) $h/9$ metre from the ground

(*b*) $7h/9$ metre from the ground

(*c*) $8h/9$ metre from the ground

(*d*) $17h/18$ metre from the ground

19. A small block slides, without friction, down an inclined plane starting from rest. Let S_n be the distance travelled from $t = (n-1)$ to $t = n$. Then $\frac{S_n}{S_{n+1}}$ is **[IIT 2004]**

(*a*) $\frac{2n-1}{2n}$ (*b*) $\frac{2n+1}{2n-1}$

(*c*) $\frac{2n-1}{2n+1}$ (*d*) $\frac{2n}{2n+1}$

20. The velocity–displacement graph of a particle moving along a straight line is shown in Fig. 7.25. The most suitable acceleration-displacement graph will be **[IIT 2005]**

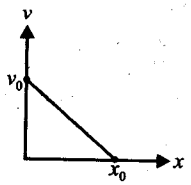


Fig. 7.25

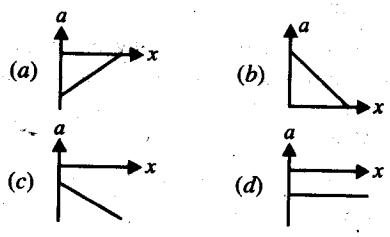


Fig. 7.26

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*d*)	**2.** (*d*)	**3.** (*c*)	**4.** (*c*)	**5.** (*b*)
6. (*d*)	**7.** (*b*)	**8.** (*c*)	**9.** (*b*)	**10.** (*b*)
11. (*a*)	**12.** (*a*)	**13.** (*c*)	**14.** (*d*)	**15.** (*b*)
16. (*c*)	**17.** (*c*)	**18.** (*c*)	**19.** (*c*)	**20.** (*a*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Time of ascent = Time of descent = 4/2 = 2 sec.

For upward journey, we have, $v = v_0 - g\,t$

or $0 = v_0 - 10 \times 2$

$\therefore \quad v_0 = 20$ m/s

2. The distance covered by the particle in the nth second (with $v_0 = 0$) is

$$S_n = \frac{1}{2}g[n^2-(n-1)^2] = \frac{1}{2}g(2n-1)$$

$$\therefore \quad S_1 = \frac{1}{2}g;\ S_2 = \frac{3}{2}g;\ S_3 = \frac{5}{2}g.$$

Therefore, choices (b) and (c) are correct.

3. For upward motion, $v = v_0 - gt$. Therefore, velocity decreases linearly with time. For downward motion, $v = v_0 + gt = 0 + gt = gt$. Again for downward motion, the velocity increases linearly with time.

4. Velocity v acquired by the ball in 2 seconds is

$$v = v_0 + at = 0 + 4.9 \times 2 = 9.8 \text{ m/s}$$

$$\therefore \quad s = 0 + \frac{1}{2} \times (4.9) \times (2)^2 = 9.8 \text{ m}$$

When the ball is released, it moves under gravity alone. Its velocity decreases till it reaches the highest point. Let it rise to a height h after its release from balloon.

$\therefore \quad (0)^2 - (9.8)^2 = -2 \times 9.8 \times h \quad \therefore \quad h = 4.9$ m

Greatest height of ball above ground

$$= 9.8 + 4.9 = 14.7 \text{ m}$$

5. Initial velocity, $v_0 = 0 \quad \therefore \quad S = \frac{1}{2}gt^2$

$$S_1 = \frac{1}{2}g\times(1)^2 = \frac{1}{2}g$$

$$S_1 + S_2 = \frac{1}{2}g\times(2)^2 = 2g; \qquad S_1 + S_2 + S_3 = \frac{1}{2}g\times(3)^2 = 4.5g$$

$$\therefore \quad S_1 = \frac{1}{2}g;\ S_2 = 2g - \frac{1}{2}g = 1.5g;\ S_3 = 4.5g - 2g = 2.5\ g$$

$$\therefore \quad S_1 : S_2 : S_3 = \frac{1}{2}g : 1.5\ g : 2.5\ g = 1 : 3 : 5$$

6. Since initial velocity in the two cases is zero, $S = \frac{1}{2}gt^2$.

Distance covered by first body in 3 sec $= \frac{1}{2} \times 9.8 \times (3)^2 = 44.1$ m

Distance covered by second body in 2 sec $= \frac{1}{2} \times 9.8 \times (2)^2 = 19.6$ m

Separation of two bodies 2 seconds after the release of second body

$$= 44.1 - 19.6 = 24.5 \text{ m}$$

7. $v_1 = \sqrt{2gh_1} = \sqrt{2\times10\times10} = \sqrt{200}$ m/s; $\qquad v_2 = -\sqrt{2gh_2} = -\sqrt{2\times10\times2.5} = -\sqrt{50}$ m/s

Average acceleration, $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_1 - v_2}{\Delta t} = \frac{\sqrt{200} + \sqrt{50}}{0.01} = 2100 \text{ m/s}^2$

8. The balls will reach the ground due to gravity. Since the initial vertical velocity of the balls is zero, the two balls will reach the ground at the same time.

9. $v_0 = 12$ m/s ; $g = 9.8$ m/s^2 ; $t = 10$ sec.

$$s = v_0 t + \frac{1}{2} g t^2 = 12 \times 10 + \frac{1}{2} \times 9.8 \times (10)^2 = 610 \text{ m}$$

10. Displacement for total flight is zero. Therefore, average velocity is zero. The speed of the body changes from v to zero during upward journey and from zero to v during downward journey. Therefore, average speed $= (v + 0)/2 = v/2$.

11. Let v_0 and a be the initial velocity and uniform acceleration respectively.

$$\therefore \quad 24 = v_0 \times 4 + \frac{1}{2} a \times 4^2 \quad \text{or} \quad 4v_0 + 8a = 24 \qquad \ldots (i)$$

$$\text{Also} \quad (24 + 64) = v_0 \times 8 + \frac{1}{2} a \times 8^2 \text{ or } 8v_0 + 32a = 88 \qquad \ldots (ii)$$

Solving eqs. (*i*) and (*ii*), $v_0 = 1$ m/s.

12. Displacement is equal to the *net* area under velocity – time graph.

$\therefore$ Displacement $= 4 \times 2 - 2 \times 2 + 2 \times 2 = 8$m

Distance travelled is equal to the total area under velocity – time graph.

$\therefore$ Distance travelled $= 4 \times 2 + 2 \times 2 + 2 \times 2 = 16$m

13. $$h = v_0 t + \frac{1}{2} g t^2 \text{ or } h = \frac{1}{2} g t^2 \quad (\because v_0 = 0) \quad \therefore \; t \propto \sqrt{h}$$

$$\therefore \quad \frac{t_1}{t_2} = \sqrt{\frac{h_1}{h_2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

14. Let the block and bullet meet after time t at a distance x vertically below the top of the cliff.

For the downward motion of the block, we have, $x = 0 + \frac{1}{2} g t^2$.

For the upward motion of the bullet, we have,

$$100 - x = 100 t - \frac{1}{2} g t^2$$

or $$100 - x = 100t - x \qquad (\because \frac{1}{2} gt^2 = x)$$

$\therefore$ $$t = 1 \text{ sec}$$

15. Total distance covered in 7 seconds is equal to the area under velocity–time graph for $t = 7$ sec.

$$\text{Total distance covered} = \frac{1}{2} \times 2 \times 10 + 2 \times 10 + \frac{1}{2} \times 2 \times 10 = 40 \text{ m}$$

Distance covered in the last 2 seconds

$$= \frac{1}{2} \times 2 \times 10 = 10 \text{ m}$$

Fraction of distance $= 10/40 = 1/4$

16. Fig. 7.27 shows the v–t graph for the motion of the car. The car accelerates for time t_1 and attains the velocity v. It moves at velocity v for time t. The car then decelerates for time t_2 to come to stop.

$v^2 - v_0^2 = 2fS$ or $v^2 - 0^2 = 2fS$ $\therefore$ $v = \sqrt{2fS}$

Total distance travelled $= \frac{1}{2}vt_1 + vt + \frac{1}{2}vt_2$

or $15S = \frac{1}{2}vt_1 + vt + \frac{1}{2}vt_2$ ... (*i*)

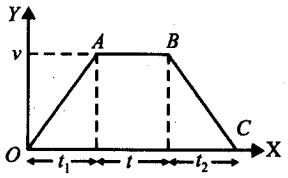


Fig. 7.27

Now $S = \frac{1}{2}ft_1^2 \quad \therefore \quad t_1 = \sqrt{\frac{2S}{f}}$

Also $0 = v - \frac{f}{2}t_2$ or $t_2 = \frac{2v}{f} = \frac{2\times\sqrt{2fS}}{f} = \sqrt{\frac{8S}{f}}$

Putting the values of v, t_1 and t_2 is eq. (*i*), we get,

$$S = \frac{1}{72}ft^2$$

17. The three balls *A*, *B* and *C* are thrown up, vertically downward and horizontally respectively from the top of a tower (point *T*) with the same speed *v* as shown in Fig. 7.28. When ball *A* returns to point *T*, it has velocity *v i.e.*, the same as that of ball *B* at point *T*. Therefore, balls *A* and *B* hit the ground with the same velocity *i.e.*, $v_A = v_B$.

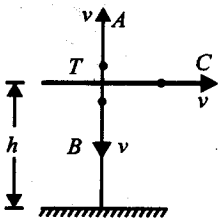


Fig. 7.28

Now $v_A = v_B = \sqrt{v^2 + 2gh}$

However, for ball *C*, the vertical velocity at point *T* is zero so that it will hit the ground with velocity v_C given by :

$$v_C^2 - v_0^2 = 2gh \quad \text{or} \quad v_C^2 - 0^2 = 2gh$$

$\therefore \quad v_C = \sqrt{2gh} \qquad \therefore \quad v_A = v_B > v_C$

18. $h = v_0T + \frac{1}{2}gT^2 \quad \text{or} \quad h = 0 \times T + \frac{1}{2}gT^2 \qquad \therefore T = \sqrt{\frac{2h}{g}}$

At time $= T/3$, $h = 0 + \frac{1}{2}g\left(\frac{T}{3}\right)^2 = \frac{1}{2}g\frac{T^2}{9} = \frac{1}{2}g\frac{2h}{g\times 9} = \frac{h}{9}$ m

$\therefore$ Position of ball from ground $= h - \frac{h}{9} = \frac{8h}{9}$ m

19. $S_n = v_0 + \frac{a}{2}(2n-1) = \frac{a}{2}(2n-1) \quad (\because v_0 = 0)$

$S_{n+1} = \frac{a}{2}[2(n+1)-1] = \frac{a}{2}[2n+1] \quad (\because v_0 = 0)$

$\therefore \qquad \frac{S_n}{S_{n+1}} = \frac{2n-1}{2n+1}$

20. From the given $v - x$ graph, we have,

Intercept $= v_0$ and slope $= -v_0/x_0$

$\therefore$ The equation of the given line is

$$v = -\frac{v_0}{x_0}x + v_0 \quad (y = mx + c)$$

$$\therefore \quad \frac{dv}{dt} = -\frac{v_0}{x_0}\frac{dx}{dt} \quad \text{or} \quad a = -\frac{v_0}{x_0}v$$

$$\therefore \quad a = -\frac{v_0}{x_0}\left(-\frac{v_0}{x_0}x + v_0\right) = \frac{v_0^2}{x_0^2}x - \frac{v_0^2}{x_0}$$

Therefore, the $a - x$ graph is a straight line with slope = v_0^2/x_0^2 (positive) and intercept is v_0^2/x_0^2 (negative). The slope and intercept reveal that graph (a) is the most suitable representation.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **Brakes are applied to a train travelling at 72 km/h. After passing over 200 m; its velocity is reduced to 36 km/h. At the same rate of retardation, how much further it will go before it is brought to rest ?**

Hint. For the first part of motion :

$$v_0 = 72 \text{ km/h} = 20 \text{ ms}^{-1}; \quad v = 36 \text{ km/h} = 10 \text{ ms}^{-1}; \quad S = 200 \text{ m}$$

Now $\quad v^2 - v_0^2 = 2aS \quad \therefore \quad a = \dfrac{v^2 - v_0^2}{2S} = \dfrac{(10)^2 - (20)^2}{2\times 200} = -0.75 \text{ ms}^{-2}$

For the second part of motion : $v_0 = 10 \text{ ms}^{-1}$; $v = 0$; $a = -0.75 \text{ ms}^{-2}$; $S = ?$

Again $\quad v^2 - v_0^2 = 2aS \quad \therefore \quad S = \dfrac{v^2 - v_0^2}{2a} = \dfrac{(0)^2 - (10)^2}{2\times(-0.75)} = 66.67 \text{m}$

2. **A body with uniform acceleration covers 76 m in the 5th second and 116 m in the 10th second. Find the distance travelled in the 15th second. What is the distance travelled in next 5 seconds ?**

Hint. The distance travelled in the nth second is given by ;

$$S_{nth} = v_0 + \frac{a}{2}(2n-1)$$

$$\therefore \quad S_{5th} = v_0 + \frac{a}{2}(2\times 5 - 1) \quad \text{or} \quad 76 = v_0 + \frac{9a}{2} \qquad ...(i)$$

Also $\quad S_{10th} = v_0 + \dfrac{a}{2}(2\times 10 - 1) \quad \text{or} \quad 116 = v_0 + \dfrac{19a}{2} \qquad ...(ii)$

Solving eqs. (i) and (ii), $v_0 = 40 \text{ ms}^{-1}$; $a = 8 \text{ ms}^{-2}$

$\therefore$ Distance covered in the 15th second is

$$S_{15th} = v_0 + \frac{a}{2}(2n-1) = 40 + \frac{8}{2}(2\times 15 - 1) = 156 \text{ m}$$

Distance covered in 10 seconds is

$$S = v_0 t + \frac{1}{2}at^2 = 40\times 10 + \frac{1}{2}\times 8\times (10)^2 = 800 \text{m}$$

Distance covered in 15 seconds is

$$S' = v_0 t' + \frac{1}{2}at'^2 = 40\times 15 + \frac{1}{2}\times 8\times (15)^2 = 1500 \text{m}$$

$\therefore$ Distance covered in next 5 seconds

$$= S' - S = 1500 - 800 = 700 \text{ m}$$

From eqs. (*i*) and (*ii*), we have, $-\frac{(19)^2 v_0^2}{400n} = -\frac{39 v_0^2}{400}$

$\therefore \quad n = \frac{(19)^2}{39} \simeq 10$

$\therefore$ Minimum number of planks required = 10 + 1= 11

6. A stone after falling freely under the influence of gravity for 1 second strikes a glass pane held horizontally. In breaking through the glass pane, the stone loses half the velocity. How far will it travel in next two seconds?

Hint. Let v ms^{-1} be the final velocity when the stone strikes the glass pane. For this part of motion, $v_0 = 0$; $t = 1$ s ; $g = 9.8$ ms^{-2}.

$\therefore \quad v = v_0 + gt = 0 + 9.8 \times 1 = 9.8$ ms^{-1}

After striking the glass pane, the stone loses half the velocity. For this part of motion, $v_0 = 4.9$ ms^{-1} ; $g = 9.8$ ms^{-2} ; $t = 2$ s ; $h = ?$

$\therefore \quad h = v_0 t + \frac{1}{2} g t^2 = 4.9 \times 2 + \frac{1}{2} \times 9.8 \times (2)^2 = 29.4$ m

7. A car accelerates from rest at a constant rate α for some time after which it decelerates at a constant rate β to come to rest. If the total time elapsed is t seconds, find (*i*) the maximum velocity reached (*ii*) the total distance travelled. **[IIT]**

Hint. Suppose the car accelerates for t_1 seconds. Then it decelerates for $(t - t_1)$ seconds.

(*i*) **Maximum velocity reached.** For accelerated motion of the car, $v_0 = 0$; $a = \alpha$; time = t_1 and $v = v_m$ (maximum velocity).

$\therefore \quad v_m = v_0 + a t_1$ or $v_m = \alpha t_1$

The velocity αt_1 becomes zero after time $(t - t_1)$.

$\therefore \quad v = v_0 + at$ or $0 = \alpha t_1 - \beta (t - t_1) \quad \therefore \quad t_1 = \frac{\beta t}{\alpha + \beta}$

$\therefore$ Maximum velocity, $v_m = \alpha t_1 = \alpha \times \frac{\beta t}{\alpha + \beta} = \frac{\alpha \beta}{\alpha + \beta} t$

(*ii*) **Total distance travelled.**

Let S_1 = distance covered by car during accelerated motion

S_2 = distance covered by car during decelerated motion

Total distance covered, $S = S_1 + S_2$

Now $v^2 - v_0^2 = 2 a S$

Here $v = v_m$; $v_0 = 0$; $a = \alpha$; $S = S_1$

$\therefore \quad v_m^2 - (0)^2 = 2 \alpha S_1 \quad \therefore \quad S_1 = \frac{v_m^2}{2\alpha}$

Again, $v^2 - v_0^2 = 2aS$

Here $v = 0$; $v = v_m$; $a = -\beta$; $S = S_2$

$\therefore \quad (0)^2 - (v_m)^2 = 2 \times (-\beta) S_2$ or $S_2 = \frac{v_m^2}{2\beta}$

Total distance covered, $S = S_1 + S_2 = \frac{v_m^2}{2\alpha} + \frac{v_m^2}{2\beta} = \frac{v_m^2}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

$$= \frac{v_m^2(\alpha+\beta)}{2\alpha\beta} \qquad ...(i)$$

Putting the value of v_m in eq. (*i*), we get,

$$S = \left(\frac{\alpha\beta}{\alpha+\beta}t\right)^2 \times \frac{\alpha+\beta}{2\alpha\beta} = \frac{\alpha\beta t^2}{2(\alpha+\beta)}$$

8. **A solid ball of density half that of water falls freely under gravity from a height of 19.6 m and then enters water. Upto what depth will the ball go ? How much time will it take to come again to the water surface ? Neglect air resistance and viscosity effects of water.** ***[Roorkee]***

Hint. Let V m^3 be the volume of the ball.

$$\text{Density of ball, } \rho = \frac{\text{density of water}}{2} = \frac{1000}{2} = 500 \text{ kg/m}^3$$

$$\text{Mass of ball, } m = V\rho = 500\ V \text{ kg}$$

The velocity v of the ball when it hits the water surface is given by ;

$$v^2 = v_0^2 + 2gh = 2gh \quad \therefore \quad v = \sqrt{2gh} = \sqrt{2\times9.8\times19.6} = 19.6 \text{ ms}^{-1} \quad (\because v_0 = 0)$$

$$\text{Upthrust of water} = V\rho_w g = V \times 1000 \times g = 1000\ Vg$$

Effective weight of the ball when in water

$$= mg - 1000\ Vg = 500\ Vg - 1000\ Vg = -500\ Vg$$

Acceleration a of the ball is given by ;

$$ma = -500\ Vg \quad \text{or} \quad 500\ V \times a = -500\ Vg \ (\because m = 500\ V)$$

$$\therefore \quad a = -g$$

The depth h upto which the ball will go in water can be found from the relation :

$$v^2 - v_0^2 = 2ah$$

Here $v = 0$; $v_0 = 19.6 \text{ ms}^{-1}$; $a = -g$

$$\therefore \quad h = \frac{v^2 - v_0^2}{2a} = \frac{(0)^2 - (19.6)^2}{2\times(-9.8)} = 19.6 \text{ m}$$

The time t' taken by the ball to come again to the water surface can be found by using the relation :

$$h = v_0 t' + \frac{1}{2}at'^2$$

Here $h = -19.6$ m ; $v_0 = 0$; $a = -9.8 \text{ ms}^{-2}$

$$\therefore \quad -19.6 = 0 \times t' + \frac{1}{2}(-9.8)\ t'^2$$

$$\therefore \quad t'^2 = \frac{2\times19.6}{9.8} = 4 \quad \text{or} \quad t' = \sqrt{4} = 2 \text{ sec}$$

9. **A rocket is fired vertically and ascends at a constant vertical acceleration. In 5 seconds, it attains an upward velocity of 150 ms^{-1}. After 5 seconds, its engine stops functioning and the rocket continues to move as a free particle and finally returns to earth. Plot the velocity-time graph of the rocket. From the graph, find (*i*) acceleration of the rocket during fuel burning (*ii*) the height reached by the rocket during the period of fuel burning (*iii*) the maximum height reached by the rocket.Take $g = 10 \text{ ms}^{-2}$.**

Hint. Figure 7.29 shows the velocity-time graph of the rocket. During the part OA of the graph, the rocket is under constant acceleration and its velocity is increasing. At point A, the velocity of the rocket is maximum (= 150 ms^{-1}). At point A, the engine stops functioning and the velocity of the rocket starts decreasing but the rocket is still rising.This part of motion is represented by graph AB. At point B, the rocket has attained maximum height and its velocity is zero. Now the rocket starts downward journey represented by graph BC. Time taken during the part AB of the graph can be found using the equation :

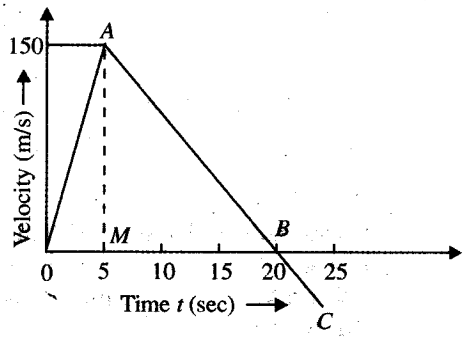


Fig. 7.29

$$v = v_0 + a\,t$$

Here $v = 0$; $v_0 = 150\ ms^{-1}$; $a = -g = -10\ ms^{-2}$

$\therefore\ 0 = 150 - 10\,t$ or $t = 150/10 = 15$ sec.

Thus, the total time for the upward journey of the rocket = 5 + 15 = 20 seconds.

(*i*) Acceleration of the rocket during fuel burning

$$= \text{Slope of line } OA = \frac{AM}{OM} = \frac{150}{5} = \mathbf{30\ ms^{-2}}$$

(*ii*) Height reached by the rocket during fuel burning

$$= \text{Area of } \Delta OAM = \frac{1}{2} \times OM \times AM = \frac{1}{2} \times 5 \times 150 = \mathbf{375\ m}$$

(*iii*) Maximum height reached by rocket

$$= \text{Area of } \Delta OAB = \frac{1}{2} \times OB \times AM = \frac{1}{2} \times 20 \times 150 = \mathbf{1500\ m}$$

10. The velocity-time graph for a car is shown in Fig. 7.30. Draw acceleration-time graph from it.

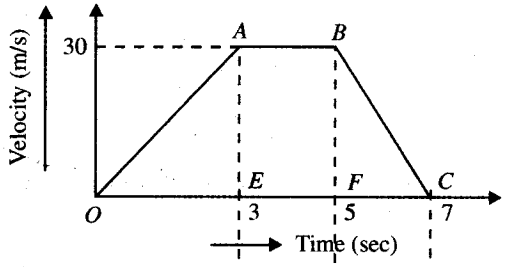


Fig. 7.30

Hint. (*i*) Referring to velocity-time graph [Fig. 7.30], during the part OA, the velocity is increasing uniformly. In other words, the car is under constant acceleration. This constant acceleration is equal to slope of the line OA.

$\therefore$ Constant acceleration during OA

$$= \frac{AE}{OE} = \frac{30}{3} = 10\ ms^{-2}$$

The acceleration-time graph for this part of motion is PQ as shown in Fig. 7.31.

(*ii*) Referring to velocity-time graph, during the part AB, the velocity is constant so that acceleration is zero. The accelertion-time graph for this part of motion is RS [See Fig. 7.31].

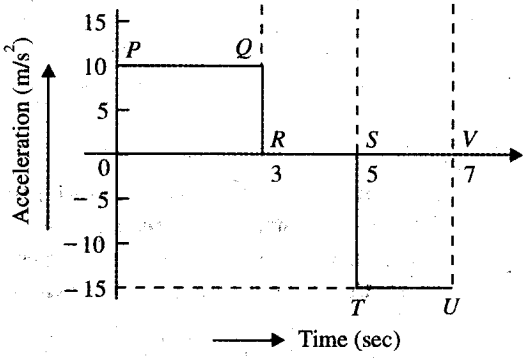


Fig. 7.31

(*iii*) Referring to velocity-time graph, during the part *BC*, the velocity is decreasing uniformly. In other words, the acceleration is constant and negative. It is given by the slope of line *BC* in Fig. 7.30.

$$\text{Constant retardation during part } BC = -\frac{BF}{FC} = -\frac{30}{2} = -15\,\text{ms}^{-2}$$

The acceleration-time graph for this part of motion is *TU* as shown in Fig. 7.31.

11. A ball falls freely on a perfectly elastic plate from a height of 3.0 m. At instant $t = 0$, the velocity of the ball is zero. Draw the velocity-time graph for the motion of the ball.

Hint. The time t taken by the ball in falling from rest through 3.0 m is given by ;

$$h = \frac{1}{2} g t^2 \quad \text{or} \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 3.0}{9.8}} = 0.78 \text{ sec}$$

The velocity v of the ball just before striking the plate is

$$v = \sqrt{2gh} = \sqrt{2\times 9.8\times 3.0} = 7.67\,\text{ms}^{-1}$$

Figure 7.32 shows the velocity-time graph for the motion of the ball. At $t = 0$, the velocity of the ball is zero. As it falls towards the plate under gravity, its velocity goes on increasing till it becomes maximum (= 7.67 ms^{-1}) after 0.78 sec. On striking the plate, it rebounds with the same velocity because the plate is perfectly elastic. Now the motion of the ball is against gravity so that its velocity goes on decreasing till it becomes zero after 0.78 sec. The ball again starts downward journey.

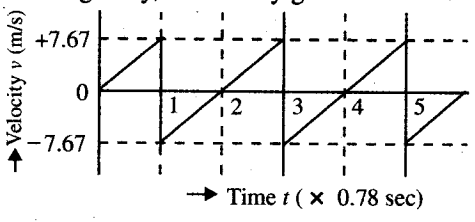


Fig. 7.32

12. From the top of a tower 98 m high, a stone is projected vertically upward with a velocity of 39.2 ms^{-1}. On its return, it misses the tower and finally strikes the ground. Calculate (*i*) the greatest elevation (*ii*) the velocity with which it strikes the ground (*iii*) the time it takes to reach the ground.

Hint. Take the upward motion as positive.

(*i*) Let h be the height from the top of the tower to which the stone rises.

Now $v^2 - v_0^2 = 2\,g\,h$

Here $v = 0$; $v_0 = 39.2\ \text{ms}^{-1}$; $g = -9.8\ \text{ms}^{-2}$

$$\therefore \quad h = \frac{v^2 - v_0^2}{2g} = \frac{(0)^2 - (39.2)^2}{2\times(-9.8)} = 78.4 \text{ m}$$

$\therefore$ Greatest elevation = 78.4 + 98 = **176.4 m**

(*ii*) The velocity v with which it strikes the ground is given by ;

$$v^2 - v_0^2 = 2\,g\,h$$

Here $v_0 = 0$; $g = -9.8\ \text{m s}^{-2}$; $h = -176.4$ m

$$\therefore \quad v^2 - 0^2 = 2 \times (-9.8) \times (-176.4)$$

or $$v = \sqrt{2\times(-9.8)\times(-176.4)} = \mathbf{58.8\ ms^{-1}}$$

(*iii*) The total time t taken to reach the ground is given by ;

$$v = v_0 + g\,t$$

Here $v = -58.8\ \text{ms}^{-1}$; $v_0 = 39.2\ \text{ms}^{-1}$; $g = -9.8\ \text{ms}^{-2}$

$$\therefore \quad t = \frac{v - v_0}{g} = \frac{(-58.8) - (39.2)}{-9.8} = 10\ \text{s}$$

13. In figure 7.33, ball 1 is released from rest at the top of a smooth inclined plane and at the same instant ball 2 is projected up the plane from the foot of the plane with such velocity that they meet half-way up the plane. Determine (i) the velocity of projection and (ii) the velocity of each ball when they meet.

Hint. (*i*) In the common time t, distance travelled by ball 1 is given by ;

$$\frac{l}{2} = 0 \times t + \frac{1}{2}(g\sin\theta)t^2 \qquad ...(i)$$

The distance travelled by ball 2 in time t is

$$\frac{l}{2} = v_2 t + \frac{1}{2}(-g\sin\theta)t^2 \qquad ...(ii)$$

Adding eqs. (*i*) and (*ii*), we have,

$$\frac{l}{2} + \frac{l}{2} = v_2 t \quad \text{or} \quad t = \frac{l}{v_2}$$

Putting $t = l/v_2$ in eq. (*i*), we have,

$$\frac{l}{2} = \frac{1}{2}(g\sin\theta)\frac{l^2}{v_2^2}$$

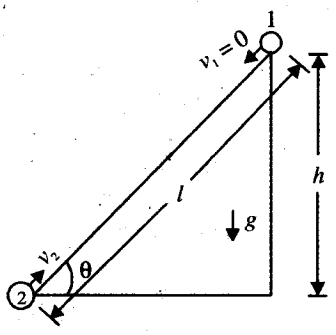


Fig. 7.33

or $\quad v_2^2 = g\,l\sin\theta$

$$\therefore \quad v_2 = \sqrt{g\,l\sin\theta} = \sqrt{gh} \qquad (\because l\sin\theta = h)$$

(*ii*) $\quad v_1'^2 = 0^2 + 2(g\sin\theta)\frac{l}{2}$ or $v_1' = \sqrt{g\ l\sin\theta} = \sqrt{gh}$

Now $\quad v_2'^2 = v_2^2 + 2(-g\sin\theta)\frac{l}{2} = gl\sin\theta - gl\sin\theta = 0$

14. A bullet is fired from a rifle with a velocity of 750 ms^{-1}. If the length of the barrel is 0. 6 m, find (*i*) the average velocity of the bullet while being accelerated (*ii*) time taken by the bullet to travel the length of the barrel (*iii*) the average acceleration.

Hint. (*i*) Initial velocity of bullet, $v_0 = 0$; Final velocity of bullet, $v = 750\ \text{ms}^{-1}$

$$\therefore \quad \text{Average velocity} = \frac{v_0 + v}{2} = \frac{0 + 750}{2} = 375\ \text{ms}^{-1}$$

(*ii*) If t is the time taken by the bullet to cross the barrel (S = 0.6 m), then,

$$S = \left(\frac{v_0 + v}{2}\right)t = 375 \times t$$

$$\therefore \quad t = \frac{S}{375} = \frac{0.6}{375} = 0.0016\ \text{s}$$

(*iii*) If a is the average acceleration, then,

$$v = v_0 + a\,t$$

or $$a = \frac{v - v_0}{t} = \frac{750 - 0}{0.0016} = 468750 \text{ ms}^{-2}$$

15. A ball after having fallen from rest under the influence of gravity for 6 s crashes through a horizontal glass plate, thereby losing two-third of its velocity. If it then reaches the ground in 2 s, find the height of the plate above the ground.

Hint. The velocity v of the ball just before striking the glass plate is given by ;

$$v = v_0 + g\,t$$

Here $v_0 = 0$; $g = -9.8 \text{ ms}^{-2}$; $t = 6$ s

$\therefore$ $$v = 0 + (-9.8) \times 6 = -58.8 \text{ ms}^{-1}$$

The velocity of the ball after passing through the glass plate, $v_1 = v/3 = -58.8/3 = -19.6 \text{ ms}^{-1}$. The height h of the glass plate above the ground is given by ;

$$h = v_1 t + \frac{1}{2} g\,t^2$$

Here $v_1 = -19.6 \text{ ms}^{-1}$; $t = 2$ s ; $g = -9.8 \text{ ms}^{-2}$; $h = -h$

$\therefore$ $$-h = -19.6 \times 2 + \frac{1}{2} \times (-9.8) \times (2)^2$$

or $$h = \mathbf{58.8 \text{ m}}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 3 : MOTION IN A STRAIGHT LINE

Q.1. In which of the following examples of motion can the body be considered approximately a point object: (*a*) a railway carriage moving without jerks between two stations. (*b*) a monkey sitting on top of a man cycling smoothly on a circular track. (*c*) a spinning cricket ball that turns sharply on hitting the ground. (*d*) a tumbling beaker that has slipped off the edge of a table?

Ans. When the size of an object is very small compared to the scale of observation, the object can be considered as a point object.

(*a*) The size of railway carriage is very small compared to the distance between the two stations. Therefore, the carriage can be considered a point object.

(*b*) The size of the monkey is very small compared to the distance covered by the cyclist. Therefore, monkey can be considered a point object.

(*c*) The size of the spinning cricket ball is comparable to the distance through which the ball may turn as it hits the ground. Therefore, cricket ball *cannot* be considered a point object.

(*d*) The size of the beaker is not negligible as compared to the height of the table. Therefore, in this case, the beaker *cannot* be considered a point object.

Q.2. The position-time ($x - t$) graphs for two children *A* and *B* returning from their school *O* to their homes *P* and *Q* respectively are shown in Fig. 3.01. Choose the correct enteries in the brackets below: (*a*) (*A/B*) lives closer to the school than (*B/A*), (*b*) (*A/B*) starts from the school earlier than (*B/A*), (*c*) (*A/B*) walks faster than (*B/A*), (*d*) *A* and *B* reach home at the (same/different) time, (*e*) (*A/B*) overtakes (*B/A*) on the road (once/twice).

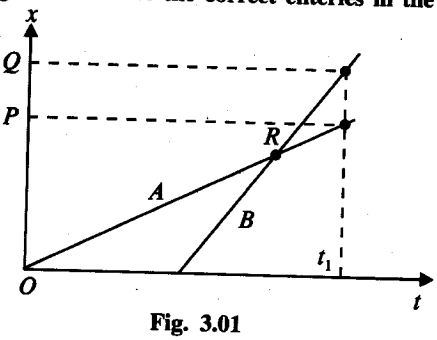


Fig. 3.01

Ans. (*a*) From the graph, it is clear that *P* is closer to *O* than *Q*. Therefore, child *A* lives closer to the school than the child *B*.

(*b*) It is clear that child *A* starts at $t = 0$, while the child *B* starts at later time.

(*c*) The $x - t$ graph for child *B* is steeper than that of child *A*. In other words, slope of x–t graph for child *B* is greater than that of child *A*. Since the slope of x–t graph is equal to the speed, child *B* walks faster than child *A*.

(*d*) It is clear from the graph that both children reach their homes at t_1, *i.e.*, they reach home at the same time.

(*e*) The x–t graphs for the children intersect only once at *R*. Therefore, *B* overtakes *A* on the road only once.

Q.3. A woman starts from her home at 9.00 A.M., walks with a speed of 5 km/h on straight road upto her office 2.5 km away, stays at the office upto 5:00 P.M. and returns home by an auto with a speed of 25 km/h. Choose suitable scale and plot the $x - t$ graph of her motion.

Ans. Time taken to reach office $= \dfrac{\text{Distance}}{\text{Speed}} = \dfrac{2.5\text{ km}}{5\text{ km/h}} = 0.5$ hr

Time taken to return from office

$= \dfrac{\text{Distance}}{\text{Speed}} = \dfrac{2.5\text{ km}}{25\text{ km/h}} = 0.1$ hr $= 6$ minutes

Therefore, the woman reaches the office at 9.30 A.M. and returns home at 5.06 P.M.

The $x - t$ graph of the motion of the woman is shown in Fig. 3.02.

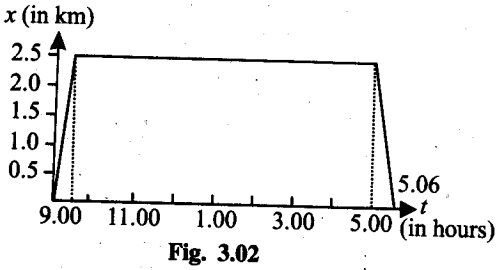


Fig. 3.02

Q.4. **A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward followed again by 5 steps forward and 3 steps backward and so on. Each step is 1 m long and requires 1 s. Plot the $x - t$ graph of his motion. Determine graphically and otherwise how long the drunkard takes to fall in a pit 13 m away from the start.**

Ans. Fig. 3.03 shows the $x - t$ graph of the motion of the drunkard. It is clear from the graph that the required time = 37 s.

We can also find the required time mathematically. Assigning the forward motion positive and backward motion negative,

$$x = \underbrace{5\text{m} + (-3\text{m})}_{\text{2 m, 8 steps}} + \underbrace{5\text{m} + (-3\text{m})}_{\text{2 m, 8 steps}} + \underbrace{5\text{m} + (-3\text{m})}_{\text{2 m, 8 steps}} + \underbrace{5\text{m} + (-3\text{m})}_{\text{2 m, 8 steps}} + \underbrace{5\text{m}}_{\text{5 steps}}$$

$\therefore$ Total number of steps = 8 + 8 + 8 + 8 + 5 = 37 steps

Since time taken for one step is 1 s, the total time taken to fall in pit 13 m away from the start = 37 × 1 = 37 s.

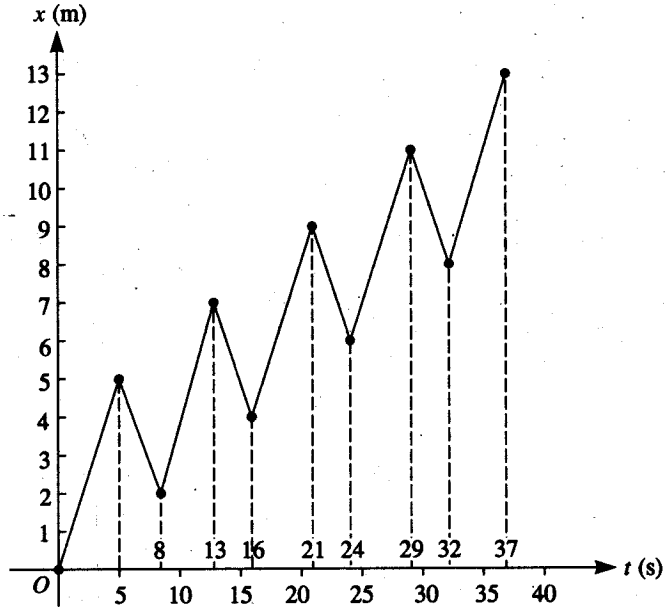


Fig. 3.03

Q.5. **A jet aeroplane travelling at the speed of 500 km/h ejects its products of combustion at the speed of 1500 km/h relative to jet plane. What is the speed of the latter w.r.t. an observer on the ground ?**

Ans. Considering the positive direction of x-axis to be the direction in which the jet plane moves, we have,

v_{JE} = Relative speed of jet plane *w.r.t.* earth = + 500 km/h

v_{CJ} = Relative speed of combustion products *w.r.t.* jet = − 1500 km/h

$\therefore$ Relative speed of combustion products *w.r.t.* earth is

$$v_{CE} = v_{CJ} + v_{JE} = (-1500) + (+500) = -1000 \text{ km/h}$$

Q.6. **A car moving along a straight highway with a speed of 126 km/h is brought to stop within a distance of 200 m. What is the retardation of the car (assumed uniform) and how long does it take for the car to stop ?**

Ans. v_0 = 126 km/h = 35 m s^{-1} ; v = 0 ; S = 200 m ; a = ? ; t = ?

$$\text{Average speed of car, } \bar{v} = \frac{v_0 + v}{2} = \frac{35 + 0}{2} = 17.5 \text{ ms}^{-1}$$

$$S = \bar{v}\ t \quad \therefore \quad t = \frac{S}{\bar{v}} = \frac{200}{17.5} = \mathbf{11.43\ s}$$

Now $v = v_0 + a\ t$

$$\therefore \quad a = \frac{v - v_0}{t} = \frac{0 - 35}{11.43} = \mathbf{-3.06\ ms^{-2}}$$

Q.7. Two trains *A* and *B* of length 400 m each are moving on two parallel tracks with a uniform speed of 72 km/h in the same direction with *A* ahead of *B*. The driver of *B* decides to overtake *A* and accelerates by 1 m s^{-2}. If after 50 s, the guard of *B* just brushes past the driver of *A*, what was the original distance between the trains?

Ans. Let x metres be the original distance between the trains. Now for each train v_0 = 72 km/h = 20 ms^{-1}.

For train *A*: $v_0 = 20\ m\ s^{-1}$; $a = 0$; $t = 50$ s

$\therefore$ Distance travelled by *A*, $S_A = v_0\ t = 20 \times 50 = 1000$ m

For train *B*: $v_0 = 20\ m\ s^{-1}$; $a = 1\ m\ s^{-2}$; $t = 50$ s

$\therefore$ Distance travelled by *B*, $S_B = v_0\ t + \frac{1}{2}\ a\ t^2 = 20 \times 50 + \frac{1}{2} \times 1 \times (50)^2 = 2250$ m

If L_A and L_B are the lengths of the trains, then the statement of the problem suggests that

$$S_B = x + S_A + L_A + L_B$$

or $$2250 = x + 1000 + 400 + 400$$

$\therefore$ $$x = \mathbf{450\ m}$$

Q.8. On a two-lane road, a car *A* is travelling with a speed of 36 km/h. Two cars *B* and *C* approach car *A* in opposite directions with a speed of 54 km/h each. At a certain instant, when the distance *AB* is equal to *AC*, both being 1 km, *B* decides to overtake *A* before *C* does. What minimum acceleration of car *B* is required to avoid an accident?

Ans. Fig 3.04 shows the conditions of the problem.

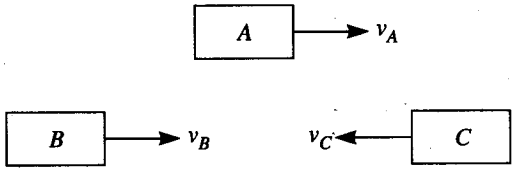


Fig. 3.04

v_A = +36 km/h = +10 ms^{-1} ; v_B = +54 km/h = +15 ms^{-1} ; v_C = −15 ms^{-1}

Relative velocity of car *C w.r.t.* car *A* = $v_C - v_A$ = (−15) − (+10) = −25 m s^{-1}

Time required by car *C* to cover 1 km to reach car *A* = $\frac{1000}{25}$ = 40 s

In order to avoid accident, the car *B* should be accelerated such that it overtakes car *A* in 40 s. Let the minimum required acceleration be '*a*'. Relative velocity of car *B w.r.t.* car *A* = $v_B - v_A$ = 15 − 10 = 5 ms^{-1}.

Now $v_0 = 5\ ms^{-1}$; $t = 40$ s ; $S = 1000$ m ; a = ?

Now, $$S = v_0 t + \frac{1}{2}\ at^2$$

or $$1000 = 5 \times 40 + \frac{1}{2} a \times (40)^2$$

or $$800\ a = 800 \quad \therefore \quad a = \mathbf{1\ ms^{-2}}$$

Q.9. Two towns *A* and *B* are connected by a regular bus service with a bus leaving in either direction every *T* minutes. A man cycling with a speed of 20 km/h in the direction *A* to *B* notices that a bus goes past him every 18 min in the direction of his motion and every 6 min

in the opposite direction. What is the period T of the bus service and with what speed (assumed constant) do the buses ply on the road ?

Ans. Let the constant speed with which the buses are plying on the road be v km/h.

$\therefore$ Speed of bus going from A to B *w.r.t.* the cyclist = $(v - 20)$ km/h

Speed of the bus going from B to A *w.r.t.* the cyclist = $(v + 20)$ km/h

The distance covered by the bus in time $T = v\,T$.

$$\therefore \quad \text{Time} = \frac{\text{Distance}}{\text{Velocity}}$$

$$\text{or} \quad \frac{18}{60} = \frac{vT}{v-20} \qquad ...(i)$$

$$\text{and} \quad \frac{6}{60} = \frac{vT}{v+20} \qquad ...(ii)$$

Dividing eq. (i) by eq. (ii), we get, $\frac{v+20}{v-20} = 3 \quad \therefore v = 40$ km/h

Putting $v = 40$ km/h in eq. (i), we get,

$$T = \frac{18(v-20)}{v} = \frac{18(40-20)}{40} = 9 \text{ min}$$

Q.10. **A player throws a ball upwards with an initial speed of 29.4 ms^{-1}. (*a*) What is the direction of acceleration during the upward motion of the ball? (*b*) What are the velocity and acceleration of the ball at the highest point of its motion? (*c*) Choose the $x = 0$ m and $t = 0$ s to be the location and time of the ball at its highest point, vertically downward to be the positive direction of X-axis, give the signs of position, velocity and acceleration of the ball during its upward and downward motion. (*d*) To what height does the ball rise and after how long does the ball return to the player's hands? [Take $g = 9.8$ ms^{-2} and neglect air resistance].**

Ans. (*a*) The ball moves under the action of gravity. Therefore, the direction of acceleration due to gravity is always vertically downwards.

(*b*) At the highest point, the velocity of the ball is zero and acceleration due to gravity is 9.8 ms^{-2} vertically downwards.

(*c*) When the highest point is chosen as the location for $x = 0$ and $t = 0$ and vertically downward direction to be the positive direction of X-axis, we have,

During upward motion. The signs of position and velocity are negative and the sign of acceleration is positive.

During downward motion. The signs of position, velocity and acceleration are positive.

(*d*) For the upward motion, $v_0 = -29.4$ ms^{-1} and $g = +9.8$ ms^{-2}.

$$\therefore \quad v^2 - v_0^2 = 2gh \quad \text{or} \quad h = \frac{-v_0^2}{2g} = \frac{-(-29.4)^2}{2\times 9.8} = -44.1 \text{ m}$$

The negative sign shows that the distance is covered in the upward direction.

If t is the time taken to reach the highest point, then,

$$v = v_0 + gt \quad \text{or} \quad 0 = -29.4 + 9.8t \quad \therefore t = 3\text{s}$$

Now for vertical motion under gravity, time of ascent = time of descent.

$\therefore$ Time taken by the ball to return to player's hands = $3 + 3 = 6$ s

Q.11. **Read each statement below carefully and state with reasons and examples, if it is true or false; A particle in one-dimensional motion (*a*) with zero speed at an instant may have non-zero acceleration at that instant (*b*) with zero speed may have non-zero velocity (*c*) with constant speed must have zero acceleration (*d*) with positive value of acceleration must be speeding up.**

(*b*) Referring to Fig. 3.06,

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{AC}{t}$$

$$\text{Average speed} = \frac{\text{Total path length}}{\text{Time}} = \frac{AB + BC}{t}$$

Since $(AB + BC) > AC$, the average speed is greater than the average velocity. If the particle moves in a straight line, displacement = total path length in a given time so that average speed will be equal to the average velocity. Thus in either case (*a*) or (*b*) the second quantity is either greater than or equal to first quantity. Equality sign is true when displacement is equal to distance.

Q.14. A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km/h. Finding the market closed, he instantly turns and walks back home with a speed of 7.5 km/h. What is the (*a*) magnitude of average velocity and (*b*) average speed of the man over the interval of time (*i*) 0 – 30 minutes (*ii*) 0 – 50 minutes (*iii*) 0 – 40 minutes?

Ans. (*i*) **Time interval 0–30 minutes.** While going to the market, the man walks at a speed of 5 km/h.

$\therefore$ Displacement in 30 min = 5/2 = 2.5 km

$$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Time}} = \frac{2.5\text{km}}{0.5\text{h}} = \mathbf{5\ km/h}$$

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{2.5\text{km}}{0.5\text{h}}\ \mathbf{5\ km/h}$$

(*ii*) **Time interval 0 – 50 minutes.** While going to the market the man walks at a speed of 5 km/h. Since the market is 2.5 km from his home, he takes 30 minutes to reach the market. While coming back from the market, he walks at a speed of 7.5 km/h so that time taken to reach home = 2.5/7.5 = 1/3 hour = (1/3) × 60 = 20 minutes. This means that in 50 minutes the man walks back home. Therefore, net displacement = 0.

$\therefore$ $$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Time}} = \frac{0}{50\text{min}} = \mathbf{0}$$

Now in 50 min, the total distance travelled = 2.5 + 2.5 = 5 km and total time taken = 50/60 hr = 5/6 hr.

$\therefore$ $$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{5}{5/6} = \mathbf{6\ km/h}$$

(*iii*) **Time interval 0–40 minutes.** In 30 minutes, the man covers a distance of 2.5 km in going from home to the market. While returning home, he covers a distance of 1.25 km in 10 minutes ($\because$ return speed is 7.5 km/h).

$\therefore$ Net displacement during time interval 0 – 40 min

= 2.5 – 1.25 = 1.25 km

$$\text{Average velocity} = \frac{\text{Net displacement}}{\text{Total time}} = \frac{1.25}{40/60} = \mathbf{1.875\ km/h}$$

Total distance travelled in 40 min = 2.5+ 1.25= 3.75 km

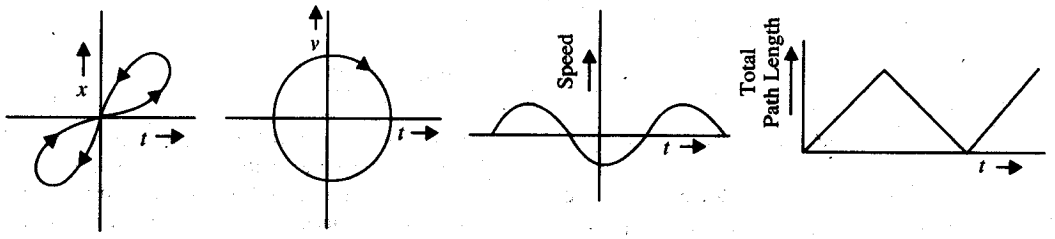
$\therefore$ $$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time}} = \frac{3.75}{40/60} = \mathbf{5.625\ km/h}$$

Q.15. In Q.13 and Q.14, we have carefully distinguished between average speed and magnitude of average velocity. No such distinction is necessary when we consider instantaneous speed and magnitude of velocity. The instantaneous speed is always equal to the magnitude of instantaneous velocity. Why?

Ans. Instantaneous velocity, $v = \frac{dx}{dt}$

In such a small interval of time (dt), the direction of motion of the particle is not supposed to change so that displacement is effectively equal to the distance. Therefore, instantaneous speed is equal to the magnitude of instantaneous velocity.

Q.16. **Look at the graphs (*a*) to (*d*) [Fig. 3.07] carefully and state, with reasons, which of these cannot possibly represent one-dimensional motion of a particle.**



(*a*) (*b*) (*c*) (*d*)

Fig. 3.07

Ans. (*a*) The graph shown in Fig. 3.07 (*a*) does not represent one-dimensional motion. It is because the particle has two positions at the same time which is not possible in one-dimensional motion.

(*b*) The graph shown in Fig. 3.07 (*b*) does not represent one-dimensional motion. It is because a particle cannot have positive and negative velocity at the same time in one-dimensional motion.

(*c*) The graph shown in Fig. 3.07 (*c*) does not represent one-dimensional motion because the particle speed can never be negative.

(*d*) The graph shown in Fig. 3.07 (*d*) does not represent one-dimensional motion. It is because the graph shows that total path length decreases after certain time. However, the total path length of a moving particle can never decrease with time.

Q.17. **Fig. 3.08 shows $x - t$ graph of one dimensional motion of a particle. Is it correct to say from the graph that the particle moves in a straight line for $t < 0$ and on a parabolic path for $t > 0$? If not, suggest a suitable physical context for this graph.**

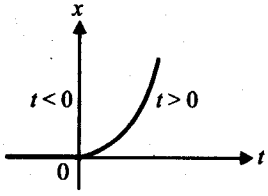


Fig. 3.08

Ans. No, because the $x - t$ graph does not represent the path taken by the particle. From the graph, at $t = 0$, $x = 0$. This graph represents the motion of a freely falling body.

Q.18. **A police van moving on a highway with a speed of 30 km/h fires a bullet at a thief's car speeding away in the same direction with a speed of 192 km/h. If the muzzle speed of bullet is 150 ms^{-1}, with what speed does the bullet hit thief's car?**

Ans. Speed of police van, $v_{PE} = 30$ km/h $= \frac{25}{3}$ ms^{-1}

Speed of bullet, $v_{BE} = 150$ ms^{-1}

Speed of thief's car, $v_{TE} = 192$ km/h $= \frac{160}{3}$ ms^{-1}

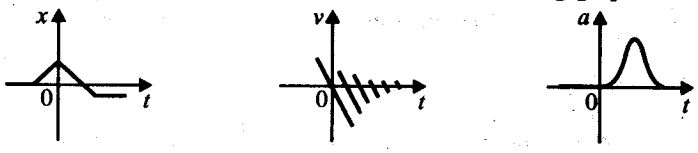
When the bullet is fired from police van, the net speed of bullet is the sum of v_{PE} and v_{BE}.

$$\therefore \quad \text{Net speed of bullet} = v_{PE} + v_{BE} = \frac{25}{3} + 150 = \frac{475}{3} ms^{-1}$$

The bullet will hit the thief's car with relative speed of the bullet *w.r.t.* thief's car. Therefore, the speed with which the bullet hits thief's car is

$$v = \frac{475}{3} - \frac{160}{3} = \frac{315}{3} = 105 \text{ ms}^{-1}$$

Q.19. **Suggest a suitable physical situation for each of the following graphs in Fig. 3.09.**



(*a*) (*b*) (*c*)

Fig. 3.09

Ans. (*a*) The $x - t$ graph in Fig. 3.09(*a*) shows that initially $x = 0$ (*i.e.*, the body is at rest). Then x increases with time, attains a constant value and again reduces to zero with time. Then x increases in the opposite direction till it again attains a constant value (*i.e.*, body comes to rest). Such a physical situation arises when a ball resting on a smooth floor is kicked. The ball rebounds from a wall with reduced speed and moves to the opposite wall which stops it.

(*b*) The $v - t$ graph in Fig. 3.09(*b*) shows that the velocity of the body changes sign again and again with passage of time and each time magnitude of velocity is reduced. Such a physical situation arises when a ball is thrown up with some initial velocity and returns back as freely falling body. On striking the ground, it rebounds with reduced speed each time it strikes the ground.

(*c*) The $a - t$ graph in Fig. 3.09(*c*) shows that initially the body moves with uniform velocity (*i.e.*, $a = 0$). Its acceleration increases for a short time and then falls to zero and thereafter, the body moves with a constant velocity. Such a physical situation arises when a cricket ball moving with a uniform speed is hit with a bat for a short interval of time.

Q.20. Figure 3.010 gives the $x - t$ plot of a particle executing one-dimensional simple harmonic motion. (You will learn about this motion in more detail in chapter 25). Give the signs of position, velocity and acceleration variables of the particle at $t = 0.3$ s, 1.2 s, –1.2 s.

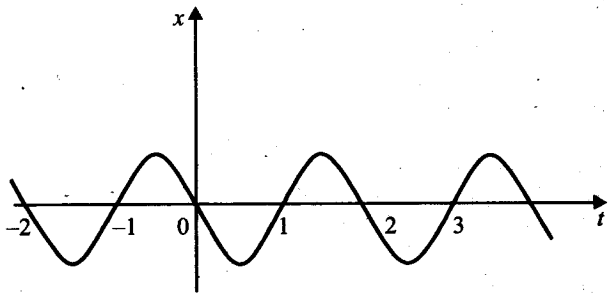


Fig. 3.010

Ans. In S.H.M., acceleration $a = -\omega^2 x$ where ω (*i.e.*, angular frequency) is constant.

(*i*) **At $t = 0.3$ s.** It is clear that x is negative. The slope of $x - t$ graph is negative so that velocity is also negative. Since $a = -\omega^2 x$ and x is negative, acceleration a is positive.

(*ii*) **At $t = 1.2$ s.** It is clear that x is positive. The slope of $x - t$ graph is positive so that velocity is also positive. Since $a = -\omega^2 x$ and x is positive, acceleration a is negative.

(*iii*) **At $t = -1.2$ s.** It is clear that x is negative. The slope of $x - t$ graph is negative. Since both x and t are negative, velocity is positive. The acceleration a is also positive.

Q.21. Fig. 3.011 shows the $x - t$ graph of a particle in one-dimensional motion. Three different equal intervals of time are shown. In which interval the average speed is greatest and in which it is the least? Give the sign of average speed for each interval.

Ans. The average speed in a small interval of time is equal to the slope of $x - t$ graph during that interval of time. It is clear from Fig. 3.011 that average speed is greatest in the time interval 3 because slope of $x - t$ graph is greatest in this time interval. The average speed is least in interval 2 because slope is least in this interval. The average speed is positive in time intervals 1 and 2 because slope of $x - t$ graph is positive for these time intervals. However, average speed in time interval 3 is negative because slope of $x - t$ graph is negative.

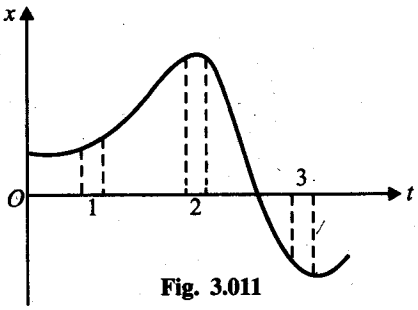


Fig. 3.011

Q.22. Fig. 3.012 gives the speed-time graph of a particle in one-dimensional motion. Three different equal intervals of time are shown. In which interval is the average acceleration greatest in magnitude? In which interval is the average speed greatest? Choosing the positive direction

as the constant direction of motion, give the signs of 'u' and 'a' in the three intervals. What are the accelerations at the points A, B, C and D?

Ans. The average acceleration in a small interval of time is equal to the slope of velocity-time graph during that interval of time. It is clear from Fig. 3.012 that slope of $v - t$ graph is maximum in interval 2 (as compared to intervals 1 and 3). Therefore, average acceleration is maximum in time interval 2. The average speed is greatest in time interval 3 because point D is at maximum on speed axis.

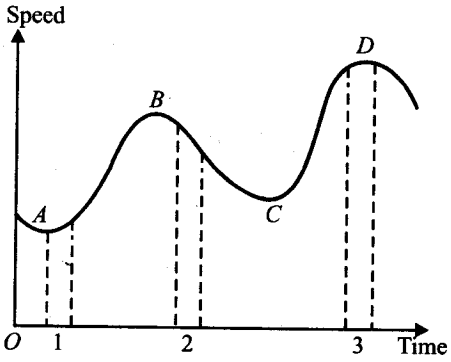


Fig. 3.012

In interval 1, the slope of velocity-time graph is positive. Therefore, acceleration as well as speed is positive. In interval 2, the slope of velocity-time graph is negative. Therefore, acceleration is negative but speed is positive. In interval 3, the velocity-time graph is parallel to the time axis. Therefore, acceleration is zero but speed is positive. At points A, B, C and D, the velocity-time graph is parallel to time axis. Therefore, acceleration is zero at these four points.

Q.23. A three wheeler starts from rest, accelerates uniformly with 1 ms^{-2} on a straight road for 10 s and then moves with uniform velocity. Plot a graph between the distance covered by the vehicle during the nth second (n = 1, 2, 3, ...) versus n. What do you expect the plot to be during accelerated motion: a straight line or a parabola?

Ans. Here $v_0 = 0$; $a = 1$ m/s^2

Distance covered in the nth second is

$$S_{nth} = v_0 + \frac{a}{2}(2n-1)$$

or $$S_{nth} = 0 + \frac{1}{2}(2n-1) = 0.5(2n - 1) \quad ...(i)$$

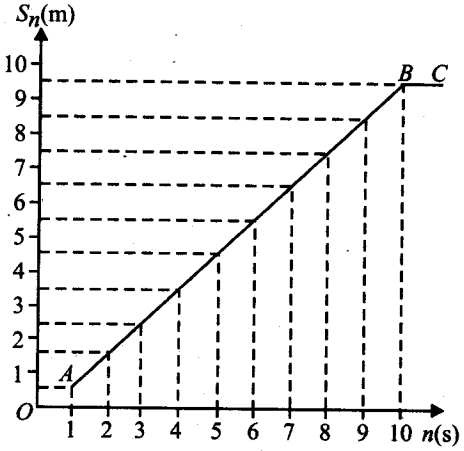


Fig. 3.013

Putting n = 1, 2, 3, ... in eq. (i), we get the following table of S_{nth} versus n.

n	1	2	3	4	5	6	7	8	9	10
S_{nth}	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

When we draw the graph between S_{nth} and n [See Fig. 3.013], we get straight line AB because $S_{nth} \propto n$. After 10 s, the three wheeler moves with uniform velocity so that graph is a straight line BC parallel to time axis.

Q.24. A boy standing on a stationary lift (open from above) throws a ball upwards with maximum initial speed he can, equal to 49 m/s. How much time does the ball take to return to his hands ? If the lift starts moving up with a uniform speed of 5 m/s and the boy again throws the ball up with the maximum speed he can, how long does the ball take to return to his hands ?

Ans. Let the time taken by the ball to reach the highest point be t. Then,

Now $v_0 = 49$ m/s ; $v = 0$; $g = -9.8$ m/s^2

$v = v_0 + gt$ or $0 = 49 - 9.8t$ $\therefore$ $t = 49/9.8 = 5$ s

Since time taken for upward journey is equal to the time taken for downward journey,

$\therefore$ Total time taken = $2t = 2 \times 5 =$ **10 s**

When the lift starts moving upward with uniform speed of 5 ms^{-1}, there is no change of relative velocity of ball *w.r.t.* boy. Therefore, the total time taken will remain the same *i.e.*, 10 s.

Q.25. On a long horizontal moving belt (Fig. 3.014), a child runs to and fro with a speed of 9 km/h (with respect to belt) between his father and mother located 50 m apart on the moving belt. The belt moves with a speed of 4 km/h. For an observer on stationary platform outside, what is the (*a*) speed of the child running in the direction of motion of the belt (*b*) speed of the child running opposite to the direction of motion of the belt and (*c*) time taken by the child in cases (*a*) and (*b*)? Which of the answers alter, if motion is viewed by one of the parents?

Ans. Fig. 3.014 shows the conditions of the problem. Considering that positive direction of *X*-axis is from left to right, we have,

v_{BE} = Speed of belt *w.r.t.* earth = 4 km/h

v_{CB} = Speed of child *w.r.t.* belt = 9 km/h

(*a*) **When child runs in the direction of motion of belt.** In this case, v_{BE}= +4 km/h and v_{CB} = +9 km/h.

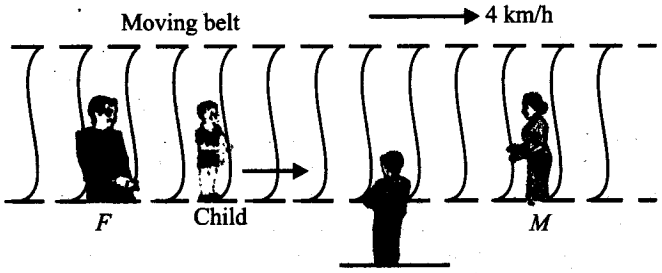


Fig. 3.014

Speed of child *w.r.t.* earth, $v_{CE} = v_{CB} + v_{BE}$ = (+9) + (+4) = **13 km/h**

(*b*) **When child runs opposite to the direction of belt.** In this case, v_{BE} = +4 km/h ; v_{CB} = –9 km/h.

Speed of child *w.r.t.* earth, $v_{CE} = v_{CB} + v_{BE}$ = (–9) + (+4) = **–5 km/h**

The negative sign shows that the child appears to be moving from right to left with a speed of 5 km/h.

(*c*) **Time taken.** Distance between parents = 50 m

Since parents and child are in the same frame of reference (*i.e.*, moving belt), the speed of the child as observed by the stationary observer will be 9 km/h (= 2.5 m/s) for the either direction of motion of the child.

$\therefore$ Time taken in cases (*a*) and (*b*) = 50/2.5= **20 s**

If motion is viewed by one of the parents, the speed of the child is 9 km/h in either direction. Therefore, answers (*a*) and (*b*) will alter. However, answer (*c*) remains unaltered.

Q.26. Two stones are thrown up simultaneously from the edge of a cliff 200 m high with initial speeds of 15 ms^{-1} and 30 ms^{-1}. Verify that the graph shown in Fig. 3.015 correctly represents the time variation of the relative position of the second stone with respect to the first. Neglect air resistance and assume that the stones do not rebound after hitting the ground. Take g = 10 ms^{-2}. Give the equations for the linear and curved parts of the plot.

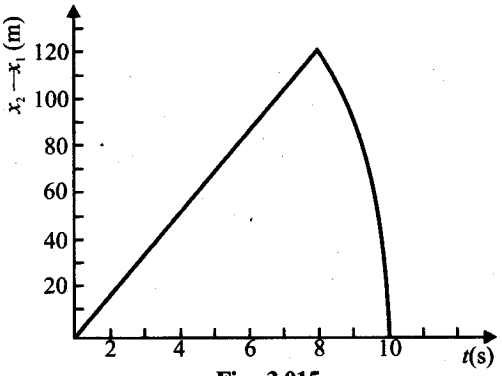


Fig. 3.015

Ans. For a uniformly accelerated motion, the position x of a body in time t is given by ;

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Motion of first stone. $x_0 = 200$ m ; $v_0 = 15$ ms^{-1} ; $a = g = -10$ ms^{-2}

If x_1 is the position (height above the ground) of the first stone after time t, then,

$$x_1 = 200 + 15t + \frac{1}{2}(-10)t^2 \quad \text{or} \quad x_1 = 200 + 15t - 5t^2 \qquad ...(i)$$

The time after which the first stone hits the ground can be found by putting $x_1 = 0$ in eq. (*i*).

$$0 = 200 + 15t - 5t^2 \quad \text{or} \quad t^2 - 3t - 40 = 0 \quad \therefore\ t = 8 \text{ s}$$

Motion of second stone. $x_0 = 200$ m ; $v_0 = 30$ ms^{-1} ; $a = g = -10$ ms^{-2}

If x_2 is the position (height above the ground) of the second stone after time t, then,

$$x_2 = 200 + 30t + \frac{1}{2}(-10)t^2 \quad \text{or} \quad x_2 = 200 + 30t - 5t^2 \qquad ...(ii)$$

The time after which the second stone hits the ground can be found by putting $x_2 = 0$ in eq. (*ii*) *i.e.*,

$$0 = 200 + 30t - 5t^2 \quad \text{or} \quad t^2 - 6t - 40 = 0 \quad \therefore\ t = 10 \text{ s}$$

From eqs. (*i*) and (*ii*), $x_2 - x_1 = 15t$ ...(*iii*)

Conclusions. The above calculations verify the correctness of the graph in Fig. 3.015 as shown below.

(*i*) Eq. (*iii*) (*i.e.*, $x_2 - x_1 = 15t$) describes the relative position of second stone *w.r.t.* first with time till the first stone hits the ground. Since first stone reaches the ground after 8 seconds, it means that upto 8 s $(x_2 - x_1) \propto t$ or graph between $(x_2 - x_1)$ and t is a straight line.

(*ii*) After 8 s, only the second stone will be in motion for 2 s. Therefore, from 8 s to 10 s, the motion of the second stone will be governed by the equation $x_2 = 200 + 30t - 5t^2$. For this reason graph is curved from 8 s to 10 s.

Q.27. **The speed-time graph of a particle moving along a fixed direction is shown in Fig. 3.016. Obtain the distance travelled by the particle between (*a*) $t = 0$ to 10 s (*b*) $t = 2$ to 6 s. What is the average speed of the particle over the intervals (*a*) and (*b*)?**

Ans. (*a*) The distance travelled by the particle from 0 to 10 s

= Area of ΔOAB from 0 to 10 s

$$= \frac{1}{2}\times 10\times 12 = 60 \text{ m}$$

Average speed = 60/10 = 6 ms^{-1}

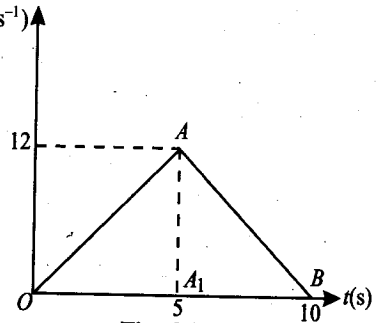


Fig. 3.016

(*b*) From time $t = 0$ s to $t = 5$ s, the particle moves with uniform acceleration a_1 given by ;

$$a_1 = \frac{v - v_0}{t} = \frac{12 - 0}{5} = 2.4 \text{ m/s}$$

Velocity of particle after 2 s is

$$v_1 = v_0 + a_1 \times 2 = 0 + 2.4 \times 2 = 4.8 \text{ m/s}$$

Distance travelled by the particle from $t = 2$ s to $t = 5$ s is

$$S_1 = v_1 t_1 + \frac{1}{2}a_1 t_1^2 \qquad \text{(Here } t_1 = 5 - 2 = 3\text{s)}$$

$$= 4.8\times 3 + \frac{1}{2}(2.4)\times 3^2 = 25.2 \text{ m}$$

From time $t = 5$ s to $t = 10$ s, the particle moves with uniform retardation given by ;

$$a_2 = \frac{0 - 12}{5} = -2.4 \text{ m/s}^2$$

For the motion of the particle from t = 5 s to t = 6 s, we have,

$$v_2 = 12 \text{ ms}^{-1},\ a_2 = -2.4 \text{ m/s}^2;\ t_2 = 1 \text{ s}$$

$\therefore$ Distance travelled by the particle from t = 5 s to t = 6 s is

$$S_2 = v_2 t_2 + \frac{1}{2} a_2 t_2^2$$

$$= 12 \times 1 + \frac{1}{2}(-2.4) \times 1^2 = 10.8 \text{ m}$$

$\therefore$ Total distance travelled, $S = S_1 + S_2 = 25.2 + 10.8 = 36$ m

$$\text{Average speed} = \frac{36}{6-2} = \frac{36}{4} = 9 \text{ ms}^{-1}$$

Q.28. **The velocity-time graph of a particle in one dimensional motion is shown in Fig. 3.017. Which of the following formulae are correct for describing the motion of the particle over the time interval t_1 and t_2.**

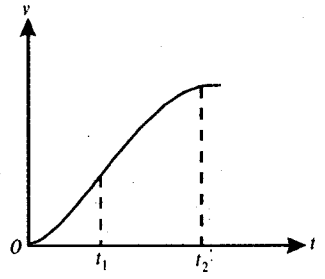


Fig. 3.017

(*i*) $x(t_2) = x(t_1) + v(t_1)(t_2 - t_1) + \frac{1}{2} a (t_2 - t_1)^2$

(*ii*) $v(t_2) = v(t_1) + a(t_2 - t_1)$

(*iii*) $v_{av} = \{r(t_2) - r(t_1)\}/(t_2 - t_1)$

(*iv*) $a_{av} = \{v(t_2) - v(t_1)\}/(t_2 - t_1)$

(*v*) $x(t_2) = x(t_1) + v_{av}(t_2 - t_1) + \frac{1}{2} a_{av} (t_2 - t_1)^2$

(*vi*) **$x(t_2) - x(t_1)$ = area under the $v - t$ curve bounded by the time axis and dotted line shown**

Ans. If we look at the $v - t$ graph over the time interval t_1 and t_2, we find that slope is neither constant nor uniform. Therefore, relations (*i*), (*ii*) and (*v*) are not correct. However, relations (*iii*), (*iv*) and (*vi*) are correct.

Unit III

KINEMATICS

(Description of motion in two and three dimensions)

- **Vectors**
- **Motion in a Plane**
- **N.C.E.R.T. Textbook Exercises**

8

Vectors

INTRODUCTION

A particle confined to a straight line (one dimensional motion) can move in only two directions *viz.* either upward or downward from the selected origin or towards right or left from the origin. We can take the motion of the particle to be positive in one of these directions and negative in the other direction. Therefore, in one dimensional motion, the vector quantities like displacements, velocities and accelerations can have either positive or negative values. We can use simple arithmetic to add or subtract vector quantities in this case. However, the situation is not that simple in two or three dimensional motion. For example, in case of two dimensional motion, the particle moves in a plane –say *XY* plane. The position of the particle is described by two numbers *viz.* x and y coordinates. Suppose a particle is at point P_1 (x_1, y_1) at any instant and after some time it is at another point P_2 (x_2, y_2). The vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ (O is the origin of the coordinate system) are not along the same line as was the case with one dimensional motion. To carry out the addition or subtraction of vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$, we require special methods. This is called *vector algebra.* Therefore, in order to study two dimensional motion (*i.e.,* motion in a plane) and three dimensional motion (*i.e.,* motion in space), we need the help of vector algebra. In this chapter, we shall discuss the various properties of vectors.

8.1. VECTOR

The physical quantities in physics are divided into two groups *viz. scalars* and *vectors.* Those physical quantities which have magnitude only are called scalars *e.g.,* mass, length, temperature, speed *etc.* The physical quantities which have magnitude and direction are called vectors *e.g.,* displacement, velocity, acceleration *etc.* If a physical quantity has both magnitude and direction but does not obey the laws of vector addition, it will not be called a vector. For example, electric current in a wire has both magnitude and direction but it does not obey the laws of vector addition. Therefore, electric current is not a vector.

A physical quantity that has both magnitude and direction and also obeys the laws of vector addition is called a vector.

A vector is represented *graphically* by a straight line with an arrowhead as shown in Fig. 8.1. The length of line (OA) represents the magnitude of the vector (on suitable scale) and the arrowhead indicates its direction.

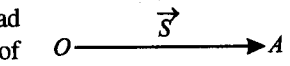


Fig. 8.1

Thus in Fig. 8.1, displacement vector $(\vec{S})$ is represented graphically. The starting point (O) of the arrowed line is called **tail of the vector** and the end (A) of the arrowed line is called **head** or **tip of the vector.**

In *writing*, a vector is represented by a single letter with an arrow on it. Thus in Fig. 8.1, displacement vector will be written as $\vec{S}$. The magnitude of the vector is represented by the same letter used for the vector without arrow. For example, the magnitude of displacement vector $\vec{S}$ is represented as S or $|\vec{S}|$.

Discussion. The following points are worth noting :

(*i*) The magnitude of a vector is a scalar and is always positive.

(*ii*) The magnitude of a vector is also called *modulus* of the vector and is represented by enclosing the vector symbol between two vertical lines. For example, the modulus of displacement vector $\vec{S}$ will be represented as $|\vec{S}|$.

(*iii*) Vectors can be added, subtracted and multiplied. *However, division of a vector by another vector is not a valid operation in vector algebra.* It is because the division of a vector by a direction is not possible.

8.2. SOME DEFINITIONS IN VECTOR ALGEBRA

(*i*) Equal vectors. *The two vectors are said to be equal if they have the same magnitude and the same direction.*

In Fig. 8.2, the two vectors $\vec{A}$ and $\vec{B}$ are equal vectors because they have the same magnitude and the same direction. For two vectors to be equal, it is not necessary that their tails should have the same starting location. Equal vectors may have different locations but they must have the same magnitude and the same direction. Thus in Fig. 8.2(*i*), the two equal vectors $\vec{A}$ and $\vec{B}$ have the same starting location (O). However, in Fig. 8.2(*ii*), the two equal vectors $\vec{A}$ and $\vec{B}$ have different starting locations (O). *This property of equal vectors is important because we can shift a vector from one location to another in a diagram provided its magnitude and direction are not changed.*

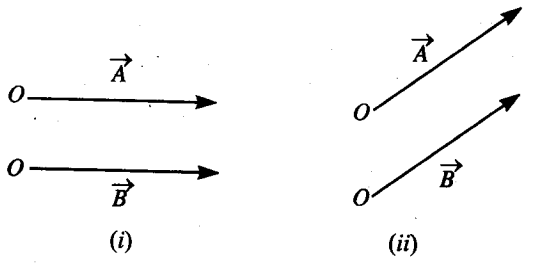


Fig. 8.2

(*ii*) Negative of a vector. *A vector is said to be negative vector of a given vector if its magnitude is the same as that of the given vector but its direction is opposite.*

The negative vector of $\vec{A}$ is represented as $-\vec{A}$ as shown in Fig. 8.3. Note that vectors $\vec{A}$ and $-\vec{A}$ have the same magnitude but opposite directions. In other words, the vectors have the same length but opposite directions. In order to find the negative of a vector, we merely reverse its arrowhead. Clearly, the addition of a vector to its own negative vector will result in zero magnitude *i.e.*,

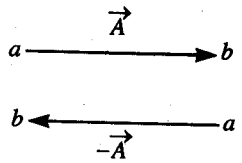


Fig. 8.3

$$\vec{A} + (-\vec{A}) = 0$$

This also applies to ordinary numbers. For example, the negative of 3 is –3 and their sum = 3 + (–3) = 0.

(*iii*) Unit vector. In vector algebra, it is often useful to represent a vector by its magnitude and a *unit vector.* A unit vector is a dimensionless vector that has a magnitude of 1 and has the same direction as that of the given vector.

A unit vector of the given vector is a vector of unit magnitude and has the same direction as that of the given vector.

For example, any vector $\vec{A}$ can be written as :

$$\vec{A} = A\hat{A}$$

Here A is the magnitude of $\vec{A}$ and $\hat{A}$ is the unit vector whose magnitude is 1 and direction is the same as that of $\vec{A}$. Note that *unit vector* $\hat{A}$ *is read as 'A cap' or 'A hat'.*

A unit vector is used to specify a given direction. Unit vectors have no other physical significance. Vectors can be conveniently written in terms of unit vectors which point along the chosen coordinate axes. In rectangular coordinate system, these unit vectors are called $\hat{i}$, $\hat{j}$ and $\hat{k}$; they point respectively along the positive X, Y and Z axes as shown in Fig. 8.4. Since unit vectors have magnitude of 1, they may be used to give direction to vectors. For example, if a force (vector) of 6 N is acting along the + X direction, we can represent the force as $6\hat{i}$ N. Similarly, if an object has a velocity (vector) of $-5\hat{j}$ ms^{-1}, it means that magnitude of velocity is 5 ms^{-1} and its direction is along the –ve Y direction. Note that the only purpose of unit vectors is to indicate the direction of a vector.

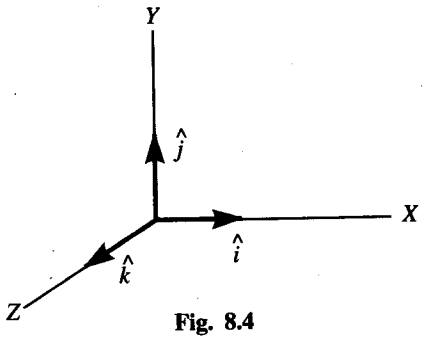


Fig. 8.4

(*iv*) **Zero vector.** *A vector that has zero magnitude is called zero vector or null vector.*

A zero vector or null vector is represented by $\vec{0}$ (arrow over the number zero). There is no need to specify the direction of a zero vector as its length is zero. The result of addition of a vector to its own negative vector is a zero vector *i.e.*, $\vec{A} + (-\vec{A}) = \vec{0}$. Similarly, when a vector is multiplied by zero, the result is a zero vector *i.e.*, $0(\vec{A}) = \vec{0}$.

Examples. (*a*) The velocity vector of a stationary object is a zero vector.

(*b*) The acceleration vector of an object moving with uniform velocity is a zero vector.

(*c*) The position vector of the origin of the coordinate axes is a zero vector.

8.3. POSITION VECTOR AND DISPLACEMENT VECTOR IN A PLANE

Consider the motion of an object in the XY plane with origin at O. Suppose at any time t_1, the object is at point $P_1(x_1, y_1)$ and at time t_2, it is at point $P_2(x_2, y_2)$.

1. **Position vector.** ***The position vector*** *of an object is the vector from the origin O of the coordinate system to the position of the object.*

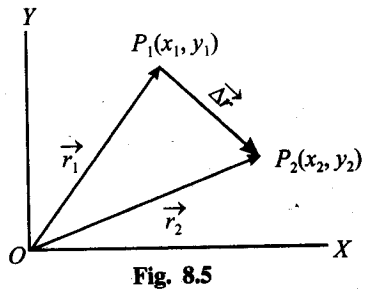


Fig. 8.5

Thus in Fig. 8.5, the position vector of the object at point P_1 is $\overrightarrow{OP_1}$ $(=\vec{r}_1)$ while the position vector of the object at point P_2 is $\overrightarrow{OP_2}$ $(=\vec{r}_2)$. The position vector of an object provides two important informations :

(*i*) *It gives straight line distance of the object from the origin O (starting point).*

(*ii*) *It gives the direction of the object w.r.t. origin.*

2. Displacement vector. Displacement is a vector quantity and is called displacement vector. We know that the displacement of an object is the straight line distance (shortest distance) between the initial position (P_1) and the final position (P_2) of the object. Thus in Fig. 8.5, $\overrightarrow{P_1P_2}$ $(= \overrightarrow{\Delta r})$ is the displacement vector. The straight line distance P_1P_2 is the magnitude of the displacement vector and the direction of the displacement vector is from P_1 to P_2.

Hence displacement vector is a vector that points from object's initial position to its final position and whose magnitude is equal to the straight line distance between the two points.

$$\text{Position vector, } \overrightarrow{OP_1} = x_1\hat{i} + y_1\hat{j}$$

$$\text{Position vector, } \overrightarrow{OP_2} = x_2\hat{i} + y_2\hat{j}$$

$$\therefore \quad \text{Displacement vector, } \overrightarrow{P_1P_2} = \text{Final position} - \text{Initial position}$$

$$= \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= (x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})$$

$$\therefore \quad \overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

8.4. ADDITION OF VECTORS

Vector addition can be done most easily by a graphical method. Suppose we want to add two vectors $\vec{P}$ and $\vec{Q}$ (suppose these are displacement vectors) as shown in Fig. 8.6(*i*). To add them graphically, we place the tail of vector $\vec{Q}$ at the head (or tip) of vector $\vec{P}$ as shown in Fig. 8.6(*ii*). Note that we can move a vector from one place to another but we cannot change the magnitude and direction of the vector while moving it. Having moved vector $\vec{Q}$, we draw another vector $\vec{R}$ from that tail of vector $\vec{P}$ to the head of vector $\vec{Q}$. This vector $\vec{R}$ represents the net displacement (or vector sum of $\vec{P}$ and $\vec{Q}$) and is called the **resultant** of vectors $\vec{P}$ and $\vec{Q}$. We can write the resultant in the form of mathematical equation as :

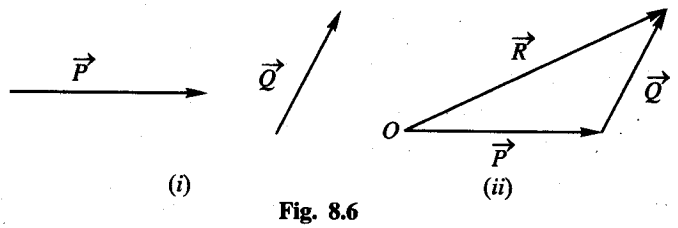


Fig. 8.6

$$\vec{R} = \vec{P} + \vec{Q}$$

*Hence the **resultant vector** of two or more vectors is the single vector which produces the same effect as is produced by the individual vectors together.* The advantage of resultant vector is that we can replace a number of vectors by a single vector. This makes mathematical analysis a simple affair.

The method of vector addition described above is called *triangle law of vector addition.* An alternative graphical method for adding two vectors $\vec{P}$ and $\vec{Q}$ is shown in Fig. 8.7. In this construction, the tails of the vectors $\vec{P}$ and $\vec{Q}$ are together and the resultant vector $\vec{R}$ is the diagonal of the parallelogram formed with the vectors $\vec{P}$ and $\vec{Q}$ as the adjacent sides. This is called *parallelogram law of vector addition.*

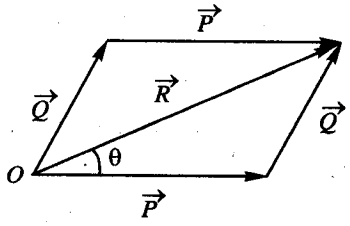


Fig. 8.7

(*i*) Triangle law of vector addition. *If two vectors acting simultaneously at a point are represented in magnitude and direction by the two sides of a triangle taken in the same order, then third or closing side of the triangle taken in the opposite order represents their resultant in magnitude and direction.*

The triangle law of vector addition is illustrated in Fig. 8.6(*ii*). Here vectors $\vec{P}$ and $\vec{Q}$ are represented in magnitude and direction by the two sides of a triangle taken in the same order. The third or closing side of the triangle taken in the opposite order represents their resultant $\vec{R}$ in magnitude and direction.

$$\therefore \qquad \vec{R} = \vec{P} + \vec{Q}$$

(*ii*) Parallelogram law of vector addition. *If two vectors acting simultaneously at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram, then the diagonal of the parallelogram passing through that point represents their resultant in magnitude and direction.*

The parallelogram law of vector addition is shown in Fig. 8.7. Here vectors $\vec{P}$ and $\vec{Q}$ are represented in magnitude and direction by the adjacent sides of the parallelogram. Then diagonal of the parallelogram passing through point O represents their resultant $\vec{R}$ in magnitude and direction.

8.5. ADDITION OF MORE THAN TWO VECTORS GRAPHICALLY

The *tail-to-tip method can be used to add any number of vectors graphically. As stated earlier, in this method the vectors are placed tip to tail one at a time *i.e.,* tail of the second vector is placed on the tip of the first vector, tail of the third vector is placed on the tip of the second vector and so on. The magnitudes and directions of the vectors should not be changed as they are moved about. The resultant vector $\vec{R}$ is represented in magnitude and direction from the tail of the first vector to the tip of the last vector. In other words, $\vec{R}$ is the closing side of the figure (of vectors) taken in the opposite order.

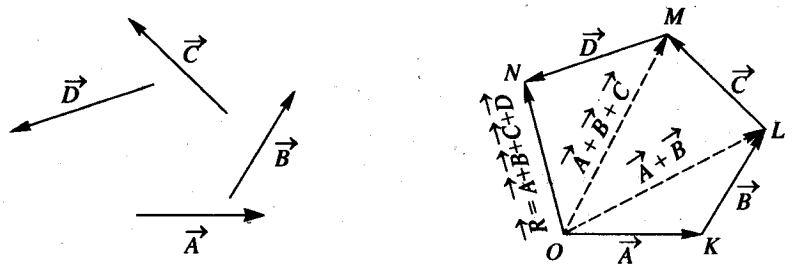


Fig. 8.8

Let us illustrate the tail-to-tip method of vector addition considering four vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ [See Fig. 8.8 (*i*)]. Applying tail-to-tip method, these vectors are represented in magnitude and direction by the sides $\overrightarrow{OK}$, $\overrightarrow{KL}$, $\overrightarrow{LM}$ and $\overrightarrow{MN}$ of the regular polygon respectively as shown in Fig. 8.8 (*ii*). Then, the resultant $\vec{R}$ is represented in magnitude and direction from the tail of the first vector to the tip of the last vector. In other words, the closing side of the polygon taken in the opposite order represents the resultant $\vec{R}$ in magnitude and direction *i.e.,* $\vec{R} = \overrightarrow{ON}$. This provides

* Note that tail-to-tip method has been explained in Art. 8.4.

us a useful rule for adding more than two vectors graphically. This is called **polygon law of vector addition**. It may be stated as under:

If a number of vectors acting simultaneously at a point are represented in magnitude and direction by the sides of a polygon taken in the same order, then closing side of the polygon taken in opposite order represents the resultant in magnitude and direction.

Proof. A little thought will show that polygon law of vector addition is an extension of triangle law of vector addition. According to triangle law of vector addition, $\overrightarrow{OL}$ is the resultant of $\vec{A}$ and $\vec{B}$ *i.e.*,

$$\overrightarrow{OL} = \vec{A} + \vec{B}$$

Likewise, $\overrightarrow{OM}$ is the resultant of $\overrightarrow{OL}$ and $\vec{C}$ *i.e.*,

$$\overrightarrow{OM} = \overrightarrow{OL} + \vec{C} = \vec{A} + \vec{B} + \vec{C}$$

$$\therefore \quad \overrightarrow{OM} = \vec{A} + \vec{C} + \vec{B}$$

Again $\overrightarrow{ON}$ is the resultant of $\overrightarrow{OM}$ and $\vec{D}$.

$$\therefore \quad \overrightarrow{ON} = \overrightarrow{OM} + \vec{D} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

$$\therefore \quad \vec{R} = \overrightarrow{ON} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

8.6. ANALYTICAL METHOD OF VECTOR ADDITION

The graphical method of adding vectors takes considerable time if done accurately. We can find the magnitude and direction of the resultant (or sum) of two vectors mathematically by using triangle or parallelogram law of vector addition.

1. Triangle law of vector addition. Let two vectors $\vec{P}$ and $\vec{Q}$ be represented in magnitude and direction by the sides $\overrightarrow{OA}$ and $\overrightarrow{AC}$ of a triangle OAC taken in the same order. Then according to triangle law of vector addition, the third or closing side of the triangle taken in the opposite order represents the resultant $\vec{R}\ (= \overrightarrow{OC})$ in magnitude and direction. Let $\vec{R}$ make an angle α with $\vec{P}$ *i.e.*, [See Fig. 8.9] $\angle AOC = \alpha$.

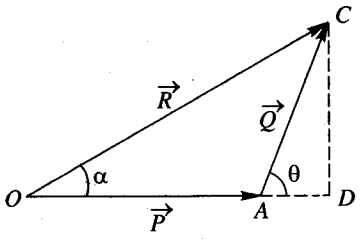


Fig. 8.9

(*i*) **Magnitude of Resultant.** From C, draw CD perpendicular to OA produced. Suppose $\angle CAD = \theta$. Note that θ is the angle between the two vectors. In the right angled triangle ODC,

$$(OC)^2 = (OD)^2 + (CD)^2 = (OA + AD)^2 + (CD)^2$$

[In right angled triangle ACD, $AD = Q\cos\theta$ and $CD = Q\sin\theta$]

$$\therefore \quad R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$= P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$= P^2 + Q^2(\sin^2\theta + \cos^2\theta) + 2PQ\cos\theta$$

$$= P^2 + Q^2 + 2PQ\cos\theta \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} \qquad \ldots (i)$$

(*ii*) Direction of resultant. From right angled triangle *ODC*,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD} = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\therefore \quad \tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta} \qquad \ldots(ii)$$

Thus, the magnitude of the resultant is given by eq. (*i*) and the direction of $\vec{R}$ with $\vec{P}$ is given by eq. (*ii*).

2. **Parallelogram law of vector addition.** Let two vectors $\vec{P}$ and $\vec{Q}$ be represented in magnitude and direction by the adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$ of the parallelogam *OACB* (See Fig. 8.10). Suppose the angle between the vectors is θ *i.e.*, $\angle AOB = \theta$. According to parallelogram law of vector addition, the diagonal $\overrightarrow{OC}$ represents the resultant $\vec{R}\,(=\overrightarrow{OC})$ in magnitude and direction. Suppose $\vec{R}$ makes angle. α with $\vec{P}$ *i.e.*, $\angle AOC = \alpha$.

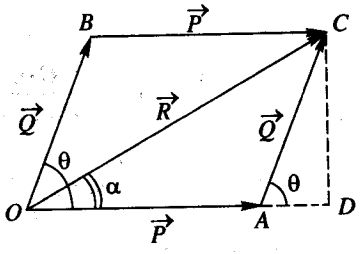


Fig. 8.10

(*i*) Magnitude of resultant. From *C*, draw *CD* perpendicular on *OA* produced. From geometry, $\angle DAC = \theta$. In right angled triangle *ODC*,

$$(OC)^2 = (OD)^2 + (CD)^2 = (OA+AD)^2 + (CD)^2$$

$$\therefore \quad R^2 = (P+Q\cos\theta)^2 + (Q\sin\theta)^2 \qquad [\because AD = Q\cos\theta\,;\ CD = Q\sin\theta]$$

$$= P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$= P^2 + Q^2(\sin^2\theta + \cos^2\theta) + 2PQ\cos\theta$$

$$= P^2 + Q^2 + 2PQ\cos\theta$$

$$\therefore \quad R = \sqrt{P^2+Q^2+2PQ\cos\theta} \qquad \ldots(iii)$$

(*ii*) Direction of resultant. From right angled triangle *ODC*,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD} = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\therefore \quad \tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta} \qquad \ldots(iv)$$

Thus, the magnitude of the resultant is given by eq. (*iii*) and the direction of $\vec{R}$ with $\vec{P}$ is given by eq. (*iv*).

Different cases.

(*a*) ***When the vectors act along the same direction*** *i.e.*, $\theta = 0°$.

$$R = \sqrt{P^2+Q^2+2PQ\cos 0°} = \sqrt{(P+Q)^2} \quad \therefore \quad R = P + Q$$

$$\tan\alpha = \frac{Q\sin 0°}{P+Q\cos 0°} = \frac{Q(0)}{P+Q(1)} = 0 \quad \therefore \quad \alpha = 0°$$

Therefore, the magnitude of the resultant of two vectors in the same direction is equal to the sum of the magnitudes of two vectors and the direction of the resultant is along the direction of $\vec{P}$ *and* $\vec{Q}$.

(*b*) ***When the vectors are at right angles*** *i.e.*, $\theta = 90°$.

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 90°} = \sqrt{P^2 + Q^2}$$

$$\tan\alpha = \frac{Q\sin 90°}{P + Q\cos 90°} = \frac{Q(1)}{P + Q(0)} = \frac{Q}{P} \quad \therefore \quad \tan\alpha = \frac{Q}{P}$$

(*c*) ***When the vectors act along opposite directions*** *i.e.*, $\theta = 180°$.

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 180°} = \sqrt{(P-Q)^2}$$

$$\therefore \quad R = P - Q$$

$$\tan\alpha = \frac{Q\sin 180°}{P + Q\cos 180°} = \frac{Q(0)}{P + Q(-1)} = 0$$

$$\therefore \quad \alpha = 0° \text{ or } 180°$$

Thus the magnitude of the resultant of two vectors acting in the opposite directions is equal to the difference in the magnitudes of the vectors. The resultant acts in the direction of the larger vector.

8.7. SOME PROPERTIES OF VECTORS

We now discuss some useful properties of vectors.

(*i*) Multiplication of a vector by a real number. *If we multiply a vector* $\vec{A}$ *by a real positive number n, we get the vector* $n\vec{A}$ *which has the same direction as* $\vec{A}$ *and magnitude nA i.e.,*

$$n\vec{A} = n(\vec{A})$$

Thus the magnitude of the vector becomes *n* times while its direction remains unchanged. For example, if we multiply vector $\vec{A}$ by 2, we get another vector $\vec{B}$ which has the same direction as that of $\vec{A}$ but twice the magnitude *i.e.*

$$\vec{B} = 2\vec{A}$$

If a vector is multiplied by a negative real number (i.e., –n), the magnitude of the vector becomes nA but direction is opposite to that of $\vec{A}$ *i.e.,*

$$-n(\vec{A}) = -n\vec{A}$$

For example, if we multiply vector $\vec{A}$ by –2, we get another vector $\vec{B}$ parallel to $\vec{A}$ but in opposite direction and of magnitude 2*A i.e.,*

$$\vec{B} = -2\vec{A}$$

(*ii*) Multiplication of a vector by a scalar. *If we multiply a vector* $\vec{A}$ *by a scalar S, the result is another vector with direction of* $\vec{A}$ *but magnitude SA. If S is negative,* $S\vec{A}$ *has a direction opposite to that of* $\vec{A}$.

If the quantities *S* and $\vec{A}$ both happen to have units, then vector $S\vec{A}$ is expressed in units indicated for such a product. For example, if $\vec{A}$ = 10 m/s north and *S* = 2 s, then $S\vec{A}$ = 10 m/s × 2 s = 20 m north.

(*iii*) Vector addition obeys commutative law. *According to this law, the resultant of the vectors remains the same in whatever order they may be added.*

In order to find the vector sum of $\vec{A}$ and $\vec{B}$, we can add (vectorially) either $\vec{A}$ to $\vec{B}$ or $\vec{B}$ to $\vec{A}$ (See Fig. 8.11). Vectors $\vec{A} + \vec{B}$ and $\vec{B} + \vec{A}$ are the same because they are parallel and have the same magnitude. Their order in the addition does not matter.

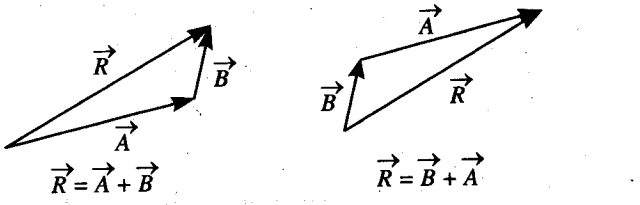


Fig. 8.11

$$\therefore \quad \vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

It is interesting to note that commutative law also holds for ordinary algebra because $3 + 2 = 2 + 3$.

(iv) **Vector addition obeys associative law.** *According to this law, the resultant of the vectors remains the same in whatever grouping they may be added.*

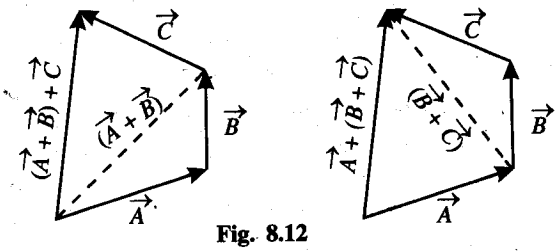


Fig. 8.12

Suppose we are to find the vector sum of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Then we can add $\vec{C}$ to the vector sum of $\vec{A} + \vec{B}$ or add $\vec{A}$ to the vector sum of $\vec{B} + \vec{C}$. Therefore, as shown in Fig. 8.12 :

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

This is quite logical. For the addition of any number of vectors, we can merely draw them in head-to-tail fashion. The sum of all vectors is then just a vector from the tail of the first vector to the head of the last vector. It is interesting to note that associative law also holds good for ordinary algebra because $(2 + 3) + 4 = 2 + (3 + 4)$.

Example 8.1. The angle between two vectors of equal magnitude is 120°. Prove that the magnitude of their resultant is equal to either of them.

Solution. $P = Q$; $\theta = 120°$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = \sqrt{P^2 + P^2 + 2P^2\cos 120°}$$

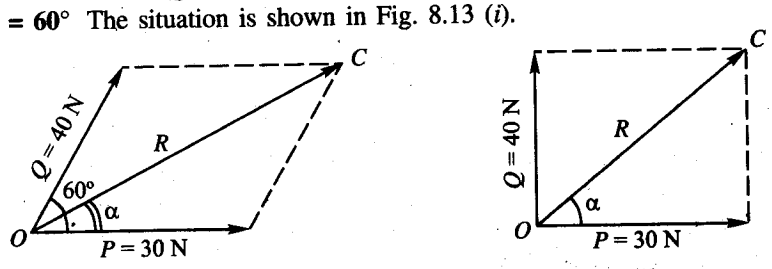
$$= \sqrt{2P^2 + 2P^2\left(\frac{-1}{2}\right)} = \sqrt{2P^2 - P^2} = P$$

$$\therefore \quad R = P$$

Example 8.2. Two forces of 30 N and 40 N are inclined to each other at an angle of 60°. Find their resultant. What will be the angle if the forces are inclined at right angles to each other?

Solution. $P = 30\ N$; $Q = 40\ N$

When $\theta = 60°$ The situation is shown in Fig. 8.13 *(i)*.



(i) Fig. 8.13 *(ii)*

$$R=\sqrt{P^2+Q^2+2PQ\cos\theta}=\sqrt{(30)^2+(40)^2+2\times30\times40\cos60^\circ}\ =60.83\ \text{N}$$

$$\tan\alpha=\frac{Q\sin\theta}{P+Q\cos\theta}=\frac{40\sin60^\circ}{30+40\cos60^\circ}=0.69 \quad \therefore \quad \theta=\tan^{-1}0.69=34.6^\circ$$

Therefore, the resultant has a magnitude of 60.83 N and makes an angle of 34.6° with 30 N force.

When $\theta = 90^\circ$. The situation is shown in Fig. 8.13 (*ii*).

$$R=\sqrt{P^2+Q^2+2PQ\cos\theta}=\sqrt{P^2+Q^2+2PQ\cos90^\circ}$$

$$=\sqrt{P^2+Q^2}=\sqrt{(30)^2+(40)^2}\ =50\ \text{N}$$

$$\tan\alpha=\frac{Q}{P}=\frac{40}{30}=1.33 \quad \therefore \quad \alpha=\tan^{-1}1.33\ =53.06^\circ$$

Therefore, the resultant has a magnitude of 50 N and makes an angle of 53.06° with 30 N force.

Example 8.3. **The resultant of two equal forces acting at right angles to each other is 1414 N. Find the magnitude of each force.**

Solution. $P = Q$; R = 1414 N ; $\theta = 90^\circ$

$$R=\sqrt{P^2+Q^2+2PQ\cos\theta}=\sqrt{P^2+P^2+2P^2\cos90^\circ}=\sqrt{2}\,P$$

$$\therefore \quad P=\frac{R}{\sqrt{2}}=\frac{1414}{\sqrt{2}}\ =1000\ \text{N} \quad \therefore \quad P\ =Q=1000\ \text{N}$$

Example 8.4. **The magnitudes of two vectors are equal and the angle between them is θ. Show that their resultant divides angle θ equally.**

Solution. $P = Q$; $\theta = \theta$

$$\tan\alpha=\frac{Q\sin\theta}{P+Q\cos\theta}=\frac{P\sin\theta}{P+P\cos\theta}=\frac{\sin\theta}{1+\cos\theta}$$

$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2} \quad \therefore \quad \alpha=\frac{\theta}{2}$$

We can also find the magnitude of the resultant.

$$R^2=\ P^2+P^2+2P^2\cos\theta=2P^2+2P^2\cos\theta=2P^2(1+\cos\theta)$$

$$=\ 2P^2\left[1+\left(2\cos^2\frac{\theta}{2}-1\right)\right]=4P^2\cos^2\frac{\theta}{2}$$

$$\therefore \quad R=\ 2P\cos\frac{\theta}{2}$$

These results are very important.

Example 8.5. **The sum of the magnitudes of two forces acting at a point is 18 N and the magnitude of the resultant is 12 N. If the resultant is at 90° with the force of smaller magnitude, what are the magnitudes of forces?**

Solution. Fig. 8.14 shows the conditions of the problem. Here $\vec{P}$ and $\vec{Q}$ are the two forces and $\vec{R}$ is their resultant. It is assumed that $\vec{P}$ is the smaller force.

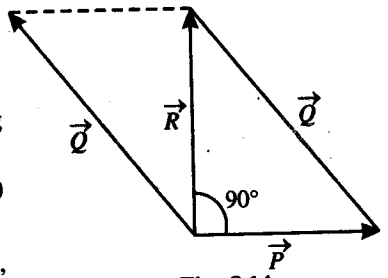


Fig. 8.14

It is given that $P + Q = 18$ N and $R = 12$ N. Referring to Fig. 8.14, we have,

$$R^2 = Q^2 - P^2 \quad \text{or} \quad R^2 = (Q + P)(Q - P)$$

$$\therefore \quad 144 = 18(Q - P) \quad \text{or} \quad Q - P = 8 \text{ N}$$

Now $P + Q = 18$ N and $Q - P = 8$ N. Therefore, $Q = 13$ N and $P = 5$ N.

Example 8.6. **A particle has a displacement of 12 m toward east and 5 m toward north and then 6 m vertically upward. Find the magnitude of the resultant displacement.**

Solution. Fig. 8.15 shows the conditions of the problem. $\overrightarrow{OB}$ is the resultant of $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

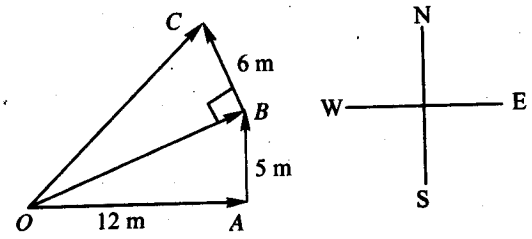


Fig. 8.15

$$\therefore \quad OB = \sqrt{(OA)^2 + (AB)^2} = \sqrt{(12)^2 + (5)^2} = 13 \text{ m}$$

Again $\overrightarrow{OC}$ is the resultant of $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

$$\therefore \quad OC = \sqrt{(OB)^2 + (BC)^2} = \sqrt{(13)^2 + (6)^2} = \sqrt{205} = 14.32 \text{ m}$$

Example 8.7. **A person moves 30 m north, then 20 m east and finally $30\sqrt{2}$ m south-west. What is the displacement from the original position?**

Solution. Figure 8.16 shows the conditions of the problem. Resultant displacement is given by ;

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$$

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along east (E) and north (N) respectively. Then,

$$\vec{S}_1 = 30\hat{j} \; ; \; \vec{S}_2 = 20\hat{i} \; ;$$

$$\vec{S}_3 = -30\sqrt{2}(\cos 45°\hat{i} + \sin 45°\hat{j}) = -30\hat{i} - 30\hat{j}$$

$$\therefore \quad \vec{S} = (30\hat{j}) + (20\hat{i}) + (-30\hat{i} - 30\hat{j}) = -10\hat{i}$$

Therefore, the resultant displacement is 10 m west.

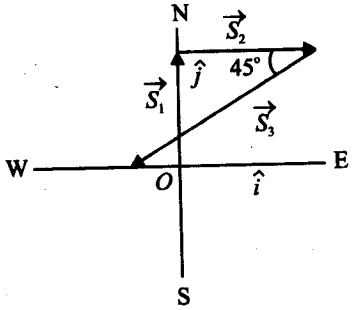


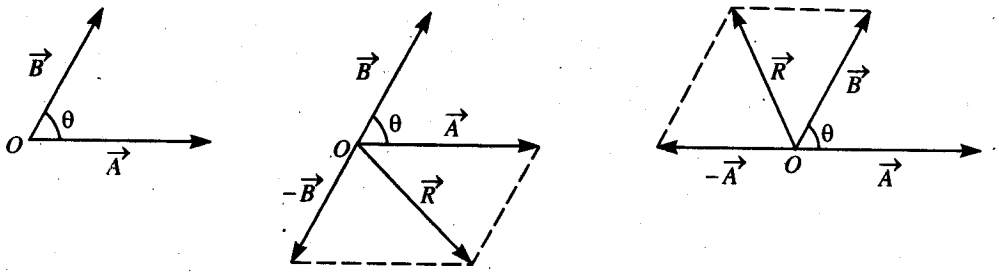
Fig. 8.16

8.8. SUBTRACTION OF VECTORS

In order to subtract vector $\vec{B}$ from vector $\vec{A}$, first reverse the direction of $\vec{B}$, thus producing $-\vec{B}$. Then add $\vec{A}$ and $(-\vec{B})$ by parallelogram law of vector addition.

$$\therefore \quad \vec{R} = \vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

Consider two vectors $\vec{A}$ and $\vec{B}$ of the same kind and inclined to each other at an angle θ as shown in Fig. 8.17. (*i*).



(*i*) (*ii*) (*iii*)

Fig. 8.17

(*i*) In order to find $\vec{A}-\vec{B}$, reverse the direction of $\vec{B}$, thus producing $-\vec{B}$ as shown in Fig. 8.17 (*ii*). Then find the sum of $\vec{A}$ and $-\vec{B}$ by parallelogram law of vector addition. The required difference is $\vec{R}$ and is shown in Fig. 8.17 (*ii*).

$$\therefore \quad \vec{R}=\vec{A}+(-\vec{B})=\vec{A}-\vec{B}$$

(*ii*) In order to find $\vec{B}-\vec{A}$, reverse the direction of $\vec{A}$, thus producing $-\vec{A}$ as shown in Fig. 8.17(*iii*). Then find the sum of $\vec{B}$ and $-\vec{A}$. The required difference is $\vec{R}$ and is shown in Fig. 8.17 (*iii*).

$$\therefore \quad \vec{R}=\vec{B}+(-\vec{A})=\vec{B}-\vec{A}$$

8.9. RELATIVE VELOCITY IN TWO DIMENSIONS

Two bodies *A* and *B* moving in a plane (two dimensional motion) may not have their motions along the same line. Consequently, their velocities $\vec{v}_A$ and $\vec{v}_B$ will be inclined at some angle as shown in Fig. 8.18.

(*i*) In order to find the relative velocity of body *A w.r.t.* to body *B* $(\vec{v}_{AB})$, we superimpose a velocity $-\vec{v}_B$ on both the bodies. As a result, the body *B* is brought to rest. Then the resultant velocity of the body *A* is $\vec{v}_A-\vec{v}_B$ and is equal to $\vec{v}_{AB}$.

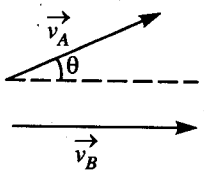


Fig. 8.18

$$\therefore \quad \vec{v}_{AB}=\vec{v}_A-\vec{v}_B=\vec{v}_A+(-\vec{v}_B)$$

Therefore, in order to find $\vec{v}_{AB}$, reverse the direction of $\vec{v}_B$ and find its sum with $\vec{v}_A$ by parallelogram law of vector addition.

(*ii*) Similarly, $\vec{v}_{BA}=\vec{v}_B-\vec{v}_A$

Again parallelogram law of vector addition can be used to find the sum of $\vec{v}_B$ and $-\vec{v}_A$.

8.10. MAGNITUDE AND DIRECTION OF RELATIVE VELOCITY

Consider two bodies *A* and *B* moving with velocities $\vec{v}_A$ and $\vec{v}_B$ respectively inclined at an angle θ.

(*i*) **Relative velocity of** *A w.r.t. B*.As discussed above, relative velocity of *A w.r.t. B* is

$$\vec{v}_{AB}=\vec{v}_A-\vec{v}_B=\vec{v}_A+(-\vec{v}_B)$$

In order to find $\vec{v}_{AB}$, we find the sum of $\vec{v}_A$ and $-\vec{v}_B$ by parallelogram law of vector addition. This is illustrated in Fig. 8.19 (*i*) where the vector sum of $\vec{v}_A$ and $-\vec{v}_B$ is $\vec{v}_{AB}(=\overrightarrow{OC})$.

Magnitude of $\vec{v}_{AB}$ is

$$v_{AB} = \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180° - \theta)}$$

or $$v_{AB} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

The angle α which $\vec{v}_{AB}$ makes with $-\vec{v}_B$ is

$$\tan\alpha = \frac{v_A \sin(180° - \theta)}{v_B + v_A \cos(180° - \theta)} \quad \text{or} \quad \tan\alpha = \frac{v_A \sin\theta}{v_B - v_A \cos\theta}$$

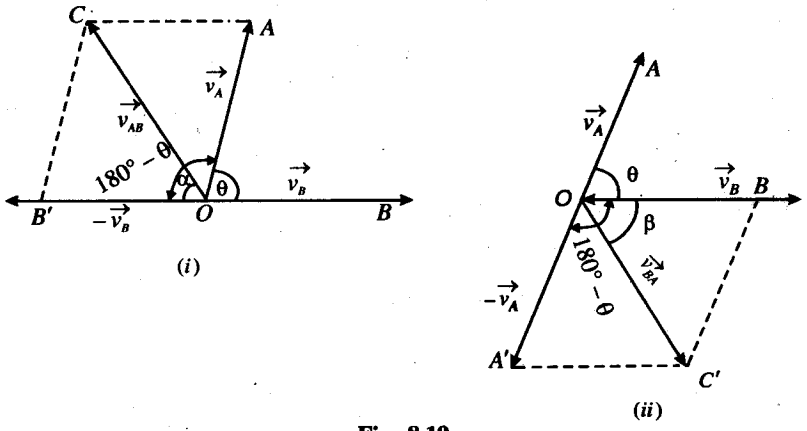


Fig. 8.19

(*ii*) Relative velocity of *B w.r.t. A*. The relative velocity of *B w.r.t. A* is

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$$

In order to find $\vec{v}_{BA}$, we find the sum of $\vec{v}_B$ and $-\vec{v}_A$ by parallelogram law of vector addition. This is illustrated in Fig. 8.19 (*ii*) where the vector sum of $\vec{v}_B$ and $-\vec{v}_A$ is $\vec{v}_{BA}$ $(=\overrightarrow{OC'})$.

Magnitude of $\vec{v}_{BA}$ is

$$v_{BA} = \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180° - \theta)}$$

or $$v_{BA} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

The angle β which $\vec{v}_{BA}$ makes with $\vec{v}_B$ is

$$\tan\beta = \frac{v_A \sin(180° - \theta)}{v_B + v_A \cos(180° - \theta)}$$

or $$\tan\beta = \frac{v_A \sin\theta}{v_B - v_A \cos\theta}$$

8.11. APPLICATIONS OF RELATIVE VELOCITY

We shall discuss a few applications of relative velocity in two dimensions.

(i) Boat to cross the river along shortest path. Fig. 8.20 shows river flowing from right to left. Point A is the starting point of the boat and point B is directly opposite to point A on the other side of the river. It is desired that boat should follow the shortest path AB. The river flow will carry the boat downstream as the boat crosses the river. In order that the boat follows the path AB, it should be rowed upstream making an angle θ with AB as shown in Fig. 8.20.

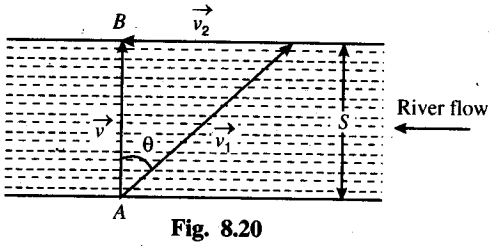


Fig. 8.20

Let $\overrightarrow{v_{BS}} = \vec{v}_1$ = Velocity of boat *w.r.t.* shore

$\overrightarrow{v_{WS}} = \vec{v}_2$ = Velocity of water *w.r.t.* shore

$\overrightarrow{v_{BW}} = \vec{v}$ = Velocity of boat *w.r.t.* water

$\therefore \quad \overrightarrow{v_{BW}} = \overrightarrow{v_{BS}} + \overrightarrow{v_{SW}} = \overrightarrow{v_{BS}} - \overrightarrow{v_{WS}}$

or $\vec{v} = \vec{v}_1 - \vec{v}_2$

The magnitude of $\vec{v}\ (=\overrightarrow{v_{BW}})$ is given by ;

$$v = \sqrt{v_1^2 - v_2^2}$$

Now the boat moves across AB with a velocity v.

$\therefore$ Time of crossing of boat, $t = \dfrac{AB}{v} = \dfrac{S}{v}$

Angle θ is given by ; $\sin\theta = v_2/v_1$

Note that angle θ is determined by the magnitudes of $\vec{v}_2$ and $\vec{v}_1$.

(ii) Boat to cross the river in shortest time. In order that the boat crosses the river in the shortest time, the river velocity $\vec{v}_2$ should help the boat velocity $\vec{v}_1$. Therefore, the boat should be rowed along AB. Now the resultant velocity $\vec{v}\ (=\overrightarrow{v_{BW}})$ is given by (See Fig. 8.21) ;

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

Magnitude of $\vec{v}$, $v = \sqrt{v_1^2 + v_2^2}$; $\tan\theta = \dfrac{v_2}{v_1}$

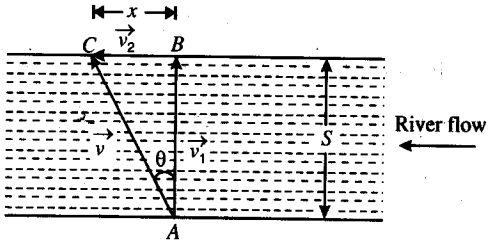


Fig. 8.21

Now the boat moves with velocity v along AC and will reach point C instead of point B. If $BC = x$, then,

$$AC = \sqrt{x^2 + S^2} \quad \text{where } S = \text{width of the river}$$

$\therefore$ Time of crossing of boat is

$$t = \frac{AC}{v} = \sqrt{\frac{x^2 + S^2}{v_1^2 + v_2^2}}$$

Now $\tan\theta = \dfrac{v_2}{v_1} = \dfrac{x}{S} \quad \therefore \quad x = \dfrac{S\,v_2}{v_1}$

(*iii*) **Relative velocity of rain *w.r.t.* man.** Consider a man walking east with velocity $\vec{v}_m$. Let the rain be falling vertically downward with velocity $\vec{v}_r$. In order to find the relative velocity of rain *w.r.t.* man $(\vec{v}_{rm})$, impress a velocity $-\vec{v}_m$ on rain as well as man. Now $\vec{v}_{rm}$ is the resultant of $-\vec{v}_m$ and $\vec{v}_r$ as shown in Fig. 8.22.

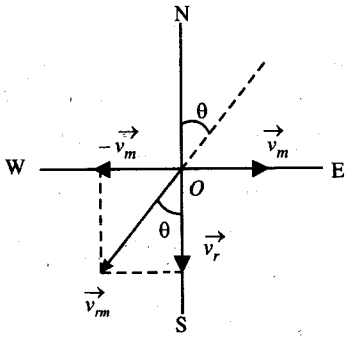


Fig. 8.22

Magnitude of $\vec{v}_{rm}$ is $v_{rm} = \sqrt{v_r^2 + v_m^2 + 2v_r v_m \cos 90°}$

or $$v_{rm} = \sqrt{v_r^2 + v_m^2}$$

If θ is the angle which $\vec{v}_{rm}$ makes with the vertical direction, then,

$$\tan\theta = \frac{v_m}{v_r} \qquad \ldots (i)$$

In order to save himself from the rain, the man should hold his umbrella in the direction of relative velocity of rain *w.r.t.* man $(\vec{v}_{rm})$. In other words, the umbrella should be held at an angle θ with the vertical direction given by eq. (*i*) above.

Example 8.8. A river 1 km wide is flowing at 3 km/h. A swimmer whose velocity in still water is 4 km/h can swim only for 15 minutes. In what direction should he strike out in order to reach the other bank? What is total distance covered?

Solution. Fig. 8.23 shows the conditions of the problem. The point O represents the starting position of the swimmer. At this point, man's velocity $\overrightarrow{OB}$ = 4 km/h and that of water current $\overrightarrow{OA}$ = 3 km/h. The resultant velocity of man will be $\overrightarrow{OB'}$ and he will reach point P on the other side of the bank.

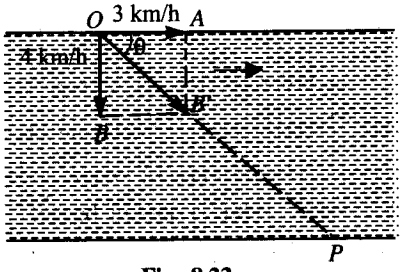


Fig. 8.23

$$\therefore \quad \tan\theta = \frac{4}{3} \text{ or } \theta = \tan^{-1}\frac{4}{3} = 53.1°$$

Magnitude of resultant velocity is

$$OB' = \sqrt{(OA)^2 + (OB)^2} = \sqrt{(3)^2 + (4)^2} = 5 \text{ km/h}$$

$\therefore$ Distance covered by the swimmer = $OB' \times \text{time} = 5 \times \frac{15}{60}$ = **1.25 km** along OB'

Example 8.9. A boatman can row with a speed of 10 km/h in still water. If the river flows steadily at 5 km/hr, in which direction should the boatman row in order to reach a point on the other bank directly opposite to the point from where he started? The width of the river is 2 km.

Solution. Suppose the river is flowing from left to right and O is the starting point. The boatman wants to reach point P directly opposite to the point O (See Fig. 8.24). If he starts along OP, he will be swayed away due to flow of water and may reach point P''. Therefore, in order to reach point P, he should start along OP' (upstream), making an angle θ with OP. The resultant velocity is now $\overrightarrow{OB'}$. $\overrightarrow{OB'} = \overrightarrow{OA} + \overrightarrow{OB}$

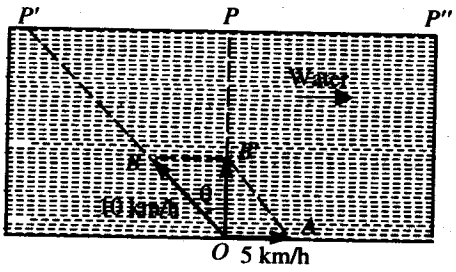


Fig. 8.24

In right angled triangle $OB'B$, $\sin\theta = \frac{BB'}{OB} = \frac{5}{10} = 0.5 \quad \therefore \quad \theta = 30°$

$\therefore$ Angle with the bank of river = 90° + θ = 90° + 30° = 120°

Therefore, the boatman should row making an angle of 120° with the bank of the river.

Example 8.10. **A man is going due east with a velocity of 3 km/h. Rain falls vertically downward with a speed of 10 km/h. Find the angle at which he should hold his umbrella so as to save himself from rain.**

Solution. Fig. 8.25 shows the conditions of the problem.

$\vec{v}_m$ = velocity of man = 3 km/h due east

$\vec{v}_r$ = velocity of rain = 10 km/h vertically downward

$\vec{v}_{rm}$ = velocity of rain *w.r.t.* man

In order to save himself from the rain, the man should hold his umbrella in the direction of relative velocity of rain w.r.t. man (*i.e.* $\vec{v}_{rm}$).

Now $\vec{v}_{rm} = \vec{v}_r - \vec{v}_m = \vec{v}_r + (-\vec{v}_m)$

In order to find $\vec{v}_{rm}$, reverse the direction of $\vec{v}_m$, thus producing $-\vec{v}_m$. Now find the sum of $\vec{v}_r$ and $-\vec{v}_m$ by parallelogram law of vector addition as shown in Fig. 8.25. The umbrella should be held at angle θ with the vertical where,

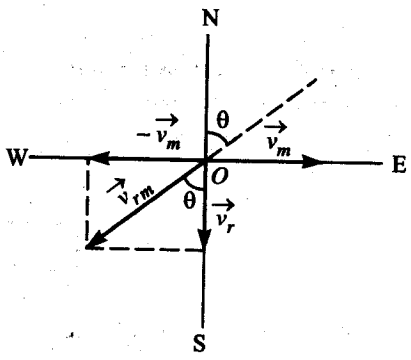


Fig. 8.25

$$\tan\theta = \frac{v_m}{v_r} = \frac{3}{10} = 0.3 \quad \therefore \quad \theta = 16.7°$$

Example 8.11. **A boy is moving with a velocity of 3 km/h along east and the rain is falling vertically with a velocity of 4 km/h. Calculate the velocity of rain *w.r.t.* the boy.**

Solution. Suppose $\vec{v}$ is the velocity of the boy and $\vec{u}$ that of rain. Then velocity of rain *w.r.t.* the boy is

$$\vec{V} = \vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

Taking x-axis along east and z-axis vertically upward (See Fig. 8.26), we have,

$$\vec{u} = -4\hat{k} \; ; \quad \vec{v} = 3\hat{i}$$

The process of splitting a single vector into two or more vectors in different directions in a plane such that their sum gives back the original vector is called ***resolution of a vector.***

The vectors into which the given vector $\vec{R}$ is resolved (or splitted) are called the **vector components** of $\vec{R}$. Fig. 8.28 shows the vector $\vec{R}$ resolved into two non-parallel vectors $\vec{A}$ and $\vec{B}$ such that :

$$\vec{R} = \vec{A} + \vec{B}$$

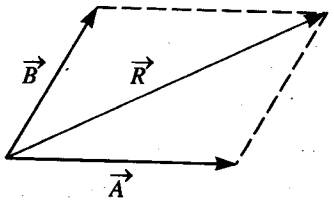


Fig. 8.28

Therefore, $\vec{A}$ and $\vec{B}$ are the vector components of $\vec{R}$. As any number of vectors can be combined to give a single equivalent vector, therefore, a vector can be resolved into any number of vector components. The resolution of a vector into its components often simplifies many problems related to vectors.

8.13. RECTANGULAR COMPONENTS OF A VECTOR IN A PLANE

When a vector in a plane is resolved (i.e., splitted) into two components at right angles to each other, the component vectors are called ***rectangular components*** *of the vector.*

Fig. 8.29(*i*) shows a vector $\vec{A}$ in the *X*-*Y* plane. It is resolved into two rectangular components $\vec{A}_x$ and $\vec{A}_y$ (along *X*-axis and *Y*-axis).

$$\therefore \quad \vec{A} = \vec{A}_x + \vec{A}_y$$

The vector components $\vec{A}_x$ and $\vec{A}_y$ are the rectangular components of vector $\vec{A}$. Therefore, we can replace vector $\vec{A}$ by $\vec{A}_x$ and $\vec{A}_y$. From the geometry of Fig. 8.29(*i*), we see that magnitudes of components $\vec{A}_x$ and $\vec{A}_y$ are related to the magnitude of $\vec{A}$ by ;

$$A_x = A \cos \theta \; ; \; A_y = A \sin \theta$$

Note that A_x and A_y are not vectors ; these are *x*-component and *y*-component respectively of vector $\vec{A}$. Therefore, we can treat them as *algebraic* quantities. Thus, we can specify vector $\vec{A}$ in a plane by scalar components A_x (= $A \cos \theta$) and A_y (= $A \sin \theta$).

Magnitude of $\vec{A}$, $A = \sqrt{A_x^2 + A_y^2} = \sqrt{(A \cos \theta)^2 + (A \sin \theta)^2}$

Direction of $\vec{A}$ with *X*-axis, $\tan \theta = \dfrac{A_y}{A_x}$

In short, we require two quantities to specify a vector such as $\vec{A}$ *viz.* either A and θ or A_x and A_y.

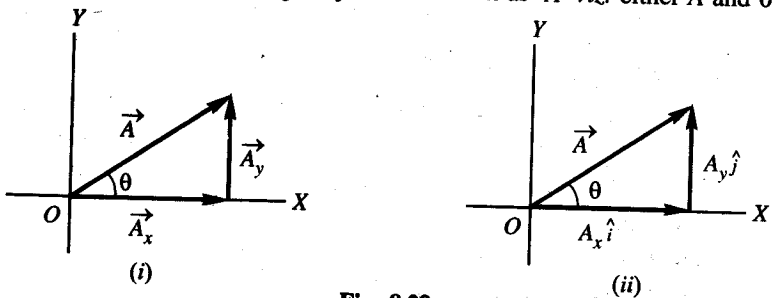


Fig. 8.29

We can express the vectors $\vec{A}_x$ and $\vec{A}_y$ in terms of unit vectors $\hat{i}$ and $\hat{j}$ [See Fig. 8.29 (*ii*)].

$$\vec{A}_x = A_x \hat{i} \; ; \; \vec{A}_y = A_y \hat{j}$$

$$\therefore \qquad \vec{A} = A_x \hat{i} + A_y \hat{j}$$

Note that A_x (= $A \cos\theta$) is not a vector ; $A_x\hat{i}$ is a vector. Similarly, A_y (= $A \sin\theta$) is not a vector ; $A_y\hat{j}$ is a vector.

8.14. VECTOR ADDITION BY RECTANGULAR COMPONENTS

Adding vectors graphically is not very accurate and is not useful for vectors in three dimensions. A far more powerful and accurate method for adding vectors is to resolve the vectors into their rectangular components.

Fig. 8.30 shows two vectors $\vec{A}$ and $\vec{B}$ in the *XY* plane. By triangle law of vectors, their vector sum is $\vec{R}$. Note that vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ are resolved into rectangular components. It is easy to see the component R_x of $\vec{R}$ along *X*-axis is equal to the algebraic sum of the components of the vectors $\vec{A}$ and $\vec{B}$ along *X*-axis *i.e.*,

$$R_x = A_x + B_x$$

Similarly, the component R_y of $\vec{R}$ along *Y*-axis is equal to the sum (algebraic) of the components of vector $\vec{A}$ and $\vec{B}$ along *Y*-axis *i.e.*,

$$R_y = A_y + B_y$$

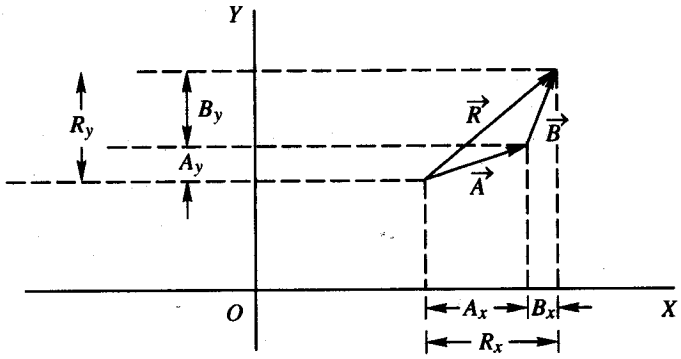


Fig. 8.30

The magnitude and direction of the resultant is given by ;

$$R = \sqrt{R_x^2 + R_y^2} \; ; \; \tan\theta = \frac{R_y}{R_x}$$

Note that angle θ (not shown) is the angle that $\vec{R}$ makes with the positive *X*-direction.

Example 8.13. Find the resultant of the following forces acting simultaneously at a point.

(*i*) a force of 50 N acting along *OX* axis

(*ii*) a force of 40 N acting at an angle of 60° to *OX* axis

(*iii*) a force of 60 N acting 330° with *OX* axis.

Solution. Fig. 8.31 (*i*) shows the conditions of the problem. Let R be the magnitude of the resultant of the forces. Resolving the forces into rectangular components, we have,

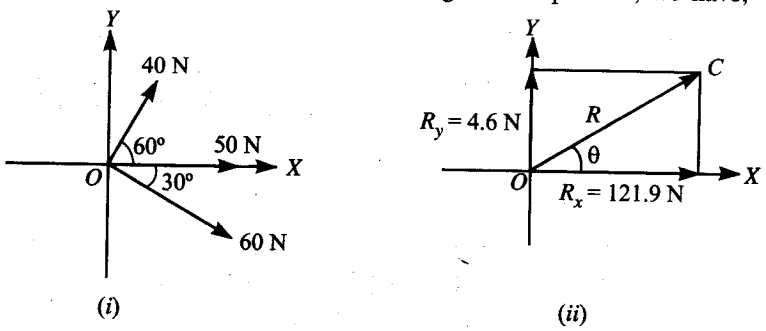


Fig. 8.31

The x-component of R, R_x = *Algebraic sum of x-components of forces

$= 50 \cos 0° + 40 \cos 60° + 60 \cos 30°$

$= 50 + 20 + 51.9 = 121.9$ N

The y-component of R, R_y = Algebraic sum of y-components of forces

$= 50 \sin 0° + 40 \sin 60° - 60 \sin 30°$

$= 0 + 34.6 - 30 = 4.6$ N

As shown in Fig. 8.31 (*ii*), R (= OC) represents the magnitude of the resultant.

$$\therefore \quad R = \sqrt{R_x^2 + R_y^2} = \sqrt{(121.9)^2 + (4.6)^2} = \mathbf{122\ N}$$

$$\tan\theta = \frac{R_y}{R_x} = 0.0377 \quad \therefore \quad \theta = \tan^{-1} 0.0377 = 2.2°$$

Note. Following the usual convention, the angle θ is measured counterclockwise from the positive X-axis.

8.15. RECTANGULAR COMPONENTS IN THREE DIMENSIONS

So far we have considered the rectangular components of a vector in a plane. The treatment can be extended to three-dimensional space in which case a vector will have three rectangular components. Consider a vector $\vec{A}$ in space. It has three rectangular components $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ as shown in Fig. 8.32.

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

Magnitude of $\vec{A}$, **$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Note that $\vec{A}_x$ is the x-component of $\vec{A}$; $\vec{A}_y$ is the y-component of $\vec{A}$ and $\vec{A}_z$ is the z-component of $\vec{A}$.

---

* An algebraic sum is one in which the sign of quantity is to be taken into account. The component along OX axis is positive while in opposite direction, it is negative. Similarly, component along OY is positive and that in the opposite direction is negative.

** Note that A_x and A_y are perpendicular to each other and their resultant is $R' = \sqrt{A_x^2 + A_y^2}$. Now A_z is $\perp$ R'. Therefore, magnitude of $A = \sqrt{R'^2 + A_z^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

We can also express vector $\vec{A}$ in terms of unit vectors as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Note that A_x is not a vector ; $A_x\hat{i}$ is a vector. Same is true for other two components.

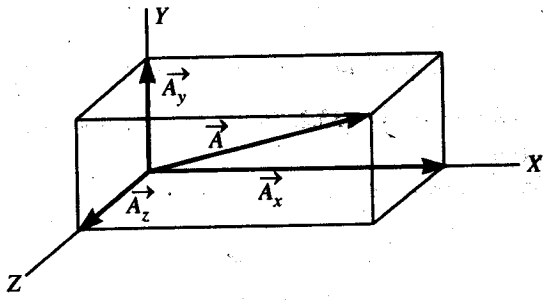


Fig. 8.32

Addition of vectors in three dimensions. To add vectors in three dimensions, each vector is resolved into three rectangular components and the treatment is carried out in a similar way as in the two dimensional case. Suppose that there are three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ in space and their sum is $\vec{R}$.

$$\therefore \quad \vec{R} = \vec{A} + \vec{B} + \vec{C}$$

The three corresponding algebraic equations are :

$$R_x = A_x + B_x + C_x \; ; \; R_y = A_y + B_y + C_y \; ; \; R_z = A_z + B_z + C_z$$

Note the economy, generality and elegance of vector algebra. A single vector equation represents three component equations.

Direction cosines of vector. If α, β and γ are the angles which vector $\vec{A}$ makes with X, Y and Z axes respectively, then,

$$\cos\alpha = \frac{A_x}{A} \quad \text{or} \quad A_x = A\cos\alpha$$

$$\cos\beta = \frac{A_y}{A} \quad \text{or} \quad A_y = A\cos\beta$$

$$\cos\gamma = \frac{A_z}{A} \quad \text{or} \quad A_z = A\cos\gamma$$

Here $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called **direction cosines** of the vector $\vec{A}$.

Now $$A^2 = A_x^2 + A_y^2 + A_z^2$$

or $$A^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$$

or $$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

Therefore, the sum of the squares of the direction cosines of a vector is always unity.

Example 8.14. **A force is represented by ;**

$$\vec{F} = (2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ newton.}$$

What is the magnitude of force?

Solution. Magnitude of force, $F = \sqrt{(2)^2 + (3)^2 + (6)^2} = \sqrt{49} = 7$ N

Example 8.15. **Two vectors are given as $\vec{A}=3\hat{i}+9\hat{j}-6\hat{k}$ and $\vec{B}=8\hat{i}-4\hat{j}+8\hat{k}$. Find $|\vec{A}+\vec{B}|$.**

Solution. $\vec{A}+\vec{B}=(3\hat{i}+9\hat{j}-6\hat{k})+(8\hat{i}-4\hat{j}+8\hat{k}) = 11\hat{i}+5\hat{j}+2\hat{k}$

$\therefore \quad |\vec{A}+\vec{B}|=\sqrt{11^2+5^2+2^2}=\sqrt{150}$

Example 8.16. **The components of a vector $\vec{A}$ are $A_x = 0$; $A_y = 22$. The components of vector $\vec{B}$ are $B_x = 33.2$ and $B_y = -33.2$. Write the vectors (*i*) in unit vector notation and (*ii*) perform the addition.**

Solution. (*i*) In unit vector notation, the vectors will be expressed as :

$$\vec{A}=0\hat{i}+22\hat{j} \; ; \; \vec{B}=33.2\hat{i}-33.2\hat{j}$$

(*ii*) $\vec{A}+\vec{B}=(0\hat{i}+22\hat{j})+(33.2\hat{i}-33.2\hat{j}) = (0+33.2)\hat{i}+(22-33.2)\hat{j} = 33.2\hat{i}-11.2\hat{j}$

Example 8.17. **Determine the vector which when added to the resultant of $\vec{A}=3\hat{i}-5\hat{j}+7\hat{k}$ and $\vec{B}=2\hat{i}+4\hat{j}-3\hat{k}$ gives unit vector along *y*-direction.**

Solution. Resultant vector, $\vec{R}=\vec{A}+\vec{B}$

$$= (3\hat{i}-5\hat{j}+7\hat{k})+(2\hat{i}+4\hat{j}-3\hat{k}) = 5\hat{i}-\hat{j}+4\hat{k}$$

Unit vector along *y*-axis = $\hat{j}$

$\therefore$ Required vector = $\hat{j}-(5\hat{i}-\hat{j}+4\hat{k}) = -5\hat{i}+2\hat{j}-4\hat{k}$

Example. 8.18. **The *x*- and *y*-components of $\vec{A}$ are 4 and 6 and those of $\vec{A}+\vec{B}$ are 10 and 9. Find the components, magnitude and direction of $\vec{B}$.**

Solution. $\vec{A} = 4\hat{i}+6\hat{j}$; $\vec{A}+\vec{B}=10\hat{i}+9\hat{j}$

$\therefore \quad \vec{B} = (10\hat{i}+9\hat{j})-\vec{A}=(10\hat{i}+9\hat{j})-(4\hat{i}+6\hat{j}) = 6\hat{i}+3\hat{j}$

Therefore, *x*- and *y*-components of $\vec{B}$ are 6 and 3.

$\therefore \quad |\vec{B}| = \sqrt{(6)^2+(3)^2}=\sqrt{45}$

Direction of $\vec{B}$, $\tan\theta=\frac{3}{6}=\frac{1}{2}$ $\therefore \theta = 26.56°$ with *x*-axis

Example 8.19. **Given $\vec{A} = (2\hat{i}+3\hat{j}+4\hat{k})$ and $\vec{B} = (3\hat{i}-5\hat{j}+\hat{k})$. Find the angle between $\vec{A}$ and $\vec{B}$.**

Solution. $\vec{A} = 2\hat{i}+3\hat{j}+4\hat{k}$ $\therefore$ $A = \sqrt{2^2+3^2+4^2}=\sqrt{29}$

$\vec{B} = 3\hat{i}-5\hat{j}+\hat{k}$ $\therefore$ $B = \sqrt{3^2+(-5)^2+1^2}=\sqrt{35}$

Resultant $\vec{R} = \vec{A}+\vec{B} = (2\hat{i}+3\hat{j}+4\hat{k})+(3\hat{i}-5\hat{j}+\hat{k})$

$= 5\hat{i}-2\hat{j}+5\hat{k}$

$\therefore \quad R = \sqrt{5^2+2^2+5^2}=\sqrt{54}$

Now $R^2 = A^2 + B^2 + 2AB\cos\theta$

$\therefore \quad \cos\theta = \dfrac{R^2 - A^2 - B^2}{2AB} = \dfrac{54 - 29 - 35}{2\times\sqrt{29}\times\sqrt{35}} = -0.1567 \quad \therefore \quad \theta = 99°$

8.16. MULTIPLICATION OF VECTORS

Vectors have direction as well as magnitude. Therefore, vector multiplication cannot follow exactly the same rules as the algebraic rules of scalar multiplication. Instead, we must *define* the operation of vector multiplication in such a way that we find it useful in physics. It is very useful to define the following two kinds of multiplication operations for vectors :

(*i*) Multiplication of one vector by a second vector so as to produce a scalar. It is called *scalar product* or *dot product* of two vectors.

(*ii*) Multiplication of one vector by a second vector so as to produce another vector. It is called *vector product* or *cross product* of two vectors.

8.17. SCALAR PRODUCT OF TWO VECTORS

Consider two vectors $\vec{A}$ and $\vec{B}$ with angle θ between them as shown in Fig. 8.33. The scalar product of vectors $\vec{A}$ and $\vec{B}$ is defined as :

$$\vec{A}.\vec{B} = AB\cos\theta$$

where A and B are the magnitudes of the vectors and θ is the angle between them when their tails touch. Since A, B and $\cos\theta$ are scalars, the product $\vec{A}.\vec{B}$ (read "A dot B") is *also scalar. The equivalent definition of scalar product is as under :

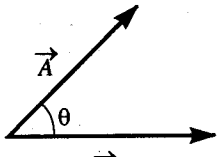


Fig. 8.33

The **scalar product** *of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of magnitude of one vector (say A) and the scalar component of the other vector ($B\cos\theta$) along the direction of the first vector $(\vec{A})$.*

Thus in Fig. 8.34 (*i*), $\vec{B}$ has a scalar component $B\cos\theta$ along the direction of $\vec{A}$.

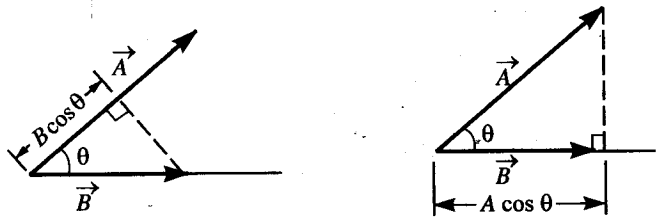


Fig. 8.34

$\therefore \quad \vec{A}.\vec{B}$ = (*A*) × scalar component of $\vec{B}$ along the direction of $\vec{A}$

$= (A)\,(B\cos\theta)$

or $\quad \vec{A}.\vec{B} = AB\cos\theta$

---

* This definition fits perfectly with our definition of work done by a constant force *i.e.*, we can write the work done by a constant force as the scalar product of force $(\vec{F})$ and displacement $(\vec{S})$.

Work done, $W = \vec{F}.\vec{S} = FS\cos\theta$

Similarly, in Fig. 8.34 (*ii*), $\vec{B}.\vec{A} = (B)\ (A\cos\theta) = AB\cos\theta$

Note that the dot or scalar product of two vectors is a scalar quantity. Each of the vector $\vec{A}$ and $\vec{B}$ has a direction but the scalar product itself does not have a direction.

8.18. PROPERTIES OF SCALAR (OR DOT) PRODUCT

The following properties of scalar product (or dot product) are worth noting :

(*i*) *For given vectors $\vec{A}$ and $\vec{B}$, the value of the scalar product depends upon the angle θ between them* (See Fig. 8.35).

For $\theta = 0°$; $\vec{A}.\vec{B} = AB\cos 0° = AB$

For $\theta = 180°$; $\vec{A}.\vec{B} = AB\cos 180° = -AB$

For $\theta = 90°$; $\vec{A}.\vec{B} = AB\cos 90° = 0$

Thus the dot product of two mutually perpendicular vectors is zero.

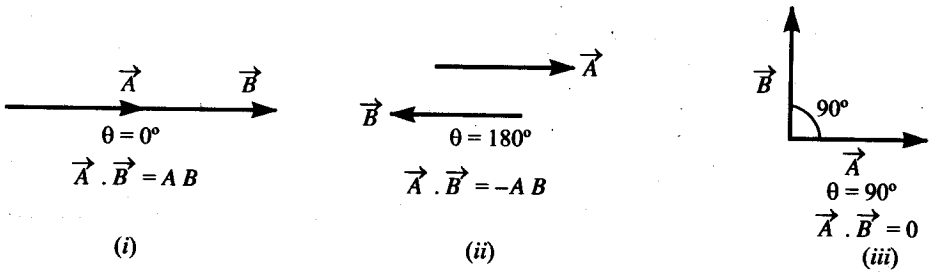


Fig. 8.35

(*ii*) *The dot product of two vectors obeys commutative law.* This directly follows from the definition of dot product.

$$\vec{A}.\vec{B} = A(B\cos\theta) = AB\cos\theta$$

$$\vec{B}.\vec{A} = B(A\cos\theta) = AB\cos\theta$$

$$\therefore \quad \vec{A}.\vec{B} = \vec{B}.\vec{A}$$

This simply means that the order of vectors in the dot product does not matter.

(*iii*) *The dot product obeys the distributive law.* If $\vec{C} = \vec{A} + \vec{B}$, then,

$$\vec{D}.\vec{C} = \vec{D}.(\vec{A} + \vec{B})$$

$$= \vec{D}.\vec{A} + \vec{D}.\vec{B}$$

or $$\vec{D}.\vec{C} = \vec{D}.\vec{A} + \vec{D}.\vec{B}$$

We can prove it by referring to Fig. 8.36. The component of vector $\vec{C}$ along the direction of $\vec{D}$ is the sum of the components of vector $\vec{A}$ and $\vec{B}$ along $\vec{D}$.

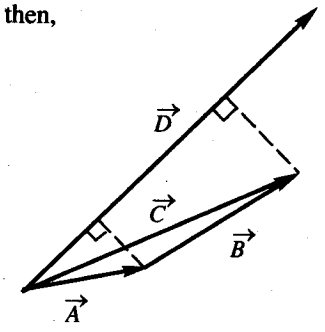


Fig. 8.36

(*iv*) *The dot product of a vector with itself gives square of its magnitude.*

$$\vec{A}.\vec{A} = (A)\ (A)\cos 0° \quad (\theta = 0°) \quad \therefore \quad \vec{A}.\vec{A} = A^2$$

Note. If $\vec{A}$ is perpendicular to $\vec{B}$, then $\vec{A}.\vec{B} = 0$. But if the converse is given *i.e.*, $\vec{A}.\vec{B} = 0$, then there are three possibilities *viz.*, $\vec{A} = 0$, $\vec{B} = 0$ or $\vec{A} \perp \vec{B}$.

Examples of dot product of two vectors. In physics, there are many physical quantities (*e.g.* work, power, magnetic flux *etc.*) that can be described as the scalar or dot product of two vectors.

(*i*) Work done, $W = FS\cos\theta = \vec{F}.\vec{S}$

Thus work done by a constant force is the dot product of force ($\vec{F}$) and displacement ($\vec{S}$).

(*ii*) Instantaneous power, $P = Fv\cos\theta = \vec{F}.\vec{v}$

Thus instantaneous power is the dot product of force ($\vec{F}$) and velocity ($\vec{v}$).

8.19. UNIT VECTORS AND THE DOT PRODUCT

We have seen that if two vectors are aligned (*i.e.*, angle θ between them is zero), their dot product is equal to the product of their magnitudes. However, if the two vectors are perpendicular, their dot product is zero. Using these definitions of dot product, we can derive the following relationships for the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

(*i*) Since $\hat{i}$ is parallel to $\hat{i}$ (*i.e.*, $\theta = 0°$) and each has a unit magnitude,

$\therefore\ \hat{i}.\hat{i} = (1)\ (1)\cos 0° = 1$. Similarly, $\hat{j}.\hat{j} = 1$ and $\hat{k}.\hat{k} = 1$.

$\therefore\ \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(*ii*) Since $\hat{i}$ and $\hat{j}$ are perpendicular and each has a unit magnitude,

$\therefore\ \hat{i}.\hat{j} = (1)\ (1)\cos 90° = 0$. Similarly, $\hat{i}.\hat{k} = 0$ and $\hat{j}.\hat{k} = 0$.

$\therefore\ \hat{i}.\hat{j} = \hat{i}.\hat{k} = \hat{j}.\hat{k} = 0$

Consider two three-dimensional vectors $\vec{A}$ and $\vec{B}$. These can be expressed in the rectangular form as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\ ;\quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\therefore\quad \vec{A}.\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

With the distributive law, it will yield,

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

This is a very useful relation.

Also $\vec{A}.\vec{B} = AB\cos\theta \quad \therefore\ \cos\theta = \dfrac{\vec{A}.\vec{B}}{AB} = \dfrac{A_xB_x + A_yB_y + A_zB_z}{AB}$

Thus we can find the angle θ between the vectors $\vec{A}$ and $\vec{B}$. Note that A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively *i.e.*,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\ ;\qquad B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Example 8.20. If $\vec{R} = \vec{A} - \vec{B}$, show that $R^2 = A^2 + B^2 - 2AB\cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.

Solution. $\vec{R} = \vec{A} - \vec{B}$

Taking dot product of $\vec{R}$ with itself, we have,

$$\vec{R}.\vec{R} = (\vec{A} - \vec{B}).(\vec{A} - \vec{B})$$

$$= \vec{A}.\vec{A} - \vec{A}.\vec{B} - \vec{B}.\vec{A} + \vec{B}.\vec{B}$$

or $$R^2 = A^2 - AB\cos\theta - BA\cos\theta + B^2$$

$\therefore$ $$R^2 = A^2 + B^2 - 2AB\cos\theta$$

Example 8.21. **Find the angle between the vectors $\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{B} = 5\hat{i} - 2\hat{j} - 3\hat{k}$.**

Solution. $$\vec{A}.\vec{B} = AB\cos\theta \quad \therefore \quad \cos\theta = \frac{\vec{A}.\vec{B}}{AB}$$

Now $$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z = 3 \times 5 + 2\,(-2) + (1)\,(-3) = 8$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(5)^2 + (-2)^2 + (-3)^2} = \sqrt{38}$$

$\therefore$ $$\cos\theta = \frac{8}{\sqrt{14 \times 38}} = 0.35 \text{ or } \theta = \cos^{-1} 0.35 = 69.5^\circ$$

Example 8.22. **The sum and difference of two vectors are perpendicular to each other. Prove that the vectors are equal in magnitude.**

Solution. Suppose the two vectors are $\vec{A}$ and $\vec{B}$. The sum and difference are $(\vec{A} + \vec{B})$ and $(\vec{A} - \vec{B})$. Since these two are perpendicular to each other, their dot product is zero *i.e.,*

$$(\vec{A} + \vec{B}).(\vec{A} - \vec{B}) = 0 \text{ or } \vec{A}.\vec{A} - \vec{A}.\vec{B} + \vec{B}.\vec{A} - \vec{B}.\vec{B} = 0$$

or $$A^2 - AB\cos\theta + BA\cos\theta - B^2 = 0 \quad \text{or} \quad A^2 - B^2 = 0 \quad \therefore \quad A = \pm B$$

Example 8.23. **The sum and difference of two vectors are equal in magnitude *i.e.* $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. Prove that vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other.**

Solution. $$|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$$

or $$|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2 \qquad \ldots (i)$$

We know that dot product of a vector with itself is equal to the square of the magnitude of the vector. Therefore, expression given in eq. (*i*) can be written as :

$$(\vec{A} + \vec{B}).(\vec{A} + \vec{B}) = (\vec{A} - \vec{B}).(\vec{A} - \vec{B})$$

or $$\vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B} = \vec{A}.\vec{A} - \vec{A}.\vec{B} - \vec{B}.\vec{A} + \vec{B}.\vec{B}$$

or $$A^2 + AB\cos\theta + AB\cos\theta + B^2 = A^2 - AB\cos\theta - AB\cos\theta + B^2$$

or $$2AB\cos\theta = -2AB\cos\theta$$

or $$4AB\cos\theta = 0 \text{ or } \cos\theta = 0 \quad \therefore \quad \theta = 90^\circ$$

Example 8.24. **If the magnitudes of two vectors are 3 and 4 and the magnitude of their scalar product is 6, find the angle between the vectors.**

Solution. Let θ be the angle between the two vectors $\vec{A}$ and $\vec{B}$. Their scalar product is

$$\vec{A}.\vec{B} = AB\cos\theta$$

Here $\vec{A}.\vec{B} = 6\ ;\ A = 3\ ;\ B = 4$

$$\therefore \qquad 6 = 3\times4\times\cos\theta \quad \text{or} \quad \cos\theta = \frac{6}{3\times4} = 0.5 \quad \therefore \quad \theta = 60^\circ$$

Example 8.25. **Prove that vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j}$ are perpendicular to each other.**

Solution. $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}\ ;\quad \vec{B} = 2\hat{i} - \hat{j}$

The two vectors are perpendicular if $\vec{A}.\vec{B} = 0$.

Now $\vec{A}.\vec{B} = (\hat{i} + 2\hat{j} + 3\hat{k}).(2\hat{i} - \hat{j})$

$$= 1 \times 2 + 2 \times (-1) = 2 - 2 = 0$$

Since the dot product of $\vec{A}$ and $\vec{B}$ is zero, the two vectors are mutually perpendicular.

Example 8.26. **The resultant vector of $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. On reversing the direction of $\vec{Q}$, the resultant vector becomes $\vec{S}$. Show that :**

$$R^2 + S^2 = 2(P^2 + Q^2)$$

Solution. Let θ be the angle between $\vec{P}$ and $\vec{Q}$.

Given : $\vec{R} = \vec{P} + \vec{Q}$ and $\vec{S} = \vec{P} - \vec{Q}$

Now $\vec{R}.\vec{R} = (\vec{P} + \vec{Q}).(\vec{P} + \vec{Q})$

or $R^2 = \vec{P}.\vec{P} + \vec{P}.\vec{Q} + \vec{Q}.\vec{P} + \vec{Q}.\vec{Q}$

or $R^2 = P^2 + PQ\cos\theta + PQ\cos\theta + Q^2$

or $R^2 = P^2 + Q^2 + 2PQ\cos\theta$ ... (*i*)

Also $\vec{S}.\vec{S} = (\vec{P} - \vec{Q}).(\vec{P} - \vec{Q})$

or $S^2 = \vec{P}.\vec{P} - \vec{P}.\vec{Q} - \vec{Q}.\vec{P} + \vec{Q}.\vec{Q}$

or $S^2 = P^2 - PQ\cos\theta - PQ\cos\theta + Q^2$

or $S^2 = P^2 + Q^2 - 2PQ\cos\theta$ ... (*ii*)

Adding eqs. (*i*) and (*ii*), we have,

$$R^2 + S^2 = 2(P^2 + Q^2)$$

Example 8.27. **A particle moves from position $(3\hat{i} + 2\hat{j} - 6\hat{k})$ to a position $(14\hat{i} + 13\hat{j} + 9\hat{k})$ in metre units and a constant force $(4\hat{i} + 2\hat{j} + 3\hat{k})$ newton acts on it. Calculate the work done by the force.**

Solution. $\vec{r_1} = 3\hat{i} + 2\hat{j} - 6\hat{k}$; $\vec{r_2} = 14\hat{i} + 13\hat{j} + 9\hat{k}$

Displacement of the particle, $\vec{S} = \vec{r_2} - \vec{r_1} = (14\hat{i} + 13\hat{j} + 9\hat{k}) - (3\hat{i} + 2\hat{j} - 6\hat{k})$

$= 11\hat{i} + 11\hat{j} + 15\hat{k}$

The work done by a constant force is equal to the dot product of force and displacement.

$\therefore$ Work done, $W = \vec{F}.\vec{S} = (4\hat{i} + 2\hat{j} + 3\hat{k}).(11\hat{i} + 11\hat{j} + 15\hat{k})$

$= 4\times11 + 2\times11 + 3\times15 = 44 + 22 + 45 = \mathbf{111\ J}$

Example 8.28. If unit vectors $\hat{A}$ and $\hat{B}$ are inclined at an angle , then prove that :

$$|\hat{A} - \hat{B}| = 2\sin\frac{\theta}{2}$$

Solution. We know that $|\hat{A} - \hat{B}|^2 = (\hat{A} - \hat{B}).(\hat{A} - \hat{B})$

or $|\hat{A} - \hat{B}|^2 = \hat{A}\cdot\hat{A} + \hat{B}\cdot\hat{B} - 2\hat{A}.\hat{B}$

$= (1)^2 + (1)^2 - 2(1)(1)\cos\theta$

$= 2 - 2\cos\theta = 2(1 - \cos\theta)$

$= 2\left[1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\right] = 4\sin^2\frac{\theta}{2}$

$\therefore$ $|\hat{A} - \hat{B}| = 2\sin\frac{\theta}{2}$

Example 8.29. Find the magnitude and direction of vector $\hat{i} + \hat{j}$.

Solution. Magnitude of vector $\hat{i} + \hat{j}$ is

$$|\hat{i} + \hat{j}| = [(1)^2 + (1)^2]^{1/2} = \sqrt{2}$$

Suppose the vector $\hat{i} + \hat{j}$ makes an angle θ with x-axis (*i.e.* with unit vector $\hat{i}$). Then,

$(\hat{i} + \hat{j}).\hat{i} = |\hat{i} + \hat{j}|\,|\hat{i}|\cos\theta$ $\quad (\because \vec{A}.\vec{B} = AB\cos\theta)$

or $\hat{i}\cdot\hat{i} + \hat{j}\cdot\hat{i} = (\sqrt{2})(1)\cos\theta$

or $1 + 0 = \sqrt{2}\cos\theta$ $\therefore$ $\cos\theta = \frac{1}{\sqrt{2}}$ or $\theta = \mathbf{45°}$

Example 8.30. Find the unit vector parallel to the resultant of vectors $\vec{A} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{B} = 4\hat{i} + 3\hat{j} - \hat{k}$.

Solution. Let $\vec{R}$ be the resultant of $\vec{A}$ and $\vec{B}$.

$\vec{R} = \vec{A} + \vec{B} = (2\hat{i} - 6\hat{j} - 3\hat{k}) + (4\hat{i} + 3\hat{j} - \hat{k}) = 6\hat{i} - 3\hat{j} - 4\hat{k}$

$\therefore$ $R = |\vec{R}| = \sqrt{6^2 + (-3)^2 + (-4)^2} = \sqrt{61}$

The unit vector parallel to $\vec{R}$ is given by ;

$$\hat{R} = \frac{\vec{R}}{R} = \frac{6\hat{i} - 3\hat{j} - 4\hat{k}}{\sqrt{61}} = \frac{6}{\sqrt{61}}\hat{i} - \frac{3}{\sqrt{61}}\hat{j} - \frac{4}{\sqrt{61}}\hat{k}$$

PROBLEMS FOR PRACTICE

1. Find the angle between the X-axis and the vector $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$. $\left[\cos^{-1}\frac{2}{\sqrt{29}}\right]$
2. Find the angle between the vectors $\vec{A} = 2\hat{i} + 4\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 2\hat{j} - 4\hat{k}$. **[90°]**
3. The magnitudes of two vectors $\vec{A}$ and $\vec{B}$ are 5 and 4 units respectively and the angle between them is 30°. Find the value of $\vec{A}.\vec{B}$. **[17.32 units]**
4. If $\vec{A} + \vec{B} = \vec{C}$ and $A^2 = B^2 + C^2$, then prove that $\vec{A}$ and $\vec{B}$ are perpendicular to each other.
5. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non zero vectors and $\vec{A}.\vec{B} = 0$ and $\vec{B}.\vec{B} = 0$, find the value of $\vec{A}.\vec{C}$. **[AC]**
6. Determine the value of a so that vectors $\vec{A} = 5\hat{i} + 7\hat{j} - 3\hat{k}$ and $\vec{B} = 2\hat{i} + 2\hat{j} - a\hat{k}$ are perpendicular. **[– 8]**
7. Prove that vectors $\vec{A} = -2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other.
8. A point of application of force $\vec{F} = 5\hat{i} - 3\hat{j} + 2\hat{k}$ is moved from $\vec{r}_1 = 2\hat{i} + 7\hat{j} + 2\hat{k}$ to $\vec{r}_2 = -5\hat{i} + 2\hat{j} + 3\hat{k}$. Find the work done. **[– 22 units]**
9. Find the angle θ between vectors $\vec{A}$ and $\vec{B}$ where $\vec{A} = -2\hat{i} + 2\hat{j}$ and $\vec{B} = 2\sqrt{3}\,\hat{i} + 2\hat{j}$. **[105°]**
10. Show that $\vec{A}$ is perpendicular to $\vec{B}$ if $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$.

8.20. VECTOR PRODUCT

Consider two vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. 8.37 (*i*) where the angle θ is the smaller of the angles between the two vectors. The vector or cross product of vectors $\vec{A}$ and $\vec{B}$ is another vector $\vec{C}$ given by :

$$\vec{C} = \vec{A} \times \vec{B}$$

It is called vector product of vectors $\vec{A}$ and $\vec{B}$ because $\vec{C}$ is itself a vector. The magnitude and direction of $\vec{C}(= \vec{A} \times \vec{B})$ are defined as under :

(*i*) The magnitude of $\vec{C}$ is given by : $C = AB\sin\theta$

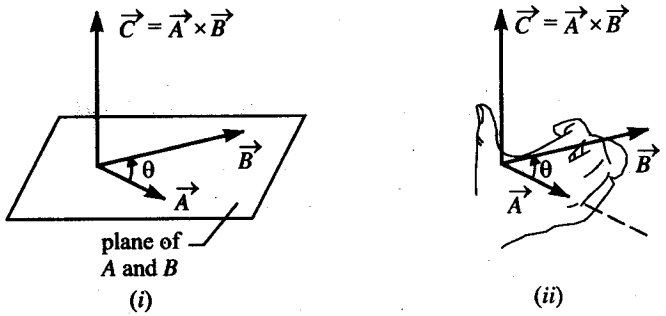


Fig. 8.37

Here A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ while θ is the smaller of the angles between vectors $\vec{A}$ and $\vec{B}$ [See Fig. 8.37 (*i*)].

(*ii*) The vector $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and its direction is given by **right-hand rule** [See Fig. 8.37 (*ii*)]. **Curl the fingers of the right hand in the sense that would rotate the first vector $\vec{A}$ into the second vector $\vec{B}$ through the smaller angle between them. The extended thumb gives the direction of $\vec{C}\,(=\vec{A}\times\vec{B})$.** Applying this rule to Fig. 8.37(*ii*), the direction of $\vec{C}$ is vertically upward.

Note. The direction of $\vec{C}\,(=\vec{A}\times\vec{B})$ can also be found by **Right hand screw rule** stated below :

If a right handed screw placed with its axis perpendicular to the plane containing the two vectors is rotated from $\vec{A}$ to $\vec{B}$ through smaller angle, then the direction of advance of the tip of the screw gives the direction of $\vec{C}\,(=\vec{A}\times\vec{B})$.

Applying this rule to Fig. 8.37(*ii*), the direction of $\vec{C}$ is vertically upward.

8.21. PROPERTIES OF VECTOR PRODUCT

The following are the important properties of vector product or cross product :

(*i*) *For given vectors $\vec{A}$ and $\vec{B}$, the value of the cross product for angle 90° between them is equal to the product of magnitudes of the two vectors i.e.,*

For $\theta = 90°$, $\vec{A} \times \vec{B} = AB \sin 90° = AB$

(*ii*) *The cross product of two vectors does not obey commutative law i.e.,* $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$. Thus referring to Fig. 8.38, the direction of vector $\vec{A}\times\vec{B}$ is opposite to that of vector $\vec{B}\times\vec{A}$. Since the magnitude in each case is $AB\sin\theta$, it follows, therefore, that

$$\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$$

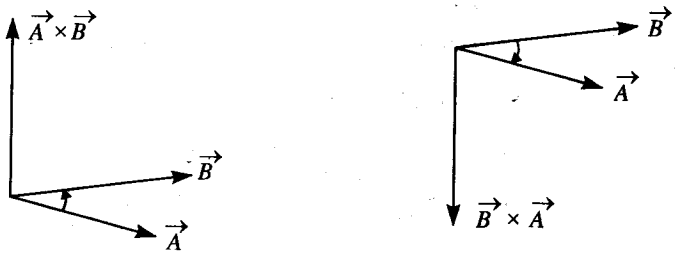


Fig. 8.38

Thus reversing the order of the vectors reverses the direction of the cross product. Therefore, commutative law does not hold for the cross product although it does for the dot product of two vectors.

(*iii*) *The cross product of a vector with itself is zero i.e.,*

$$\vec{A}\times\vec{A} = 0 \qquad (\because\ \theta = 0°)$$

(*iv*) Suppose two vectors $\vec{A}$ and $\vec{B}$ are parallel or antiparallel. The angle θ between them is either 0° or 180°. Then $\vec{A}\times\vec{B} = 0$. *The cross product of parallel (or antiparallel vectors) is zero.*

8.22. UNIT VECTORS AND THE CROSS PRODUCT

The cross product of two vectors can be evaluated in terms of the rectangular components. To do so, we must consider the cross product of the unit vectors.

(*i*) Let us evaluate $\hat{i}\times\hat{i}$. The result is zero. It is because the two vectors are parallel ($\sin\theta = 0$) and each has a unit magnitude.

$\therefore \quad \hat{i}\times\hat{i} = (1)\ (1)\ \sin 0° = 0$

Similarly, $\hat{j}\times\hat{j} = 0$ and $\hat{k}\times\hat{k} = 0$

$\therefore \quad \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(*ii*) To evaluate $\hat{i}\times\hat{j}$, refer to Fig. 8.41 which shows the unit vectors on *XYZ* – *coordinate system. The magnitude of each unit vector is 1 and so the magnitude of $\hat{i}\times\hat{j}$ is 1 *i.e.*, $|\hat{i}\times\hat{j}| = (1)\ (1)\ \sin 90° = 1$. The direction of $\hat{i}\times\hat{j}$ is given by the right-hand rule and from Fig. 8.41, it is along the positive *Z*-axis. But this is just the unit vector $\hat{k}$. Therefore, we have $\hat{i}\times\hat{j} = \hat{k}$. Simply reversing the order of the unit vectors gives $\hat{j}\times\hat{i} = -\hat{k}$.

$\therefore \quad \hat{i}\times\hat{j} = -(\hat{j}\times\hat{i}) = \hat{k}$

Similarly, $\hat{j}\times\hat{k} = -(\hat{k}\times\hat{j}) = \hat{i}$

and $\hat{k}\times\hat{i} = -(\hat{i}\times\hat{k}) = \hat{j}$

Consider three dimensional vectors $\vec{A}$ and $\vec{B}$. These vectors can be expressed in terms of rectangular vectors as:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\ ;\ \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

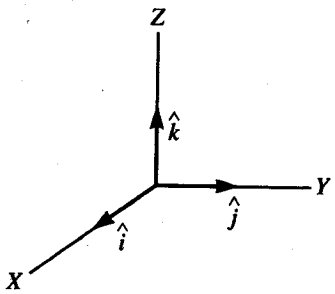


Fig. 8.41

We form cross product $\vec{C} = \vec{A}\times\vec{B}$, expressing $\vec{A}$ and $\vec{B}$ in terms of their components.

$\therefore \quad \vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

The distributive property allows us to carry out the multiplication and we get,

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k} \qquad \text{... (}i\text{)}$$

In determinant form, this can be written as :

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Another way to express the result is

$$\vec{A}\times\vec{B} = \vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k} \qquad \text{... (}ii\text{)}$$

---

* The coordinate system shown in Fig. 8.41 is called *right-handed coordinate system*. The unit vectors $\hat{i}, \hat{j}, \hat{k}$ associated with the *X*, *Y* and *Z* axes in that order are connected by the right-hand rule $\hat{i}\times\hat{j} = \hat{k}$.

By comparison with eq. (*i*), the components of $\vec{C}$ are :

$$C_x = A_yB_z - A_zB_y \; ; \; C_y = A_zB_x - A_xB_z \; ; \; C_z = A_xB_y - A_yB_x$$

Example 8.31. Find the cross product of vectors $\vec{A} = 2\hat{i}$ and $\vec{B} = -2\hat{i} + 4\hat{j}$.

Solution. $\vec{A} \times \vec{B} = (2\hat{i}) \times (-2\hat{i} + 4\hat{j})$

$$= -4(\hat{i} \times \hat{i}) + 8(\hat{i} \times \hat{j})$$

$$= 0 + 8\hat{k} = 8\hat{k}$$

Note that in this example we see again that the plane of $\vec{A}$ and $\vec{B}$ (the $\hat{i} - \hat{j}$ plane) is perpendicular to the direction of $\vec{A} \times \vec{B}$ (the $\hat{k}$ direction).

Example 8.32. Determine the area of the parallelogram whose adjacent sides are $2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} - \hat{j}$.

Solution. Let $\vec{A}$ and $\vec{B}$ be the vectors representing the adjacent sides of the parallelogram.

Here $\vec{A} = 2\hat{i} + \hat{j} + 3\hat{k}$; $\vec{B} = \hat{i} - \hat{j}$

The area of parallelogram is equal to the magnitude of the cross product of $\vec{A}$ and $\vec{B}$.

Now $\vec{A} \times \vec{B} = (2\hat{i} + \hat{j} + 3\hat{k}) \times (\hat{i} - \hat{j})$

$$= 2(\hat{i} \times \hat{i}) - 2(\hat{i} \times \hat{j}) + (\hat{j} \times \hat{i}) - (\hat{j} \times \hat{j}) + 3(\hat{k} \times \hat{i}) - 3(\hat{k} \times \hat{j})$$

$$= 0 - 2\hat{k} - \hat{k} - 0 + 3\hat{j} + 3\hat{i}$$

$\therefore$ $\vec{A} \times \vec{B} = 3\hat{i} + 3\hat{j} - 3\hat{k}$

$\therefore$ Magnitude of the area of the parallelogram

$$= \sqrt{(3)^2 + (3)^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3} \text{ sq units}$$

Example 8.33. Show that vectors $\vec{A} = \hat{i} - 5\hat{j}$ and $\vec{B} = 2\hat{i} - 10\hat{j}$ are parallel to each other.

Solution. If the two vectors are parallel, then their cross product must be zero *i.e.*, $\vec{A} \times \vec{B} = 0$.

Now $\vec{A} \times \vec{B} = (\hat{i} - 5\hat{j}) \times (2\hat{i} - 10\hat{j})$

$$= 2(\hat{i} \times \hat{i}) - 10(\hat{i} \times \hat{j}) - 10(\hat{j} \times \hat{i}) + 50(\hat{j} \times \hat{j})$$

$$= 2(0) - 10(\hat{k}) - 10(-\hat{k}) + 50(0) = 0$$

Hence vectors $\vec{A}$ and $\vec{B}$ are parallel to each other.

Example 8.34. Find a unit vector perpendicular to both the vectors $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Solution. By definition, the cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector that is perpendicular to both $\vec{A}$ and $\vec{B}$.

Now $\vec{A} \times \vec{B} = (3\hat{i} + \hat{j} + 2\hat{k}) \times (2\hat{i} - 2\hat{j} + 4\hat{k}) = 8\hat{i} - 8\hat{j} - 8\hat{k}$

$\therefore$ $|\vec{A} \times \vec{B}| = \sqrt{(8)^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$

$\therefore$ Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \frac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$$

PROBLEMS FOR PRACTICE

1. Find the vector product of $\hat{i} + \hat{j} + \hat{k}$ and $-3\hat{i} + \hat{j} - 2\hat{k}$. **[$3\hat{i} + \hat{j} - 4\hat{k}$]**
2. Prove that vectors $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = 6\hat{i} + 9\hat{j} - 3\hat{k}$ are parallel.
3. If $\vec{A} = 18\hat{i} + 17\hat{j} - 7\hat{k}$ and $\vec{B} = 5\hat{i} - 3\hat{j} + \hat{k}$, find $|\vec{A} \times \vec{B}|$. **[32 units]**
4. Show that $(\vec{A} + \vec{B}) \times (\vec{A} - \vec{B}) = 2(\vec{B} \times \vec{A})$.
5. The adjacent sides of a parallelogram are represented by $4\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} - 3\hat{j} + 2\hat{k}$. Find the area of the parallelogram. **[$\sqrt{171}$ sq units]**

CONCEPTUAL QUESTIONS

Q.1. $\vec{A}$ and $\vec{B}$ are two vectors, can $\vec{A} + \vec{B}$ be zero?

Ans. $\vec{A} + \vec{B}$ will be zero if $\vec{A}$ and $\vec{B}$ have same magnitude but are opposite in direction.

Q.2. When are two three-dimensional vectors equal?

Ans. Suppose $\vec{A}$ and $\vec{B}$ are three dimensional vectors. These vectors will be equal if their corresponding rectangular components are equal *i.e.*

$\vec{A} = \vec{B}$ if $A_x = B_x$, $A_y = B_y$ and $A_z = B_z$

Q.3. What is the difference between A_x and $\vec{A}_x$?

Ans. A_x means the scalar component of vector $\vec{A}$ along X-axis. $\vec{A}_x$ is the vector component of $\vec{A}$ along X-axis.

Q.4. What is the importance of scalar product and vector product?

Ans. The scalar product and vector product are very useful in physics. It is because many quantities can be described as the scalar product or vector product of two vectors.

Q.5. Distinguish between cross product and dot product of two vectors $\vec{A}$ and $\vec{B}$.

Ans. The following are the distinguishing features between the cross product and dot product:

(*i*) The cross product of two vectors is a vector ; the dot product is a scalar.

(*ii*) The magnitude of the cross product is $AB \sin\theta$; the magnitude of dot product is $AB \cos\theta$.

(*iii*) The cross product of two vectors is zero when the vectors are parallel ; the dot product of two vectors is zero when the vectors are perpendicular.

(*iv*) The cross product does not obey commutative law ; the dot product does.

Q.6. A scalar may be a pure number or a physical quantity. What is the effect of multiplication of a vector by a scalar?

Ans. (*i*) If we multiply vector $\vec{A}$ by a pure number n, the new vector is $n\vec{A}$. It means that the magnitude of new vector is n times the magnitude of vector $\vec{A}$. The new vector has the same direction as $\vec{A}$ if n is positive and the opposite direction if n is negative. Note that physical nature of the original vector has not changed in this case.

(*ii*) If we multiply a vector by a scalar (physical quantity), the physical nature of product differs from the original vector. For example,

mass × acceleration = Force

or $m\vec{a} = \vec{F}$

Here $\vec{a}$ is an acceleration (vector) and m is mass (positive scalar). Note that physical nature of $\vec{F}$ is different from the original vector $\vec{a}$.

Q.7. What is the necessary condition for adding vectors?

Ans. The necessary condition for adding (or subtracting) the vectors is that the vectors must be of same kind. We cannot add displacement vector to velocity vector.

Q.8. Can three vectors (*i*) lying in a plane (*ii*) not lying in a plane give zero resultant?

Ans. Consider three vectors $\vec{A}, \vec{B}$ and $\vec{C}$. If resultant of these vectors is zero, then,

$$\vec{A}+\vec{B}+\vec{C}=0 \qquad \text{or} \qquad \vec{C}=-(\vec{A}+\vec{B})$$

Therefore, they will produce zero resultant if $\vec{C}=-(\vec{A}+\vec{B})$ or vector sum of any two vectors is equal and opposite to the third vector. The vectors can give this result only if they lie in the same plane.

The three vectors cannot give zero resultant if they do not lie in the same plane.

Q.9. What are coplanar vectors?

Ans. Those vectors which act in the same plane are called coplanar vectors.

Q.10. What is the importance of vectors?

Ans. The laws of physics are generally expressed in vector form due to the following reasons:

(*i*) By the use of vectors, many physics laws can be expressed in compact form.

(*ii*) The analysis of many physcis laws is greatly simplified by the use of vectors.

Q.11. What are the characteristics of vector addition?

Ans. (*i*) Only vectors of the same nature can be added.

(*ii*) Vector addition obeys commutative law. It is because

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

(*iii*) Vector addition obeys associative law. It is because

$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$$

Q.12. What is the condition for zero resultant vector for more than three vectors acting simultaneously on a particle?

Ans. If a number of vectors acting simultaneously on a particle can be represented in magnitude and direction by the sides of a polygon taken in the same order, their resultant vector is zero.

Q.13. Walking of a man is an example of resolution of a vector. Discuss.

Ans. When a man walks, he presses the ground with his feet backward at an angle θ. The reaction R of the ground on the feet of the man is in the opposite direction. The reaction R is resolved into two rectangular components *viz* vertical component $R \sin\theta$ and horizontal component $R \cos\theta$. The vertical component $R \sin\theta$ balances the weight of the man while the horizontal component $R \cos\theta$ helps the man to walk forward [See Fig. 8.42].

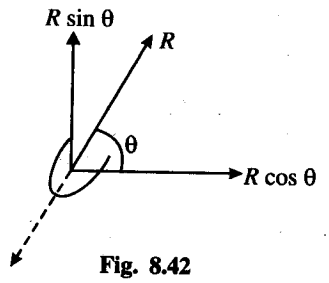


Fig. 8.42

Q.14. If $\vec{A} \times \vec{B} = \vec{C} \times \vec{B}$, show that $\vec{C}$ need not be equal to $\vec{A}$. When will $\vec{A}$ be equal to $\vec{C}$?

Ans. $\vec{A} \times \vec{B} = \vec{C} \times \vec{B}$ or $(\vec{A} - \vec{C}) \times \vec{B} = 0$

In other words, the cross product of $(\vec{A} - \vec{C})$ and $\vec{B}$ is zero. This is possible if any of the following three possibilities occur :

(*i*) $\vec{A} - \vec{C} = 0$ or $\vec{A} = \vec{C}$

(*ii*) $\vec{B} = 0$

(*iii*) $(\vec{A} - \vec{C})$ and $\vec{B}$ are parallel (*i.e.*, $\theta = 0°$)

From point (*iii*) above, we have, $\vec{A} = \vec{C} + m\vec{B}$ where m is a non-zero real number. The first statement is true if $\vec{B}$ is a null vector [point (*i*)] or $\vec{A}$ is equal to $(\vec{C} + m\vec{B})$ [point (*iii*)]. On the other hand, the second statement is true if $\vec{B}$ is not a null vector and $\vec{A}$ is not equal to $(\vec{C} + m\vec{B})$.

VERY SHORT ANSWER QUESTIONS

Q.1. Which of the following is not a vector quantity? Mass, acceleration, momentum, force.

Ans. Mass.

Q.2. Is a quantity which has a magnitude and direction always a vector? Give examples.

Ans. No. For example, electric current possesses both magnitude and direction but is a scalar quantity. For a quantity to be a vector, it must obey the laws of vector algebra.

Q.3. Pick up the only vector quantity from the following: temperature, power, area, gravitational potential, coefficient of friction.

Ans. Only area is the vector quantity.

Q.4. Give an example of a physical quantity (*i*) which has neither unit nor direction (*ii*) has a direction but not a vector (*iii*) can be either a vector or scalar.

Ans. (*i*) Specific gravity (*ii*) current (*iii*) area.

Q.5. How is a vector represented?

Ans. A vector is represented by a straight line with an arrowhead. The length of the line represents the magnitude of the vector (on suitable scale) and the arrowhead indicates its direction.

Q.6. What condition should be satisfied for two vectors to be equal?

Ans. The two vectors will be equal if they have (*i*) the same magnitude and (*ii*) the same direction.

Q.7. Can you add velocity vector to the displacement vector?

Ans. No. It is because only vectors of the same kind can be added or subtracted.

Q.8. **The magnitude of the sum of two vectors is equal to the magnitude of difference between them. What is the angle between the two vectors?**

Ans. $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

$\therefore \quad A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$ or $4AB\cos\theta = 0$ or $\theta = 90°$.

Q.9. **What is the property of two vectors if $\vec{A} + \vec{B} = \vec{A} - \vec{B}$?**

Ans. It means $\vec{B}$ is a null vector.

$$\vec{A} + \vec{B} = \vec{A} - \vec{B} \quad \text{or} \quad 2\vec{B} = 0 \quad \therefore \quad \vec{B} = 0$$

Q.10. **When can three vectors be added to give null vector?**

Ans. When the three vectors are lying in the same plane and are represented in magnitude and direction by the three sides of a triangle taken in the same order.

Q.11. **How can a vector be tripled?**

Ans. By multiplying the vector by 3 (scalar) or by adding two similar vectors (vectors of same magnitude and direction) to the given vector.

Q.12. **Does vector addition hold for any two vectors?**

Ans. No. The vector addition (or subtraction) holds for two vectors of the same kind.

Q.13. **What is the result of multiplying a vector by a real number n?**

Ans. When a vector $\vec{A}$ is multiplied by a real positive number n, the result is $n\vec{A}$ *i.e.,* resultant vector has magnitude nA and has the same direction as that of $\vec{A}$. For negative real number, magnitude is nA but direction opposite to that of $\vec{A}$.

Q.14. **Why vectors cannot be added algebraically?**

Ans. Vectors possess magnitude as well as direction. In scalar addition, we have to add magnitudes of scalars. But vector addition requires the use of vector algebra.

Q.15. **Why is $\vec{A} \times \vec{A} = 0$?**

Ans. The angle between similar vectors is zero. Therefore, $\vec{A} \times \vec{A} = AA\sin\theta = A^2\sin 0° = 0$.

Q.16. **What is the number of minimum forces that can give zero resultant?**

Ans. Two, provided they are equal in magnitude but act in opposite directions.

Q.17. **What is the minimum number of unequal forces that can give zero resultant?**

Ans. Three, provided they act in the same plane and can be represented in magnitude and direction by the sides of a triangle taken in the same order.

Q.18. **Will $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ be in the same or different planes?**

Ans. They will be in the same plane.

Q.19. **Is commutative law applicable for vector subtraction?**

Ans. No, because $\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$.

Q.20. **Is associative law applicable for vector subtraction?**

Ans. Yes, because $(\vec{A} + \vec{B}) - \vec{C} = \vec{A} + (\vec{B} - \vec{C})$.

Q.21. **Can the magnitude of the resultant of two vectors be less than the magnitude of any of the vector?**

Ans. Yes, if the angle between them is more than 90° but less than 270°. It is because in that case $\cos\theta$ is negative.

Q.22. **What is the unit of a unit vector?**

Ans. It has no units.

Q.23. Can the scalar product of two vector quantities be negative?

Ans. The scalar product of two vectors can be negative if angle between them is between 90° to 270°.

Q.24. What is the maximum and minimum value of the resultant of two forces?

Ans. The resultant is maximum when both forces act in the same direction and is minimum when they act in opposite directions.

Q.25. Is the magnitude and direction of $(\vec{A}-\vec{B})$ is the same as that of $(\vec{B}-\vec{A})$?

Ans. The magnitude is the same but direction is opposite.

Q.26. Is it possible to add vectors representing force of 5 N and 200 dynes?

Ans. Yes, it is because both vectors represent force. However, they must be represented on the same scale.

Q.27. If a vector is added or subtracted from a vector, the resultant is a vector. Is this also true in case of multiplication of two vectors?

Ans. This may or may not be true. In case of cross product of two vectors, the result is a vector quantity. But in case of dot product of two vectors, the result is a scalar quantity.

Q.28. Consider a vector $\vec{F}=3\hat{i}-4\hat{j}$. What is a unit vector perpendicular to $\vec{F}$?

Ans. It will be $\hat{k}$ (*i.e.*, along *Z*-axis) because $\vec{F}$ lies in the *X-Y* plane.

SHORT ANSWER QUESTIONS

Q.1. Can two vectors of the same magnitude have resultant equal to either of them?

Ans. Yes when the two vectors are inclined at 120° angle.

$P^2 = P^2 + P^2 + 2P.\ P \cos\theta$ or $\cos\theta = -1/2 \quad \therefore \quad \theta = 120°$

Q.2. In the above question, what will be the magnitude of the difference of the two vectors?

Ans. In this case θ will be 60°.

$$\therefore \qquad R = \sqrt{P^2+P^2+2\,P.P.\cos 60°} = \sqrt{2P^2+2P^2\times(0.5)} = \sqrt{3}P$$

Q.3. Can any of the two rectangular components of a given vector have a magnitude greater than the vector itself?

Ans. No. The rectangular components of any vector $\vec{A}$ have values $A\cos\theta$ and $A\sin\theta$. Since the value of cos θ or sin θ cannot be greater than 1, the values of $A\cos\theta$ and $A\sin\theta$ cannot be greater than A.

Q.4. Show that $\vec{P}+\vec{Q}=\vec{Q}+\vec{P}$.

Ans. $R^2 = P^2 + Q^2 + 2P.\ Q\cos\theta$. If we interchange $\vec{P}$ and $\vec{Q}$, the magnitude and direction of resultant remains the same. Therefore, $\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$.

Q.5. Prove that the magnitude of the resultant of two vectors $\vec{A}$ and $\vec{B}$ cannot be greater than $(A + B)$ and smaller than $(A - B)$ or $(B - A)$.

Ans. $R = \sqrt{A^2+B^2+2AB\cos\theta}$

$R_{max} = \sqrt{A^2+B^2+2AB\cos 0°} = (A+B)$; $R_{min} = \sqrt{A^2+B^2+2AB\cos 180°} = (A-B)$ or $(B-A)$

Q.6. A boat is moving with a velocity of $(3\hat{i}+4\hat{j})$ *w.r.t.* ground. The water in the river is moving with a velocity $(-3\hat{i}-4\hat{j})$ *w.r.t* ground. What is the relative velocity of boat *w.r.t.* water?

Ans. $v_{BG} = (3\hat{i}+4\hat{j})$; $v_{WG} = (-3\hat{i}-4\hat{j})$

$\therefore$ Velocity of boat *w.r.t.* water, $v_{BW} = v_{BG} - v_{WG} = (3\hat{i}+4\hat{j}) - (-3\hat{i}-4\hat{j}) = 6\hat{i}+8\hat{j}$

Q.7. Give two examples of (*i*) dot product and (*ii*) vector product of two vectors.

Ans. Refer to Art. 8.18 and Art. 8.21.

Q.8. There are two displacements, one of magnitude 3 m and another of magnitude 4 m. How the displacement vectors should be added to get a resultant displacement of 7 m?

Ans. $R = \sqrt{A^2+B^2+2AB\cos\theta}$. Here $A = 3$ m ; $B = 4$ m and $R = 7$ m.

$\therefore \quad 7 = \sqrt{3^2+4^2+2\times3\times4\cos\theta}$ or $\cos\theta = 1 \quad \therefore \quad \theta = 0°$

Q.9. A force is represented by : $\vec{F} = (2\hat{i}+3\hat{j}+6\hat{k})$N. What is the magnitude of force?

Ans. Magnitude of force $= \sqrt{2^2+3^2+6^2} = \sqrt{49} = 7$ N

Q.10. At what angle the two forces $A + B$ and $A - B$ act so that their resultant is $\sqrt{3A^2+B^2}$?

Ans. Here $P = A + B$; $Q = A - B$ and $R = \sqrt{3A^2+B^2}$

$R^2 = (A+B)^2 + (A-B)^2 + 2(A+B)(A-B)\cos\theta$

or $3A^2 + B^2 = 2(A^2+B^2) + 2(A^2-B^2)\cos\theta$ or $\cos\theta = 1/2 \quad \therefore \theta = 60°$

Q.11. What is the magnitude of the resultant of two forces, one 6N due east and the other 8N due north?

Ans. Resultant $= \sqrt{6^2+8^2} = 10$N $(\because \theta = 90°)$

Q.12. What conclusions you draw from the following equations? (*i*) $\vec{A}+\vec{B}=\vec{C}$ and $A + B = C$ (*ii*) $\vec{A}+\vec{B}=\vec{A}-\vec{B}$ (*iii*) $\vec{A}+\vec{B}=\vec{C}$ and $A^2 + B^2 = C^2$.

Ans. (*i*) In this case, the two vectors are in the same direction because $A + B = C$.

(*ii*) In this case, $\vec{B}$ must be zero because $\vec{A} + \vec{B} = \vec{A} - \vec{B}$ or $\vec{B} = 0$.

(*iii*) In this case, $\vec{A}$ is perpendicular to $\vec{B}$ because resultant $C = \sqrt{A^2+B^2}$.

Q.13. A force on a body is 100 N, 60° west of south. Find the north and east components of the force.

Ans. Fig. 8.43 shows the conditions of the problem. It is clear that force F (= 100 N) makes an angle of 120° with north direction and 150° with east direction.

$\therefore F_{north} = 100\cos 120° = -50$ N; $F_{east} = 100\cos 150° = -86.6$ N

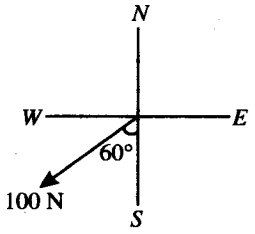


Fig. 8.43

Q.14. Find the magnitude of the resultant of two forces impressed on a particle: 3 N due north-east and 4N south-east.

Ans. Resultant $= \sqrt{3^2+4^2} = 5$ N $(\because \theta = 90°)$

Q.15. What is the maximum number of components into which a vector can be resolved?

Ans. A vector can be resolved into any number of components (infinite). However, a vector can be resolved into only two rectangular components.

Q.16. What is the magnitude and direction of $(\hat{i}+\hat{j})$?

Ans. $|\hat{i}+\hat{j}| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$. Suppose the vector $(\hat{i}+\hat{j})$ makes an angle θ with the direction of $\hat{i}$. Then, $\tan\theta = 1/1 = 1$ or $\theta = 45°$.

Q.17. If $\vec{A} = 2\hat{i}+4\hat{j}-5\hat{k}$, find (*i*) magnitude of $\vec{A}$ (*ii*) the direction cosines of $\vec{A}$.

Ans. (*i*) $|\vec{A}| = \sqrt{(2)^2+(4)^2+(-5)^2} = \sqrt{45}$

(*ii*) $\cos\alpha = 2/\sqrt{45}$; $\cos\beta = 4/\sqrt{45}$; $\cos\gamma = -5/\sqrt{45}$

Q.18. Two forces of 5 N and 10 N are inclined to each other at an angle of 120°. What is the angle which the resultant makes with 10N?

Ans. Here $P = 10$ N ; $Q = 5$ N ; $\theta = 120°$; $\alpha = ?$

$$\tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta} = \frac{5\sin 120°}{10+5\cos 120°} = \frac{1}{\sqrt{3}} \quad \therefore \alpha = 30°$$

Q.19. If the magnitudes of vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are 3, 4 and 5 units respectively and if $\vec{A}+\vec{B}=\vec{C}$, find (*i*) angle between $\vec{A}$ and $\vec{B}$ (*ii*) angle between $\vec{A}$ and $\vec{C}$.

Ans. (*i*) Since $3^2 + 4^2 = 5^2$ (*i.e.*, $A^2 + B^2 = C^2$), the angle between $\vec{A}$ and $\vec{B}$ is 90°.

(*ii*) If θ is the angle between $\vec{A}$ and $\vec{C}$, then $\cos\theta = A/C = 3/5$. Therefore, $\theta = \cos^{-1}(3/5)$.

Q.20. Determine the vector which when added to the resultant of $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ gives unit vector along *y*-direction.

Ans. $\vec{A}+\vec{B} = (3\hat{i} - 5\hat{j} + 7\hat{k}) + (2\hat{i} + 4\hat{j} - 3\hat{k}) = 5\hat{i} - \hat{j} + 4\hat{k}$

The unit vector along *y*-direction = $\hat{j}$. Therefore, required vector = $\hat{j} - (5\hat{i} - \hat{j} + 4\hat{k}) = -5\hat{i} + 2\hat{j} - 4\hat{k}$.

Q.21. What is the angle between $\vec{A}$ and $\vec{B}$, if $\vec{A}$ and $\vec{B}$ denote the adjacent sides of a parallelogram drawn from a point and the area of the parallelogram is $\frac{1}{2}AB$?

Ans. Area of parallelogram = $|\vec{A} \times \vec{B}| = AB\sin\theta$

But it is given that area of parallelogram = $AB/2$.

$\therefore$ $AB\sin\theta = AB/2$ or $\sin\theta = 1/2$ $\therefore$ $\theta = 30°$.

Q.22. Can a vector be multiplied with both dimensional and non-dimensional scalars?

Ans. Yes. When a vector is multiplied with a dimensional scalar, the resultant vector will have different dimensions. For example, if velocity vector is multiplied with mass (a dimensional scalar), the resultant vector has the dimensions of momentum. When a vector is multiplied by a non-dimensional scalar, it will be a vector having dimensions as that of the given vector.

LONG ANSWER QUESTIONS

1. Explain (*i*) negative vectors (*ii*) equal vectors. **[Refer to Art. 8.2]**
2. Explain (*i*) unit vector and (*ii*) null vector. **[Refer to Art. 8.2]**
3. What do you mean by resultant of two vectors ? How will you add two vectors graphically? **[Refer to Art. 8.4]**
4. State parallelogram law of vector addition. Find analytically the magnitude and direction of the resultant of two vectors when they are inclined at an angle θ. **[Refer to Arts. 8.4 and 8.6]**
5. State and explain triangle law and polygon law of addition of vectors. **[Refer to Arts. 8.4, 8.5 and 8.6]**
6. Explain dot product and vector product of two vectors. **[Refer to Arts. 8.17 and 8.20]**
7. Explain rectangular components of a vector in a plane. **[Refer to Art. 8.13]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Vectors are quantities that have both magnitude and direction. Scalars are quantities that have only magnitude.
2. Vectors are used to describe motion in two or three dimensions.
3. Vectors can be added using either the *triangle method* or *parallelogram method*. In the triangle method [See Fig: 8.44 (*i*)], the vector $\vec{R} = \vec{P} + \vec{Q}$ runs from the tail of $\vec{P}$ to the tip of $\vec{Q}$. In parallelogram method [See Fig. 8.44 (*ii*)], $\vec{R}$ is the diagonal of the parallelogram having $\vec{P}$ and $\vec{Q}$ as its adjacent sides. The order in which we add vectors does not change the resultant vector *i.e.*,

$$\vec{R} = \vec{P} + \vec{Q} = \vec{Q} + \vec{P}$$

We can find the magnitude and direction of the resultant vector analytically.

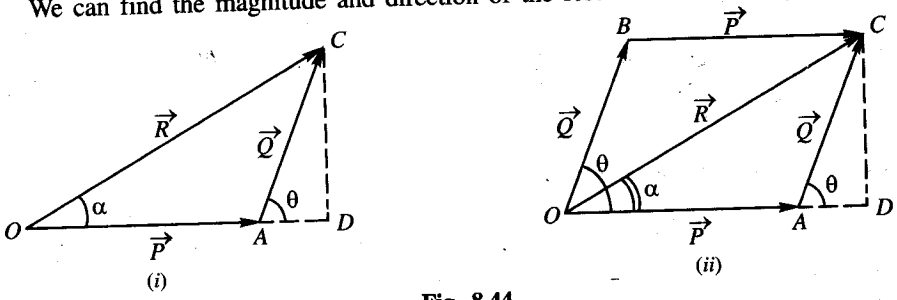


Fig. 8.44

Magnitude of resultant in Fig. 8.44 (*i*) or Fig 8.44 (*ii*) is

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

The direction which the resultant vector $\vec{R}$ makes with $\vec{P}$ is

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

4. A vector in a plane can also be described by giving its components on a chosen set of coordinates. The x component of vector $\vec{A}$ is equivalent to its projection along x-axis *i.e.*, $A_x = A\cos\theta$ as shown in Fig. 8.45. Here θ is the angle (measured CCW) that vector $\vec{A}$ makes with positive direction of x-axis. Similarly, the y component of $\vec{A}$ is its projection along the y-axis *i.e.*, $A_y = A\sin\theta$.

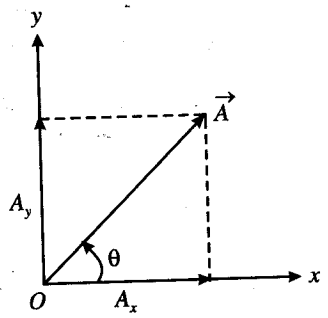


Fig. 8.45

5. The vectors in a plane can be added by using the method of components. Thus to find the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \ldots$, we add algebraically the x components of the vectors to find R_x and then add y components algebraically to find R_y.

$$R_x = A_x + B_x + C_x + ... ; \quad R_y = A_y + B_y + C_y + ...$$

Magnitude of resultant, $R = \sqrt{R_x^2 + R_y^2}$

The angle θ (measured CCW) that resultant vector makes with positive x-axis is

$$\tan\theta = \frac{R_y}{R_x}$$

6. If a vector $\vec{A}$ in a plane has an x component equal to A_x and y component equal to A_y, the vector can be represented in unit vector form as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

In this notation, $\hat{i}$ is unit vector in the positive x direction and $\hat{j}$ is a unit vector in the positive y direction. Since $\hat{i}$ and $\hat{j}$ are unit vectors, $|\hat{i}| = |\hat{j}| = 1$.

7. A vector $\vec{A}$ in three dimensions can be represented as $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$.

8. Dot or scalar product of two vectors. The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as a scalar quantity equal to the product of the magnitudes of two vectors and cosine of smaller angle between them *i.e.*

$$\vec{A}.\vec{B} = AB\cos\theta$$

where θ is the smaller angle between $\vec{A}$ and $\vec{B}$ as shown in Fig. 8.46. Note that $\vec{A}$ and $\vec{B}$ need not have the same units. Note in Fig. 8.46, $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$ *i.e.*, $\vec{A}.\vec{B}$ =(Magnitude of $\vec{A}$) (Projection of $\vec{B}$ onto $\vec{A}$).

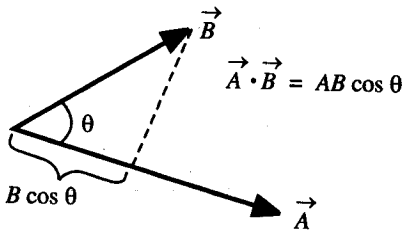


Fig. 8.46

The following properties of dot product are worth noting :

(*i*) $$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

i.e. order of the dot product can be reversed.

(*ii*) The dot product obeys the *distributive law of multiplication* so that :

$$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$

(*iii*) The scalar or dot product of unit vectors are:

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 ; \quad \hat{i}.\hat{j} = \hat{i}.\hat{k} = \hat{j}.\hat{k} = 0$$

(*iv*) The two vectors $\vec{A}$ and $\vec{B}$ can be expressed in component form as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} ; \quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Using the property of dot product of unit vectors, we have,

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

In special case when $\vec{A} = \vec{B}$, we have,

$$\vec{A}.\vec{A} = A_x^2 + A_y^2 + A_z^2 = A^2$$

9. Cross or vector product. Consider two vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. 8.47 where angle θ is the smaller angle between the two vectors. The vector or cross product of vectors $\vec{A}$ and $\vec{B}$ is another vector $\vec{C}$ given by ; $\vec{C} = \vec{A} \times \vec{B}$

It is called vector product of $\vec{A}$ and $\vec{B}$ because $\vec{C}$ is itself a vector.

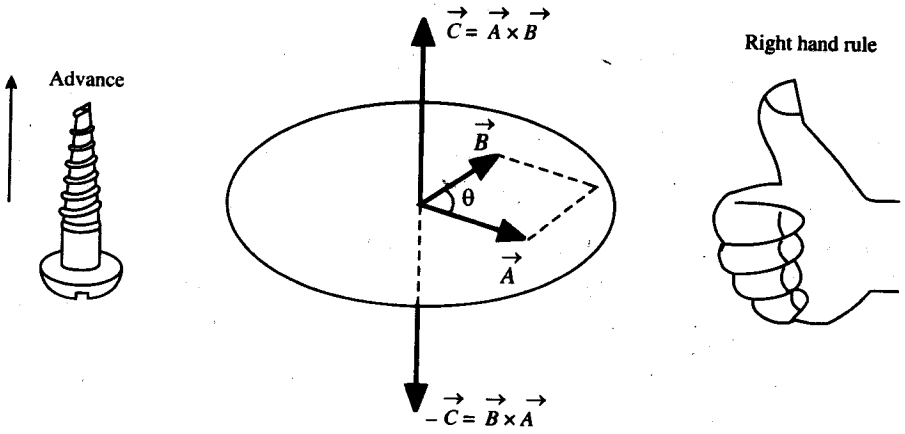


Fig. 8.47

Magnitude of $\vec{C}$. The magnitude of $\vec{C}$ is given by ;

$$C = |\vec{C}| = AB\sin\theta$$

Note that the quantity $AB\sin\theta$ is equal to the area of the parallelogram formed by $\vec{A}$ and $\vec{B}$ as shown in Fig. 8.47.

Direction and sense of C. The direction of $\vec{C}$ $(= \vec{A} \times \vec{B})$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ as shown in Fig. 8.47 and its sense is determined by the advance of a right-handed screw when turned from $\vec{A}$ to $\vec{B}$ through the *smaller* angle θ. A more convenient rule to use for the direction of $\vec{A} \times \vec{B}$ is the right-hand rule illustrated in Fig. 8.47. The four fingers of the right hand are pointed along $\vec{A}$ and then "wrapped" into $\vec{B}$ through the angle θ. The direction of the erect right thumb is the direction of $\vec{A} \times \vec{B}$. Because of the notation, $\vec{A} \times \vec{B}$ is often read "A cross B", hence the term *cross product.* Some properties of the vector product which follow from its definition are :

(*i*) Unlike the scalar product, the order in which the two vectors are multiplied in a cross product is important *i.e.*

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Therefore, if you change the order of cross product, you must change the sign. You can easily verify this relation with right-hand rule (See Fig. 8.47).

(*ii*) If $\vec{A}$ is parallel to $\vec{B}$ (*i.e.* $\theta = 0°$ or $180°$), then $\vec{A} \times \vec{B} = 0$. It follows, therefore, $\vec{A} \times \vec{A} = 0$.

(*iii*) If $\vec{A}$ is perpendicular to $\vec{B}$, then $|\vec{A} \times \vec{B}| = AB$.

(*iv*) It is also important to note that vector product obeys the *distributive law i.e.*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(*v*) The derivative of cross product w.r.t. some variable such as t is given by ;

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$$

Here it is important to preserve the multiplication order of $\vec{A}$ and $\vec{B}$.

(*vi*) The cross products of unit vectors are given below :

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 ; \quad \hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$$

$$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} ; \quad \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

Note that signs are interchangeable. For example,

$$\hat{i} \times (-\hat{j}) = -\hat{i} \times \hat{j} = -\hat{k}$$

(*vii*) The cross product of any two vectors $\vec{A}$ and $\vec{B}$ can be expressed in the following determinant form :

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Expanding this determinant gives the result :

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. If $\vec{P} \cdot \vec{Q} = |\vec{P} \times \vec{Q}|$, then the angle between $\vec{P}$ and $\vec{Q}$ is **[AFMC – Pune 2001]**

(*a*) 0 (*b*) $\frac{\pi}{2}$

(*c*) $\frac{\pi}{4}$ (*d*) π

2. If $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ will be **[CBSE PMT 2001]**

(*a*) 90° (*b*) 120°

(*c*) 0° (*d*) 60°

3. Two equal vectors have a resultant equal to either of them. The angle between them is **[AIIMS 2001]**

(*a*) 60° (*b*) 90°

(*c*) 100° (*d*) 120°

4. A force $\vec{F} = 3\hat{i} + c\hat{j} + 2\hat{k}$ acting on a particle causes displacement $\vec{S} = -4\hat{i} + 2\hat{j} + 3\hat{k}$ in its own direction. If work done is 6 J, then value of '*c*' is **[Pb PMT 2001]**

(*a*) 0 (*b*) 1

(*c*) 6 (*d*) 12

5. $\vec{P} \cdot \vec{Q} = PQ$, then angle between $\vec{P}$ and $\vec{Q}$ is **[AIIMS 1999]**

(*a*) 0° (*b*) 30°

(*c*) 45° (*d*) 60°

6. The vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. The positive value of 'a' is **[AFMC-Pune 2000]**

(a) 3 (b) 4

(c) 9 (d) 13

7. For a body, angular velocity $\vec{\omega} = \hat{i} - 2\hat{j} + 3\hat{k}$ and radius vector $\vec{r} = \hat{i} + \hat{j} + \hat{k}$. Then linear velocity is **[CBSE PMT 1999]**

(a) $-5\hat{i} + 2\hat{j} + 3\hat{k}$ (b) $-5\hat{i} + 2\hat{j} - 3\hat{k}$

(c) $-5\hat{i} - 2\hat{j} + 3\hat{k}$ (d) $-5\hat{i} - 2\hat{j} - 3\hat{k}$

8. The force acting on a body of 10 kg is $(2\hat{i} + \hat{j} - \hat{k})$ N. If the body is initially at rest, then velocity at the end of 20 sec will be (in m/s) **[Pb. PMT 1997]**

(a) $2\sqrt{6}$ (b) $6\sqrt{2}$

(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

9. The angle between two vectors of magnitude 12 and 18 units when their resultant is 24 units is **[AIIMS 1997]**

(a) 63.50° (b) 75.52°

(c) 82.30° (d) 89.16°

10. The angle between two vectors $\vec{A} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{B} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ will be **[AIIMS 1996]**

(a) zero (b) 45°

(c) 90° (d) 180°

11. A bus is going due north at a speed of 50 km/h. It makes 90° left turn without changing its speed. The change in the velocity of the bus is **[CBSE PMT 1990]**

(a) 50 km/h towards west

(b) 50 km/h towards south-west

(c) 70.7 km/h towards south-west

(d) 70.7 km/h towards north-west

12. Two vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. Then the angle between the two vectors will be **[CBSE PMT 1991]**

(a) 45° (b) 60°

(c) 75° (d) 90°

13. A force $\vec{F} = (6\hat{i} - 8\hat{j} + 10\hat{k})$ N acting on a mass produces an acceleration of 1 m/s^2. What will be the mass of the body? **[CBSE PMT 1996]**

(a) $10\sqrt{2}$ kg (b) $2\sqrt{10}$ kg

(c) 10 kg (d) 20 kg

14. Two quantities A and B have different dimensions. Which mathematical operation given below is physically meaningful? **[CPMT 1997]**

(a) A/B (b) $A + B$

(c) $A - B$ (d) none

15. The sum of two forces at a point is 16 N. If their resultant is normal to the smaller force and has a magnitude of 8N, then the forces are **[Pb. PMT 1995]**

(a) 6N and 10N (b) 8N and 8N

(c) 4N and 12N (d) 2N and 14N

16. A boat is sent across a river with a velocity of 8 km/h. If the resultant velocity of boat is 10 km/h, then velocity of river is **[CBSE PMT 1994]**

(a) 12.8 km/h (b) 6 km/h

(c) 8 km/h (d) 10 km/h

17. Which of the following cannot be the resultant of the vectors of magnitudes 5 and 10? **[AFMC-Pune 1996]**

(a) 7 (b) 8

(c) 5 (d) 2

18. The vector sum of two forces is perpendicular to their vector difference. In that case, the forces **[CBSE PMT 2003]**

(a) are not equal to each other in magnitude

(b) cannot be predicted

(c) are equal to each other

(d) are equal to each other in magnitude

19. An aeroplane flies 400 m due north and then 300 m due south and then flies 1200 m upwards, the net displacement is: **[AFMC 2004]**

(a) greater than 1200 m

(b) less than 1200 m

(c) 1400 m

(d) 1500 m

20. Minimum number of unequal vectors which can give zero resultant are: **[AFMC 2005]**

(a) two (b) three

(c) four (d) more than four

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*c*)	**2.** (*b*)	**3.** (*d*)	**4.** (*c*)	**5.** (*a*)
6. (*a*)	**7.** (*a*)	**8.** (*a*)	**9.** (*b*)	**10.** (*c*)
11. (*d*)	**12.** (*d*)	**13.** (*a*)	**14.** (*a*)	**15.** (*a*)
16. (*b*)	**17.** (*d*)	**18.** (*d*)	**19.** (*a*)	**20.** (*b*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. $\vec{P}.\vec{Q} = PQ\cos\theta$ and $|\vec{P}\times\vec{Q}| = PQ\sin\theta$

As per problem, $PQ\cos\theta = PQ\sin\theta$ or $\tan\theta = 1$ $\therefore$ $\theta = \pi/4$

2. Let angle between $\vec{A}$ and $\vec{B}$ be θ.

$\therefore$ $A^2 + B^2 + 2AB\cos\theta = A^2$ or $A^2 + A^2 + 2A^2\cos\theta = A^2$

or $2A^2(1+\cos\theta) = A^2$ or $\cos\theta = -\frac{1}{2}$ $\therefore$ $\theta = 120°$

3. Solution is the same as MCQ No. 2.

4. Work done = $\vec{F}.\vec{S}$ or $6 = (3\hat{i} + c\hat{j} + 2\hat{k}).(-4\hat{i} + 2\hat{j} + 3\hat{k})$

or $6 = -12 + 2c + 6$ $\therefore$ $c = 6$

5. $\vec{P}.\vec{Q} = PQ$ ... given or $PQ\cos\theta = PQ$ or $\cos\theta = \frac{PQ}{PQ} = 1$ $\therefore$ $\theta = 0°$

6. We know that if two vectors are perpendicular to each other, then their dot product is zero.

$\therefore$ $\vec{P}.\vec{Q} = 0$ or $(a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0$ or $a^2 - 2a - 3 = 0$

On solving this equation, $a = 3$ or -1. Therefore, positive value of 'a' is 3.

7. We know that linear velocity $\vec{v}$ is the cross product of angular velocity $(\vec{\omega})$ and radius vector $(\vec{r})$.

$$\therefore \quad \vec{v} = \vec{\omega}\times\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k})\times(\hat{i} + \hat{j} + \hat{k}) \quad \text{or} \quad \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

or $\vec{v} = (-2-3)\hat{i} + (3-1)\hat{j} + (1+2)\hat{k}$ or $\vec{v} = -5\hat{i} + 2\hat{j} + 3\hat{k}$

8. Acceleration, $\vec{a} = \frac{\vec{F}}{m} = \frac{2\hat{i} + \hat{j} - \hat{k}}{10}$ m/s^2. Here $\vec{u} = 0$; t = 20 sec ; $\vec{v}$ = ?

$$\vec{v} = \vec{u} + \vec{a}t = 0 + \left(\frac{2\hat{i} + \hat{j} - \hat{k}}{10}\right)\times 20 = 4\hat{i} + 2\hat{j} - 2\hat{k}$$

$\therefore$ Magnitude of velocity, $v = \sqrt{4^2 + 2^2 + (-2)^2} = 2\sqrt{6}$ m/s

9. $R^2 = A^2 + B^2 + 2AB\cos\theta$

or $(24)^2 = (12)^2 + (18)^2 + 2\times12\times18\cos\theta$

or $\cos\theta = 0.25$ or $\theta = 75.52°$

10. $\vec{A}.\vec{B} = (3\hat{i} + 4\hat{j} + 5\hat{k}).(3\hat{i} + 4\hat{j} - 5\hat{k}) = 9 + 16 - 25 = 0$

But $\vec{A}.\vec{B} = AB\cos\theta \quad \therefore \quad \cos\theta = \dfrac{\vec{A}.\vec{B}}{AB} = \dfrac{0}{AB} = 0$ or $\theta = 90°$

11. Change in velocity = $\sqrt{(50)^2 + (50)^2}$ = 70.7 km/h
The direction of this change in velocity is north-west.

12. Let θ be the angle between the two vectors.

$|\vec{A} + \vec{B}|^2 = A^2 + B^2 + 2AB\cos\theta$ and $|\vec{A} - \vec{B}|^2 = A^2 + B^2 - 2AB\cos\theta$

As per the condition in the problem,

$$A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta \quad \therefore \quad 4AB\cos\theta = 0$$

or $\cos\theta = \dfrac{0}{4AB} = 0 \quad \therefore \quad \theta = 90°$

13. $\vec{F} = 6\hat{i} - 8\hat{j} + 10\hat{k}$

Magnitude of force, $F = \sqrt{(6)^2 + (-8)^2 + (10)^2} = 10\sqrt{2}$ N

$\therefore$ Mass = $\dfrac{F}{a} = \dfrac{10\sqrt{2}}{1} = 10\sqrt{2}$ kg

14. The quantities having different dimensions can only be multiplied or divided but cannot be added or subtracted.

15. Let P and Q be the forces and R their resultant. $P + Q = 16$ N and $R = 8$N.
If P is the smaller force, then,

$$R^2 = Q^2 - P^2 = (Q+P)(Q-P) \text{ or } (8)^2 = (16)(Q-P) \therefore Q - P = 4$$

On solving, P = 6N, Q = 10N

16. River velocity = $\sqrt{10^2 - 8^2}$ = 6 km/h

17. The resultant must lie between (10 + 5) and (10 – 5) *i.e.* between 15 and 5.

18. Let $\vec{A}$ and $\vec{B}$ be the two forces.

Vector sum : $\vec{F}_1 = \vec{A} + \vec{B}$; Vector difference : $\vec{F}_2 = \vec{A} - \vec{B}$

Since $\vec{F}_1$ is perpendicular to $\vec{F}_2$, $\vec{F}_1 . \vec{F}_2 = 0$.

$\therefore \quad (\vec{A} + \vec{B}) . (\vec{A} - \vec{B}) = 0$

or $A^2 - AB + AB - B^2 = 0 \quad \therefore \quad |\vec{A}| = |\vec{B}|$

19. Fig. 8.48 shows the conditions of the problem.

OA = 400 m (due north)

AB = 300 m (due south)

$\therefore$ $OB = OA - AB$

$= 400 - 300 = 100$ m

Also BC = 1200 m (upward)

It is clear from Fig. 8.48 that net displacement is OC. In the right angled triangle OBC,

$$OC = \sqrt{OB^2 + BC^2} = \sqrt{(100)^2 + (1200)^2} = 1204 \text{ m}$$

Hence net displacement is greater than 1200 m.

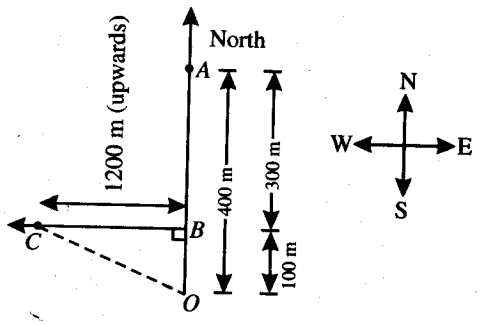


Fig. 8.48

20. According to triangle law of vectors, the minimum number of unequal vectors required to give zero resultant is 3.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. The angle between two vectors $-2\hat{i}+3\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}-4\hat{k}$ is **[EAMCET 1990]**

(a) 0° (b) 90°

(c) 180° (d) none

2. Find the torque of a force $\vec{F} = -3\hat{i}+\hat{j}+5\hat{k}$ acting at the point $\vec{r} = 7\hat{i}+3\hat{j}+\hat{k}$. **[Pb CET 1998]**

(a) $14\hat{i}-38\hat{j}+16\hat{k}$ (b) $4\hat{i}+4\hat{j}+16\hat{k}$

(c) $-14\hat{i}+38\hat{j}-16\hat{k}$ (d) $-21\hat{i}+3\hat{j}-5\hat{k}$

3. If vectors $\vec{P}, \vec{Q}$ and $\vec{R}$ have magnitudes 5, 12 and 13 units and $\vec{P}+\vec{Q}=\vec{R}$, then angle between $\vec{Q}$ and $\vec{R}$ is **[Haryana CET 1998]**

(a) $\cos^{-1}\frac{5}{12}$ (b) $\cos^{-1}\frac{5}{13}$

(c) $\cos^{-1}\frac{12}{13}$ (d) $\cos^{-1}\frac{7}{13}$

4. The radius vector is $2\hat{i}+\hat{j}+\hat{k}$ while linear momentum is $2\hat{i}-3\hat{j}+\hat{k}$. The angular momentum is **[BHU 1997]**

(a) $2\hat{i}-4\hat{k}$ (b) $4\hat{i}-8\hat{k}$

(c) $2\hat{i}-4\hat{j}+2\hat{k}$ (d) $-4\hat{i}+8\hat{k}$

5. If $\vec{A} = 5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{B} = 2\hat{i}+2\hat{j}-c\hat{k}$ are perpendicular vectors, then value of c is **[CET 1991]**

(a) − 2 (b) 8

(c) − 7 (d) − 8

6. Find the unit vector perpendicular to $3\hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}-2\hat{j}+4\hat{k}$. **[Haryana CET 1996]**

(a) $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}-\hat{k})$ (b) $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

(c) $(\hat{i}-\hat{j}-\hat{k})$ (d) $\sqrt{3}\,\hat{i}-\hat{j}-\hat{k}$

7. Two forces $\vec{F_1} = 5\hat{i}+10\hat{j}-20\hat{k}$ and $\vec{F_2} = 10\hat{i}-5\hat{j}-15\hat{k}$ act on a single point. The angle between $\vec{F_1}$ and $\vec{F_2}$ is **[Haryana CET 1998]**

(a) 60° (b) 90°

(c) 45° (d) 30°

8. The relative velocity of a ship moving at V m/s East with reference to another ship moving at V m/s North is **[Haryana CET 1993]**

(a) $\sqrt{2}$ V SE (b) $\sqrt{2}$ V NE

(c) $\sqrt{2}$ V NW (d) $\sqrt{2}$ V SW

9. The angle between $\vec{A}$ and $\vec{B}$ is θ. The value of triple product $\vec{A}\cdot\vec{B}\times\vec{A}$ is **[Haryana CET 1996]**

(a) $A^2B\cos\theta$ (b) $A^2B\sin\theta$

(c) A^2B (d) zero

10. The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of

resultant is 12. If the resultant is at 90° with the force of smaller magnitude, then their magnitudes are **[Roorkee 1992]**

(*a*) 3, 15 (*b*) 4, 14

(*c*) 5, 13 (*d*) 6, 12

11. Three forces start acting simultaneously on a particle moving with velocity $\vec{v}$. These forces are represented in magnitude and direction by the three sides of a triangle *ABC* (See Fig. 8.49). The particle will now move with velocity: **[AIEEE 2003]**

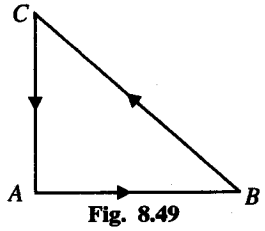


Fig. 8.49

(*a*) less than $\vec{v}$ (*b*) greater than $\vec{v}$

(*c*) $|\vec{v}|$ in the direction of largest force *BC*

(*d*) $\vec{v}$, remaining unchanged

12. Let $\vec{F}$ be the force acting on a particle having position vector $\vec{r}$ and $\vec{\tau}$ be the torque of this force about the origin. Then: **[AIEEE 2003]**

(*a*) $\vec{r}.\vec{F}=0$ and $\vec{F}.\vec{\tau}\neq 0$

(*b*) $\vec{r}.\vec{\tau}\neq 0$ and $\vec{F}.\vec{\tau}=0$

(*c*) $\vec{r}.\vec{\tau}\neq 0$ and $\vec{F}.\vec{\tau}\neq 0$

(*d*) $\vec{r}.\vec{\tau}=0$ and $\vec{F}.\vec{\tau}=0$

13. The coordinates of a moving particle at any time t are given by $x = \alpha t^3$ and $y = \beta t^3$. The speed of the particle at time t is given by : **[AIEEE 2003]**

(*a*) $3t\sqrt{\alpha^2+\beta^2}$ (*b*) $3t^2\sqrt{\alpha^2+\beta^2}$

(*c*) $t^2\sqrt{\alpha^2+\beta^2}$ (*d*) $\sqrt{\alpha^2+\beta^2}$

14. If $\vec{A}\times\vec{B}=\vec{B}\times\vec{A}$, then the angle between $\vec{A}$ and $\vec{B}$ is : **[AIEEE 2004]**

(*a*) π (*b*) $\pi/3$

(*c*) $\pi/2$ (*d*) $\pi/4$

15. Two forces are acting at a point. The magnitude of each force is 2N and the magnitude of their resultant is also 2N. Then the angle between the two forces is **[DCE 2004]**

(*a*) 90° (*b*) 60°

(*c*) 120° (*d*) 0°

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*b*)	**5.** (*d*)
6. (*a*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*c*)
11. (*d*)	**12.** (*d*)	**13.** (*b*)	**14.** (*a*)	**15.** (*c*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. $\cos\theta=\dfrac{\vec{A}.\vec{B}}{AB}$. Now $\vec{A}.\vec{B}=(-2\hat{i}+3\hat{j}+\hat{k}).(\hat{i}+2\hat{j}-4\hat{k}) = -2+6-4=0$

$\therefore \quad \cos\theta=\dfrac{0}{AB}=0$ or $\theta = 90°$

2. Torque $= \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix} = \hat{i}(15-1)+\hat{j}(-3-35)+\hat{k}(7+9) = 14\hat{i}-38\hat{j}+16\hat{k}$

3. The conditions of the problem suggest $(13^2=5^2+12^2)$ that the angle between $\vec{P}$ and $\vec{Q}$ is 90°. If θ is the angle between $\vec{Q}$ and $\vec{R}$, then $\cos\theta=12/13$.

4. Angular momentum is given by ;

$$\vec{L} = \vec{r} \times \vec{p} = (2\hat{i} + \hat{j} + \hat{k}) \times (2\hat{i} - 3\hat{j} + \hat{k}) = 4\hat{i} - 8\hat{k}$$

5. The scalar product of perpendicular vectors is zero.

$\therefore \vec{A}.\vec{B} = 0$ or $A_xB_x + A_yB_y + A_zB_z = 0$ or $10 + 14 + 3c = 0 \quad \therefore \quad c = -8$

6. $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$; $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

The unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is $\vec{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

$$\vec{A} \times \vec{B} = (4+4)\hat{i} + (4-12)\hat{j} + (-6-2)\hat{k} = 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$$|\vec{A} \times \vec{B}| = \sqrt{8^2 + 8^2 + 8^2} = 8\sqrt{3} \quad \therefore \quad \vec{n} = \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

7. $$\cos\theta = \frac{\vec{F_1}.\vec{F_2}}{F_1F_2} = \frac{(5\hat{i} + 10\hat{j} - 20\hat{k}).(10\hat{i} - 5\hat{j} - 15\hat{k})}{(\sqrt{5^2 + 10^2 + (-20)^2})\sqrt{10^2 + (-5)^2 + (-15)^2}}$$

$$= \frac{5 \times 10 - 10 \times 5 + (-20) \times (-15)}{(\sqrt{525})(\sqrt{350})} = 0.7$$

$\therefore \quad \theta \simeq 45°$

8. Fig. 8.50 shows the conditions of the problem.

Relative velocity, $V_R = \sqrt{V^2 + V^2} = \sqrt{2}$ V SE

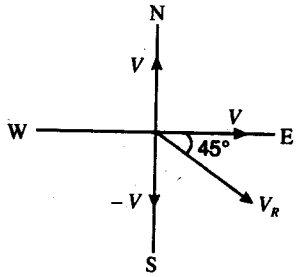


Fig. 8.50

9. $\vec{B} \times \vec{A}$ is a vector which is perpendicular to $\vec{A}$ and $\vec{B}$. Its dot product with $\vec{A}$ will be zero as $\vec{A}$ and $\vec{B} \times \vec{A}$ are perpendicular to each other.

10. Fig. 8.51 shows the conditions of the problem.

$R = 12$; $Q + P = 18$; $Q^2 = P^2 + R^2$ or $Q^2 - P^2 = R^2 = 144$

$\therefore \quad (Q - P)(Q + P) = 144$

or $$(Q - P) = \frac{144}{Q+P} = \frac{144}{18} = 8$$

$Q + P = 18$ and $Q - P = 8$. On solving, $Q = 13$ and $P = 5$.

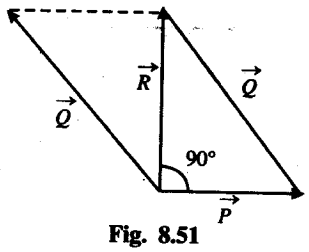


Fig. 8.51

11. Since the three forces are represented in magnitude and direction by the three sides of a triangle in the same order, the resultant force is zero. Therefore, the particle velocity $(\vec{v})$ will remain unchanged.

12. $\vec{\tau} = \vec{r} \times \vec{F}$ means that $\vec{r}, \vec{F}$ and $\vec{\tau}$ are perpendicular to each other.

$\therefore \quad \vec{\tau}.\vec{r} = 0$ and $\vec{F}.\vec{\tau} = 0$

13. $x = \alpha t^3$; $y = \beta t^3$

$\therefore\ v_x = \dfrac{dx}{dt} = \dfrac{d}{dt}(\alpha t^3) = 3\alpha t^2\ ;\ v_y = \dfrac{dy}{dt} = \dfrac{d}{dt}(\beta t^3) = 3\beta t^2$

$\therefore$ Resultant velocity $= \sqrt{v_x^2 + v_y^2} = \sqrt{(3\alpha t^2)^2 + (3\beta t^2)^2} = 3t^2\sqrt{\alpha^2 + \beta^2}$

14. $\vec{A} \times \vec{B} = AB \sin\theta\ ;\ \vec{B} \times \vec{A} = BA \sin\theta = -AB \sin\theta$

Now $\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$ or $AB \sin\theta = -AB \sin\theta$

$\therefore\ AB \sin\theta + AB \sin\theta = 0$ or $2AB \sin\theta = 0$

Since A and B are not zero, $\sin\theta = 0$ or $\theta = 0$ or π

15. $R^2 = F_1^2 + F_2^2 + 2F_1F_2 \cos\theta$

or $2^2 = 2^2 + 2^2 + 2 \times 2 \times 2 \cos\theta$

$\therefore\ \cos\theta = -4/8 = -1/2$ or $\theta = 120°$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **At what angle do the two forces $(P + Q)$ and $(P - Q)$ act so that the resultant is $\sqrt{3P^2 + Q^2}$?**

Hint. Let $X = P + Q$ and $Y = P - Q$. Now $R = \sqrt{3P^2 + Q^2}$. Suppose the angle between X and Y is θ.

Now $R^2 = X^2 + Y^2 + 2XY\cos\theta$

or $3P^2 + Q^2 = (P+Q)^2 + (P-Q)^2 + 2(P+Q)(P-Q)\cos\theta$

or $3P^2 + Q^2 = 2P^2 + 2Q^2 + 2(P^2 - Q^2)\cos\theta$

or $P^2 - Q^2 = 2(P^2 - Q^2)\cos\theta$

$\therefore\ \cos\theta = \dfrac{P^2 - Q^2}{2(P^2 - Q^2)} = \dfrac{1}{2}$ or $\theta = 60°$

2. **A ball is given simultaneously two velocities, one of 10 m/s due east and the other 20 m/s due north-west. Calculate the resultant velocity of the ball.**

Hint. Figure 8.52 shows the conditions of the problem.

$v_1 = 10\ \text{ms}^{-1}$; $v_2 = 20\ \text{ms}^{-1}$; $\theta = 135°$

The resultant velocity v is given by ;

$v^2 = v_1^2 + v_2^2 + 2v_1v_2\cos\theta$

$= (10)^2 + (20)^2 + 2\times10\times20\cos135° = 217.2$

$\therefore\ v = \sqrt{217.2} = 14.73\ \text{ms}^{-1}$

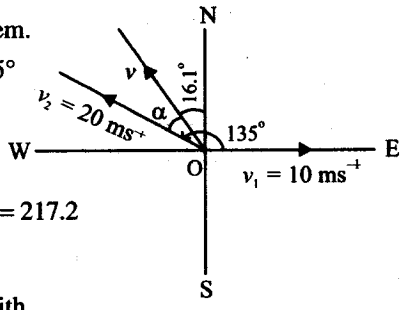


Fig. 8.52

Suppose α is the angle which the resultant makes with velocity v_2 (= 20 ms^{-1}). Then,

$\tan\alpha = \dfrac{v_1 \sin\theta}{v_2 + v_1\cos\theta} = \dfrac{10\sin135°}{20 + 10\cos135°} = 0.5518$

$\therefore\ \alpha = \tan^{-1} 0.5518 = 28.9°$

$\therefore$ Angle which v makes with north = 45° – 28.9° = 16.1°

Hence, the resultant velocity is **14.73 ms^{-1} and it acts 16.1° west of north.**

3. **A river is flowing from west to east at a speed of 5 m/min. In what direction should a man on the south bank of the river capable of swimming at 10 m/min in still water should swim to cross the river (i) in shortest time (ii) along shortest path. [I.I.T.]**

Hint. (*i*) Suppose the man is at point O on the southern bank of the river and B is a point, opposite to O, on the other side of the bank [See Fig. 8.53]. Suppose the man swims with velocity v in a direction OA making an angle θ with OB as shown in Fig. 8.53. The velocity v can be resolved into two rectangular components, *viz.*, $v\cos\theta$ along OB and $v\sin\theta$ acting opposite to river flow. Only the component $v\cos\theta$ is helping the swimmer to cross the river. If d is the width of the river, then time t taken by the swimmer to cross the river is given by;

$$t = \frac{d}{v\cos\theta}$$

The time t will be minimum when $\cos\theta$ is maximum *i.e.*, $\cos\theta = 1$ or $\theta = 0°$. Therefore, in order to cross the river in minimum time, the man should swim **perpendicular to the flow of river.**

$$t_{min} = \frac{d}{v}$$

Note that due to the flow of water, the man will reach point C on the opposite bank where

$$BC = v_w \times t_{min} = v_w \times \frac{d}{v}$$

Here v_w = velocity of river flow

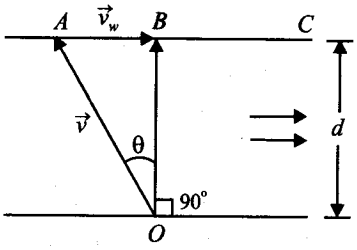


Fig. 8.53

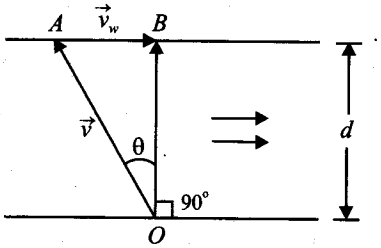


Fig. 8.54

(*ii*) The shortest path between the two banks of the river is OB [See Fig. 8.54]. If the river water had been still, the man could directly go from O to B. In order to cross the flowing river along OB, the man should swim along OA making an angle θ such that the horizontal component $v\sin\theta$ of v cancels the velocity v_w of river flow. Note that $v\sin\theta$ and v_w act in opposite directions.

$\therefore$ $$v_w = v\sin\theta$$

or $$\sin\theta = \frac{v_w}{v} = \frac{5}{10} = \frac{1}{2}$$

or $$\theta = \mathbf{30° \text{ west of north}}$$

The time t' taken by the man to cross the river is

$$t' = \frac{d}{v\cos\theta} = \frac{t_{min}}{\cos\theta}$$

Note that time taken in this case is not minimum.

4. **Find the angle between X-axis and the force represented by ;**

$$\vec{F} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

Hint. Magnitude of force is given by ;

$$F = \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

$$\therefore \quad \cos\theta = \frac{F_x}{F} = \frac{2}{\sqrt{29}} \quad \therefore \ \theta = \cos^{-1}\frac{2}{\sqrt{29}}$$

5. **A force of $7\hat{i} + 6\hat{k}$ newton makes a body move on a rough plane with a velocity $3\hat{j} + 4\hat{k}$ ms^{-1}. Calculate the power in watt.**

Hint. Power = $\vec{F}.\vec{v} = (7\hat{i} + 6\hat{k}).(3\hat{j} + 4\hat{k}) = 6 \times 4 = 24$ W

6. **The resultant of two vectors $\vec{u}$ and $\vec{v}$ is perpendicular to $\vec{u}$ and its magnitude is equal to half of the magnitude of vector $\vec{v}$. Find out the angle between $\vec{u}$ and $\vec{v}$.**

Hint. Figure 8.55 shows the conditions of the problem.

$$|\vec{v}| = v \ ; \quad |\vec{R}| = \frac{v}{2}$$

$$\therefore \quad \sin\theta = \frac{v/2}{v} = \frac{1}{2}$$

or $\quad \theta = 30°$

$\therefore$ Angle between $\vec{u}$ and $\vec{v}$ = 180° − 30° = 150°

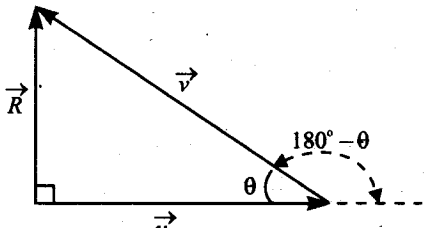


Fig. 8.55

7. **(*i*) For two vectors $\vec{A}$ and $\vec{B}$ if $\vec{A} + \vec{B} = \vec{C}$ and $A + B = C$, then prove that vectors $\vec{A}$ and $\vec{B}$ are parallel to each other.**

(*ii*) If $\vec{A} + \vec{B} = \vec{C}$ and $A^2 + B^2 = C^2$, prove that vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

Hint. (*i*) $\vec{A} + \vec{B} = \vec{C}$

Taking the self-dot product of both sides, we have,

$$(\vec{A} + \vec{B}).(\vec{A} + \vec{B}) = \vec{C}.\vec{C} \quad \text{or} \quad \vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B} = \vec{C}.\vec{C}$$

Now $\vec{A}.\vec{B} = \vec{B}.\vec{A} = AB\cos\theta$ where θ is the angle between vectors $\vec{A}$ and $\vec{B}$.

$$\therefore \quad A^2 + AB\cos\theta + AB\cos\theta + B^2 = C^2$$

or $\quad A^2 + B^2 + 2AB\cos\theta = C^2 \qquad \ldots (i)$

It is given that $A + B = C$ or $A^2 + B^2 + 2AB = C^2 \qquad \ldots (ii)$

Comparing eqs. (*i*) and (*ii*), $\cos\theta = 1$ or $\theta = 0°$. Hence, vectors $\vec{A}$ and $\vec{B}$ are parallel and act in the same direction.

(*ii*) $\quad \vec{A} + \vec{B} = \vec{C}$ or $(\vec{A} + \vec{B}).(\vec{A} + \vec{B}) = \vec{C}.\vec{C}$

As found above, $A^2 + B^2 + 2AB\cos\theta = C^2 \qquad \ldots (i)$

It is given that $A^2 + B^2 = C^2$. ... (*ii*)

Comparing eqs. (*i*) and (*ii*), $2AB\cos\theta = 0$. Since A and B are not zero, $\cos\theta = 0$ or $\theta = 90°$. Hence, $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

8. A particle is moving eastward with a velocity of 5 ms^{-1}. In 10 seconds, the velocity changes to 5 ms^{-1} northwards. What is the average acceleration in this time? [I.I.T.]

Hint. Figure 8.56 shows the conditions of the problem.

Initial velocity, $\vec{v}_1 = 5\ ms^{-1}$ along east $= 5\,\hat{i}$

Final velocity, $\vec{v}_2 = 5\ ms^{-1}$ along north $= 5\hat{j}$

Change in velocity, $\vec{\Delta v} = \vec{v}_2 - \vec{v}_1 = 5\hat{j} - 5\,\hat{i}$

$$\therefore \quad |\vec{\Delta v}| = \sqrt{5^2 + (-5)^2} = 5\sqrt{2}$$

The direction of $\vec{\Delta v}$ is north-west.

Average acceleration, $\vec{a} = \dfrac{\vec{\Delta v}}{\Delta t} = \dfrac{5\sqrt{2}}{10} = \dfrac{1}{\sqrt{2}}\ ms^{-1}$ north-west

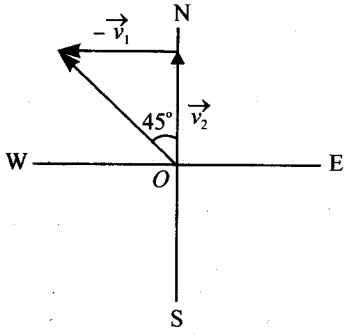


Fig. 8.56

9. Find the resultant of three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ shown in Fig. 8.57. Radius of the circle is R.

Hint. $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = R$

The resultant of $\overrightarrow{OA} + \overrightarrow{OC}$ is along $\overrightarrow{OB}$ (bisector) and its magnitude

$$= 2R\cos 45° = R\sqrt{2}$$

The resultant of $(\overrightarrow{OA} + \overrightarrow{OC})$ and $\overrightarrow{OB}$ is along $\overrightarrow{OB}$ and its magnitude

$$= R\sqrt{2} + R = R(1 + \sqrt{2})$$

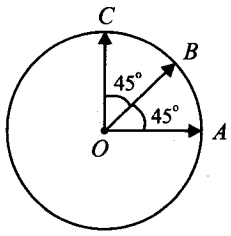


Fig. 8.57

10. If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 7\hat{i} + 24\hat{j}$, then find the vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$.

Hint. Let the required vector be

$$\vec{C} = C_x\hat{i} + C_y\hat{j}$$

It is given that $\vec{C}$ is parallel to $\vec{A}$. Therefore, $\dfrac{C_y}{C_x} = \dfrac{4}{3}$. ... (*i*)

It is also given that $|\vec{C}| = |\vec{B}|$. $\therefore \quad |\vec{C}| = \sqrt{(7)^2 + (24)^2} = \sqrt{625}$

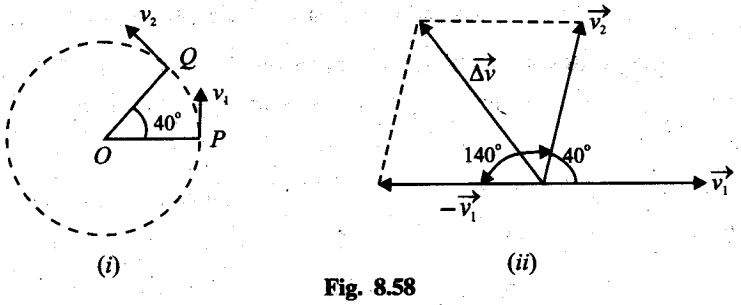
or $$\sqrt{C_x^2 + C_y^2} = \sqrt{625} \quad \text{or} \quad C_x^2 + C_y^2 = 625$$ ... (*ii*)

Solving eqs. (*i*) and (*ii*), we get, $C_x = 15$; $C_y = 20$

$$\therefore \quad \vec{C} = 15\hat{i} + 20\hat{j}$$

11. A particle is moving in a circle of radius r centred at O with constant speed v. What is the change in velocity in moving from P to Q? Given that $\angle POQ = 40°$.

Hint. Fig. 8.58 (*i*) shows the conditions of the problem. Suppose the velocity of the particle is $\vec{v}_1$ at P and $\vec{v}_2$ at Q. Note that magnitudes of the velocities are the same ; only direction changes *i.e.*, $|\vec{v}_1| = |\vec{v}_2| = v$. Hence, the particle has acceleration.



(*i*) (*ii*)

Fig. 8.58

Referring to Fig. 8.58 (*ii*), change in velocity $\vec{\Delta v}$ is given by ;

$$\vec{\Delta v} = \vec{v}_2 - \vec{v}_1$$

Magnitude of $\overrightarrow{\Delta v} = 2v\cos\dfrac{140°}{2} = 2v \cos 70° = 2v \sin 20°$

12. An airplane is flying with velocity of 70 km/h in north-east direction. The wind is blowing 30 km/h from north to south. What is the resultant displacement of the airplane in 4 hours?

Hint. Suppose $\vec{v}_1$ is the velocity of airplane and $\vec{v}_2$ that of the wind. Then the resultant velocity $\vec{v}$ of the airplane is given by (See Fig. 8.59) ;

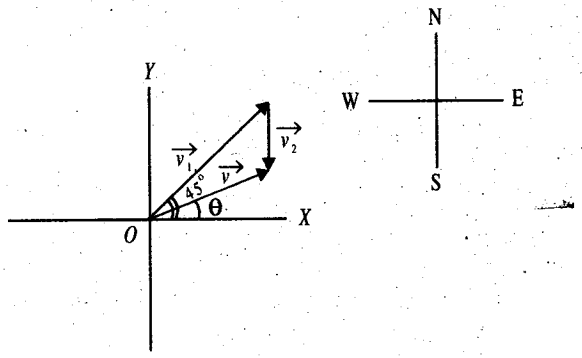


Fig. 8.59

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

Now $\vec{v}_1$ = 70 km/h north-east

$\vec{v}_2$ = 30 km/h north to south

Taking X-axis along east and Y-axis along north, we have,

$$\vec{v}_1 = 70\cos 45°\,\hat{i} + 70\cos 45°\,\hat{j} = \frac{70}{\sqrt{2}}(\hat{i}+\hat{j}) \text{ and } \vec{v}_2 = -30\hat{j}$$

$$\therefore \quad \vec{v} = \vec{v}_1 + \vec{v}_2 = \frac{70}{\sqrt{2}}(\hat{i}+\hat{j}) - 30\hat{j} = (49.5\hat{i} + 19.5\hat{j}) \text{ km/h}$$

$\therefore$ Displacement in time t (= 4 hours) is

$$\vec{S} = \vec{v}\,t = (49.5\hat{i} + 19.5\hat{j}) \times 4 = (198\hat{i} + 78\hat{j}) \text{ km}$$

Magnitude of displacement is given by ;

$$|\vec{S}| = \sqrt{(198)^2 + (78)^2} = 212 \text{ km}$$

If θ is the angle made by the resultant displacement with X-axis (*i.e.*, east), then,

$$\tan\theta = \frac{78}{198} = 0.394 \quad \therefore \ \theta = 21.5^\circ$$

13. A vector $\vec{A}$ has magnitude of 6 units and is directed along positive X-axis. Another vector $\vec{B}$ has magnitude of 4 units and lies in the XY-plane, making an angle of 30° with positive X-axis and an angle of 60° with positive Y-axis. Find the vector product of $\vec{A}$ and $\vec{B}$.

Hint. Taking unit vector $\hat{i}$ along positive X-axis and unit vector $\hat{j}$ along positive Y-axis, the two vectors can be expressed as :

$$\vec{A} = 6\hat{i} \ ; \ \vec{B} = 4\cos 30^\circ \hat{i} + 4\cos 60^\circ \hat{j} = 2\sqrt{3}\,\hat{i} + 2\hat{j}$$

$$\therefore \quad \vec{A} \times \vec{B} = 6\hat{i} \times (2\sqrt{3}\hat{i} + 2\hat{j}) = 12\hat{k}$$

Thus, the vector $\vec{A} \times \vec{B}$ has a magnitude of 12 units and is directed along positive Z-axis.

9

Motion in a Plane

INTRODUCTION

In one dimensional motion, the object moves in a straight line (*e.g.*, along *X*-axis or *Y*-axis). Therefore, only one number (x or y) is required to specify the position of the object. But in two dimensional motion, the object moves in a plane —say *XY* plane. Consequently, we require two numbers (x, y) to specify the position of the object in the plane. However, in three dimensional motion, the object moves in space and we need three numbers (x, y, z) to locate the position of the object in space. It is because the space we live in is three dimensional. Two dimensional motion is very important due to two principal reasons. First, many interesting phenomena are associated with two dimensional motion. Secondly, this type of motion can be easily illustrated on a paper or blackboard. For this reason, most of our discussion in this chapter will be confined to two dimensional motion.

In one dimensional motion, we can handle vector quantities (*e.g.*, displacement, velocity, acceleration *etc.*) quite easily because their directional aspect can be taken care of by +ve and –ve signs. This is due to the fact that in one dimension, only two directions are possible. However, the situation is not that simple in two dimensional motion (*i.e.*, motion in a plane) or three dimensional motion (*i.e.*, motion in space). Here we have to use vectors to deal with vector quantities. We shall use the properties of vectors discussed in the previous chapter to describe the motion of an object in a plane.

9.1. MOTION IN A PLANE

When an object moves in a plane, say *XY* plane, we require two rectangular coordinates (x, y) to specify its position in the plane. Consider the simple case of an object moving along a straight line *AB* in the *XY* plane as shown in Fig. 9.1. When the object is at point *A*, its coordinates are (x_1, y_1) and when it is at point *B*, its coordinates are (x_2, y_2). Thus we can locate the position of an object in a plane by rectangular coordinates (x, y). However, we can also locate the position of an object in a plane by position vector of the object. Thus in Fig. 9.1, the position vector of the object at point *A* is $\overrightarrow{OA}$ $(= \vec{r}_1)$ and at point *B*, the position vector of the object is $\overrightarrow{OB}$ $(= \vec{r}_2)$. The two methods are equivalent and are frequently used to describe the motion of an object in a plane.

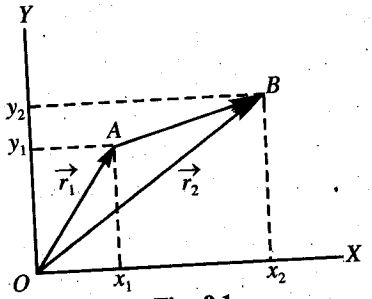


Fig. 9.1

Uniform velocity. *An object is said to be moving with a uniform velocity in a plane if it undergoes equal displacements in equal intervals of time, however small these time intervals may be.*

Note that uniform velocity means that the magnitude as well as direction of the velocity remains the same. This is possible only if the object moves in a straight line with constant speed in the plane. *Therefore, we arrive at an important conclusion that the path of an object moving with uniform velocity in a plane is a straight line.*

9.2. MOTION WITH UNIFORM VELOCITY IN A PLANE

Consider an object moving with a uniform velocity $\vec{v}$ in the *XY* plane as shown in Fig. 9.2. Suppose at $t = 0$, the object is at point $A(x_0, y_0)$ and the position vector of the object is $\overrightarrow{OA}$ $(= \vec{r}_0)$. Let at time t, the object be at point $B(x, y)$ and its position vector be $\overrightarrow{OB}$ $(= \vec{r})$.

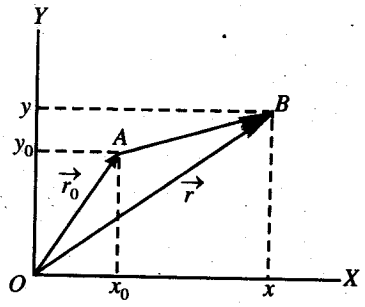


Fig. 9.2

(*i*) **Displacement.** The displacement of the object in the time interval $(t - 0)$ is $\overrightarrow{AB}$. According to triangle law of vectors,

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

or $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$\therefore$ Displacement, $$\overrightarrow{AB} = \vec{r} - \vec{r}_0 \qquad ...(i)$$

Eq. (*i*) gives the displacement $\overrightarrow{AB}$ of the object in terms of its position vectors.

We can also express displacement $\overrightarrow{AB}$ in terms of rectangular components (or coordinates).

Now $$\vec{r}_0 = x_0\hat{i} + y_0\hat{j} \; ; \; \vec{r} = x\hat{i} + y\hat{j}$$

$\therefore$ Displacement, $$\overrightarrow{AB} = \vec{r} - \vec{r}_0 = (x\hat{i} + y\hat{j}) - (x_0\hat{i} + y_0\hat{j})$$

or $$\overrightarrow{AB} = (x - x_0)\hat{i} + (y - y_0)\hat{j} \qquad ...(ii)$$

Eq. (*ii*) gives the displacement $\overrightarrow{AB}$ of the object in terms of rectangular components.

(*ii*) **Velocity.** The velocity of the object is given by ;

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time interval}}$$

or $$\vec{v} = \frac{\vec{r} - \vec{r}_0}{t - 0} = \frac{\vec{r} - \vec{r}_0}{t}$$

$\therefore$ $$\vec{v} = \frac{\vec{r} - \vec{r}_0}{t} \qquad ...(iii)$$

Eq. (*iii*) gives the velocity $\vec{v}$ of the object in terms of its position vectors. Since the object is moving with uniform velocity ($\vec{v}$), the magnitude as well as direction of the velocity is the same at every instant.

(*iii*) **Equation of motion of object.** From eq. (*iii*), we have,

$$\vec{r} - \vec{r}_0 = \vec{v}\,t$$

or $$\vec{r} = \vec{r_0} + \vec{v}\,t \qquad ...(iv)$$

Eq. (*iv*) gives the position ($\vec{r}$) of the object moving with uniform velocity ($\vec{v}$) at any time t. Note that $\vec{r_0}$ is the position vector of the object at $t = 0$.

In terms of rectangular components. Let us express eq. (*iv*) in terms of rectangular components (or coordinates). Suppose v_x and v_y are the magnitudes of the rectangular components of $\vec{v}$ along X-axis and Y-axis respectively. Then,

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \quad \text{where } v = \sqrt{v_x^2 + v_y^2}$$

Since magnitude of $\vec{v}$ is constant, v_x = constant and v_y = constant.

Now $$\vec{r} = \vec{r_0} + \vec{v}\,t$$

Here $$\vec{r} = x\hat{i} + y\hat{j} \;;\; \vec{r_0} = x_0\hat{i} + y_0\hat{j} \;;\; \vec{v} = v_x\hat{i} + v_y\hat{j}$$

$\therefore$ $$x\hat{i} + y\hat{j} = (x_0\hat{i} + y_0\hat{j}) + (v_x\hat{i} + v_y\hat{j})t$$

or $$x\hat{i} + y\hat{j} = (x_0 + v_x t)\hat{i} + (y_0 + v_y t)\hat{j} \qquad ...(v)$$

The coefficients $\hat{i}$ and $\hat{j}$ on both sides of the eq. (*v*) must be equal.

$\therefore$ $$x = x_0 + v_x t \qquad ...(vi)$$

$$y = y_0 + v_y t \qquad ...(vii)$$

Thus eq. (*iv*) can be replaced by two scalar equations (*vi*) and (*vii*). Now eqs. (*vi*) and (*vii*) represent motion along two perpendicular straight lines—motion along X-axis with uniform velocity v_x and motion along Y-axis with uniform velocity v_y.

Therefore, two dimensional motion (i.e., motion in a plane) with uniform velocity is a combination of two one-dimensional uniform motions along perpendicular directions.

(*iv*) **Path of the object.** An object moving with a uniform velocity in a plane follows a straight line path. We have already given the physical explanation for it. Let us now turn to mathematical explanation.

Now $$x = x_0 + v_x t \;;\; y = y_0 + v_y t$$

It is clear that $t = \dfrac{x - x_0}{v_x}$. Putting this value of t in the equation for y, we have,

$$y = y_0 + \frac{v_y}{v_x}(x - x_0)$$

It is clear that this is the equation of a straight line having slope = v_y / v_x. *Therefore, the path of the object moving with uniform velocity in a plane is a straight line.*

9.3. MOTION WITH UNIFORM ACCELERATION IN A PLANE

We now discuss the motion of an object moving with uniform acceleration $\vec{a}$ in a plane. *An object is said to be moving with uniform acceleration in a plane if its velocity vector changes by equal amounts in equal intervals of time, however small these time intervals may be.* Note that uniform acceleration means that magnitude as well as direction of acceleration remains the same. As we shall see, the path of the object moving with uniform acceleration in a plane is parabolic.

(*i*) **Acceleration.** Suppose at $t = 0$, the velocity of the object is $\vec{v}_0$ and at $t = t$, its velocity is $\vec{v}$. Then uniform acceleration $\vec{a}$ of the object is

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time interval}}$$

or $$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t-0} = \frac{\vec{v} - \vec{v}_0}{t}$$

$\therefore$ $$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t} \quad ...(i)$$

Eq. (*i*) gives the expression for the uniform acceleration $\vec{a}$ of the object. Since the object is moving with uniform (constant) acceleration, the magnitude as well as direction of acceleration remains constant at every instant of motion.

(*ii*) **Equation of motion.** From eq. (*i*), we have,

$$\vec{v} - \vec{v}_0 = \vec{a}t$$

or $$\vec{v} = \vec{v}_0 + \vec{a}t \quad ...(ii)$$

Eq. (*ii*) gives the velocity $(\vec{v})$ of the object moving with uniform acceleration $(\vec{a})$ at any time t. Note that $\vec{v}_0$ is the velocity of the object at $t = 0$.

In terms of rectangular components. Let us express eq. (*ii*) in terms of rectangular components (or coordinates). Suppose a_x and a_y are the magnitudes of the rectangular components of $\vec{a}$ along *X*-axis and *Y*-axis respectively. Then,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} \quad \text{where} \quad a = \sqrt{a_x^2 + a_y^2}$$

Since magnitude of $\vec{a}$ is constant, a_x = constant and a_y = constant.

Now $$\vec{v} = \vec{v}_0 + \vec{a}t$$

Here $$\vec{v} = v_x\hat{i} + v_y\hat{j};\ \vec{v}_0 = v_{x0}\hat{i} + v_{y0}\hat{j};\ \vec{a} = a_x\hat{i} + a_y\hat{j}$$

$\therefore$ $$v_x\hat{i} + v_y\hat{j} = (v_{x0}\hat{i} + v_{y0}\hat{j}) + (a_x\hat{i} + a_y\hat{j})t$$

or $$v_x\hat{i} + v_y\hat{j} = (v_{x0} + a_xt)\hat{i} + (v_{y0} + a_yt)\hat{j} \quad ...(iii)$$

The coefficients of $\hat{i}$ and $\hat{j}$ on both sides of eq. (*iii*) must be the same.

$\therefore$ $$v_x = v_{x0} + a_xt \quad ...(iv)$$

$$v_y = v_{y0} + a_yt \quad ...(v)$$

Thus eq. (*ii*) can be replaced by two equations (*iv*) and (*v*). Now eqs. (*iv*) and (*v*) represent motion along two perpendicular straight lines—motion along *X*-axis with uniform acceleration a_x and motion along *Y*-axis with uniform acceleration a_y.

Therefore, two dimensional motion (i.e, motion in a plane) with uniform acceleration is a combination of two one dimensional uniformly accelerated motions along perpendicular directions.

(*iii*) **Expression for displacement for uniform acceleration.** Suppose at $t = 0$, the velocity of the object is $\vec{v}_0$ and at $t = t$, it is $\vec{v}$. The average velocity $\vec{v}_{av}$ during the time interval $0 - t$ is

$$\vec{v}_{av} = \left(\frac{\vec{v}_0 + \vec{v}}{2}\right) = \left(\frac{\vec{v}_0 + \vec{v}_0 + \vec{a}t}{2}\right) \qquad \left(\because \vec{v} = \vec{v}_0 + \vec{a}t\right)$$

$$\therefore \quad \vec{v}_{av} = \vec{v}_0 + \frac{1}{2}\vec{a}t$$

If $\vec{r}_0$ and $\vec{r}$ are the position vectors of the object at $t = 0$ and $t = t$ respectively, then $\vec{v}_{av}$ is given by ;

$$\vec{v}_{av} = \frac{\vec{r} - \vec{r}_0}{t - 0} = \frac{\vec{r} - \vec{r}_0}{t}$$

or $$\vec{r} - \vec{r}_0 = \vec{v}_{av}\, t$$

or $$\vec{r} = \vec{r}_0 + \vec{v}_{av}\, t = \vec{r}_0 + (\vec{v}_0 + \frac{1}{2}\vec{a}t)t \qquad \left(\because \vec{v}_{av} = \vec{v}_0 + \frac{1}{2}\vec{a}t\right)$$

$$\therefore \quad \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \qquad \ldots (vi)$$

Eq. (*vi*) can be written in terms of rectangular components as :

$$x\hat{i} + y\hat{j} = \left(x_0\hat{i} + y_0\hat{j}\right) + \left(v_{x0}\hat{i} + v_{y0}\hat{j}\right)t + \frac{1}{2}\left(a_x\hat{i} + a_y\hat{j}\right)t^2$$

or $$x\hat{i} + y\hat{j} = \left(x_0 + v_{x0}t + \frac{1}{2}a_x t^2\right)\hat{i} + \left(y_0 + v_{y0}t + \frac{1}{2}a_y t^2\right)\hat{j}$$

$$\therefore \quad x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \qquad \ldots (vii)$$

and $$y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2 \qquad \ldots (viii)$$

Note that eqs. (*vii*) and (*viii*) are similar to that of the position-time relation for the motion of the object with uniform acceleration in one dimension.

(*iv*) **Path of the object.** If we eliminate t from eqs. (*vii*) and (*viii*), we get second degree equation in x and y. Therefore, path of the object will be parabolic. *Therefore, path of an object moving with uniform acceleration in a plane is parabolic.*

Example 9.1. The position vector of a particle at $t = 0$ is $\vec{r}_1 = -3\hat{i} + 2\hat{j}$ and later at $t = t$, it is $\vec{r}_2 = 9\hat{i} + 2\hat{j}$. What is the displacement from $\vec{r}_1$ to $\vec{r}_2$?

Solution. $$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \left(9\hat{i} + 2\hat{j}\right) - \left(-3\hat{i} + 2\hat{j}\right) = 12\,\hat{i}$$

Example 9.2. A particle moves in the *XY* plane in such a way that its coordinates x and y vary with time as:

$$x = t^3 - 32\,t \; ; \qquad y = 5\,t^2 + 12$$

Here x and y are in metres and t in seconds. Find the position, velocity and acceleration of the particle at $t = 3$ s.

Solution. The position of the particle is given by ;

$$\vec{r} = x\hat{i} + y\hat{j} = \left(t^3 - 32t\right)\hat{i} + \left(5t^2 + 12\right)\hat{j}$$

$$= [(3)^3 - (32 \times 3)]\,\hat{i} + [5\,(3)^2 + 12]\,\hat{j} = -69\,\hat{i} + 57\,\hat{j}\ \text{m}$$

The two rectangular components of the velocity $\vec{v}$ are :

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(t^3 - 32t) = 3t^2 - 32$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(5t^2 + 12) = 10t$$

At t = 3s, $v_x = 3 \times 3^2 - 32 = -5$ ms^{-1}

At t = 3s, $v_y = 10 \times 3 = 30$ ms^{-1}

$$\therefore \quad \vec{v} = -5\hat{i} + 30\hat{j} \text{ ms}^{-1}$$

The two rectangular components of the acceleration $\vec{a}$ are :

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(3t^2 - 32) = 6t \; ; \; a_y = \frac{dv_y}{dt} = \frac{d}{dt}(10t) = 10$$

At t = 3s, $a_x = 6 \times 3 = 18$ ms^{-2}

At t = 3s, $a_y = 10$ ms^{-2}

$$\therefore \quad \vec{a} = 18\hat{i} + 10\hat{j} \text{ ms}^{-2}$$

9.4. PROJECTILE

*A **projectile** is an object that is thrown into air and then moves under the influence of gravity *alone.*

Since the projectile moves horizontally as well as vertically, it is a case of two dimensional motion *i.e.,* motion in a plane. Therefore, two coordinates are required to specify the position of the projectile at any time. The projectile motion is the most familiar case of motion in a plane with constant acceleration. This constant acceleration has a vertically downward direction and its magnitude is g (= 9.8 ms^{-2}).

Examples. The following are the examples of projectile motion :

(*i*) A body dropped from the window of a moving train.

(*ii*) A bullet fired from the gun.

(*iii*) An athlete doing the high jump.

(*iv*) A cricket ball thrown in any direction.

(*v*) A hammer thrown by an athlete.

Projectile motion. Let us first discuss the salient features of projectile motion. Fig. 9.3 shows a projectile projected from the origin O with an initial velocity $\vec{v_0}$ at angle θ above the horizontal (*i.e.,* X-axis). The projectile rises to the highest point B and then descends. It finally strikes the ground at point D at some horizontal distance from its point of projection O. The angle θ is called the *angle of projection* and the horizontal distance OD (= R) is called the *horizontal range* of the projectile. The total time taken by the projectile in describing the path OBD is called the

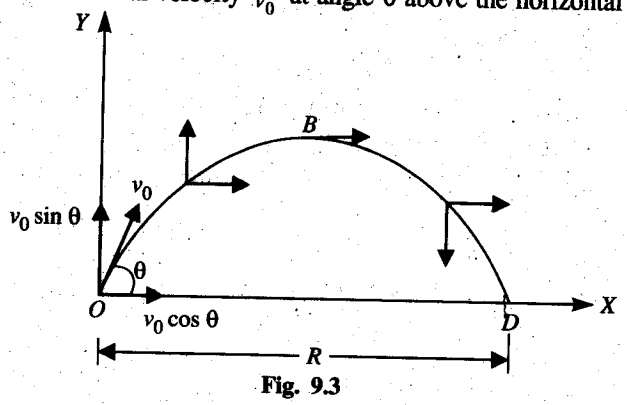


Fig. 9.3

* For simplicity, we neglect air resistance.

time of flight. The path *OBD* followed by the projectile is called its *trajectory.* It will be shown that trajectory of the projectile is parabolic.

The initial velocity $\vec{v_0}$ of the projectile can be resolved into two rectangular components *viz.* $v_0 \cos\theta$ along the horizontal (*i.e.*, *X*-axis) and $v_0 \sin\theta$ along the vertical (*i.e.*, *Y*-axis). The horizontal component $v_0 \cos\theta$ remains constant because there is no acceleration in the horizontal direction. However, the vertical component $v_0 \sin\theta$ is under the action of gravity and its value goes on changing during the projectile motion. Thus projectile motion is a combination of the following two independent perpendicular motions :

(*i*) *Motion along the horizontal with constant velocity* ($v_0 \cos\theta$).

(*ii*) *Motion along the vertical under gravity (i.e., motion under uniform acceleration g).*

9.5. MATHEMATICAL ANALYSIS OF PROJECTILE MOTION

Suppose a projectile is projected from the origin *O* with an initial velocity $\vec{v_0}$ at an angle θ above the horizontal as shown in Fig. 9.4. At $t = 0$, the rectangular components of $\vec{v_0}$ are $v_{x0} = v_0 \cos\theta$ along the horizontal and $v_{y0} = v_0 \sin\theta$ along the vertical. Also at $t = 0$, $x_0 = 0$ and $y_0 = 0$ because the projectile is at the origin.

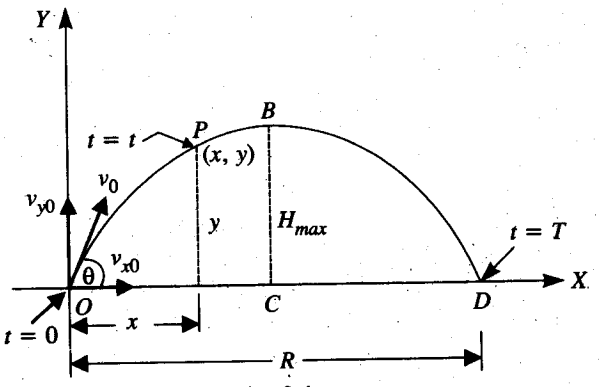


Fig. 9.4

Suppose at any time *t*, the projectile is at point *P*(*x*, *y*). This means that at time *t*, the projectile has covered a horizontal distance *x* from the origin *O* and has attained a height *y* above the origin.

Motion along the horizontal. The horizontal component v_x ($= v_{x0} = v_0 \cos\theta$) of the velocity is constant because there is no horizontal acceleration (*i.e.*, $a_x = 0$). At time *t*, the projectile is at point *P*(*x*, *y*) and its position along *X*-axis is

$$x = x_0 + v_x t + \frac{1}{2} a_x t^2$$

or $$x = v_x t \qquad (\because\ x_0 = 0\ ,\ a_x = 0)$$

$\therefore$ $$x = (v_0 \cos\theta)\, t \qquad (\because\ v_x = v_{x0} = v_0 \cos\theta)\ \ ...(i)$$

The horizontal velocity of the projectile at point *P* is v_x ($= v_0 \cos\theta$); it remains the same throughout.

Motion along the vertical. The initial vertical component of velocity (*i.e.*, at $t = 0$) is $v_0 \sin\theta$ and it changes at constant rate with time due to constant gravitational acceleration (*g*). Note that constant acceleration in the motion is always $-g$* whether the projectile goes up or comes down (*i.e.*, $a_y = -g$). At time *t*, the projectile is at point *P* (*x*, *y*) and its position along *Y*- axis is given by ;

$$y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2$$

---

* Here we choose *y* to be positive in the upward direction. Since *g* always acts downward whether motion of the projectile is upward or downward, $a_y = -g$.

or $$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \qquad [\because v_{y0} = v_0 \sin\theta; y_0 = 0 \text{ and } a_y = -g]$$

$$\therefore \quad y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \qquad ...(ii)$$

At time t, the projectile is at $P(x, y)$ and the vertical component of its velocity at this time is given by ;

$$v_y = v_{y0} - gt$$

or $$v_y = v_0 \sin\theta - gt \qquad ...(iii)$$

Trajectory of projectile. The path or trajectory of the projectile (air friction neglected) is a parabola. To show this, we rewrite eq. (*ii*).

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

From eq. (*i*), $t = x/v_0 \cos\theta$. Putting this value of t in the above equation, we have,

$$y = (v_0 \sin\theta)\frac{x}{v_0 \cos\theta} - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta}\right)^2$$

or $$y = x\tan\theta - \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x^2$$

From this equation, we see that y as a function of x has the form :

$$y = ax - bx^2 \qquad \text{... Equation of parabola}$$

where $a\ (= \tan\theta)$ and $b\left(= \frac{g}{2v_0^2 \cos^2\theta}\right)$ are constants for any given projectile. *Therefore, the path or the trajectory of the projectile is parabolic.*

9.6. PARAMETERS OF PROJECTILE MOTION

Consider the projectile motion shown in Fig. 9.4 (Refer back). The important parameters of the projectile motion are :

1. Time of flight (T)
2. Maximum height attained (H_{max})
3. Horizontal range (R)

The first two parameters (*i.e.*, T and H_{max}) are determined from the vertical motion of the projectile while the third (*i.e.*, R) is calculated from the horizontal motion of the projectile.

1. Time of flight (T). *It is the time elapsing from the launching to the time the projectile returns to the ground again.* Obviously, it is the time for which the projectile remains in air before coming to original height (ground). The position y of the projectile along Y-axis at any time t is

$$y = (v_0 \sin\theta)\, t - \frac{1}{2}g\, t^2$$

When projectile returns to ground, its vertical displacement is zero (*i.e.*, $y = 0$).

$$\therefore \quad 0 = (v_0 \sin\theta)\, t - \frac{1}{2}g\, t^2$$

or $$0 = t\left(v_0 \sin\theta - \frac{1}{2}gt\right)$$

$$\therefore \quad t = 0 \ ; \quad t = \frac{2v_0 \sin\theta}{g}$$

The first solution for t (*i.e.*, $t = 0$) corresponds to the projecting point O. The second solution for t is the time of flight (T) of the projectile.

$$\therefore \quad \text{Time of flight, } T = \frac{2v_0 \sin\theta}{g}$$

Note. The time taken by the projectile to go from the starting point O to the highest point B is called the *time of ascent.* On the other hand, the time taken by the projectile to go from the highest point B to the level of point O (*i.e.*, point D in Fig. 9.4) is the *time of descent.* Since the trajectory of the projectile is *symmetrical about the mid-point of its flight, the two times are equal *i.e.*,

$$\text{Time of ascent} = \text{Time of descent} = \frac{T}{2} = \frac{v_0 \sin\theta}{g}$$

2. Maximum height attained (H_{max}). *It is the maximum height to which the projectile rises above the launching point (origin).* The position of the projectile along Y-axis at any time t is given by;

$$y = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$$

$$\text{When } t = \frac{T}{2} = \frac{v_0 \sin\theta}{g}, \text{ then, } y = H_{max}.$$

$$\therefore \quad H_{max} = (v_0 \sin\theta)\frac{(v_0 \sin\theta)}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2$$

or
$$H_{max} = \frac{v_0^2 \sin^2\theta}{2g}$$

3. Horizontal Range (R). *It is the horizontal distance travelled by the projectile before returning to the ground (i.e., original height).* In other words, it is the horizontal distance travelled by the projectile during time T (= time of flight). The position of the projectile along X-axis at any time t is given by ;

$$x = (v_0 \cos\theta)\, t$$

$$\text{When } t = T = \frac{2v_0 \sin\theta}{g} \; ; \; x = R$$

$$\therefore \quad R = (v_0 \cos\theta)\left(\frac{2v_0 \sin\theta}{g}\right) = v_0^2 \left(\frac{2\sin\theta\cos\theta}{g}\right)$$

or
$$R = \frac{v_0^2 \sin 2\theta}{g} \quad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$

Note that the horizontal range (R) of a projectile depends upon the projection angle (θ) and initial speed (v_0). For fixed θ, $R \propto v_0^2$; doubling the projection speed will increase the range four times.

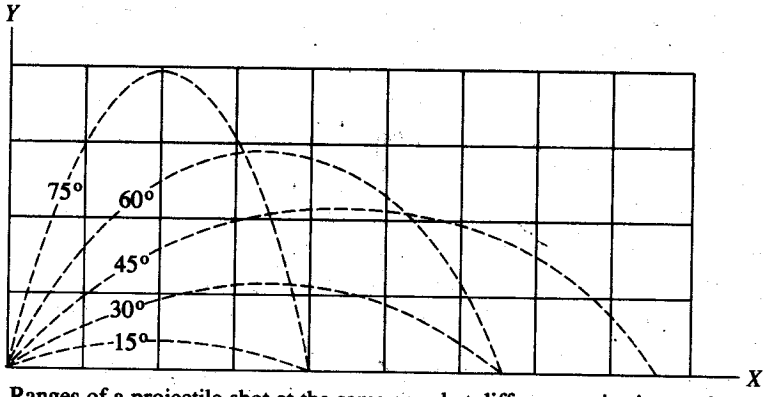
Condition for maximum horizontal range. For a given initial speed v_0, the horizontal range will be maximum when $\sin 2\theta = 1$ or $\theta = 45°$.

$$\therefore \quad \text{Maximum range, } R_{max} = \frac{v_0^2}{g} \qquad \text{...when } \theta = 45°$$

---

* This symmetry is due to the fact that the acceleration due to gravity (g) is the same for upward and downward motion.

Hence for a given initial speed, the horizontal range of the projectile will be maximum when the angle of projection is 45° (neglecting air friction).

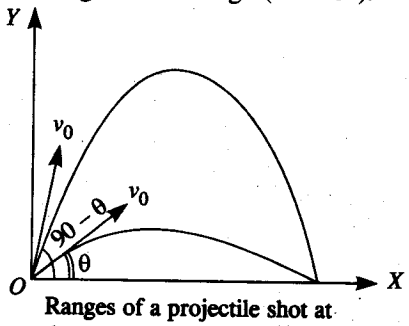


Ranges of a projectile shot at the same speed at different projection angles

Fig. 9.5

Fig. 9.5 shows the paths of several projectiles, all with the same initial speed (v_0) but with different projection angles. Note that horizontal range is maximum for the projectile which is launched at a projection angle of 45°. You may see that for small projection angles ($\theta < 45°$), the horizontal range is *small. It is also small when the projection angles are **large ($\theta > 45°$).

. Two projection angles for same R. The reader may note another remarkable thing in Fig. 9.5 above. For the given initial speed v_0, there are two projection angles θ and $(90° - \theta)$ for which the horizontal range of the projectile is the same. Thus referring to Fig. 9.5, the horizontal range of the projectile is the same for $\theta = 30°$ and $(90° - \theta) = 90° - 30° = 60°$. In other words, if two projectiles are launched at projection angles of 30° and 60° with same initial speed, their horizontal range will be the same. Does this result surprise you? We can prove it mathematically by referring to Fig. 9.6. For projection angle θ, the horizontal range is



Ranges of a projectile shot at the same speed at different projection angles.

Fig. 9.6

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 2\sin\theta\cos\theta}{g} \quad \ldots (i)$$

For projection angle $(90° - \theta)$, the horizontal range is

$$R = v_0^2\left[\frac{2\sin(90°-\theta)\cos(90°-\theta)}{g}\right]$$

or $$R = v_0^2\left(\frac{2\cos\theta\sin\theta}{g}\right)$$

---

* If the projection angle is small, the horizontal component of velocity ($v_0 \cos\theta$) is relatively high but the time of flight ($T = 2v_0 \sin\theta/g$) will be very short. As a result, R [$= (v_0 \cos\theta)T$] will be small.

**. In this case, the time of flight is long but horizontal component of velocity is small. Again horizontal range will be small.

This is just the same equation as eq. (*i*). Although projection angles θ and $(90° - \theta)$ give the same horizontal range (of course, for same initial speed v_0), the two paths differ in time of flight (T) and maximum height (H_{max}) attained. This is not surprising because the values of T and H_{max} depend upon the angle of projection (θ).

9.7. VELOCITY OF PROJECTILE AT ANY TIME

During the flight of the projectile, it is under the action of two velocities *viz.*

(*i*) a constant horizontal velocity v_{x0}.

(*ii*) uniformly changing velocity along the vertical or y-axis.

Therefore, the velocity v of the projectile changes with time. At $t = 0$, the vertical component of velocity is $v_0 \sin\theta$ and goes on decreasing uniformly till it becomes zero at the highest point. From now onward, the vertical component of velocity goes on increasing till it becomes $v_0 \sin\theta$ when projectile hits the ground [See Fig. 9.7].

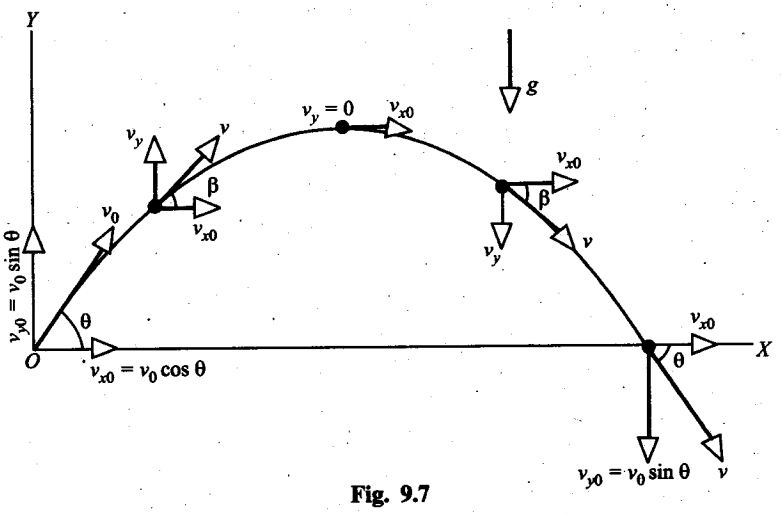


Fig. 9.7

At any time t, the vertical component of velocity is

$$v_y = v_0 \sin\theta - gt$$

Horizontal component of velocity, $v_x = v_{x0} = v_0 \cos\theta$

Since the velocities v_y and v_{x0} are always at right angles to each other, the projectile velocity is given by ;

$$v = \sqrt{(v_{x0})^2 + (v_y)^2} = \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}$$

$$= \sqrt{v_0^2\cos^2\theta + v_0^2\sin^2\theta + g^2t^2 - 2v_0gt\sin\theta}$$

$$= \sqrt{v_0^2(\cos^2\theta + \sin^2\theta) + g^2t^2 - 2v_0gt\sin\theta}$$

$$\therefore \quad v = \sqrt{v_0^2 + g^2t^2 - 2v_0gt\sin\theta}$$

Since v is tangent to the path at any instant, the angle β which it makes with the horizontal at the considered instant is

This is just the same equation as eq. (*i*). Although projection angles θ and $(90° - \theta)$ give the same horizontal range (of course, for same initial speed v_0), the two paths differ in time of flight (T) and maximum height (H_{max}) attained. This is not surprising because the values of T and H_{max} depend upon the angle of projection (θ).

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During the flight of the projectile, it is under the action of two velocities *viz.*

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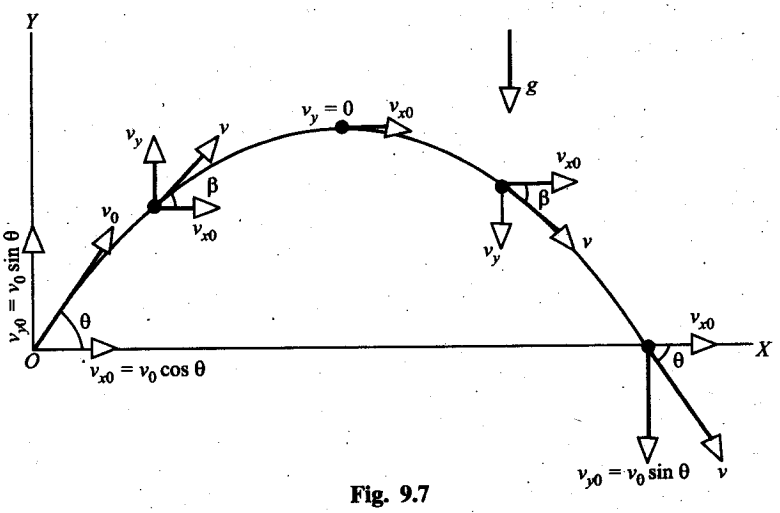

Fig. 9.7

At any time t, the vertical component of velocity is

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Horizontal component of velocity, $v_x = v_{x0} = v_0 \cos\theta$

Since the velocities v_y and v_{x0} are always at right angles to each other, the projectile velocity is given by ;

$$v = \sqrt{(v_{x0})^2 + (v_y)^2} = \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}$$

$$= \sqrt{v_0^2\cos^2\theta + v_0^2\sin^2\theta + g^2t^2 - 2v_0gt\sin\theta}$$

$$= \sqrt{v_0^2(\cos^2\theta + \sin^2\theta) + g^2t^2 - 2v_0gt\sin\theta}$$

$$\therefore \quad v = \sqrt{v_0^2 + g^2t^2 - 2v_0gt\sin\theta}$$

Since v is tangent to the path at any instant, the angle β which it makes with the horizontal at the considered instant is

$$\tan\beta = \frac{v_y}{v_{x0}} = \frac{v_0\sin\theta - gt}{v_0\cos\theta} = \frac{v_0\sin\theta}{v_0\cos\theta} - \frac{gt}{v_0\cos\theta}$$

$$\therefore \quad \tan\beta = \tan\theta - \frac{gt}{v_0\cos\theta}$$

Note that angle β goes on changing with time t.

9.8. SPECIAL CASES OF PROJECTILE MOTION

So far we have considered the general case of projectile motion *i.e.*, projectile launched from the origin at an angle θ above the horizontal. However, the following two special cases of projectile motion will be discussed :

1. Projectile fired at an angle with the vertical.
2. Projectile fired horizontally.

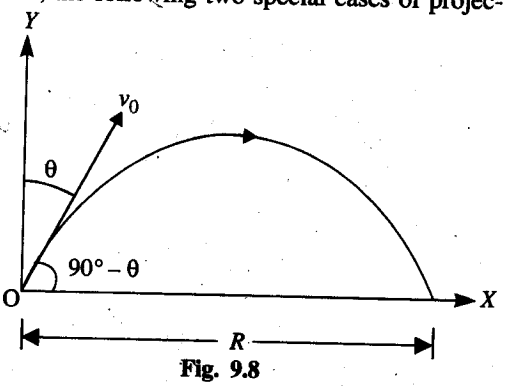

Fig. 9.8

1. Projectile fired at an angle with the vertical. Fig. 9.8 shows a projectile fired with initial velocity v_0 at an angle θ with the vertical. Therefore, firing angle with the horizontal is $90° - \theta$. In order to find the various parameters of projectile motion, change θ to $90° - \theta$ in the expressions derived in Arts. 9.5, 9.6 and 9.7.

(*i*) **Trajectory of projectile.** Changing θ to $90° - \theta$ in Art. 9.5, we have,

$$y = x\tan(90° - \theta) - \left(\frac{g}{2v_0^2\cos^2(90° - \theta)}\right)x^2$$

$$= x\cot\theta - \left(\frac{g}{2v_0^2\sin^2\theta}\right)x^2$$

$$\therefore \quad y = x\cot\theta - \frac{gx^2}{2v_0^2\sin^2\theta}$$

It is clear that the *trajectory of the projectile is parabolic.*

(*ii*) **Time of flight.** Changing θ to $90° - \theta$, time of flight is given by;

$$\text{Time of flight, } T = \frac{2v_0\sin(90° - \theta)}{g} = \frac{2v_0\cos\theta}{g}$$

(*iii*) **Maximum height attained.** Changing θ to $90° - \theta$, maximum height attained is given by ;

$$\text{Max. height attained, } H_{max} = \frac{v_0^2\sin^2(90° - \theta)}{2g} = \frac{v_0^2\cos^2\theta}{2g}$$

(*iv*) **Horizontal range.** Changing θ to $90° - \theta$, horizontal range is given by;

$$\text{Horizontal range, } R = \frac{v_0^2\sin 2(90° - \theta)}{g} = \frac{v_0^2\sin(180° - 2\theta)}{g} = \frac{v_0^2\sin 2\theta}{g}$$

(*v*) **Velocity of projectile at any time *t*.** Changing θ to $90° - \theta$, the velocity (v) of the projectile is given by ;

$$v = \sqrt{v_0^2 + g^2t^2 - 2v_0gt\sin(90° - \theta)}$$

or $$v = \sqrt{v_0^2 + g^2t^2 - 2v_0gt\cos\theta}$$

The angle β which v makes with horizontal is

$$\tan\beta = \tan(90° - \theta) - \frac{gt}{v_0\cos(90° - \theta)}$$

or $$\tan\beta = \cot\theta - \frac{gt}{v_0\sin\theta}$$

2. Projectile fired horizontally. Suppose a projectile is fired horizontally with velocity v_0 ($= v_x = v_{x0}$) at a height H above the ground (See Fig. 9.9). The horizontal velocity v_0 ($= v_x = v_{x0}$) remains constant throughout the projectile motion. The vertical downward velocity is zero at the time of firing the projectile and goes on increasing uniformly with time till the projectile hits the ground. Let us choose the firing point as the origin. In order to find the various expressions for this projectile motion, we should take vertical displacement and velocity *negative, $\theta = 0°$ in all the expressions derived in Arts. 9.5, 9.6 and 9.7.

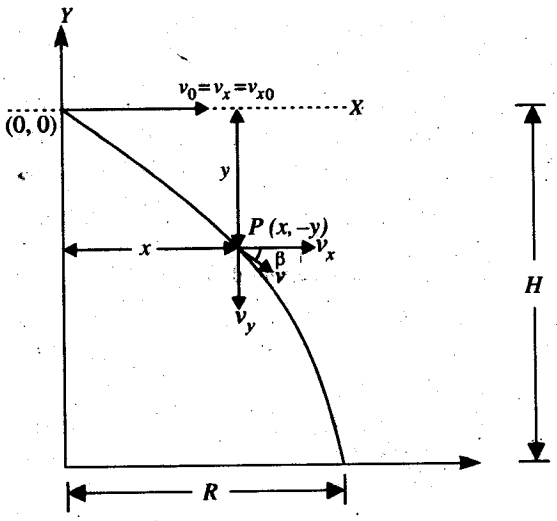

Fig. 9.9

(*i*) Motion along the horizontal. Suppose at any time t, the projectile is at point P $(x, -y)$. The horizontal displacement covered by the projectile is

$$x = (v_0\cos 0°)\,t = v_0\,t \qquad ...(i)$$

(*ii*) Motion along the vertical. At the considered point P, the vertical displacement is $-y$.

$$\therefore \quad -y = (v_0\sin 0°)\,t - \frac{1}{2}gt^2$$

or $$y = \frac{1}{2}gt^2 \qquad ...(ii)$$

Note that time of flight (T) and maximum height (H_{max}) attained by the projectile are determined by the vertical motion of the projectile.

(*iii*) Trajectory of projectile

$$y = \frac{1}{2}gt^2$$

From eq. (*i*), $t = x/v_0$

$$\therefore \quad y = \frac{1}{2}g\left(\frac{x}{v_0}\right)^2 = \left(\frac{1}{2}\frac{g}{v_0^2}\right)x^2$$

or $$y = kx^2 \quad \left(\because \frac{1}{2}\frac{g}{v_0^2} = \text{constant} = k\right)$$

---

* Since motion is downward.

This is the equation of parabola. *Therefore, the path or trajectory of the projectile is parabolic.*

(*iv*) **Time of flight** (T). At the considered point P, the vertical displacement is

$$-y = (v_0 \sin 0°)t - \frac{1}{2} gt^2$$

At $t = T$ = time of flight, $y = H$.

$$\therefore \quad -H = (v_0 \sin 0°)\, T - \frac{1}{2} gT^2 = -\frac{1}{2} gT^2$$

$$\therefore \quad \text{Time of flight, } T = \sqrt{\frac{2H}{g}}$$

(*v*) **Horizontal range** (R). At the considered point P, the horizontal displacement is

$$x = (v_0 \cos 0°)\, t$$

At $t = T$, $x = R$ = horizontal range.

$$\therefore \quad \text{Horizontal range, } R = (v_0 \cos 0°)\, T = v_0\, T = v_0 \sqrt{\frac{2H}{g}}$$

$$\therefore \quad R = v_0 \sqrt{\frac{2H}{g}}$$

(*vi*) **Velocity of projectile at time** t. At any time t, velocity v of the projectile is

$$v = \sqrt{v_0^2 + g^2 t^2 - 2v_0 gt \sin 0°}$$

or

$$v = \sqrt{v_0^2 + g^2 t^2}$$

The angle β which it makes with the horizontal at the considered instant is

$$\tan \beta = \frac{-v_y}{v_{x0}} = \frac{-(v_0 \sin 0° - gt)}{v_0 \cos 0°} = \frac{gt}{v_0}$$

Example 9.3. How far will a stone travel over level ground if it is thrown upward at an angle of 30° with the horizontal and with a speed of 12 ms^{-1}? What maximum range could be achieved with the same initial speed?

Solution. Horizontal range, $R = \frac{v_0^2}{g} \sin 2\theta$

Here $v_0 = 12$ ms^{-1}; $\theta = 30°$; $g = 9.8$ ms^{-2}

$$\therefore \quad R = \frac{(12)^2}{9.8} \sin (2 \times 30°) = 12.7 \text{ m}$$

The horizontal range will be maximum when projection angle $\theta = 45°$ *i.e.*, $\sin 2\theta = 1$.

$$\therefore \quad R_{max} = \frac{v_0^2}{g} = \frac{(12)^2}{9.8} = 14.7 \text{ m}$$

Note that the horizontal range is increased by about 16%.

Example 9.4. A ball is thrown with a speed of 17 ms^{-1} at a projection angle of 58° above the horizontal. Determine (*i*) the time of maximum height (*ii*) the maximum height above the release point. Neglect air friction.

Solution. Here $v_0 = 17\ ms^{-1}$; $\theta = 58°$; $g = 9.8\ ms^{-2}$

(*i*) The time (t_m) taken by the ball to reach maximum height is

$$t_m = \frac{v_0 \sin\theta}{g} = \frac{17 \times \sin 58°}{9.8} = 1.5\ s$$

(*ii*) Maximum height (H_{max}) attained by the ball is

$$H_{max} = \frac{v_0^2 \sin^2\theta}{2g} = \frac{(17)^2 \times (\sin 58°)^2}{2 \times 9.8} = 11\ m$$

Example 9.5. An aeroplane travelling horizontally at 78 ms^{-1} at a height of 210 m drops a bomb to hit the target.

(*i*) At what horizontal distance from the target should the bomb be dropped ?

(*ii*) Calculate the velocity of the bomb as it reaches the ground.

Solution. Fig. 9.10 shows the conditions of the problem. We have placed the origin of the coordinate system at the ground directly under the position of the plane when it drops the bomb.

(*i*) The time t taken by the bomb to reach the ground will be determined from the vertical motion of the bomb.

$$y - y_0 = v_{y0}\, t + \frac{1}{2} a_y\, t^2$$

Here $y - y_0 = -210$ m ; $v_{y0} = 0$;

$$a_y = -g = -9.8\ ms^{-2}$$

$$\therefore \quad -210 = -\frac{1}{2} \times 9.8 \times t^2$$

or $$t = \sqrt{\frac{210 \times 2}{9.8}} = 6.5\ s$$

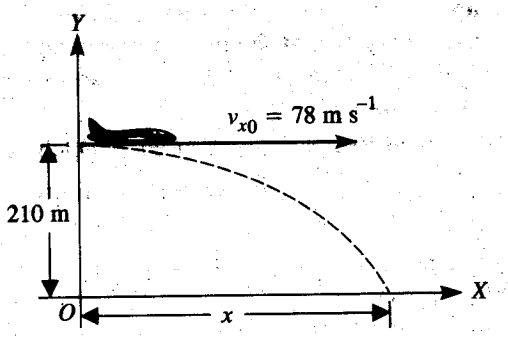

Fig. 9.10

Horizontal range, $R = x = {}^{*}v_x \times t = 78 \times 6.5 = 507$ m

(*ii*) The initial horizontal component (v_{x0}) of bomb's velocity remains 78 ms^{-1}. The initial vertical component (v_{y0}) of bomb's velocity changes with time. At any time t, its value is

$$v_y = v_{y0} - gt = 0 - 9.8 \times 6.5 = -64\ ms^{-1}$$

$\therefore$ Magnitude of bomb's velocity (v) when it reaches the ground is

$$v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(78)^2 + (-64)^2} = 100\ ms^{-1}$$

Example 9.6. An object is thrown horizontally with a velocity of 10 ms^{-1} from the top of a 20 m high building as shown in Fig. 9.11. Where does the object strike the ground?

Solution. In order to find time t taken by the object to strike the ground, we consider the vertical motion of the object. Taking the downward direction **positive,

$$v_{y0} = 0\ ;\ a_y = g = 9.8\ ms^{-2}\ ;\ y - y_0 = 20\ m$$

---

* $v_{x0} = v_x = 78\ ms^{-1}$. The horizontal speed of the bomb remains 78 ms^{-1} till it hits the ground. It is because there is no acceleration in the horizontal direction.

** It is arbitrary whether we choose y to be positive or negative in the upward or downward direction, but we must be consistent about it throughout the problem.

Now, $y - y_0 = v_{y0}\, t + \frac{1}{2} a_y\, t^2$

or $20 = 0 \times t + \frac{1}{2}\, g\, t^2$

$\therefore \quad t = \sqrt{\frac{20 \times 2}{g}} = \sqrt{\frac{40}{9.8}} = 2.02 \text{ s}$

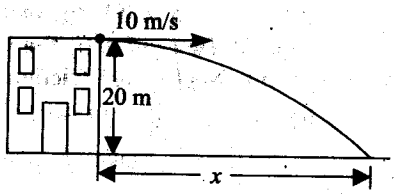

Fig. 9.11

$\therefore$ Horizontal range, $R = x = v_x \times t = 10 \times 2.02 = \mathbf{20.2}$ m

Example 9.7. A rifle shoots a bullet with a muzzle velocity of 500 ms^{-1} at a small target 50 m away. How high above the target must the rifle be aimed at so that the bullet will hit the target? (Take g = 10 ms^{-2})

Solution. Time taken by the bullet to reach the target is $t = \frac{\text{distance}}{\text{velocity}} = \frac{50}{500} = 0.1$ s

Vertical fall of bullet due to gravity is $y = \frac{1}{2} g t^2 = \frac{1}{2} \times 10 \times (0.1)^2 = 0.05 \text{ m} = 5 \text{ cm}$

Therefore, the rifle must be aimed **5 cm** above the target for the bullet to hit the target.

Example 9.8. A bomb is dropped on an enemy post by an aeroplane flying with a horizontal velocity of 60 km/h and at a height of 490 m. How far the aeroplane must be from the enemy post at the time of dropping the bomb so that it may directly hit the target? What is the trajectory of the bomb as seen by an observer on earth and as seen by person sitting in the plane?

Solution. Horizontal speed of bomb, $v_{x0} = 60$ km/h

The time taken by the bomb to hit the target depends upon vertical motion alone. Let this time be t.

Now $h = \frac{1}{2} g\, t^2$

$\therefore \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 490}{9.8}} = \sqrt{100} = 10 \text{s}$

$\therefore$ Distance of aeroplane from enemy post at the time of dropping the bomb is

$$R = v_{x0} \times t = 60 \times \frac{10}{60 \times 60} \qquad \left(\because 10\text{s} = \frac{10}{60 \times 60}\text{h}\right)$$

$$= \frac{1}{6} \text{ km} = \frac{1}{6} \times 1000 \text{ m} = \frac{500}{3} \text{ m}$$

The trajectory of the bomb as seen by an observer on the earth is a parabola. At each instant during the motion of the bomb, the horizontal velocity of the bomb is the same as the horizontal velocity of the aeroplane. Therefore, the falling bomb will always remain below the aeroplane. Consequently, the observer sitting in the plane will observe the bomb falling vertically.

Example 9.9. The horizontal range of a projectile is $4\sqrt{3}$ times of its maximum height. Calculate the angle of projection.

Solution. If v_0 is the initial velocity and θ is the angle of projection, then,

$$\text{Horizontal range, } R = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 (2 \sin\theta \cos\theta)}{g}$$

$$\text{Maximum height, } H_{max} = \frac{v_0^2 \sin^2\theta}{2g}$$

It is given that $R = 4\sqrt{3}\, H_{max}$

or $$\frac{v_0^2(2\sin\theta\cos\theta)}{g} = 4\sqrt{3}\times\frac{v_0^2\sin^2\theta}{2g}$$

or $$\frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}} \quad \text{or} \quad \tan\theta = \frac{1}{\sqrt{3}} \quad \therefore \theta = 30°$$

Example 9.10. Two tall buildings are 200 m apart. With what speed must a ball be thrown horizontally from a window of one building 2 km above the ground so that it will enter a window 40 m from the ground in the other?

Solution. Figure 9.12 shows the conditions of the problem. Here A and B are the two tall buildings having windows W_1 and W_2 respectively. The window W_1 is 2 km (= 2000 m) above the ground while window W_2 is 40 m above the ground. We want to throw the ball from window W_1 with such a horizontal speed (v_{x0}) so that it enters the window W_2. Note that the horizontal range of the ball is R = 200 m. Let t sec be the time taken by the ball to reach from window W_1 to window W_2. This time will depend upon the vertical motion (downward) alone.

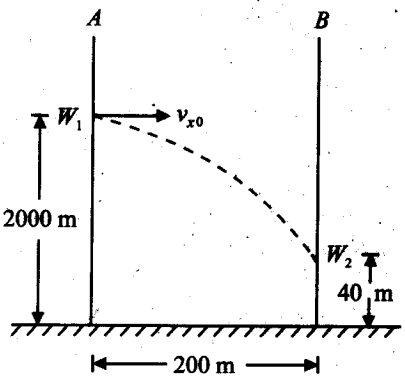

Fig. 9.12

For vertical motion :

$$h = 2000 - 40 = 1960 \text{ m};$$

$$g = 9.8 \text{ ms}^{-2};\ v_{y0} = 0$$

$$\therefore \quad h = \frac{1}{2}gt^2 \quad \text{or} \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times1960}{9.8}} = 20 \text{ s}$$

Now $$R = v_{x0}\, t \quad \therefore \quad v_{x0} = \frac{R}{t} = \frac{200}{20} = \mathbf{10\ ms^{-1}}$$

Example 9.11. A body is thrown horizontally from the top of a tower and strikes the ground after 3 seconds at an angle of 45° with the horizontal. Find (*i*) the height of the tower (*ii*) the speed with which the body was projected.

Solution. Fig. 9.13 shows the conditions of the problem. The horizontal and vertical motions of the ball are independent of each other. Note that horizontal velocity v_{x0} remains the same throughout the journey of the ball. However, the vertical velocity (downward) increases with time.

(*i*) The height of the tower h (= OA) will be determined from the vertical motion (downward) of the ball. The initial vertical velocity of the ball is zero *i.e.*, $v_{y0} = 0$.

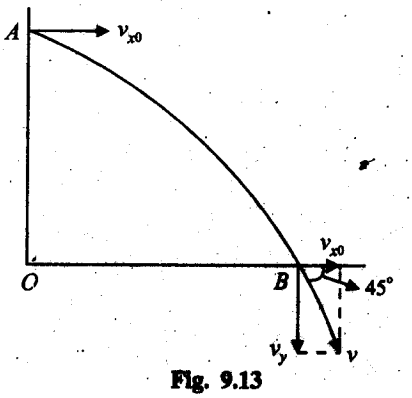

Fig. 9.13

$$\therefore \quad h = v_{y0}\, t + \frac{1}{2}\, g\, t^2$$

$$= 0 \times t + \frac{1}{2} \times 9.8 \times (3^2) = \mathbf{44.1\ m}$$

(*ii*) The ball strikes the ground at point *B*. At point *B*, the ball has two velocities (*a*) horizontal velocity v_{x0} and (*b*) vertical velocity $v_y = v_{y0} + gt = 0 + gt = gt$. Referring to Fig. 9.13, $\alpha = 45°$.

Now $\tan\alpha = \dfrac{v_y}{v_{x0}} = \dfrac{gt}{v_{x0}}$ $\therefore$ $v_{x0} = \dfrac{gt}{\tan\alpha} = \dfrac{9.8\times3}{\tan 45°} = 29.4 \text{ ms}^{-1}$

Example 9.12. **A grasshopper can jump a maximum horizontal distance of 1 m. If it spends minimum time on the ground, what is the speed of travel along the road?**

Solution. Horizontal range, $R = \dfrac{v_0^2 \sin 2\theta}{g}$; Maximum range, $R_{max} = \dfrac{v_0^2}{g}$ (For $\theta = 45°$)

$\therefore$ $v_0 = \sqrt{g R_{max}} = \sqrt{9.8\times1} = 3.13 \text{ ms}^{-1}$

Velocity along road = $v_0 \cos\theta = 3.13 \times \cos 45° = 2.21 \text{ ms}^{-1}$

Example 9.13. **From the top of a building 45 m high, a stone is thrown at an angle of 30° to the horizontal with an initial velocity of 20 m/s. Find (*i*) the time of flight (*ii*) the distance from the foot of the building where it strikes the ground.**

Solution. Fig. 9.14 shows the conditions of the problem. The x and y components of initial velocity v_0 (= 20 m/s) are :

$v_{x0} = v_0 \cos\theta = 20 \cos 30° = 17.3$ m/s

$v_{y0} = v_0 \sin\theta = 20 \sin 30° = 10$ m/s

(*i*) Let us choose the firing point as the origin. The time of flight is determined by vertical motion only. For vertical motion,

$$h = v_{y0}t + \frac{1}{2}gt^2$$

Here $h = -45$ m; $v_{y0} = 10$ m/s; $g = -9.8$ m/s^2; $t = T$

$$\therefore \quad -45 = 10\,T - \frac{1}{2} \times 9.8\,T^2$$

or $$-45 = 10T - 4.9\,T^2$$

Solving this quadratic equation, $T = 4.22$ s (The negative value of T has no physical meaning since time intervals are always positive).

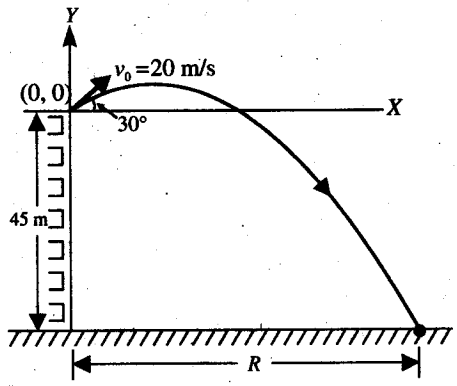

Fig. 9.14

(*ii*) Horizontal range, $R = v_{x0} \times T = 17.3 \times 4.22 = 73$ m

Example 9.14. **A boy wants to throw a letter wrapped over a stone to his friend across the street 40 m wide. The boy's window is 10m below friend's window. How should he throw the ball ?**

Solution. Fig. 9.15 shows the conditions of the problem. The boy's window is at O and friend's window is at A. Let the boy throw the stone with a velocity v_0 making an angle θ with the horizontal so as to enter the window at A. The stone will follow the parabolic path with A as the highest point on the trajectory of stone.

$\therefore$ $\dfrac{R}{2} = 40$ or Horizontal range, $R = 2 \times 40 = 80$ m

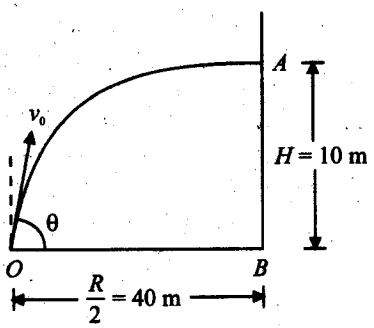

Fig. 9.15

Maximum height attained, H = 10 m

Now $$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 2\sin\theta\cos\theta}{g} \quad ...(i)$$

and $$H = \frac{v_0^2 \sin^2\theta}{2g} \quad ... (ii)$$

$$\therefore \quad \frac{H}{R} = \frac{\tan\theta}{4} \text{ or } \tan\theta = 4\times\frac{H}{R} = 4\times\frac{10}{80} = \frac{1}{2} \quad \therefore \theta = 26.56^\circ$$

The projection velocity v_0 can be found by putting the value of θ in eq. (*ii*).

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

$$\therefore \quad v_0^2 = \frac{2gH}{\sin^2\theta} = \frac{2\times 9.8\times 10}{\sin^2 26.56^\circ} = 980 \text{ or } v_0 = \sqrt{980} = 31.3 \text{ ms}^{-1}$$

Example 9.15. **A body is projected with a velocity of 40 ms^{-1}. After 2 s, it crosses a vertical pole of height 20.4 m. Calculate the angle of projection and the horizontal range.**

Solution. Let θ be the angle of projection. For vertical motion,

$$h = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$$

or $$20.4 = (40 \sin\theta) \times 2 - \frac{1}{2} \times 9.8 \times (2)^2$$

or $$20.4 = 80 \sin\theta - 19.6$$

$$\therefore \quad \sin\theta = \frac{20.4 + 19.6}{80} = \frac{40}{80} = \frac{1}{2} \quad \therefore \quad \theta = 30^\circ$$

Horizontal range, $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(40)^2 \times \sin 60^\circ}{9.8} = 141.4$ m

PROBLEMS FOR PRACTICE

1. A gun has a muzzle velocity of 60 ms^{-1}. At what angle should it be aimed to strike a target 320 m away? Ignore air resistance. **[30.3° or 59.7°]**
2. A tiger leaps horizontally from a 16 m high rock with a speed of 7 ms^{-1}. How far from the base of the rock will he land ? **[12.65 m]**
3. What is the least velocity with which a cricket ball can be thrown through a distance of 100 m? **[31.3 m s^{-1}]**
4. An aeroplane travelling at 150 km/h wants to drop supplies to flood victims isolated on a patch of land 250 m below. The supplies should be dropped how many seconds before the plane is directly overhead ? **[7.1 s]**
5. A ball thrown horizontally at 22 ms^{-1} from the roof of a building lands 36 m from the base of the building. How high is the building ? **[12.9 m]**
6. A football is kicked at an angle of 37° with a velocity of 20 m s^{-1}. Calculate (*i*) the maximum height (*ii*) the time of flight. (*iii*) How far away it hits the ground ? **[(*i*) 7.35 m (*ii*) 2.45 s (*iii*) 39.2 m]**
7. An athlete executing a long jump leaves the ground at an angle of 30° and travels 8.9 m. What was the take off speed? **[10 m s^{-1}]**

9.9. NON-UNIFORM MOTION IN A PLANE

An object is said to have *non-uniform motion* in a plane if its velocity (and hence acceleration) changes from instant to instant. In such a case, we have to deal with *instantaneous velocity* and *instantaneous acceleration*. Obviously, the path of the object will not be a straight line (as for uniform motion) rather the object will follow a curved path.

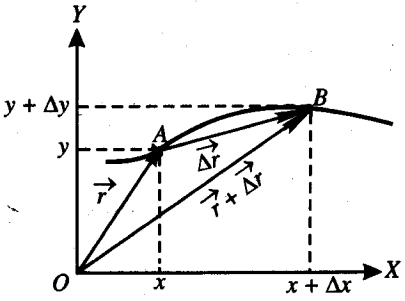

Fig. 9.16

Consider an object moving in the *XY* plane with a non-uniform velocity. Clearly, the object will follow a curved path as shown in Fig. 9.16. Suppose at time t, the object is at point $A(x, y)$ and its position vector is $\vec{r}$. Let at time $(t + \Delta t)$ its position be $B(x + \Delta x, y + \Delta y)$ and the position vector $\vec{r} + \vec{\Delta r}$. The displacement of the object in the small time interval Δt is $\vec{\Delta r} = (\Delta x\,\hat{i} + \Delta y\,\hat{j})$.

1. Instantaneous velocity vector. The average velocity of the object between points A and B is

$$\vec{v}_{av} = \frac{\overrightarrow{\Delta r}}{\Delta t}$$

The instantaneous velocity $\vec{v}$ of the object at point A is the limit of the average velocity as the time interval Δt approaches zero *i.e.*,

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{\overrightarrow{dr}}{dt}$$

$$\therefore \quad \vec{v} = \frac{\overrightarrow{dr}}{dt}$$

We can also express instantaneous velocity vector $\vec{v}$ in terms of its rectangular components as under :

$$\vec{v} = \frac{\overrightarrow{dr}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j}) = \frac{*dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

or $$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

or $$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

The coefficients are the scalar components of $\vec{v}$.

$$v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}$$

The following points may be noted :

(*i*) The magnitude of the velocity is $v = \sqrt{v_x^2 + v_y^2}$.

(*ii*) The direction in which the object is heading at any time may be described in terms of angle θ between velocity vector and X-axis as :

$$\tan\theta = \frac{v_y}{v_x}$$

---

* Note that $\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = 0$ since these are unit vectors and are constant in magnitude and direction.

Average angular velocity, $\bar{\omega} = \dfrac{\text{Angle turned}}{\text{Time taken}} = \dfrac{\Delta\theta}{\Delta t} = \dfrac{\theta_2 - \theta_1}{t_2 - t_1}$

The SI unit of angular velocity is rad/s and its dimensional formula is $[M^0L^0T^{-1}]$.

Instantaneous angular velocity (ω). *The angular velocity of an object moving along a circular path at any instant of time is called its instantaneous angular velocity.* It is denoted by ω. It is given by the limit of $\Delta\theta/\Delta t$ as Δt approaches zero *i.e.*,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The linear counterpart of angular velocity (ω) is the linear velocity (v).

(*iii*) **Average angular acceleration ($\bar{\alpha}$).** *The average angular acceleration of an object moving along a circular path is defined as the time rate of change of its angular velocity.*

$$\text{Average angular acceleration, } \bar{\alpha} = \frac{\text{Change in angular velocity}}{\text{Time taken}}$$

If the change in the angular velocity of an object is $\Delta\omega(=\omega_2 - \omega_1)$ during the time interval Δt $(=t_2 - t_1)$, then average angular acceleration $\bar{\alpha}$ of the object is

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

The SI unit of angular acceleration is rad/s^2 and its dimensional formula is $[M^0L^0T^{-2}]$.

Instantaneous angular acceleration (α). *The angular acceleration of an object moving along a circular path at any instant of time is called its instantaneous angular acceleration.* It is denoted by α. It is given by the limit of $\Delta\omega/\Delta t$ as Δt approaches zero *i.e.*,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

The linear counterpart of angular acceleration (α) is the linear acceleration (a).

(*iv*) **Time period.** *The time period of an object moving along a circular path is defined as the time taken by it to complete one revolution.* It is denoted by T and its unit is second.

For example, if an object completes 120 revolutions in 30 seconds, its time period is $T = 30/120 = 0.25$ s. It means that the object will complete one cycle in 0.25 s.

(*v*) **Frequency.** *The frequency of an object moving along a circular path is defined as the number of revolutions completed by it in one second.* It is denoted by f and its SI unit is s^{-1} or Hertz (Hz).

Thus in the above case, the object completes 120 revolutions in 30 seconds. Therefore, the frequency of the object is $f = 120/30 = 4$ Hz. It means that the object will complete 4 revolutions in one second.

(*vi*) **Relation between T and f.** Suppose an object executing circular motion has frequency f. It means that the object completes f revolutions in 1 second. Therefore, the time taken to complete one revolution is $1/f$. But this is the time period of the object.

$$\therefore \qquad T = \frac{1}{f} \quad \text{or} \quad f = \frac{1}{T} \quad \text{or} \quad fT = 1$$

All these three equations are equivalent.

(*vii*) **Relation between ω, f and T.** When an object executing circular motion completes one revolution, angular displacement $\theta = 2\pi$ radians and time taken is T(= time period; by definition).

$$\therefore \quad \text{Angular velocity, } \omega = \frac{\theta}{T} = \frac{2\pi}{T} = 2\pi f \qquad \left(\because f = \frac{1}{T}\right)$$

$$\therefore \quad \omega = \frac{2\pi}{T} = 2\pi f$$

9.11. RELATION BETWEEN LINEAR AND ANGULAR QUANTITIES

As an object moves in a circle, it not only sweeps out an angle θ but it also covers an arc of length s as shown in Fig. 9.18. We know that angle in radians is related to arc length s and radius r as :

$$s = r\theta \qquad \text{...}\theta \text{ in radians}$$

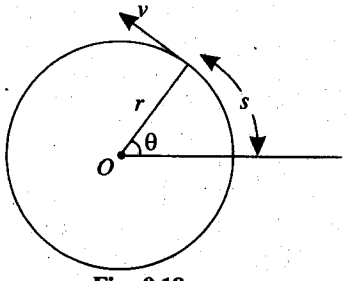

Fig. 9.18

Thus there is a definite relation between linear and angular quantities.

(*i*) Relation between linear and angular velocity. As an object moves along the circumference of the circle, it has linear velocity v which is always tangent to the circular path at every instant as shown in Fig. 9.18. The relation between the linear velocity v and angular velocity ω of the object can be found as under :

$$v = \frac{ds}{dt} = \frac{d}{dt}(s) = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$$

$$\therefore \quad v = r\omega \qquad \text{...}(i) \text{ [}\omega \text{ in rad/s]}$$

Eq. (*i*) is an important relationship between the circular motion of an object and the linear motion that results from rotation.

(*ii*) Relation between linear and angular acceleration. The relation between linear acceleration a and angular acceleration α can be found as under:

We know that : $v = \omega r$

Now, $$a = \frac{dv}{dt} = \frac{d}{dt}(\omega r) = r\frac{d\omega}{dt} = r\alpha$$

$$\therefore \quad a = r\alpha \qquad \text{...}(ii)$$

Eq. (*ii*) is again a useful result.

Note. In vector form : $\vec{v} = \vec{\omega} \times \vec{r}$; $\vec{a} = \vec{\alpha} \times \vec{r}$. Note that these are the cross products of two vectors.

9.12. UNIFORM CIRCULAR MOTION

An important case of two dimensional motion (*i.e.*, motion in a plane) is that of an object moving in a circular path at constant speed. This is called uniform circular motion.

When an object moves in a circular path at constant speed, the object is said to be in uniform circular motion.

The word "uniform" refers to the constant speed. It means that in uniform circular motion, the object covers equal distances along the circumference in equal intervals of time *i.e.*, the speed is constant. Although the magnitude of velocity (*i.e.*, speed) remains constant, the direction of the velocity is changing continuously. Therefore, the object is undergoing acceleration. This is called *centripetal acceleration* and is directed radially towards the centre of the circle as shown in Fig. 9.19. Note that centripetal means "centre-seeking". If an object moves in a circle of radius r with constant speed v, then it can be proved that the magnitude of centripetal acceleration $a_c = v^2/r$.

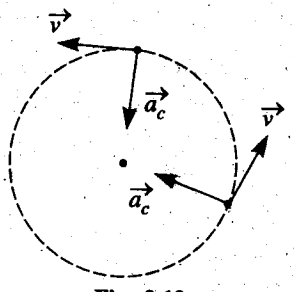

Fig. 9.19

9.13 CENTRIPETAL ACCELERATION

The acceleration acting on the object executing uniform circular motion is called ***centripetal acceleration.*** It is denoted by a_c.

In uniform circular motion, the magnitude of velocity (*i.e.*, speed) remains constant but direction of velocity changes continuously. Therefore, centripetal acceleration acting on the object describes the rate of turning *i.e.*, it represents the rate of change of the *direction* of the motion. We now determine the direction and magnitude of the centripetal acceleration.

Consider an object moving with a constant speed v around a circle of radius r as shown in Fig. 9.20(*i*). At point A, its velocity $\vec{v}_A$ is in the direction of the tangent to the path at point A. After a short time Δt, the object is at point B and its velocity $\vec{v}_B$ is in the direction of the tangent to the path at point B. During the time interval Δt, the object moves from A to B, covering a small distance Δl along the arc which subtends a small angle $\Delta\theta$ at the centre O. Note that magnitudes of velocity at points A and B are the same (since speed is constant). However, directions of velocity differ at points A and B. Hence the object has acceleration. The change in velocity vector ($\overrightarrow{\Delta v}$) is

$$\overrightarrow{\Delta v} = \vec{v}_B - \vec{v}_A$$

The change in velocity ($\overrightarrow{\Delta v}$) is shown in Fig. 9.20(*ii*). The acceleration of the object is given by ;

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{\overrightarrow{dv}}{dt}$$

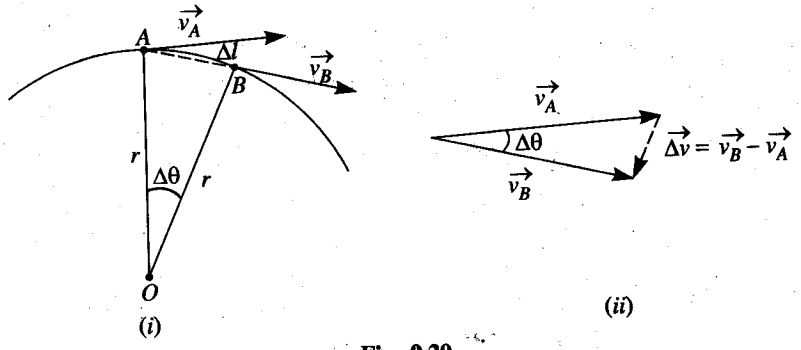

Fig. 9.20

(*i*) Direction of acceleration. In Fig. 9.20(*ii*), we find that when Δt is very small (approaching zero), Δl and $\Delta\theta$ are also very small. Consequently, $\vec{v}_B$ will be almost parallel to $\vec{v}_A$ and $\overrightarrow{\Delta v}$ will be essentially perpendicular to them. In the limit $\Delta t \to 0$, $\overrightarrow{\Delta v}$ will be exactly perpendicular to $\vec{v}_A$. Thus $\overrightarrow{\Delta v}$ is along the radius of the circle. By definition, $\vec{a}$ is always in the direction of $\overrightarrow{\Delta v}$. Therefore, acceleration 'a' must point towards the centre of the circle. Therefore, this acceleration is called *centripetal acceleration* (a_c).

Hence when an object moves with a constant speed around a circle, the centripetal acceleration is always directed radially towards the centre of the circle.

(*ii*) **Magnitude of a_c.** The vectors $\vec{v}_B, \vec{v}_A$ and $\overrightarrow{\Delta v}$ in Fig. 9.20(*ii*) form a triangle that is geometrically *similar to the triangle *ABO* in Fig. 9.20(*i*). Thus we can write

$$\frac{\Delta v}{v} = \frac{\Delta l}{r}$$

or $$\Delta v = \frac{v}{r}\Delta l$$

This becomes an exact equality when $\Delta t \to 0$ for then Δl equals the chord *AB*. In order to get the magnitude of centripetal acceleration a_c, we use the above relation for Δv and write

$$a_c = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{v\Delta l}{r\Delta t}$$

$$= \frac{v}{r} \lim_{\Delta t \to 0} \frac{\Delta l}{\Delta t} = \frac{v}{r} \times v \qquad \left(\because \lim_{\Delta t \to 0} \frac{\Delta l}{\Delta t} = v\right)$$

$\therefore$ Centripetal acceleration, $a_c = \frac{v^2}{r}$

Thus when an object moves in a circle of radius r with a constant speed v, the magnitude of the centripetal acceleration is v^2/r and it is directed radially towards the centre of the circle.

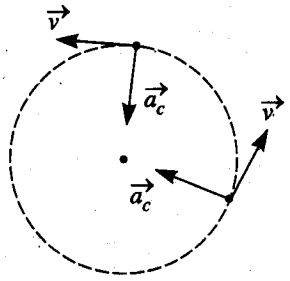

Fig. 9.21

Since v and r are constant, the magnitude of centripetal acceleration a_c ($= v^2/r$) is constant. However, the direction of a_c is not constant; it is changing continuously as shown in Fig. 9.21. Note that linear velocity $\vec{v}$ and $\vec{a}_c$ are perpendicular to each other at every instant. Further, the directions of $\vec{v}$ and $\vec{a}_c$ are changing continuously. Since centripetal acceleration always acts radially towards the centre of the circle, it is sometimes called *radial acceleration.* The magnitude of a_c can be given in the following equivalent formulas :

Centripetal acceleration, $a_c = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r = (2\pi f)^2 r = 4\pi^2 f^2 r$

9.14. PROJECTILE MOTION VERSUS UNIFORM CIRCULAR MOTION

(*i*) Both are two dimensional motions.

(*ii*) In projectile motion, the acceleration is constant in both magnitude and direction but the velocity changes in both magnitude and direction. However, in uniform circular motion, both velocity and acceleration are constant in magnitude but both change their directions continuously.

(*iii*) In projectile motion, $\vec{a}$ ($= \vec{g}$) makes various angles with $\vec{v}$ during the course of the motion. However, in case of uniform circular motion, $\vec{a}_c$ is perpendicular to $\vec{v}$ at every instant.

(*iv*) For projectile motion, $\vec{a}$ ($= \vec{g}$) is constant in both magnitude and direction. Therefore, kinematic equations for constant acceleration can be used for this type of motion. However, these equations *cannot* be used for uniform circular motion because the acceleration is not constant in direction.

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* $OA = OB = r$ and $v_A = v_B = v$, the velocity-vector triangle and Δ *ABO* are isosceles (two sides are equal). Also since $\vec{v}_A \perp OA$ and $\vec{v}_B \perp OB$, the angle between $\vec{v}_A$ and $\vec{v}_B$ is $\Delta\theta$. Hence the two triangles are similar.

Example 9.16. **Calculate the angular speed of flywheel making 120 revolutions/minute.**

Solution. Angular speed, $\omega = \theta/t$

Here $\theta = 120 \times *2\pi$ rad ; t = 1 minute = 60 seconds

$$\therefore \quad \omega = \frac{120 \times 2\pi}{60} = 4\pi \text{ rad/s}$$

Example 9.17. **The earth makes one revolution in 24 hours. Calculate (*i*) angular speed of earth (*ii*) tangential speed of an object at the equator due to this rotational motion. The radius of the earth is 6.38×10^6 m.**

Solution. (*i*) Angular speed, $\omega = \theta/t$

Here θ = 1 revolution = 2π rad ; t = 24 hours = $24 \times 60 \times 60$ s

$$\therefore \quad \omega = \frac{2\pi}{24 \times 60 \times 60} = 7.3 \times 10^{-5} \text{ rad/s}$$

(*ii*) Tangential speed of the object is

$$v = \omega r$$

Here, $\omega = 7.3 \times 10^{-5}$ rad/s ; $r = 6.38 \times 10^6$ m

$$\therefore \quad v = (7.3 \times 10^{-5}) \times (6.38 \times 10^6) = 460 \text{ ms}^{-1}$$

Thus a person standing on the equator travels in a circular path at a speed of 460 ms^{-1}. Fortunately, the earth's atmosphere rotates at about the same speed ; otherwise the earth would be very windy !

Example 9.18. **A satellite is put into a circular orbit 200 km above the earth's surface where the acceleration due to gravity is 9.2 ms^{-2}. Determine (*i*) satellite's speed and (*ii*) period (time to make one revolution). The radius of earth is 6400 km.**

Solution. (*i*) Radius of satellite's orbit, r = 6400 + 200 = 6600 km = 6.6×10^6 m

The satellite is accelerated (centripetally) towards earth's **centre with an acceleration a_c = 9.2 ms^{-2}.

Now, $$a_c = \frac{v^2}{r}$$

$\therefore$ Speed of satellite, $v = \sqrt{a_c r} = \sqrt{9.2 \times 6.6 \times 10^6} = 7.8 \times 10^3$ ms^{-1}

(*ii*) Time period, $T = \frac{2\pi r}{v} = \frac{2\pi \times 6.6 \times 10^6}{7.8 \times 10^3} = 5.3 \times 10^3$ s

Example 9.19. **The moon's nearly circular orbit about the earth has a radius of about 3.85×10^8 m and a period of 27.3 days. Determine the acceleration of the moon toward the earth.**

Solution. The linear speed v of the moon in its orbit about the earth is

$$v = \frac{2\pi r}{T}$$

Here $r = 3.85 \times 10^8$ m ; T = 27.3 days = $27.3 \times 24 \times 3600$ s

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* In one revolution, the angular displacement is 2π rad.

** If the satellite were not so accelerated, it would fly off in a straight line.

$$\therefore \quad v = \frac{2\pi\times(3.85\times10^8)}{27.3\times24\times3600} = 1.02 \times 10^3 \text{ m s}^{-1}$$

Now $$a_c = \frac{v^2}{r} = \frac{(1.02\times10^3)^2}{3.85\times10^8} = 2.7 \times 10^{-3} \text{ ms}^{-2}$$

The moon is accelerating (centripetally) towards earth's centre with this acceleration. If the moon were not so accelerated, it would fly off in a straight line.

PROBLEMS FOR PRACTICE

1. A stone is swung with a constant speed in a horizontal circle on a 2 m chord. The time period is 1 s. What is the angular speed ? What is the linear speed ? What is the radial acceleration? **[6.28 rad/s; 12.6 ms^{-1} ; 79 ms^{-2}]**
2. What is the centripetal acceleration of a car driving at 40 km/h on a circular track of radius 20 m? **[6.2 m s^{-2}]**
3. An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km/h. Compare its centripetal acceleration with the acceleration due to gravity. **[a_c = 6.38 g]**
4. Jupiter's natural satellite I_o has an average orbit of radius 4.22 × 10^8 m and a period of 42.5 h. Calculate (*i*) tangential speed (*ii*) angular speed and (*iii*) centripetal acceleration of I_o. **[(*i*) 1.73 × 10^4 m s^{-1} (*ii*) 4.11 × 10^{-5} rad/s (*iii*) 0.712 ms^{-2}]**

9.15. MOTION OF AN OBJECT IN THREE DIMENSIONAL SPACE

So far, we have been considering the motion of an object in two dimensions *i.e.*, in a plane. In order to specify the position of an object in a plane, we require two numbers (x, y). If an object is not confined to a plane but moves in three dimensional space, three numbers (x, y and z) are needed to specify the position of the object. Therefore, three dimensional motion is just an extension of the two dimensional motion.

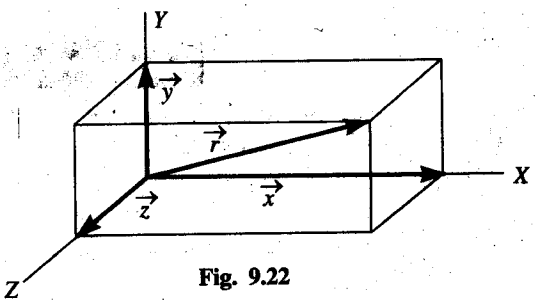

Fig. 9.22

(*i*) The position vector $\vec{r}$ of an object moving in space (See Fig. 9.22) is given by ;

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Here x, y and z are the coordinates specifying the position of the object.

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

(*ii*) The instantaneous velocity $\vec{v}$ of an object moving in space is given by ;

$$\vec{v} = \frac{\vec{dr}}{dt} = \frac{d}{dt}(\vec{r})$$

$$= \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\therefore \quad \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The three scalar components of $\vec{v}$ are :

$$v_x = \frac{dx}{dt} ; v_y = \frac{dy}{dt} ; v_z = \frac{dz}{dt}$$

(*iii*) The instantaneous acceleration $\vec{a}$ of an object moving in space is given by ;

$$\vec{a} = \frac{\overrightarrow{dv}}{dt} = \frac{d}{dt}(\vec{v})$$

$$= \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

$$\therefore \quad \vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

The three scalar components of $\vec{a}$ are :

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} ; \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} ; \quad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

CONCEPTUAL QUESTIONS

Q.1. Give two practical examples of two dimensional motion.

Ans. The first example is the projectile motion. Two coordinates are required to describe projectile's motion since it moves horizontally as well as vertically. The second example is the uniform circular motion *i.e.*, motion in a circle at constant speed.

Q.2. In projectile motion, what is the relationship between the directions of acceleration and velocity?

Ans. In projectile motion (non-vertical), $\vec{a}$ (= g) is constant in both magnitude and direction (downward) and makes various angles with $\vec{v}$ during the course of motion.

Q.3. In the uniform circular motion, what is the relationship between the directions of acceleration and velocity?

Ans. In uniform circular motion, $\vec{a}_c$ and $\vec{v}$ are perpendicular to each other since $\vec{v}$ is tangential to the circle and $\vec{a}_c$ points toward the centre. Note that directions of both $\vec{a}_c$ and $\vec{v}$ change continuously but at every instant $\vec{v} \perp \vec{a}_c$.

Q.4. Can you cite any practical example of uniformly accelerated motion in two dimensions.

Ans. The projectile motion is an example of uniformly accelerated motion in two dimensions. In this motion, acceleration is constant in both magnitude and direction.

Q.5. What assumptions are made in the treatment of projectile motion ?

Ans. To treat projectile motion, we make the following two assumptions :

(*i*) We neglect the variation in the direction or magnitude of gravitational acceleration (g). This assumption is equivalent to neglecting the curvature of the earth and is valid for projectiles whose horizontal and vertical motions are small compared with earth's radius.

(*ii*) We neglect air resistance. For dense objects moving relatively slowly, it is almost negligible. But for object whose surface area to mass is large (*e.g.*, ping-pong ball) or objects moving at high speeds, air resistance may dramatically alter the motion.

Q.6. Give two practical examples of uniform circular motion.

Ans. Driving around a curve at constant speed, you temporarily describe a circular arc and are, therefore, temporarily in uniform circular motion.As you swing a hockey stick, you produce motion that is circular and that may be approximately uniform.

Q.7. In uniform circular motion, centripetal acceleration is $a_c = v^2/r$ where v is the constant speed of the body and r is the radius of the circle. How do you explain this relation physically ?

Ans. Centripetal acceleration, $a_c = v^2/r$. An increase in speed v means that the time Δt for a given change in direction of the velocity becomes shorter. Not only that, but the associated change $\overrightarrow{\Delta v}$ in velocity is larger. These two effects combine to give an acceleration that depends on the square of speed. On the other hand, an increase in radius with a fixed speed simply increases the time Δt associated with a given change in velocity. Therefore, acceleration is inversely proportional to the radius.

Note. Although the form of equation does not look familiar, we can check that dimensions of R.H.S. are that of acceleration.

$$[a] = \frac{[v^2]}{[r]} = \frac{[L^2T^{-2}]}{[L]} = [LT^{-2}]$$

Q.8. In uniform circular motion, what is the relation between time period and frequency?

Ans. Consider a particle describing uniform circular motion. Let T be its time period and f be its frequency. The particle completes f revolutions in 1 second. The particle will complete 1 revolution (*i.e.* time period) in $1/f$ second.

$$\therefore \qquad T = \frac{1}{f} \text{ or } f = \frac{1}{T}$$

Q.9. What is the relation between angular velocity (ω), frequency (f) and time period (T) in uniform circular motion ?

Ans. Consider a particle describing uniform circular motion. Let its anugular velocity, time period and frequency be ω, T and f respectively. In one complete revolution, the angular displacement is 2π and time = T (time period).

$$\therefore \qquad \omega = \frac{2\pi}{T} = 2\pi f \qquad \left(\because \frac{1}{T} = f\right)$$

Q.10. A bob of mass 100 g hung from the ceiling of a string 1.5 m long is set into oscillations. The speed of the bob at mean position is 2 ms^{-1}. What is the trajectory of the bob if string is cut when bob is (*i*) at one of its extreme position (*ii*) at its mean position?

Ans. (*i*) If the string is cut when the bob is at one of its extreme position, the bob will fall vertically downward. The reason is simple. When the bob is at the extreme position, it comes to rest for a moment. If the string is now cut, it will fall vertically downward like a freely falling body.

(*ii*) In the second case, the bob will follow a parabolic path. When the bob is at the mean position, it has only horizontal velocity. If the string is now cut, the bob will become a projectile fired horizontally and, therefore, follow a parabolic path.

Q.11. The magnitude and direction of the acceleration of a body are constant. Will the path of the body necessarily be a straight line?

Ans. If the magnitude as well as direction of the acceleration of a body is constant, it is not necessary that the path of the body is a straight line. For example, in projectile motion,

the projectile is under g which has constant magnitude (9.8 m/s^2) and constant direction (vertically downward). The path of projectile is parabolic and not a straight line.

Q.12. A body is dropped freely from the window of a train. Will the time of the free fall be equal, if (*i*) the train is stationary (*ii*) the train moves with a constant velocity (*iii*) the train moves with an acceleration?

Ans. In all the three cases, the time of free fall of the body will be the same. It is because the motion of train affects only the horizontal motion and not motion along the vertical. For vertically downward motion, the parameters (g and h) are the same in the three cases.

Q.13. A body slides down a smooth inclined plane when released from the top, while another body falls freely from the same point. Which one will strike the ground earlier?

Ans. For a freely falling body, the acceleration due to gravity g acts on it. But when a body slides down a smooth inclined plane, its acceleration is $g \sin \theta$ where θ is the inclination of the plane with the horizontal. Since $g > g \sin \theta$, the body falling freely will reach earlier ($v = v_0 + gt$).

Q.14. A person sitting in a moving train throws a ball vertically upwards. How does the ball appear to move to an observer (*i*) inside the train (*ii*) outside the train?

Ans. (*i*) To the observer sitting inside the train, the ball will appear to move straight vertically upwards and then downwards. It is because in this case the ball has only one velocity acting vertically.

(*ii*) To an observer outside the train (*e.g.*, standing on the ground outside the train), the ball will appear to move along the parabolic path. It is because in this case, the ball is acted upon by vertical as well as horizontal velocities.

Q.15. The velocity at the maximum height of a projectile is half its initial velocity of projection v_0. What is the horizontal range of the projectile?

Ans. It is given that $v_\theta \cos \theta = v_0/2$. Therefore, $\theta = 60°$.

$$\therefore \text{Horizontal range, } R = \frac{v_0^2}{g} \sin 2\theta = \frac{v_0^2}{g} \sin 2\times 60° = \frac{\sqrt{3}\, v_0^2}{2g}$$

VERY SHORT ANSWER QUESTIONS

Q.1. Give three examples of motion in a plane (two dimensional motion).

Ans. (*i*) Crawling of a snake (*ii*) A bag dropped from an aeroplane flying horizontally (*iii*) A gun shot fired at an angle with the horizontal.

Q.2. Give two examples of three dimensional motion.

Ans. (*i*) Flight of an insect (*ii*) Molecular motion.

Q.3. Give three examples of projectile motion.

Ans. (*i*) A piece of stone thrown in any direction (*ii*) A body dropped from the window of a moving train (*iii*) A bomb released from a flying aeroplane.

Q.4. What is the nature of projectile motion?

Ans. It is a case of two dimensional motion (*i.e.*, motion in a plane) with constant acceleration.

Q.5. For a given projection angle (θ), what will be the effect on the horizontal range if initial velocity is doubled?

Ans. It will be four times the initial horizontal range ($\because R = v_0^2 \sin 2\theta/g$).

Q.6. What is a projectile?

Ans. A projectile is an object that is thrown into air and then moves under the influence of gravity alone (for simplicity, we neglect air resistance).

Q.7. A particle is projected at an angle of 45° with the horizontal with a velocity of 12 ms^{-1}. What will be its horizontal range?

Ans. Since projection angle $\theta = 45°$, the horizontal range will be maximum and $= v_0^2/g = 12^2/9.8 =$ 14.7 m.

Q.8. Is projectile motion two dimensional motion?

Ans. Yes. Since the projectile moves horizontally as well as vertically, it is a case of two dimensional motion *i.e.,* two coordinates are required to specify the position of the projectile at any time.

Q.9. Is rocket flight an example of projectile?

Ans. No. A rocket is driven by engines into space and does not move under gravity alone.

Q.10. What is the characteristic of projectile motion?

Ans. A projectile motion consists of two perpendicular motions *viz.* (*i*) a horizontal motion with constant velocity (*ii*) motion along vertical under gravity *i.e.,* with constant acceleration g (= 9.8 ms^{-2}).

Q.11. What is the angle of projection for a projectile thrown parallel to the horizontal?

Ans. It is 0°.

Q.12. Is there any point on the projectile path at which the acceleration is perpendicular to the velocity?

Ans. Yes. At the highest point of the parabolic path, the projectile has only horizontal velocity because the vertical component of velocity is zero. However, acceleration due to gravity is acting vertically downward.

Q.13. In projectile motion, when is the kinetic energy of projectile maximum and minimum?

Ans. The K.E. of the projectile is maximum at the point of projection or point of reaching the ground and is minimum at the highest point.

Q.14. Do the time of flight, horizontal range and maximum height of a projectile depend on its mass?

Ans. No because there is no mass term in their equations.

Q.15. For a given initial velocity v_0 of the projectile when is its (*i*) horizontal range maximum (*ii*) height attained maximum?

Ans. (*i*) The horizontal range is maximum when projection angle $\theta = 45°$ because $R = v_0^2 \sin 2\theta/g$. (*ii*) The height attained is maximum when $\theta = 90°$ because $H_{max} = v_0^2 \sin^2 \theta/2g$.

Q.16. What is the angular velocity of the hour hand of a watch?

Ans. For hour hand of watch, the time taken to complete one revolution is 12 hours. Therefore, its angular velocity $= 2\pi/12 = \pi/6$ rad/hour.

Q.17. What is the angular velocity of earth about its own axis?

Ans. The time taken by earth to complete one revolution about its own axis is 24 hours. Therefore, its angular velocity $= 2\pi/24 = \pi/12$ rad/hour.

Q.18. In uniform circular motion, which quantities (*i*) are constant (*ii*) not constant?

Ans. (*i*) In uniform circular motion, speed and magnitude of acceleration are constant (*ii*) Velocity and acceleration are not constant in uniform circular motion.

Q.19. What does the word "uniform" indicate in uniform circular motion?

Ans. In uniform circular motion, the word "uniform" means constant speed. Although the magnitude of velocity (*i.e.,* speed) remains constant in this type of motion, the direction of velocity is changing continuously.

Q.20. Is uniform circular motion two dimensional motion?

Ans. Yes, because it is a motion in a plane.

Q.21. What is the angle between velocity vector and acceleration vector in uniform circular motion?

Ans. In uniform circular motion, velocity vector and acceleration vector are perpendicular to each other at every instant.

Q.22. What is the direction of velocity vector of a particle in uniform circular motion?

Ans. At every point, the velocity vector is tangent to the circular path.

SHORT ANSWER QUESTIONS

Q.1. What is the angle of projection for which horizontal range and maximum height are equal?

Ans. $\frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 \sin^2\theta}{2g}$ or $2 \sin\theta \cos\theta = \frac{\sin^2\theta}{2}$ $\therefore \theta = \tan^{-1}(4)$

Q.2. How does projectile motion differ from uniform circular motion?

Ans. In projectile motion, the acceleration (= g) is constant in both magnitude and direction but the velocity changes in both magnitude and direction. However, in uniform circular motion, both velocity and acceleration are constant in magnitude but both change their directions continuously.

Q.3. A body is projected with a speed v_0 at an angle to the horizontal to have maximum horizontal range. What is its velocity at the highest point?

Ans. For maximum horizontal range, projection angle $\theta = 45°$. At the highest point, the velocity has only horizontal component = $v_0 \cos\theta = v_0 \cos 45° = v_0/\sqrt{2}$.

Q.4. A bullet is dropped from a certain height and at the same time another bullet is fired horizontally from the same height. Which one will hit the ground earlier and why?

Ans. Both will hit the ground at the same time. It is because in both cases, the initial vertical downward velocity is zero, vertical downward acceleration $g(= 9.8\ ms^{-2})$ is the same and the vertical height is the same.

Q.5. Prove that the maximum horizontal range is four times the maximum height attained by the projectile.

Ans. For maximum horizontal range, projection angle $\theta = 45°$.

$$\therefore \quad R_{max} = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 \sin 2\times 45°}{g} = \frac{v_0^2}{g}$$

Now $$H_{max} = \frac{v_0^2 \sin^2\theta}{2g} = \frac{v_0^2 \sin^2 45°}{2g} = \frac{v_0^2}{4g}$$

Therefore, $R_{max}/H_{max} = 4$ or $R_{max} = 4\ H_{max}$.

Q.6. Can we apply constant acceleration kinematic equations to uniform circular motion?

Ans. No. It is because in uniform circular motion, direction of acceleration is not constant, although magnitude of acceleration is constant.

Q.7. A projectile with kinetic energy of 500 J is fired for maximum horizontal range. What is the K.E. of the projectile at the highest point?

Ans. Initial K.E., $K = mv_0^2/2 = 500$ J. For maximum horizontal range, projection angle $\theta = 45°$. We know that at the highest point, the projectile has only horizontal velocity = $v_0 \cos\theta = v_0 \cos 45° = v_0/\sqrt{2}$.

$\therefore$ K.E. at highest point = $\frac{1}{2}m(v_0/\sqrt{2})^2 = \frac{1}{2}mv_0^2 \times \frac{1}{2} = K/2 = 250$ J

Q.8. In long jumping, what factors determine the span of jump?

Ans. $H_{max} = \frac{v_0^2 \sin^2\theta}{2g}$ and $R = \frac{v_0^2 \sin 2\theta}{g}$ $\therefore \frac{R}{H_{max}} = \frac{2 \sin 2\theta}{\sin^2\theta} = \frac{2\times 2 \sin\theta \cos\theta}{\sin^2\theta}$

Therefore, $R = 4\ H_{max} \cot\theta$. Now $H_{max} \propto v_0$ (velocity of jump). It follows that the length of jump depends upon v_0 and θ.

Q.9. Two particles are thrown with same velocity at an angle of 30° and 60° with the horizontal. Which parameter (time of flight, horizontal range and maximum height) will remain same?

Ans. The horizontal range will remain the same. It is because horizontal range is the same for projection angles θ and $(90° - \theta)$ [*i.e.*, 30° and (90° − 30°)].

Q.10. At which point of projectile motion (*i*) K.E. is maximum (*ii*) P.E. is maximum (*iii*) K.E. is minimum (*iv*) velocity is minimum (*v*) velocity is maximum.

Ans. (*i*) Place of projection (*ii*) At the highest point (*iii*) At the highest point (*iv*) At the highest point (*v*) Place of projection or at a point where it hits the ground.

Q.11. The greatest height to which a man can throw a stone is *h*. What will be the greatest distance upto which he can through the stone?

Ans. $H_{max} = v_0^2/2g = h$ (for $\sin\theta = 1$). Therefore, maximum horizontal range $= v_0^2/g = 2h$.

Q.12. If both speed of a body and radius of its circular path are doubled, what will happen to centripetal acceleration?

Ans. Centripetal acceleration, $a_c = v^2/r$. If both v and r are doubled, the new centripetal acceleration is $a_c' = (2v)^2/2r = 2\times v^2/r = 2a_c$.

Q.13. Prove that the horizontal range of a projectile is the same for projection angles of (45° + θ) and (45° – θ).

Ans. Horizontal range, $R = v_0^2 \sin 2\theta/g$

For projection angle $(45° + \theta)$, $R_1 = \dfrac{v_0^2 \sin(90° + 2\theta)}{g} = \dfrac{v_0^2 \cos 2\theta}{g}$

For projection angle $(45° - \theta)$, $R_2 = \dfrac{v_0^2 \sin(90° - 2\theta)}{g} = \dfrac{v_0^2 \cos 2\theta}{g}$

Therefore, $R_1 = R_2$ *i.e.*, horizontal range is the same for projection angles (45° + θ) and (45° – θ).

Q.14. A particle moves with a constant angular speed of certain radius *r* and is acted upon by a certain centripetal force *F*. If the angular speed is doubled, keeping the radius the same, what will be the new centripetal force?

Ans. Centripetal force, $F_c = \dfrac{mv^2}{r} = \dfrac{m(\omega r)^2}{r} = m\omega^2 r$

If ω is doubled, $F_c' = m(2\omega)^2 r \quad \therefore F_c'/F_c = 4$ or $F_c' = 4F_c$

Q.15. A particle of mass *m* is moving in a circular path of constant radius *r* such that its centripetal acceleration a_c is varying with time as $a_c = k^2 r t^2$ where *k* is a constant. What is the power delivered to the particle by the force acting on it?

Ans. $a_c = k^2 r t^2$ or $\dfrac{v^2}{r} = k^2 r t^2 \quad \therefore v = krt$

Tangential force, $F = m\dfrac{dv}{dt} = m\dfrac{d}{dt}(krt) = mkr$

$\therefore$ Power, $P = F \times v = (mkr)(krt) = mk^2 r^2 t$

LONG ANSWER QUESTIONS

1. Derive an expression for displacement in two dimensions with uniform acceleration. **[Refer to Art. 9.3]**
2. Derive an expression for displacement in two dimensions with uniform velocity. **[Refer to Art. 9.2]**
3. A projectile is fired with a velocity v_0 making an angle θ with the horizontal. Find expressions for (*i*) time of flight (*ii*) maximum height attained and (*iii*) horizontal range. **[Refer to Art. 9.6]**
4. A projectile is fired with velocity v_0 making an angle θ with the horizontal. Show that its path is parabolic. **[Refer to Art. 9.5]**
5. What is centripetal acceleration? Find its magnitude and direction in case of uniform circular motion. **[Refer to Art. 9.13]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Projectile motion is two dimensional motion with constant acceleration.
2. It is useful to think of projectile motion as the superposition of two motions: (*i*) uniform motion in the *X* direction with velocity v_x, which remains constant throughout the motion (*ii*) motion in the vertical direction, subject to a constant downward acceleration of magnitude $g = 9.8\ m/s^2$.
3. Fig. 9.23 shows a projectile fired from the origin at $t = 0$ with an initial velocity v_0 making an angle θ with the positive *X*-axis.

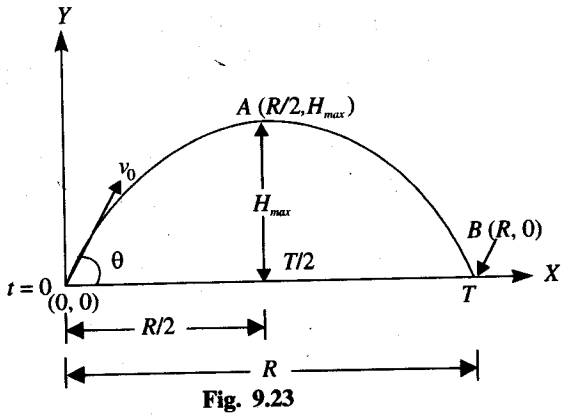

Fig. 9.23

Time of flight, $T = \dfrac{2v_0 \sin\theta}{g}$

Max. height, $H_{max} = \dfrac{v_0^2 \sin^2\theta}{2g}$

Horizontal range, $R = \dfrac{v_0^2 \sin 2\theta}{g}$

(*i*) The highest point is $A\ (R/2,\ H_{max})$ while point $B\ (R,\ 0)$ represents the condition when projectile returns to ground.

(*ii*) During the entire motion of the projectile, it is subjected to a constant acceleration $g\ (= 9.8\ ms^{-2})$ acting vertically downward.

(*iii*) At any time *t* during projectile motion, the horizontal and vertical components of velocity are :

$$v_x = v_0 \cos\theta = \text{constant} \ ; \ v_y = v_0 \sin\theta - gt$$

(*iv*) The velocity of the projectile is maximum (v_0) at the starting point (*i.e.* origin) and goes on decreasing (component v_y decreases but v_x is constant) till it becomes minimum ($v_0 \cos\theta$) at the highest point *A*. At point *A*, $v_y = 0$ and the velocity of the projectile is $v_0 \cos\theta$ acting horizontally.

(*v*) At any time *t* during projectile motion, the velocity *v* of the projectile is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}$$

$$= \sqrt{v_0^2 + g^2t^2 - 2v_0gt\sin\theta}$$

The angle β which *v* makes with horizontal is

$$\tan\beta = \frac{v_y}{v_x} = \frac{v_0\sin\theta - gt}{v_0\cos\theta} = \tan\theta - \frac{gt}{v_0\cos\theta}$$

(*vi*) The linear momentum of the projectile at the highest point is $mv_0 \cos\theta$ and it is mv_0 at the lowest point.

(*vii*) The K.E. of the projectile is maximum at the starting point (*i.e.* origin) and lowest at the highest point.

4. The horizontal range of the projectile is maximum when the angle of projection $\theta = 45°$.

5. The horizontal range of the projectile is the same for projection angles of θ and $(90° - \theta)$.
6. The projectile attains maximum height when the projection angle $\theta = 90°$.
7. When the projectile is fired for maximum horizontal range R_{max} ($\theta = 45°$), the maximum height attained by the projectile is $R_{max}/4$.
8. Consider a projectile fired horizontally with velocity v_0 at a height H above the ground. Let us choose starting point as the origin. Initially, the vertical velocity is zero. If T is the time of flight, then,

$$-H = -\frac{1}{2}gT^2 \text{ or } T = \sqrt{\frac{2H}{g}}$$

Horizontal range, $R = v_0 \times T = v_0\sqrt{\frac{2H}{g}}$

At any time t, $v_x = v_0$ and $v_y = gt$. Therefore, velocity of projectile at time t is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + g^2t^2}$$

The angle β which $\vec{v}$ makes with horizontal is $\tan\beta = \frac{v_y}{v_x} = \frac{gt}{v_0}$

9. If a projectile of mass m is fired with velocity v_0 making an angle θ with the horizontal ground, then change in its linear momentum at the point of landing on the ground is $2mv_0 \sin\theta$ acting vertically downward.
10. When a body moves in a circular path, it has tangential velocity as well as angular velocity. The magnitudes of tangential velocity and angular velocity are related as :

$$v = \omega r$$

11. When a body is moving in a circular path, the rate of change of angular velocity (ω) is called angular accelerations α *i.e.* $\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$

where θ is the angular displacement.

If the body moves with constant angular velocity, then angular acceleration is zero.

12. If a body moves in a circle of radius r with constant speed v, it has centripetal acceleration a_c that acts along the radius towards the centre of the circle.

$$a_c = \frac{v^2}{r}$$

13. If the speed of a body revolving in a circle is changing, then there will be tangential acceleration a_t as well as centripetal acceleration a_c.

$$a_t = \frac{dv}{dt} \text{ and } a_c = \frac{v^2}{r}$$

(*i*) The tangential acceleration arises from the change in magnitude of the velocity whereas the centripetal acceleration arises from the change in direction of the velocity.

(*ii*) The a_t and a_c are always perpendicular to each other and their directions change continuously. The total vector acceleration $\vec{a}$ is the vector sum of $\vec{a_t}$ and $\vec{a_c}$.

$$\therefore \quad \vec{a} = \vec{a_t} + \vec{a_c}$$

The magnitude of $\vec{a}$ is given by ; $a = \sqrt{a_t^2 + a_c^2}$

14. In projectile motion, the kinetic energy at the highest point = $K\cos^2\theta$ where K is the kinetic energy with which the projectile is fired.
15. The maximum height H_{max} to which a projectile rises when fired from the ground at an angle θ with the horizontal is $H_{max} = 0.25\ R\tan\theta$ where R is the horizontal range.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Two projectiles are projected with the same velocity. If one is projected at an angle of 30° and the other at 60° to the horizontal, then the ratio of maximum heights reached is **[AIIMS 2001]**
   (a) 3 : 1 (b) 1 : 3
   (c) 1 : 2 (d) 2 : 1
2. An aeroplane moving horizontally with a speed of 720 km/h drops a food packet while flying at a height of 396.9 m. The time taken by the food packet to reach the ground and its horizontal range are (Take $g = 9.8\ m/s^2$) **[AFMC-Pune 2001]**
   (a) 3 sec and 2000 m
   (b) 5 sec and 500 m
   (c) 8 sec and 1500 m
   (d) 9 sec and 1800 m
3. A stone tied to the end of string 80 cm long is whirled in a horizontal circle with a constant speed. If stone makes 25 revolutions in 14 sec, what is the magnitude of acceleration of the stone? **[AIIMS 2001]**
   (a) 680 cm/s^2 (b) 720 cm/s^2
   (c) 860 cm/s^2 (d) 990 cm/s^2
4. A particle is projected making an angle of 45° with the horizontal having kinetic energy K. The kinetic energy at the highest point will be **[CBSE PMT 2001]**
   (a) $\frac{K}{\sqrt{2}}$ (b) $\frac{K}{2}$
   (c) $2K$ (d) K
5. Two particles having masses M and m are moving in a circular path having radii R and r respectively. If their periods are same, then the ratio of angular velocities will be **[CBSE PMT 2000]**
   (a) $\frac{r}{R}$ (b) $\frac{R}{r}$
   (c) 1 (d) $\sqrt{R/r}$
6. The angular speed of a fly-wheel making 120 revolutions/minute is **[Pb PMT 1999]**
   (a) π rad/sec (b) 2π rad/sec
   (c) 4π rad/sec (d) $4\pi^2$ rad/sec
7. An object is projected vertically upward with a velocity $\sqrt{4gR/3}$ from earth. The velocity of the object at the maximum height reached by it will be **[AFMC-Pune 2000]**
   (a) $\sqrt{\frac{2gR}{3}}$ (b) $\sqrt{\frac{gR}{3}}$
   (c) $\sqrt{2gR}$ (d) zero
8. The angular velocity of the seconds-hand in a watch is **[AFMC-Pune 2000]**
   (a) 0.053 rad/sec (b) 0.210 rad/sec
   (c) 0.105 rad/sec (d) 0.42 rad/sec
9. A body is projected at such an angle that the horizontal range is three times the greatest height attained. The angle of projection is **[AIIMS 1998]**
   (a) 25° 8' (b) 33° 7'
   (c) 42° 8' (d) 53.1°
10. Two particles of same mass and with same velocity are thrown at an angle of 60° and 30° with the horizontal. Which will remain the same ? **[CBSE PMT 2000]**
    (a) Time of flight
    (b) Range of projectile
    (c) Max. height acquired
    (d) All of them
11. A bomber plane is moving horizontally with a speed of 500 m/s and bomb released from it strikes the ground in 10 sec. Then angle at which it strikes the ground is (Take $g = 10\ m/s^2$) **[Pb PMT 1998]**
    (a) $\tan^{-1} 5$ (b) $\tan^{-1} 1$
    (c) $\tan^{-1} 1/5$ (d) $\sin^{-1} 1/5$
12. Two particles are projected with the same initial velocity, one making angle θ with the horizontal and other making angle θ with the vertical. If their common range is R, then product of their times of flight is directly proportional to **[CBSE PMT 1999]**

(a) R (b) R^2
(c) $1/R$ (d) R^3

13. A bomb is dropped from an aeroplane moving horizontally at a constant speed. When air resistance is taken into consideration, then the bomb **[AFMC-Pune 1999]**
(a) flies with the aeroplane
(b) falls on earth ahead of aeroplane
(c) falls on earth behind aeroplane
(d) falls on earth exactly below aeroplane

14. The speed of a boat is 5 km/hr in still water. It crosses a river of width 1 km along the shortest possible path in 15 minutes. The velocity of the river is **[CBSE PMT 1998]**
(a) 1 km/hr (b) 3 km/hr
(c) 4 km/hr (d) 5 km/hr

15. A boat is sent across a river with a velocity of 8 km/hr. If the resultant velocity of boat is 10 km/hr, then velocity of river is **[CBSE PMT 1994]**
(a) 12.8 km/hr (b) 6 km/hr
(c) 8 km/hr (d) 10 km/hr

16. The maximum horizontal range of a projectile is 400 m. The maximum height attained by it will be: **[AFMC Pune 2005]**
(a) 100 m (b) 200 m
(c) 400 m (d) 800 m

17. For a given angle of projection, if the time of flight of a projectile is doubled, the horizontal range will increase to: **[Manipal 2003]**
(a) four times (b) thrice
(c) once (d) twice

18. If retardation produced by air resistance of projectile is one-tenth of acceleration due to gravity, the time to reach maximum height: **[Manipal 2003]**
(a) decreases to 11 percent
(b) increases to 11 percent
(c) decreases to 90 percent
(d) increases to 90 percent

19. Two stones are projected with same velocity v at an angle θ and $(90 - \theta)$. If H and H_1 are the greatest heights in the two paths, what is the relation between R, H and H_1? **[Manipal 2005]**
(a) $R = 4\sqrt{H H_1}$ (b) $R = \sqrt{HH_1}$
(c) $R = 4HH_1$ (d) None of the above

20. The direction of the angular velocity vector is along **[AIIMS 2004]**
(a) inward radius (b) axis of rotation
(c) outward radius
(d) tangent to the circular path

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (b)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (c)
6. (c)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (b)
11. (c)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (b)
16. (a)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (b)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Maximum height reached by a projectile is

$$H = \frac{v_0^2 \sin^2\theta}{2g} \quad \text{or} \quad H \propto \sin^2\theta$$

$$\therefore \quad \frac{H_1}{H_2} = \left(\frac{\sin\theta_1}{\sin\theta_2}\right)^2 = \left(\frac{\sin 30^\circ}{\sin 60^\circ}\right)^2 = \frac{1}{3} \quad \text{or} \quad H_1 : H_2 = 1 : 3$$

2. Horizontal velocity, $v_x = 720$ km/h $= 200$ m/s
At starting, vertical velocity, $v_y = 0$

$\therefore 396.9 = 0 + \frac{1}{2}gt^2$ or $t^2 = \frac{396.9 \times 2}{9.8} = 81$ $\therefore t = 9$ sec

Horizontal range = $v_x \times t = 200 \times 9 = 1800$ m

3. Time period, $T = \frac{\text{No. of revolutions}}{\text{Time taken}} = 25/14 = 1.78$ sec

Angular velocity, $\omega = \frac{2\pi}{T} = \frac{2\pi}{1.78} = 3.52$ rad/sec

$\therefore$ Acceleration, $a_c = \omega^2 \times l = (3.52)^2 \times 80 = 990$ cm/s^2

4. Initial K. E., $K = \frac{1}{2}mv_0^2$

At the highest point, there is only horizontal velocity $v_0 \cos\theta$.

$\therefore$ K.E. at the highest point $= \frac{1}{2}m(v_0 \cos\theta)^2 = \frac{1}{2}mv_0^2 \cos^2\theta$

$= K \cos^2\theta = K \times (\cos 45°)^2 = K/2$

5. Angular velocity, $\omega = \frac{2\pi}{T}$ $\therefore \omega_1 = \frac{2\pi}{T_1}$ and $\omega_2 = \frac{2\pi}{T_2}$

Since $T_1 = T_2$ $\therefore \frac{\omega_1}{\omega_2} = 1$

6. Revolutions/minute, $N = 120$

$\therefore$ Angular velocity, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 120}{60} = 4\pi$ rad/sec

7. The velocity of the object goes on decreasing due to g acting vertically downward. Therefore, velocity of the object at the maximum height will be zero.

8. The time period of seconds–hand $T = 60$ sec.

$\therefore$ Angular velocity, $\omega = \frac{2\pi}{T} = \frac{2\pi}{60} = 0.105$ rad/sec

9. Greatest height, $H = \frac{v_0^2 \sin^2\theta}{2g}$; Horizontal range, $R = \frac{v_0^2 \sin 2\theta}{g}$

It is given that $R = 3H$

$\therefore$ $\frac{v_0^2 \sin 2\theta}{g} = 3 \times \frac{v_0^2 \sin^2\theta}{2g}$

or $2\sin\theta\cos\theta = 3 \times \frac{\sin^2\theta}{2}$ or $\tan\theta = 4/3$ $\therefore \theta = 53.1°$

10. Only horizontal range will remain the same.

Horizontal range, $R = \frac{v_0^2 \sin 2\theta}{g}$

For particle 1, $R_1 = \frac{v_0^2 \sin 120°}{g}$; For particle 2, $R_2 = \frac{v_0^2 \sin 60°}{g}$

$\therefore$ $R_1 = R_2$

11. The horizontal velocity of bomb $v_x = 500$ m/s and remains the same throughout the journey. At the instant of dropping the bomb, $v_x = 500$ m/s and vertical velocity is zero.

$\therefore$ $h = 0 + \frac{1}{2}gt^2$ or $h = \frac{1}{2} \times 10 \times (10)^2 = 500$ m

Now $(v_y)^2 - (0)^2 = 2gh$ $\therefore$ $v_y = \sqrt{2gh} = \sqrt{2\times10\times500} = 100$ m/s

$\therefore$ $\tan\theta = \dfrac{v_y}{v_x} = \dfrac{100}{500} = 1/5$ or $\theta = \tan^{-1} 1/5$

12. For the particle projected at an angle θ with the horizontal,

$$T_1 = \frac{2v_0 \sin\theta}{g}$$

For the particle projected at angle θ with the vertical,

$$T_2 = \frac{2v_0 \sin(90° - \theta)}{g} = \frac{2v_0 \cos\theta}{g}$$

$\therefore$ $$T_1 T_2 = \left(\frac{2v_0 \sin\theta}{g}\right)\left(\frac{2v_0 \cos\theta}{g}\right) = \frac{2v_0^2 \sin 2\theta}{g^2}$$

$\therefore$ $T_1 T_2 \propto R$

13. Due to air resistance, the horizontal velocity of bomb decreases while the aeroplane continues to move at a constant speed. Therefore, bomb **falls on earth behind the aeroplane.**

14. Velocity of boat with respect to water is $V_{BW} = 5$ km/hr. The velocity of water *w.r.t.* shore V_{WS} is to be found out. The magnitude of velocity of boat *w.r.t.* shore is (See Fig. 9.24)

$$V_{BS} = \frac{\text{Distance}}{\text{Time}} = \frac{1\text{km}}{1/4} = 4\text{km/hr}$$

Now $\vec{V}_{BS} = \vec{V}_{BW} + \vec{V}_{WS}$

$\therefore$ Magnitude of $V_{WS} = \sqrt{(V_{BW})^2 - (V_{BW})^2}$

$= \sqrt{(5)^2 - (4)^2} = 3$ km/hr

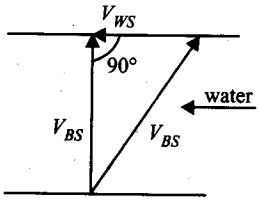

Fig. 9.24

15. Here $V_{BW} = 10$ km/hr, $V_{BS} = 8$ km/hr

$\therefore$ $V_{WS} = \sqrt{(10)^2 - (8)^2} = \mathbf{6\ km/hr}$

16. Horizontal range, $R = \dfrac{v_0^2 \sin 2\theta}{g}$

The horizontal range will be maximum when $\theta = 45°$.

$\therefore$ $R_{max} = \dfrac{v_0^2}{g}$ or $400 = \dfrac{v_0^2}{g}$

Maximum height attained, $H_{max} = \dfrac{v_0^2 \sin^2 45°}{2g} = \dfrac{v_0^2}{g} \times \dfrac{(1/\sqrt{2})^2}{2} = \dfrac{400\times(1/2)}{2} = \mathbf{100\ m}$

17. Time of flight, $T = \dfrac{2v_0 \sin\theta}{g}$ or $T^2 = \dfrac{4v_0^2 \sin^2\theta}{g^2}$

Horizontal range, $R = \dfrac{v_0^2 \sin 2\theta}{g} = \dfrac{v_0^2 \times 2\sin\theta\cos\theta}{g}$

$\therefore$ $T^2 = \dfrac{2R\tan\theta}{g}$ or $R \propto T^2$

Therefore, if T is doubled, R will become four times.

18. Suppose t_m is the time taken to reach maximum height in the absence of air resistance.

$$\therefore \quad t_m = \frac{v_0 \sin\theta}{g} \qquad ...(i)$$

If a is the retardation caused by air resistance, then total retardation = $g + a$. Therefore, time taken (t'_m) to reach maximum height is

$$t'_m = \frac{v_0 \sin\theta}{g+a} = \frac{v_0 \sin\theta}{g+g/10} = \frac{10\, v_0 \sin\theta}{11\, g} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $\frac{t'_m}{t_m} = \frac{10}{11}$ or t'_m = 90% of t_m

Therefore, time will decrease to 90%.

19. For projection angle θ, the horizontal range (R) and height attained (H) are :

$$R = \frac{2v^2 \sin\theta\cos\theta}{g}; \quad H = \frac{v^2 \sin^2\theta}{2g}$$

For projection angle ($90° - \theta$), the height attained (H_1) is

$$H_1 = \frac{v^2 \sin^2(90°-\theta)}{2g} = \frac{v^2 \cos^2\theta}{2g}$$

$$\therefore \quad HH_1 = \frac{v^2 \sin^2\theta}{2g} \times \frac{v^2 \cos^2\theta}{2g} = \frac{v^4 \sin^2\theta \cos^2\theta}{4g^2}$$

Now $$R^2 = \frac{4v^4 \sin^2\theta\cos^2\theta}{g^2} = 16\, HH_1 \text{ or } R = 4\sqrt{HH_1}$$

20. We know that the direction of the angular velocity is along the axis of rotation. It is found by the right hand rule.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A body is revolving with a uniform speed v in a circle of radius r. The tangential acceleration is **[Haryana CET 2001]**
   (*a*) v/r^2 (*b*) v^2/r
   (*c*) v/r (*d*) zero

2. A boat is moving with a velocity of $3\hat{i}+4\hat{j}$ *w.r.t.* ground. The water in the river is moving with a velocity of $-3\hat{i}-4\hat{j}$ *w.r.t.* ground. The relative velocity of boat *w.r.t.* water is **[EAMCET 1991]**
   (*a*) $8\hat{j}$ (*b*) $-6\hat{i}-8\hat{j}$
   (*c*) $6\hat{i}+8\hat{j}$ (*d*) $5\sqrt{2}$

3. A body is thrown with a velocity of 9.8 ms^{-1} making an angle of 30° with the horizontal. It will hit the ground after a time **[Orissa JEE 1991]**
   (*a*) 3 s (*b*) 2 s
   (*c*) 1.5 s (*d*) 1 s

4. A person aims a gun at a bird from a point at a horizontal distance of 100 m. If the gun can impart a speed of 500 ms^{-1} to the bullet, at what height above the bird must he aim his gun in order to hit it ? (Take g = 10 m/s^2) **[Pb CET 1992]**
   (*a*) 10 cm (*b*) 20 cm
   (*c*) 50 cm (*d*) 100 cm

5. A train of 150 m length is going towards north direction at a speed of 10ms^{-1}. A parrot flies at a speed of 5 ms^{-1} towards south direction parallel to the railway track. The time taken by the parrot to cross the train is **[BHU 1998]**
   (*a*) 12 s (*b*) 8 s
   (*c*) 15 s (*d*) 10 s

6. For angle of projection of a projectile at angles ($45° + \theta$) and ($45° - \theta$), the horizontal ranges described by the projectile are in the ratio of (if $\theta \geq 45°$) **[MP PET 1993]**
   (*a*) 2 : 1 (*b*) 1 : 2
   (*c*) 1 : 1 (*d*) 2 : 3

7. An aeroplane moving horizontally with a speed of 180 km/hr drops a food packet while flying at a height of 490 m. The horizontal range is **[Haryana CET 1996]**

(*a*) 500 m (*b*) 980 m

(*c*) 670 m (*d*) 180 m

8. The horizontal range is four times the maximum height attained by a projectile. The angle of projection is **[MP PET 1994]**

(*a*) 90° (*b*) 60°

(*c*) 45° (*d*) 30°

9. At the top of the trajectory of a projectile, the acceleration is **[Manipal 1995]**

(*a*) maximum (*b*) minimum

(*c*) zero (*d*) g

10. A motor car is moving with a speed of 30 m/s on a circular track of radius 500 m. Its speed is increasing at the rate of 2 m/s^2. What will be its resultant acceleration ? **[Rajasthan PET 1996]**

(*a*) 2.5 m/s^2 (*b*) 2.7 m/s^2

(*c*) 2 m/s^2 (*d*) 4.5 m/s^2

11. The range of a projectile is R when the angle of projection is 30°. Then the value of the other angle of projection for the same range is **[Pb CET 1998]**

(*a*) 45° (*b*) 60°

(*c*) 50° (*d*) 40°

12. A stone thrown at an angle θ to the horizontal, reaches a maximum height h. The time of flight of the stone is **[EAMCET 1998]**

(*a*) $\frac{\sqrt{2h\sin\theta}}{g}$ (*b*) $\frac{2\sqrt{2h\sin\theta}}{g}$

(*c*) $2\sqrt{\frac{2h}{g}}$ (*d*) $\sqrt{\frac{2h}{g}}$

13. A ball is projected horizontally with a velocity of 4 m/s The velocity of the ball after 0.7 s (g = 10 ms^{-2}) is **[Haryana CET 1991]**

(*a*) 11 ms^{-1} (*b*) 10 ms^{-1}

(*c*) 8 ms^{-1} (*d*) 3 ms^{-1}

14. A stone released with zero velocity from the top of the tower reaches the ground in 4 seconds. The height of the tower is about **[BHU 1998]**

(*a*) 20 m (*b*) 40 m

(*c*) 80 m (*d*) 160 m

15. For a body moving with constant speed in a horizontal circle, which of the following remains constant ? **[MP PET 1997]**

(*a*) velocity (*b*) acceleration

(*c*) centripetal force (*d*) kinetic energy

16. A boat covers a given distance in 6 hours moving downstream of a river. It covers the same distance in 10 hours moving upstream. What would it take to cover the same distance in still water ? **[Haryana CET 1994]**

(*a*) 10 s (*b*) 7 .5 s

(*c*) 5 s (*d*) 2.5 s

17. Which of the following statements is false for a particle moving in a circle with a constant angular speed? **[AIEEE 2004]**

(*a*) The velocity vector is tangent to the circle.

(*b*) The acceleration vector is tangent to the circle.

(*c*) The acceleration vector points to the centre of the circle.

(*d*) The velocity and acceleration vectors are perpendicular to each other.

18. A ball is thrown from a point with a speed v_0 at an angle of projection θ. From the same point and at the same instant, a person starts running with a constant speed $v_0/2$ to catch the ball. Will the person be able to catch the ball? If yes, what should be the angle of projection? **[AIEEE 2004]**

(*a*) Yes, 60° (*b*) Yes, 30°

(*c*) No (*d*) Yes, 45°

19. A boy playing on the roof of a 10 m high building throws a ball with a speed of 10 m/s at an angle of 30° with the horizontal. How far from the throwing point will the ball be at the height of 10 m from the ground?

[g = 10 m/s^2, sin 30° = 1/2, cos 30° = $\sqrt{3}/2$] **[AIEEE 2003]**

(*a*) 5.20 m (*b*) 4.33 m

(*c*) 2.60 m (*d*) 8.66 m

20. An object is moving in a circle of radius 100 m with a constant speed of 31.4 m/s. What is its average speed for one complete revolution? **[DCE 2004]**

(*a*) 3.14 m/s (*b*) 31.4 m/s

(*c*) zero m/s (*d*) $\sqrt{2}\times31.4$ m/s

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*d*)	**2.** (*c*)	**3.** (*d*)	**4.** (*b*)	**5.** (*d*)
6. (*c*)	**7.** (*a*)	**8.** (*c*)	**9.** (*d*)	**10.** (*b*)
11. (*b*)	**12.** (*c*)	**13.** (*c*)	**14.** (*c*)	**15.** (*d*)
16. (*b*)	**17.** (*b*)	**18.** (*a*)	**19.** (*d*)	**20.** (*b*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Since there is no change in speed, the tangential acceleration is zero. There will be only centripetal acceleration ($= v^2 / r$) due to continuous change in the direction of velocity.

2. Velocity of boat *w.r.t.* ground, $\vec{V}_{BG} = 3\hat{i} + 4\,\hat{j}$

   Velocity of water *w.r.t.* ground, $\vec{V}_{WG} = -3\hat{i} - 4\,\hat{j}$

   $\therefore$ Velocity of boat *w.r.t* water, $\vec{V}_{BW} = \vec{V}_{BG} + \vec{V}_{GW} = \vec{V}_{BG} - \vec{V}_{WG}$

   $$= (3\hat{i} + 4\hat{j}) - (-3\hat{i} - 4\hat{j}) = 6\hat{i} + 8\hat{j}$$

3. Time of flight, $T = \dfrac{2v_0 \sin\theta}{g}$

   $$= \frac{2 \times 9.8 \times \sin 30°}{9.8} = 1 \text{ sec}$$

4. Fig. 9.25 shows the conditions of the problem. The bird is at point B and the person is at point O. The distance OB = 100 m. Let the gun be fired on the bird with a velocity v_0 making an angle θ with the horizontal.

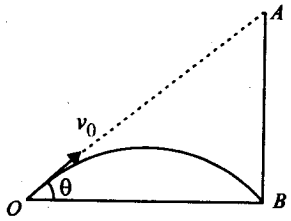

Fig. 9.25

$$OB = \frac{v_0^2 \sin 2\theta}{g} \text{ or } 100 = \frac{(500)^2 \times \sin 2\theta}{10}$$

$$\therefore \quad \sin 2\theta = \frac{100 \times 10}{(500)^2} = \frac{1}{250} \text{ or } \quad 2\theta = 14' \quad \therefore \theta = 7' = \frac{7}{60} \times \frac{\pi}{180} \text{ rad}$$

Now $\theta = \dfrac{AB}{OB} \quad \therefore \quad AB = \theta \times OB = \dfrac{7}{60} \times \dfrac{\pi}{180} \times (100 \times 100) = 20 \text{ cm}$

5. Impress a velocity 10 ms^{-1} toward south on both train and parrot. It means the velocity of parrot is 10 + 5 = 15 ms^{-1} towards south. Therefore, time taken by the parrot to cross the train = 150/15 = 10 sec.

6. Horizontal angle, $R = v_0^2 \sin 2\theta / g$

   For projection angle of 45° – θ, $R_1 = \dfrac{v_0^2 \sin(90° - 2\theta)}{g} = \dfrac{v_0^2 \cos 2\theta}{g}$

   For projection angle of 45° + θ, $R_2 = \dfrac{v_0^2 \sin(90° + 2\theta)}{g} = \dfrac{v_0^2 \cos 2\theta}{g}$

   $\therefore \quad R_1 = R_2 \quad$ or $R_1 : R_2 = 1 : 1$

7. The time of flight T is determined by the vertical motion of the food packet. At the instant of dropping the packet, the vertical velocity is zero.

$\therefore \quad h = 0 + \frac{1}{2} gT^2 \quad$ or $\quad T = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 490}{9.8}} = 10 \text{ sec}$

Horizontal velocity, v_x = 180 km/hr = 50 m /s

$\therefore$ Horizontal range, $R = v_x \times T = 50 \times 10 = 500$ m

8. $R = 4H$

or $\frac{v_0^2 \sin 2\theta}{g} = 4 \times \frac{v_0^2 \sin^2 \theta}{2g}$ or $\tan\theta = 1$ or $\theta = 45°$

9. The acceleration is g acting vertically downward throughout the motion of the projectile. At the top, vertical velocity is zero and horizontal velocity is $v_0 \cos\theta$ (θ = projection angle).

10. Centripetal acceleration, $a_c = v^2/r = (30)^2/500 = 1.8$ m/s^2. The tangential acceleration, a_t = 2 m /s^2. Since the two accelerations are perpendicular to each other,

$\therefore$ Resultant acceleration, $a = \sqrt{a_c^2 + a_t^2} = \sqrt{(1.8)^2 + (2)^2} = 2.7$ m/s^2

11. The horizontal range of a projectile is the same for projection angles θ and $(90° - \theta)$.

12. $h = \frac{v_0^2 \sin^2 \theta}{2g}$ or $v_0^2 \sin^2 \theta = 2gh$ or $v_0 \sin\theta = \sqrt{2gh}$

Time of flight, $T = \frac{2v_0 \sin\theta}{g} = \frac{2\sqrt{2gh}}{g} = 2\sqrt{\frac{2h}{g}}$

Alternatively : $h = 0 + \frac{1}{2} gt^2$ or $t = \sqrt{\frac{2h}{g}}$

Time of flight, $T = 2t = 2\sqrt{\frac{2h}{g}}$

13. The horizontal velocity v_x (= 4 ms^{-1}) remains the same throughout the journey. At the instant of projection, vertical velocity is zero. After time t (0.7s) it will be

$$v_y = 0 + gt = 10 \times 0.7 = 7 \text{ ms}^{-1} \therefore v = \sqrt{v_x^2 + v_y^2} = \sqrt{(4)^2 + (7)^2} = 8 \text{ ms}^{-1}$$

14. $h = 0 + \frac{1}{2} gt^2 = \frac{1}{2} \times 10 \ (4)^2 = 80$ m

15. In uniform circular motion, the magnitude of velocity v (*i.e.*, speed) remains constant. Therefore, kinetic energy ($mv^2/2$) does not change.

16. Let S be the given distance. If V_{BG} and V_{WG} are the velocities of boat and water *w.r.t.* ground, then for downstream motion, resultant velocity is $V_{BG} + V_{WG}$ and for upstream motion, it is $V_{BG} - V_{WG}$.

$\therefore \quad 6 = \frac{S}{V_{BG} + V_{WG}}$; $10 = \frac{S}{V_{BG} - V_{WG}}$

On solving, V_{BG} = 8 S/60.

$\therefore$ For covering distance S in still water, time taken t is

$$t = \frac{S}{V_{BG}} = S \div \frac{8S}{60} = 7.5 \text{ s}$$

17. For a particle moving in a circle with constant angular speed, velocity vector is always tangent to the circle and the acceleration vector always points towards the centre of circle or is always along the radius of the circle. Therefore, velocity vector will be perpendicular to the acceleration vector. But **acceleration vector is not tangent to the circle** [option (*b*)].

18. The man will catch the ball if the horizontal component of velocity of the ball is equal to the constant speed of man *i.e.*,

$$v_0 \cos\theta = \frac{v_0}{2} \text{ or } \cos\theta = \frac{1}{2} \quad \therefore \theta = 60°$$

19. The statement of the problem suggests that we have to find the horizontal range (R) of the ball.

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(10)^2 \times \sin 2\times 30°}{10} = 8.66 \text{ m}$$

20. The time (t) taken to complete one revolution is

$$t = \frac{2\pi r}{\text{Speed}} = \frac{2\pi \times 100}{31.4} = 20 \text{ s}$$

$$\therefore \quad \text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}} = \frac{2\pi r}{20} = \frac{2\pi \times 100}{20} = 31.4 \text{ m/s}$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. A boy throws a ball horizontally with a velocity of 8 ms^{-1} from the top of a building. The ball falls to the ground 12 m away from the building. What is the height of the building ?

Hint. Horizontal velocity, $v_{x0} = 8$ ms^{-1}; Horizontal range, $R = 12$ m

$\therefore$ Time of flight, $T = \frac{R}{v_{x0}} = \frac{12}{8} = 1.5 \text{ s}$

Height of building, $h = \frac{1}{2} gT^2 = \frac{1}{2} \times 9.8 \times (1.5)^2 = 11$ m

2. A boy can throw a ball 50 m vertically upwards. Find the greatest horizontal distance he can throw.

Hint. Let the upward projection velocity be v_0 ms^{-1}.

$\therefore$ $v^2 - v_0^2 = 2gh$ or $0^2 - v_0^2 = 2 \times (-g) \times 50$ or $v_0^2 = 100\, g$

Now Maximum horizontal range, $R_{max} = \frac{v_0^2}{g} = \frac{100g}{g} = 100$ m

3. The ball is thrown horizontally from the top of the building with a speed of 40 m/s. Find the velocity and position of the ball at the end of 4 seconds.

Hint. The horizontal and vertical velocities of the ball are independent of each other.

For vertical motion : Initial velocity, $v_{y0} = 0$; $t = 4$ s

$\therefore$ Vertical velocity after 4 s is given by ;

$$v_y = 0 + gt = 0 + 9.8 \times 4 = 39.2 \text{ } ms^{-1}$$

For horizontal motion : Uniform horizontal velocity, $v_{x0} = 40$ ms^{-1}

$\therefore$ Resultant velocity v after 4 s is given by ;

$$v = \sqrt{v_{x0}^2 + v_y^2} = \sqrt{(40)^2 + (39.2)^2} = 56.005 \text{ ms}^{-1}$$

If v makes an angle α with the vertical, then,

$$\tan \alpha = \frac{v_{x0}}{v_y} = \frac{40}{39.2} = 1.019 \quad \therefore \alpha = 45.54°$$

The horizontal distance travelled by the ball in 4 s is

$$x = v_{x0} t = 40 \times 4 = 160 \text{ m}$$

The vertical distance (downward) travelled by the ball in 4s is

$$y = \frac{1}{2} g t^2 = \frac{1}{2} \times 9.8 \times (4)^2 = 78.4 \text{ m}$$

Thus, the ball is 78.4 m below and 160 m horizontally away from the starting point.

4. **The height y and distance x along the horizontal plane of a projectile on a certain planet (with no surrounding atmosphere) are given by;**

$$\mathbf{y = (8\,t - 5\,t^2)m \; ; \; x = 6\,t \; m}$$

where t is in seconds. Find (*i*) the velocity with which the projectile was projected (*ii*) the angle with the horizontal at which projectile was projected (*iii*) acceleration due to gravity.

Hint. $x = 6\,t$ m ; $y = (8\,t - 5\,t^2)$ m

The x- and y-components of velocity are :

$$v_x = \frac{dx}{dt} = 6 \text{ ms}^{-1} \; ; \; v_y = \frac{dy}{dt} = (8 - 10t) \text{ ms}^{-1}$$

(*i*) The initial x- and y-components of velocity are obtained by putting $t = 0$.

$\therefore$ $v_{x0} = 6 \text{ ms}^{-1}$; $v_{y0} = 8 - 10 \times 0 = 8 \text{ ms}^{-1}$

$\therefore$ Initial velocity, $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2} = \sqrt{6^2 + 8^2} = 10 \text{ ms}^{-1}$

(*ii*) Angle of projection with the horizontal is given by;

$$\theta = \tan^{-1} \frac{v_{y0}}{v_{x0}} = \tan^{-1} \frac{8}{6} = 53.1°$$

(*iii*) Acceleration due to gravity is along the vertical or y-axis.

$\therefore$ $$a_y = \frac{d}{dt}(v_y) = \frac{d}{dt}(8 - 10t) = -10 \text{ ms}^{-2}$$

$\therefore$ Acceleration due to gravity on the planet is 10 ms^{-2} downward.

5. **Prove that the maximum horizontal range is four times of maximum height attained by a projectile fired along the required oblique direction.**

Hint. The horizontal range of a projectile fired at the initial velocity v_0 at an angle θ with the horizontal is

$R = \frac{v_0^2 \sin 2\theta}{g}$. For maximum horizontal range ; $\theta = 45°$.

$\therefore$ Maximum horizontal range, $R_{max} = \frac{v_0^2}{g}$ ... (*i*)

Maximum height attained by the projectile at $\theta = 45°$ is

$$H_{max} = \frac{v_0^2 \sin^2\theta}{2g} = \frac{v_0^2 \sin^2 45°}{2g} = \frac{v_0^2}{4g} \quad ...(ii)$$

From eqs. (*i*) and (*ii*), $R_{max} = 4H$

6. From the top of a building 19.6 m high, a ball is projected horizontally. After how long does it strike the ground. If the line joining the point of projection to the point where it hits the ground makes an angle of 45° with the horizontal, what is the initial velocity of the ball?

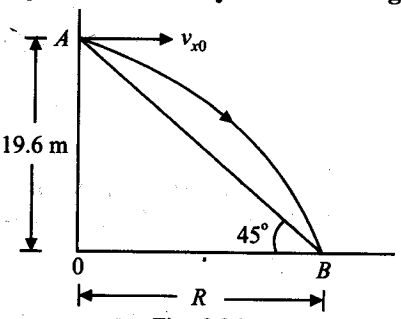

Fig. 9.26

Hint. Fig. 9.26 shows the conditions of the problem. The ball is projected horizontally at point *A* with a velocity of v_{x0} with *OA*= 19.6 m. It is clear that horizontal range is *R*= *OB* = *OA* = 19.6 m. The time taken by the ball to hit the ground (point *B*) will depend upon vertical motion of the ball alone. Let it be *t*. Now *h*= 19.6 m, $g = 9.8\ ms^{-2}$.

$\therefore$ $h = \frac{1}{2}gt^2$ or $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times19.6}{9.8}} = 2$

Also $R = v_{xo}\, t$ $\therefore$ $v_{x0} = R/t = 19.6/2 = 9.8$ ms

7. A shell is fired at an angle of 30° to the horizontal with a velocity of 196 ms^{-1}. Find (*i*) the total time of flight (*ii*) horizontal range and (*iii*) maximum height attained.

Hint. $v_0 = 196\ ms^{-1}$; $\theta = 30°$; $g = 9.8\ ms^{-2}$

(*i*) Time of flight, $T = \frac{2v_0 \sin\theta}{g} = \frac{2\times196\times\sin 30°}{9.8} = 20$ s

(*ii*) Horizontal range, $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(196)^2 \times \sin 60°}{9.8} = 3394.72$ m

(*iii*) Maximum height attained, $H_{max} = \frac{v_0^2 \sin^2\theta}{2g} = \frac{(196)^2 \times \sin^2 30°}{2\times9.8} = 490$ m

8. The compass of an aeroplane indicates that it is heading due north and its air speed indicator shows that it is moving relative to air at a speed of 120 km/h. If there is a wind blowing from west to east at a speed of 50 km/h *w.r.t* earth, find the velocity of plane *w.r.t.* earth.

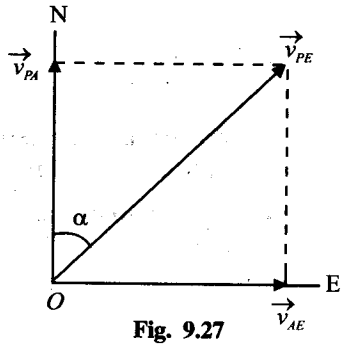

Fig. 9.27

Hint. Let $\vec{v}_{PE}$ = velocity of plane *w.r.t.* earth

$\vec{v}_{AE}$ = velocity of air *w.r.t.* earth

$\vec{v}_{PA}$ = velocity of plane *w.r.t.* air

$\therefore$ $\vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE}$

Now $\vec{v}_{PA}$ points north while $\vec{v}_{AE}$ points east. Referring

to Fig. 9.27, the magnitude of $\vec{v}_{PE}$ is given by ;

$$v_{PE} = \sqrt{v_{PA}^2 + v_{AE}^2} = \sqrt{(120)^2 + (50)^2} = 130 \text{ km/h}$$

$$\alpha = \tan^{-1}\frac{v_{AE}}{v_{PA}} = \tan^{-1}\frac{50}{120} = 22.6° \text{ East of north}$$

9. An aeroplane flying in a horizontal direction with a velocity of 720 km/h drops a bomb at a height of 1960 m. Find (*i*) the time taken by the bomb to reach the ground (*ii*) the magnitude and direction of resultant velocity with which the bomb strikes the ground.

Hint. (*i*) The time taken by the bomb to reach the ground can be determined from the vertical motion. Let it be t seconds. The initial vertical velocity = 0. Therefore, for vertical motion ;

$$h = \frac{1}{2}gt^2 \text{ or } t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times1960}{9.8}} = 20 \text{ s}$$

(*ii*) The initial horizontal velocity v_{x0} (= 720 km /h = 200 ms^{-1}) of the bomb remains the same throughout the journey. The downward vertical velocity (initially zero) increases with time.

Downward vertical velocity, $v_y = 0 + gt = 9.8 \times 20 = 196$ ms^{-1}

$\therefore$ Magnitude of the resultant hitting velocity

$$= \sqrt{v_{x0}^2 + v_y^2} = \sqrt{(200)^2 + (196)^2} = 280.02 \text{ ms}^{-1}$$

Let the resultant hitting velocity make an angle α with the horizontal.

$$\therefore \quad \alpha = \tan^{-1}\frac{v_y}{v_{x0}} = \tan^{-1}\frac{196}{200} = 44.42°$$

10. A gun throws a shell with a muzzle speed of 98 ms^{-1}. When the elevation is 45°, the range is found to be 900 m. How much is the range decreased by air resistance ?

Hint. Horizontal range, $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(98)^2 \times \sin 90°}{9.8} = 980 \text{ m}$

The above formula is valid if air resistance is neglected. In actual practice, air resistance decreases the horizontal range.

$\therefore$ Decrease in range due to air resistance = 980 – 900 = 80 m

11. A gun kept on a straight horizontal road is used to hit a car, travelling along the same road away from the gun with a uniform speed of 72 km/h. The car is at a distance of 500 m from the gun when the gun is fired at an angle of 45° with the horizontal. Find (*i*) the distance of the car from the gun when the shell hits it (*ii*) the speed of projection of the shell from the gun. **[IIT]**

Hint. Fig. 9.28 shows the conditions of the problem. The car is travelling at a constant speed = 72 km/h = 20 ms^{-1}. When the car is at a distance of 500 m from the gun, the shell is fired with velocity v_0 making an angle of 45° with the horizontal. Let t be the time taken by the shell to hit the car. In this time, the car has travelled a horizontal distance $x = 20 \times t = 20t$.

$\therefore$ Horizontal range of shell, $R = (500 + 20t)$ m ...(*i*)

Note that t is the time of flight of the shell and is given by ;

$$t = \frac{2v_0 \sin\theta}{g} = \frac{2v_0 \sin 45^\circ}{g} = \frac{2v_0 \times 1/\sqrt{2}}{g} = \frac{\sqrt{2}v_0}{g}$$

$$\therefore \quad R = 500 + \frac{20 \times \sqrt{2} v_0}{g} \qquad \ldots (ii)$$

Also $$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 \sin 90^\circ}{g} = \frac{v_0^2}{g} \qquad \ldots (iii)$$

From eqs. (*ii*) and (*iii*), $\frac{v_0^2}{g} = 500 + \frac{20\sqrt{2}\, v_0}{g}$.

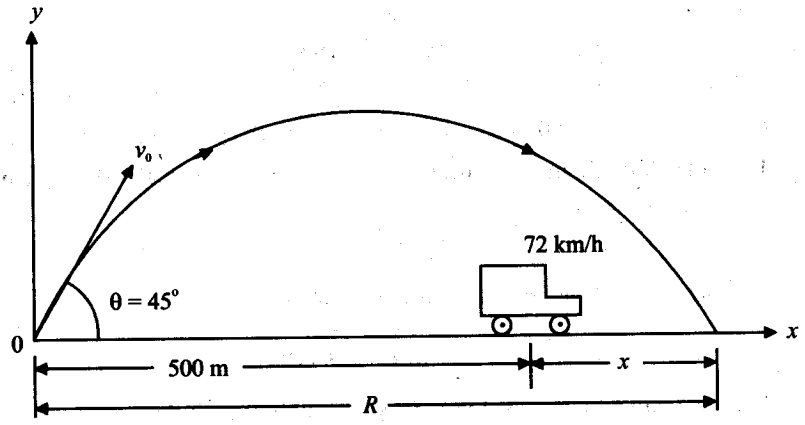

Fig. 9.28

(*ii*) Putting the value of g and solving for v_0, we get, $v_0 = 85.56$ ms^{-1}.

(*i*) Range, $R = \frac{v_0^2}{g} = \frac{(85.56)^2}{9.8} = 746.9$ m

12. **A ball falling freely from a given height H hits an inclined plane in its path at a height h. As a result of this impact, the direction of velocity of the ball becomes horizontal. For what value of h/H, the ball will take maximum time to reach the ground.** **[IIT]**

Hint. Fig. 9.29 shows the conditions of the problem. The ball falls freely from point P at a height H above the ground. It strikes at point A of the inclined plane AC, the point A being at a height h from the ground. After striking at A, the velocity becomes horizontal. The motion of the body from A to B is projectile motion.

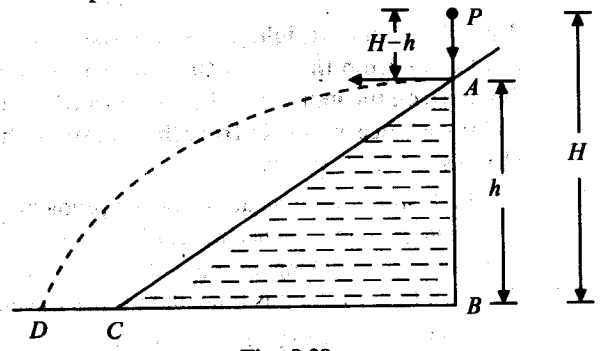

Fig. 9.29

Motion from P to A. The initial velocity of the ball is zero and it traverses a height

$(H - h)$ in going from P to A. If t_1 is the time taken, then,

$$H - h = \frac{1}{2} g t_1^2 \quad \text{or} \quad t_1 = \sqrt{\frac{2(H-h)}{g}} \qquad ...(i)$$

Motion from A to D. At point A, the initial vertical velocity is again zero. The time t_2 taken by the ball in reaching from A to D can be found from its vertical motion.

$$h = \frac{1}{2} g t_2^2 \quad \text{or} \quad t_2 = \sqrt{\frac{2h}{g}} \qquad ...(ii)$$

Total time t taken by the ball is given by ;

$$t = t_1 + t_2 = \sqrt{\frac{2(H-h)}{g}} + \sqrt{\frac{2h}{g}}$$

or
$$t = \sqrt{\frac{2}{g}} \left[(H-h)^{1/2} + h^{1/2}\right]$$

The time t taken will be maximum for height h if $dt/dh = 0$.

$$\therefore \quad \left(\sqrt{\frac{2}{g}}\right)\left[-\frac{1}{2}(H-h)^{-1/2} + \frac{1}{2}h^{-1/2}\right] = 0$$

or
$$-(H-h)^{-1/2} + h^{-1/2} = 0$$

or
$$(H-h)^{-1/2} = h^{-1/2}$$

or
$$H - h = h \quad \therefore \quad \frac{h}{H} = \frac{1}{2}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 4 : MOTION IN A PLANE

Q.1. State, for each of the following physical quantities, if it is a scalar or a vector: volume, mass, speed, acceleration, density, number of moles, velocity, angular frequency, displacement, angular velocity.

Ans. Scalars. Volume, mass, speed, density, number of moles, angular frequency.

Vectors. Acceleration, velocity, displacement, angular velocity.

Q.2. Pick out the two scalar quantities in the following list: force, angular momentum, work, current, linear momentum, electric field, average velocity, magnetic moment, relative velocity.

Ans. Of the given physical quantities, only *work* and *current* are scalar quantities.

Q.3. Pick out the only vector quantity in the following list: temperature, pressure, impulse, time, power, total path length, energy, gravitational potential, coefficient of friction, charge.

Ans. In the given list, only *impulse* is the vector quantity.

Q.4. State with reasons, whether the following algebraic operations with scalar and vector physical quantities are meaningful: (*a*) adding any two scalars (*b*) adding a scalar to a vector of the same dimensions (*c*) multiplying any vector by any scalar (*d*) multiplying any two scalars (*e*) adding any two vectors (*f*) adding a component of a vector to the same vector.

Ans. (*a*) No, because only scalars of the same kind can be added. Thus temperature (scalar) cannot be added to length (scalar).

(*b*) No, because a scalar cannot be added to a vector.

(*c*) Yes. For example, $\vec{F} = m\vec{a}$ [*i.e.* mass (scalar) multiplied by acceleration (vector)] gives rise to force.

(*d*) Yes. For example, $W = Pt$ [*i.e.* power (scalar) multiplied by time (scalar)] gives rise to work done (W).

(*e*) No, because only vectors of the same kind can be added. Thus force (vector) cannot be added to velocity (vector).

(*f*) No, because adding a component of a vector to the same vector is not a meaningful operation.

Q.5. Read each statement below carefully and state with reasons, if it is true or false: (*a*) The magnitude of a vector is always a scalar (*b*) each component of a vector is always a scalar (*c*) the total path length is always equal to the magnitude of the displacement vector of a particle (*d*) the average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particle over the same interval of time (*e*) three vectors not lying in a plane can never add up to give a null vector.

Ans. (*a*) True because magnitude is a pure number.

(*b*) False because the component of a vector is also a vector.

(*c*) False because total path length may be greater than or equal to the magnitude of the displacement.

(*d*) True because total path length may be greater than or equal to the magnitude of the displacement.

(*e*) True. The resultant of two vectors will lie in the plane containing the two vectors. As the third vector is not in the plane of the resultant of the two vectors, they cannot give rise to a null vector.

Q.6. Establish the following vector inequalities geometrically or otherwise.

(*a*) $|\vec{A}+\vec{B}| \leq |\vec{A}|+|\vec{B}|$ (*b*) $|\vec{A}+\vec{B}| \geq \left||\vec{A}|-|\vec{B}|\right|$

(*c*) $|\vec{A}-\vec{B}| \leq |\vec{A}|+|\vec{B}|$ (*d*) $|\vec{A}-\vec{B}| \geq \left||\vec{A}|-|\vec{B}|\right|$

When does the equality sign above apply?

Ans. Consider two vectors $\vec{A}$ and $\vec{B}$ represented in magnitude and direction by $\overrightarrow{OQ}$ and $\overrightarrow{OP}$ respectively. The addition of these vectors can be done by the parallelogram law of vectors as shown in Fig. 4.01 (*i*). The resultant is $\overrightarrow{OR}$ *i.e.,*

$$\overrightarrow{OR} = \vec{A} + \vec{B} \quad \text{or} \quad OR = |\vec{A} + \vec{B}|$$

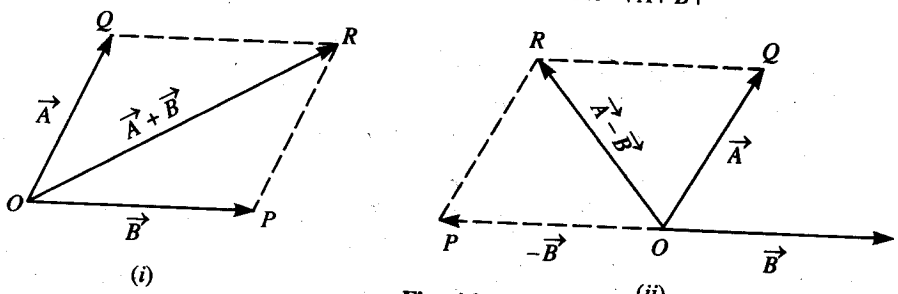

Fig. 4.01

Fig. 4.01 (*ii*) shows the subtraction of vector $\vec{B}$ from vector $\vec{A}$. The resultant is $\overrightarrow{OR}$ *i.e.,*

$$\overrightarrow{OR} = \vec{A} - \vec{B} \quad \text{or} \quad OR = |\vec{A} - \vec{B}|$$

(*a*) Consider ΔOPR in Fig. 4.01 (*i*). Since the one side of a triangle is always smaller than the sum of the other two sides,

$$\therefore \quad OR < OP + PR \quad \text{or} \quad OR < OQ + OP \qquad (\because PR = OQ)$$

or

$$|\vec{A} + \vec{B}| < |\vec{A}| + |\vec{B}|$$

When vectors $\vec{A}$ and $\vec{B}$ are along a straight line in the same direction, then,

$$|\vec{A} + \vec{B}| = |\vec{A}| + |\vec{B}|$$

$$\therefore \quad |\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$$

Note that sign of equality will apply for only one case *i.e.,* when $\vec{A}$ and $\vec{B}$ are along a straight line in the same direction.

(*b*) Consider ΔOPR in Fig. 4.01 (*i*). Since the side of a triangle can never be less than the difference of the other two sides,

$$\therefore \quad OR > |PR - OP|$$

or

$$OR > |OQ - OP| \qquad (\because PR = OQ)$$

The modulus $PR - OP$ (or $OQ - OP$) has been taken because the L.H.S. is always positive but the R.H.S. may be negative in case PR (or OQ) is less than OP.

or

$$|\vec{A} + \vec{B}| > \left||\vec{A}| - |\vec{B}|\right|$$

When the vectors $\vec{A}$ and $\vec{B}$ are along a straight line but in opposite directions, then,

$$|\vec{A} + \vec{B}| = \left||\vec{A}| - |\vec{B}|\right|$$

$$\therefore \quad |\vec{A} + \vec{B}| \geq \left||\vec{A}| - |\vec{B}|\right|$$

Note that equality sign is valid for only one case.

(*c*) Consider ΔOPR in Fig. 4.01 (*ii*).

$$OR < PR + OP$$

or $$|\vec{A}-\vec{B}|<|\vec{A}|+|-\vec{B}|$$

or $$|\vec{A}-\vec{B}|<|\vec{A}|+|\vec{B}|$$

In case vectors $\vec{A}$ and $\vec{B}$ are along a straight line in the opposite directions, then,

$$|\vec{A}-\vec{B}|=|\vec{A}|+|\vec{B}|$$

$\therefore$ $$|\vec{A}-\vec{B}|\leq|\vec{A}|+|\vec{B}|$$

(*d*) Consider ΔOPR in Fig. 4.01 (*ii*).

$$OR + OP > PR$$

or $$OR > |PR-OP|$$

The modulus $PR - OP$ has been taken because the L.H.S. is always positive but the R.H.S. may be negative in case PR is less than OP.

or $$|\vec{A}-\vec{B}|>\left||\vec{A}|-|\vec{B}|\right|$$

In case vectors $\vec{A}$ and $\vec{B}$ are along a striaght line in the same direction, then,

$$|\vec{A}-\vec{B}|=\left||\vec{A}|-|\vec{B}|\right|$$

$\therefore$ $$|\vec{A}-\vec{B}|\geq\left||\vec{A}|-|\vec{B}|\right|$$

Q.7. Given $\vec{A}+\vec{B}+\vec{C}+\vec{D}=\mathbf{0}$, which of the following statements are correct?

(*a*) $\vec{A},\vec{B},\vec{C}$ and $\vec{D}$ must each be a null vector.

(*b*) The magnitude of $(\vec{A}+\vec{C})$ equals the magnitude of $(\vec{B}+\vec{D})$.

(*c*) The magnitude of $\vec{A}$ can never be greater than the sum of the magnitudes of $\vec{B},\vec{C}$ and $\vec{D}$.

(*d*) $\vec{B}+\vec{C}$ must lie in the plane of $\vec{A}+\vec{D}$, if $\vec{A}$ and $\vec{D}$ are not collinear and in the line of $\vec{A}$ and $\vec{D}$, if they are collinear.

Ans. (*a*) The statement is not correct. It is because $\vec{A}+\vec{B}+\vec{C}+\vec{D}$ can be zero in many ways other than $\vec{A},\vec{B},\vec{C}$ and $\vec{D}$ being each a null vector.

(*b*) The statement is correct as proved below:

$\vec{A}+\vec{B}+\vec{C}+\vec{D}=0$ or $\vec{A}+\vec{C}=-(\vec{B}+\vec{D})$ $\therefore$ $\left|\vec{A}+\vec{C}\right|=\left|\vec{B}+\vec{D}\right|$

(*c*) The statement is correct as proved below:

$\vec{A}+\vec{B}+\vec{C}+\vec{D}=0$ $\quad\therefore\quad$ $\vec{A}=-(\vec{B}+\vec{C}+\vec{D})$

(*d*) The statement is correct as proved below:

$\vec{A}+\vec{B}+\vec{C}+\vec{D}=0$ or $\vec{A}+(\vec{B}+\vec{C})+\vec{D}=0$

The resultant of three vectors $\vec{A}, (\vec{B}+\vec{C})$ and $\vec{D}$ can be zero only if $(\vec{B}+\vec{C})$ lies in the plane of $\vec{A}$ and $\vec{D}$. If $\vec{A}$ and $\vec{D}$ are collinear, then $(\vec{B}+\vec{C})$ must be in the line of $\vec{A}$ and $\vec{D}$, only then the vector sum of all the vectors will be zero.

Q.8. Three girls skating on a circular ice ground of radius 200 m start from a point P on the edge of the ground and reach a point Q diametrically opposite to P following different paths as shown in Fig. 4.02. What is the magnitude of the displacement vector for each? For which girl is this equal to the actual length of path skated?

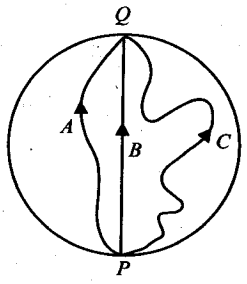

Fig. 4.02

Ans. The magnitude of displacement = PQ = Diameter of the circle = 400 m and is the same for each girl.

For girl B, the magnitude of displacement is equal to the actual length of path skated.

Q.9. A cyclist starts from the centre O of a circular park of radius 1 km, reaches the edge P of the park, then cycles along the circumference, and returns to the centre along QO as shown in Fig 4.03. If the round trip takes 10 min, what is the (a) net displacement (b) average velocity and (c) average speed of the cyclist?

Ans. (a) Since the cyclist returns to the starting point, net displacement is zero.

(b) Average velocity $= \dfrac{\text{Net displacement}}{\text{Time taken}} = \dfrac{0}{10\,\text{min}} = 0$

(c) Total distance travelled $= OP + \widehat{PQ} + QO$

$$= 1 + \frac{1}{4}(2\pi)\times 1 + 1 = 3.57 \text{ km}$$

$\therefore$ Average speed $= \dfrac{\text{Total distance travelled}}{\text{Total time taken}} = \dfrac{3.57}{10/60} = 21.42$ km/h

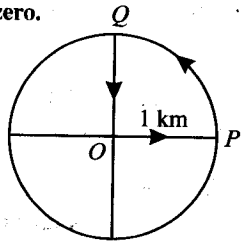

Fig. 4.03

Q.10. On an open ground, a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.

Ans. Fig. 4.04 shows the conditions of the problem. It is clear that the path is a regular hexagon. Let the motorist start from point A.

Third turn. The motorist will take the third turn at D. The displacement vector at D is $\overrightarrow{AD}$.

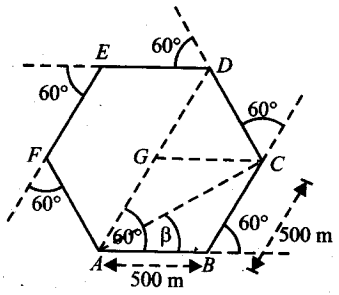

Fig. 4.04

$$\therefore \left|\overrightarrow{AC}\right| = \sqrt{AB^2 + BC^2 + 2AB \times BC \times \cos 60°}$$

$$= \sqrt{(500)^2 + (500)^2 + 2\times 500 \times 500 \cos 60°} = 500\sqrt{3} \text{ m}$$

The angle between AC and CD is 90°.

$$\therefore \left|\overrightarrow{AD}\right| = \sqrt{AC^2 + CD^2} = \sqrt{(500\sqrt{3})^2 + (500)^2} = 1000 \text{ m}$$

Total path length from A to D = AB + BC + CD = 500 + 500 + 500 = 1500 m. The direction of $\overrightarrow{AD}$ is 60° to the initial direction.

Sixth turn. The motorist takes the sixth turn at A (*i.e.* starting position). Therefore, displacement is $_0$ and total path length = $6 \times 500 = 3000$ m.

Eighth turn. The motorist takes the eighth turn at C. Now $\left|\overrightarrow{AC}\right| = 500\sqrt{3}$ m and total path length $= 8 \times 500 = 4000$ m. The direction of $\overrightarrow{AC}$ (*i.e.* angle β) is 30° to the left of the initial direction.

Comparison of magnitude of displacement with total path length is

(*i*) 3rd turn : $\dfrac{\text{Displacement}}{\text{Path length}} = \dfrac{1000\text{ m}}{1500\text{ m}} = 0.666$

(*ii*) Sixth turn : $\dfrac{0\text{ m}}{3000\text{ m}} = 0$

(*iii*) Eighth turn : $\dfrac{500\sqrt{3}\text{ m}}{4000\text{ m}} = 0.216$

Q.11. **A passanger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 min. What is (*a*) the average speed of the taxi (*b*) the magnitude of average velocity? Are the two equal?**

Ans. Actual path length = 23 km ; Displacement = 10 km ; Time taken = 28/60 h

(*a*) $\text{Average speed} = \dfrac{\text{Actual path length}}{\text{Time taken}} = \dfrac{23}{28/60} = 49.3$ km/h

(*b*) $\text{Magnitude of average velocity} = \dfrac{\text{Displacement}}{\text{Time taken}} = \dfrac{10}{28/60} = 21.4$ km/h

The average speed is not equal to the magnitude of average velocity.

Q.12. **Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella to protect herself from the rain?**

Ans. Fig. 4.05 shows the conditions of the problem. The rain is falling vertically downward with velocity $\vec{v}_R$ and is represented by vector $\overrightarrow{OA}$. The woman rider is moving north to south with velocity $\vec{v}_W$ and is represented by vector $\overrightarrow{OB}$.

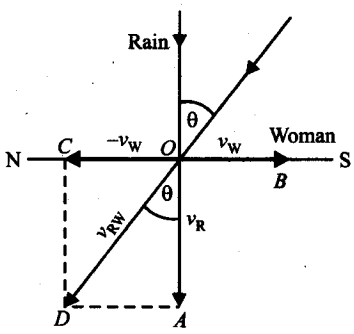

Fig. 4.05

Here $\vec{v}_R = 30$ ms^{-1} ...Vertically downward

$\vec{v}_W = 10$ ms^{-1} ...North to South

In order to save herself from the rain, the woman rider should hold umbrella in the direction of relative velocity of rain *w.r.t.* woman rider (*i.e.* $\overrightarrow{v_{RW}}$).

Now $\overrightarrow{v_{RW}} = \vec{v}_R - \vec{v}_W = \vec{v}_R + (-\vec{v}_W)$

To find $\overrightarrow{v_{RW}}$, reverse the direction of $\vec{v}_W$, thus producing $-\vec{v}_W$ $(= \overrightarrow{OC})$. The vector $\overrightarrow{OD}$ (= diagonal of parallelogram $OCDA$) represents $\overrightarrow{v_{RW}}$. Now $\overrightarrow{v_{RW}}$ makes an angle θ with the vertical.

$\therefore \quad \tan\theta = \dfrac{AD}{OA} = \dfrac{OC}{OA} = \dfrac{10}{30} = \dfrac{1}{3} \quad \therefore \quad \theta = 18.43°$ *with vertical towards south*

Q.13. A man can swim with a speed of 4 km/hr in still water. How long does he take to cross a river 1 km wide if the river flows steadily at 3 km/hr and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank?

Ans. River velocity, $v_r = 3$ km/h

$$\text{Time to cross river, } t = \frac{\text{Width of river}}{\text{Speed of man}} = \frac{1\text{ km}}{4\text{ km/h}} = \frac{1}{4}\text{h} = 15\text{ min}$$

Distance moved along the river in time t

$$= v_r \times t = 3\text{ km/h} \times \frac{1}{4}\text{h} = 0.75\text{ km} = 750\text{ m}$$

Q.14. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the *N-E* direction. If the boat starts moving at a speed of 51 km/h to north, what is the direction of the flag on the mast of the boat?

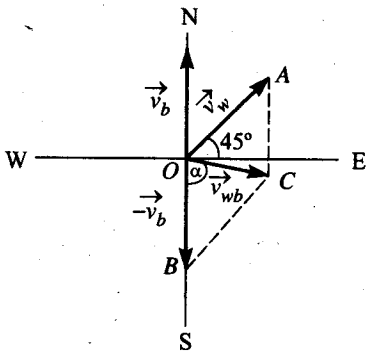

Fig. 4.06

Ans. Fig. 4.06 shows the conditions of the problem. When the boat is stationary, the flag flutters along *N-E* direction. Therefore, wind is blowing due *N-E* because flag flutters in the direction of the wind.

$\vec{v}_b$ = velocity of boat = 51 km/h toward north

$\vec{v}_w$ = velocity of wind = 72 km/h along *N-E* direction

When the boat starts moving, the flag will flutter in the direction of relative velocity of wind w.r.t. boat (*i.e.*, $\vec{v}_{wb}$).

Now $$\vec{v}_{wb} = \vec{v}_w - \vec{v}_b = \vec{v}_w + (-\vec{v}_b)$$

In order to find $\vec{v}_{wb}$, reverse the direction of $\vec{v}_b$, thus producing $-\vec{v}_b$. Now find the sum of $\vec{v}_w$ and $-\vec{v}_b$ by parallelogram law of vector addition as shown in Fig. 4.06. Obviously, the flag will flutter in the direction of $\overrightarrow{OC}$ (= $\vec{v}_{wb}$). Suppose $\overrightarrow{OC}$ makes an angle α with $-\vec{v}_b$.

$$\therefore \quad \tan\alpha = \frac{v_w \sin\theta}{v_b + v_w\cos\theta} = \frac{v_w \sin 135°}{v_b + v_w \cos 135°} \qquad (\because\ \theta = 135°)$$

$$= \frac{72\times 0.707}{51 - 72\times 0.707} = 530.25 \quad \therefore \quad \alpha \simeq 90°$$

Therefore, the flag is almost along east.

Q.15. The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 ms^{-1} can go without hitting the ceiling of the hall?

Ans. We should adjust the projection angle θ so that $H_{max} = 25$ m.

Now $$H_{max} = \frac{v_0^2 \sin^2\theta}{2g}$$

or $$\sin^2\theta = \frac{2gH_{max}}{v_0^2}$$

Here $g = 9.8$ ms^{-2}; $H_{max} = 25$ m ; $v_0 = 40$ ms^{-1}

$$\therefore \quad \sin^2\theta = \frac{2\times 9.8\times 25}{(40)^2} = 0.306$$

or $\sin\theta = \sqrt{0.306} = 0.554$

or $\theta = \sin^{-1} 0.554 = 33.6°$

$\therefore$ Maximum horizontal range for $\theta = 33.6°$ is

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(40)^2 \times \sin(2\times 33.6°)}{9.8} = 150.5 \text{ m}$$

Q.16. A cricketer can throw a ball to a maximum horizontal distance of 100 m. With the same speed, how much high above the ground can the cricketer throw the same ball?

Ans. The maximum horizontal range is

$$R_{max} = \frac{v_0^2}{g} \quad \text{or} \quad v_0 = \sqrt{R_{max} \times g} = \sqrt{100g}$$

For the upward motion of the ball, we have,

$$v^2 - v_0^2 = -2gh \quad \therefore \quad (0)^2 - 100g = -2gh \quad \text{or} \quad h = \frac{-100g}{-2g} = 50 \text{ m}$$

Q.17. A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude and direction of the acceleration of the stone?

Ans. Magnitude of acceleration, $a_c = \omega^2 r$

Here $\omega = \frac{\theta}{t} = \frac{2\pi\times 14}{25} = \frac{28\pi}{25}$ rad/s ; r = 80 cm = 0.8 m

$\therefore$ $$a_c = \left(\frac{28\pi}{25}\right)^2 \times 0.8 = 9.89 \text{ ms}^{-2}$$

The acceleration is directed radially toward the centre of the circle.

Q.18. An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km/h. Compare its centripetal acceleration with the acceleration due to gravity.

Ans. Here r = 1 km = 1000 m ; v = 900 km/h = $\frac{900\times 10^3}{60\times 60}$ = 250 ms^{-1}

Centripetal acceleration, $a_c = \frac{v^2}{r} = \frac{(250)^2}{1000} = 62.5 \text{ ms}^{-2}$

Now $$\frac{a_c}{g} = \frac{62.5}{9.8} = 6.38 \quad \therefore \quad a_c = 6.38 \text{ g}$$

Q.19. Read each statement below carefully and state, with reasons, if it is true or false:

(*a*) The net acceleration of a particle in circular motion is always along the radius of the circle towards the centre.

(*b*) The velocity vector of a particle at a point is always along the tangent to the path of the particle at that point.

(*c*) The acceleration vector of a particle in uniform circular motion averaged over one cycle is a null vector.

Ans. (*a*) The statement is false. The net acceleration of a particle is towards the centre *only* in case of a uniform circular motion (*i.e.* circular motion with constant speed).

(*b*) True. The velocity vector of a particle at a point is along the tangent to the path of the particle at that point in linear or circular or curvilinear motion.

(*c*) True. It is because the net displacement in one complete cycle in uniform circular motion is zero. Therefore, average acceleration over one cycle is zero.

Q.20. The position of a particle is given by $\vec{r} = 3.0t\,\hat{i} - 2.0t^2\,\hat{j} + 4.0\hat{k}$ m, where t is in seconds and the coefficients have the proper units for $\vec{r}$ to be in metres. (a) Find the $\vec{v}$ and $\vec{a}$ of the particle. (b) What is the magnitude and direction of velocity of the particle at t = 2 s?

Ans. (a) $$\text{Velocity, } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(3.0t\,\hat{i} - 2.0t^2\,\hat{j} + 4.0\hat{k})$$

$$= (3.0\,\hat{i} - 4.0t\,\hat{j}) \text{ ms}^{-1}$$

$$\text{Acceleration, } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(3.0\,\hat{i} - 4.0t\,\hat{j}) = -4.0\,\hat{j} \text{ ms}^{-2}$$

(b) At time t = 2 s, $\vec{v} = (3.0\,\hat{i} - 4.0\times 2\,\hat{j}) = (3.0\,\hat{i} - 8.0\,\hat{j})$ ms^{-1}

Its magnitude is $|\vec{v}| = \sqrt{(3.0)^2 + (-8)^2} = \sqrt{73} = 8.54$ ms^{-1}

The angle θ which $\vec{v}$ makes with x-axis is given by :

$$\tan\theta = \frac{v_y}{v_x} = \frac{-8}{3} = -2.667 = -\tan 69.5°$$

$\therefore$ $\theta = 69.5°$ *below the x-axis*

Q.21. A particle starts from the origin at t = 0 with a velocity of $10.0\,\hat{j}$ ms^{-1} and moves in the x-y plane with a constant acceleration of $(8.0\,\hat{i} + 2.0\,\hat{j})$ ms^{-2}. (a) At what time is the x-coordinate of the particle 16 m? What is the y-coordinate of the particle at that time? (b) What is the speed of the particle at that time?

Ans. The position of the particle is given by ;

$$\vec{r}(t) = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 = 10.0\,\hat{j}t + \frac{1}{2}(8.0\,\hat{i} + 2.0\,\hat{j})t^2$$

$$= 10.0\,\hat{j}t + 4.0\,\hat{i}t^2 + 1.0\,\hat{j}t^2$$

$$= 4.0\,\hat{i}t^2 + (10.0t + 1.0t^2)\,\hat{j}$$

$\therefore$ $x(t) = 4.0\ t^2$; $y(t) = 10.0\ t + 1.0\ t^2$

(a) When $x(t)$ = 16 m ; $16 = 4.0t^2$ or $t = \sqrt{16/4.0} = 2$ s

When t = 2 s ; $y = 10.0 \times 2 + 1.0 \times 2^2 = 24$ m

(b) $$\text{Velocity, } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(4.0\,\hat{i}t^2) + (10.0t + 1.0t^2)\,\hat{j}\right]$$

$$= 8.0\,\hat{i}t + (10.0 + 2.0t)\,\hat{j}$$

When t = 2 s ; $\vec{v} = (16\,\hat{i} + 14\,\hat{j})$ ms^{-1}

$\therefore$ $$\text{Speed, } |\vec{v}| = \sqrt{16^2 + 14^2} = \sqrt{452} = 21.26 \text{ ms}^{-1}$$

Q.22. $\hat{i}$ and $\hat{j}$ are unit vectors along x-axis and y-axis respectively.

(a) What is the magnitude and direction of the vectors $(\hat{i} + \hat{j})$ and $(\hat{i} - \hat{j})$?

(b) What are the components of a vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the directions of $(\hat{i} + \hat{j})$ and $(\hat{i} - \hat{j})$?

Ans. (a) The magnitude of vector $\hat{i} + \hat{j}$ is given by ;

$$\left|\hat{i} + \hat{j}\right| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

Suppose the vector $\hat{i} + \hat{j}$ makes an angle θ_1 with the direction of $\hat{i}$ (*i.e.*, *x*-axis). Then,

$$\cos\theta_1 = \frac{(\hat{i} + \hat{j}).\hat{i}}{\left|\hat{i} + \hat{j}\right|\left|\hat{i}\right|} = \frac{1}{(\sqrt{2})(1)} = \frac{1}{\sqrt{2}} \qquad \therefore \quad \theta_1 = 45°$$

The magnitude of vector $\hat{i} - \hat{j}$ is given by ;

$$\left|\hat{i} - \hat{j}\right| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

Suppose the vector $\hat{i} - \hat{j}$ makes an angle θ_2 with the direction of $\hat{i}$ (*i.e.*, *x*-axis). Then,

$$\cos\theta_2 = \frac{(\hat{i} - \hat{j}).\hat{i}}{\left|\hat{i} - \hat{j}\right|\left|\hat{i}\right|} = \frac{1}{\sqrt{2}} \qquad \therefore \quad \theta_2 = 45°$$

Note that here $\theta_2 = -45°$ with $\hat{i}$.

(b) $$\vec{A} = 2\hat{i} + 3\hat{j}$$

Let $\hat{a}$ be the unit vector along the direction of $(\hat{i} + \hat{j})$. Then,

$$\hat{a} = \frac{\hat{i} + \hat{j}}{\left|\hat{i} + \hat{j}\right|} = \frac{\hat{i} + \hat{j}}{\sqrt{1^2 + 1^2}} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

The magnitude of the component of $\vec{A}$ along $(\hat{i} + \hat{j})$

$$= \vec{A}.\hat{a} = \left(2\hat{i} + 3\hat{j}\right).\frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) = \frac{(2+3)}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

$\therefore$ Component of $\vec{A}$ along $(\hat{i} + \hat{j}) = (\vec{A}.\hat{a})\hat{a} = \frac{5}{\sqrt{2}}\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right) = \frac{5}{2}(\hat{i} + \hat{j})$

Let $\hat{b}$ be the unit vector along the direction of $(\hat{i} - \hat{j})$. Then,

$$\hat{b} = \frac{\hat{i} - \hat{j}}{\left|\hat{i} - \hat{j}\right|} = \frac{\hat{i} - \hat{j}}{\sqrt{(1)^2 + (-1)^2}} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

The magnitude of the component of $\vec{A}$ along $(\hat{i} - \hat{j})$

$$= \vec{A}.\hat{b} = (2\hat{i} + 3\hat{j}).\frac{\hat{i} - \hat{j}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$\therefore$ Component of $\vec{A}$ along $(\hat{i} - \hat{j}) = (\vec{A}.\hat{b})\hat{b} = -\frac{1}{\sqrt{2}}\frac{\hat{i} - \hat{j}}{\sqrt{2}} = -\frac{1}{2}(\hat{i} - \hat{j})$

Q.23. For an arbitrary motion in space, which of the following relations are true :

(*a*) $\vec{v}_{average} = \frac{1}{2}[\vec{v}(t_1) + \vec{v}(t_2)]$

(*b*) $\vec{v}_{average} = [\vec{r}(t_2) - \vec{r}(t_1)]/(t_2 - t)$

(*c*) $\vec{v}(t) = \vec{v}(0) + \vec{a}\,t$

(*d*) $\vec{r}(t) = \vec{r}(0) + \vec{v}(0)t + (1/2)\vec{a}\,t^2$

(*e*) $\vec{a}_{average} = [\vec{v}(t_2) - \vec{v}(t_1)]/(t_2 - t_1)$

Ans. The relations (*b*) and (*e*) are true. The relations (*a*), (*c*) and (*d*) are false because they are true only in case of uniformly accelerated motion.

Q.24. Read each statement below carefully and state with reasons and examples, if it is true or false. A scalar quantity is one that (*a*) is conserved in a process (*b*) can never take negative values (*c*) must be dimensionless (*d*) does not vary from one point to another in space (*e*) has the same value for observers with different orientations of axes.

Ans. (*a*) False because energy (scalar quantity) is not conserved during inelastic collisions.

(*b*) False because temperature (scalar quantity) can be negative.

(*c*) False because mass (scalar quantity) has dimensions.

(*d*) False because gravitational potential energy (scalar quantity) varies from point to point in space.

(*e*) True because orientation of axes does not change the value of a scalar quantity.

Q.25. An aircraft is flying at a height of 3400 m above the ground .If the angle subtended at a ground observation point by the aircraft at positions 10 s apart is 30°, what is the speed of the aircraft?

Ans. Fig. 4.07 shows the conditions of the problem. Here *A* is the initial position of aircraft and *B* is the final position after 10 s. Further, *O* is the point of observation.

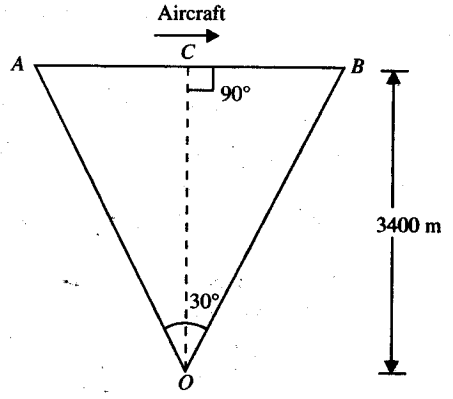

Fig. 4.07

$\angle AOB = 30°$; $OC = 3400$ m ; $\angle AOC = 15°$

$\tan 15° = \frac{AC}{OC}$

$\therefore$ $AC = OC \tan 15° = 3400 \tan 15° = 911$ m

$\therefore$ Distance $AB = 2 \times AC = 2 \times 911 = 1822$ m

Speed of aircraft

$$= \frac{\text{Distance } AB}{\text{Total time}} = \frac{1822}{10} = 182.2 \text{ m/s}$$

Q.26. A vector has magnitude and direction. (*i*) Does it have a location in the space? (*ii*) Can it vary with time? (*iii*) Will two equal vectors $\vec{a}$ and $\vec{b}$ at different locations in space necessarily have identical physical effects? Give examples in support of your answer.

Ans. (*i*) In general, a vector has no definite location in space. Provided the magnitude and direction of a vector do not change, it remains unaffected when displaced in space. However, a position vector has a definite location in space.

(*ii*) A vector can vary with time *e.g.* the velocity vector of an accelerated particle varies with time.

(*iii*) Two equal vectors located at different locations may not have the same physical effect. For example, a force of 5N applied to a mass of 1 kg has different effects on earth and moon.

Q.27. A vector has both magnitude and direction. Does that mean anything that has magnitude and direction is necessarily a vector? The rotation of a body can be specified by the direction of the axis of rotation and the angle of rotation about the axis. Does that make any rotation a vector?

Ans. No. The necessary condition is that the physical quantity should obey the laws of vector addition. For example, electric current has magnitude as well as direction but is a scalar quantity. It is because electric current does not obey the laws of vector addition.

The finite rotation of a body about an axis does not obey the laws of vector addition. Therefore, any rotation is not a vector. However, the small rotation of a body (*i.e.* small angle of rotation) is a vector because it obeys the laws of vector addition.

Q.28. Can you associate vectors with (*a*) the length of a wire bent into a loop (*b*) a plane area (*c*) a sphere? Explain.

Ans. (*a*) We cannot associate a vector with the length of a wire bent into a loop.

(*b*) In many problems in physics, we associate a vector with a plane area. Such a vector is called area vector and its direction is represented by outward drawn normal to the area.

(*c*) We cannot associate a vector with the volume of a sphere. However, vector can be associated with the area of a sphere.

Q.29. A bullet fired at an angle of 30° with the horizontal hits the ground 3 km away. By adjusting its angle of projection, can one hope to hit a target 5 km away? Assume the muzzle speed to be fixed and neglect air friction.

Ans. For the first case, $R = \dfrac{v_0^2 \sin 2\theta}{g}$

or $$\frac{v_0^2}{g} = \frac{R}{\sin 2\theta} = \frac{3\times 1000}{\sin 60°} = 2000\sqrt{3}\ \text{m}$$

For the second case, initial speed v_0 is the same. The maximum horizontal range that can be attained is

$$R_{max} = \frac{v_0^2}{g} = 2000\sqrt{3}\ \text{m} = 3464\ \text{m} = 3.464\ \text{km}$$

But the target is 5 km away. Therefore, the target cannot be hit.

Q.30. A fighter plane flying horizontally at an altitude of *1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 m s^{-1} to hit the plane? At what minimum altitude should the pilot fly the plane to avoid being hit ? (Take g = 10 ms^{-2})

Ans. Fig. 4.08 shows the conditions of the problem.

Here, v_x = 720 km/h = $\dfrac{720\times 1000}{60\times 60} = 200\ \text{ms}^{-1}$

$v_0 = 600\ \text{ms}^{-1}$; y = 1.5 km = 1500 m

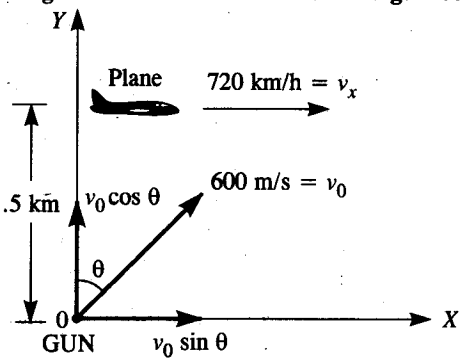

Fig. 4.08

When the plane is just above the gun, let the shell be fired at an angle θ with the vertical so as to hit the plane in time t seconds. This means that in t seconds the horizontal distance travelled by the plane (= $v_x \times t$) will equal to the horizontal distance covered by the shell (= **$v_0 \sin\theta \times t$).

* Note that shell (projectile) travels horizontally as well as vertically. Although the two motions are independent of each other, they are linked by common time t. The statement of the problem suggests that the maximum height attained by the shell will be more than 1.5 km.

** Since shell makes angle θ with the vertical, its horizontal velocity is $v_0 \sin\theta$.

$\therefore \quad v_x \times t = v_0 \sin\theta \times t$

or $\quad \sin\theta = \dfrac{v_x}{v_0} = \dfrac{200}{600} = \dfrac{1}{3}$

$\therefore \quad \theta = \sin^{-1}\left(\dfrac{1}{3}\right) = 19.5°$ *with vertical*

Since the shell is fired at an angle of 19.5° with the vertical, the maximum height to which it rises is

$$H_{max} = \frac{v_0^2 \sin^2(90° - \theta)}{2g} = \frac{v_0^2 \cos^2\theta}{2g}$$

$$= \frac{(600)^2}{2\times 10} \times \frac{8}{9} \qquad \left(\because \sin\theta = \frac{1}{3};\ \cos^2\theta = \frac{8}{9}\right)$$

$$= 16000 \text{ m} = 16 \text{ km}$$

Therefore, the pilot should fly the plane at a minimum height of 16 km to avoid being hit by the shell.

Q.31. **A cyclist is riding with a speed of 27 km/h. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at a constant rate of 0.5 ms^{-2}. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn?**

Ans. If the speed of an object moving in a circle is changing, then there will be a tangential acceleration $\vec{a}_T$ as well as the centripetal acceleration $\vec{a}_c$. The tangential acceleration arises from the change in the magnitude of the velocity whereas the centripetal acceleration arises from the change in the direction of the velocity. The tangential acceleration always points in a direction tangent to the circle and is in the direction of motion (parallel to $\vec{v}$) if the speed is increasing. If the speed is decreasing, $\vec{a}_T$ points antiparallel to $\vec{v}$. In *either case, $\vec{a}_T$ and $\vec{a}_c$ are always perpendicular to each other. The total vector acceleration $\vec{a}$ is given by ;

$$\vec{a} = \vec{a}_T + \vec{a}_c$$

The magnitude of total acceleration is

$$a = \sqrt{a_T^2 + a_c^2}$$

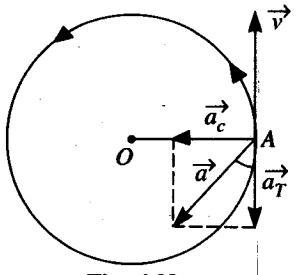

Fig. 4.09

Fig. 4.09 shows the conditions of the problem. Since the speed of the bicycle is decreasing at constant rate, $\vec{a}_T$ is antiparallel to $\vec{v}$.

Now $\quad a_c = \dfrac{v^2}{r}$

Here, $\quad v = 27 \text{ km/h} = \dfrac{27\times 1000}{60\times 60} = 7.5 \text{ ms}^{-1}$; $r = 80$ m

$\therefore \quad a_c = \dfrac{(7.5)^2}{80} = 0.7 \text{ ms}^{-2}$; $\quad a_T = 0.5 \text{ ms}^{-2}$ (given)

$\therefore \quad$ Total acceleration, $a = \sqrt{a_T^2 + a_c^2} = \sqrt{(0.5)^2 + (0.7)^2} = 0.86 \text{ ms}^{-2}$.

If θ is the angle between $\vec{a}$ and $\vec{a}_T$, then,

$$\tan\theta = \frac{a_c}{a_T} = \frac{0.7}{0.5} = 1.4 \quad \therefore \quad \theta = \tan^{-1} 1.4 = 54.5°$$

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* Note that direction of $\vec{a}_T$ and $\vec{a}_c$ change continuously as the body moves along the circular path.

Q.32 (*a*) Show that for a projectile, the angle between the velocity and the *x*-axis as a function of time is given by $\theta_{(t)} = \tan^{-1}\left(\frac{v_{0y} - gt}{v_{0x}}\right)$ where the various symbols have their usual meanings.

(*b*) Show that projection angle θ_0 for a projectile launched from the origin is given by $\theta_0 = \tan^{-1}\left(\frac{4h_m}{R}\right)$ where the symbols have their usual meanings.

Ans. (*a*) Fig. 4.010 shows the conditions of the problem. The initial velocity at the origin O is v_0 (at $t = 0$) and its components along OX and OY axes are v_{0x} and v_{0y} respectively. Let the projectile go from O to P in time t. Suppose the velocity $\vec{v}$ of projectile at point P has components v_x and v_y along OX and OY axes respectively. Then,

$$v_y = v_{0y} - gt \quad \text{and} \quad v_x = v_{0x}$$

If θ is the angle which the resultant velocity $\vec{v}$ makes with OX axis, then,

$$\tan\theta = \frac{v_y}{v_x} = \frac{v_{0y} - gt}{v_{0x}} \therefore \quad \theta = \tan^{-1}\left(\frac{v_{0y} - gt}{v_{0x}}\right)$$

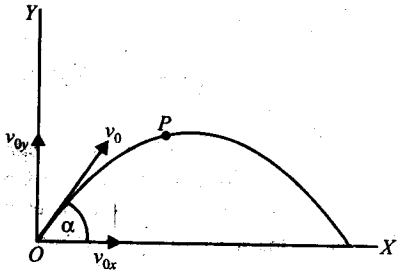


Fig. 4.010

(*b*) For the projection angle θ_0, we have,

$$\text{Maximum vertical height, } h_m = \frac{v_0^2 \sin^2\theta_0}{2g}$$

$$\text{Horizontal range, } R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{v_0^2}{g}(2\sin\theta_0\cos\theta_0)$$

$$\therefore \quad \frac{h_m}{R} = \frac{\tan\theta_0}{4} \quad \text{or} \quad \theta_0 = \tan^{-1}\left(\frac{4h_m}{R}\right)$$

Unit IV

LAWS OF MOTION

- **Force**
- **Friction and Dynamics of Circular Motion**
- **N.C.E.R.T. Textbook Exercises**

10

Force

INTRODUCTION

In the previous chapters, we have discussed how motion is described in terms of velocity and acceleration. We now investigate the cause of motion. What causes a body at rest to move? What causes a moving body to come to a stop? What causes a body to accelerate or decelerate? What is involved when an object moves in a circle? Our answer is that in each case, a *force* is required. The relation between force and motion is described in Newton's three laws of motion. These laws are very simple statements and enable us to describe the future (or past) motion of a body if we know the forces acting on it. Since basic cause of motion is the force, we shall discuss this important physical quantity first before dealing with Newton's laws of motion.

10.1. FORCE

In everyday language, a force is defined as under :

A **force** *is a push or pull which produces or tends to produce, stops or tends to stop motion in a body.*

This is the **intuitive definition of force.* For example, a rubber ball lying on the ground is at rest but when it is pushed, it begins to move. Similarly, when you apply your hand to a moving ball, it is stopped. Sometimes, this push or pull tends to produce or stop motion but fails to do so *e.g.*, a man cannot move or stop a truck.

The above notion of force is all right in everyday conversation but not for the applications of Newton's laws of motion to problems in physics. We have coined a clear definition of force. We know that a body at rest or moving with a constant velocity (*i.e.*, with constant speed in a straight line) has zero acceleration. If the velocity of the body is changing (in magnitude or direction or both), it is accelerating. It has been found experimentally that if a body has zero acceleration, no *net* force is acting on it. On the other hand, force is required to produce acceleration in a body.

Thus a **force** *is an agent that produces or tends to produce acceleration (or retardation) in a body.*

For example, a body moving with a variable speed in a straight line is accelerating because the magnitude of the velocity is changing. Therefore, force is required to cause this acceleration. Similarly, a body moving with a constant speed in a circle is accelerating because the direction of the velocity is changing continuously. Therefore, force is required to produce this acceleration *i.e.*, to produce continuous change in the direction of the velocity.

(*i*) Since force has magnitude as well as direction, it is a *vector quantity*.

* Idea obtained by your feelings rather than by considering the facts.

(*ii*) If several forces act simultaneously on the same object, it is the net force that determines the motion of the object. *The* **net force** *is the vector sum of all the forces acting on the object.* We often call the net force as the *resultant force* or the *unbalanced force.*

10.2. INERTIA

The tendency of a body to maintain its state of rest or of uniform motion in a straight line is called the **inertia** *of the body.*

In other words, objects do not change their state of rest or motion unless acted upon by some net external force. Thus if a body is at rest, it tends to remain at rest. We say that inertia of the body tends to keep it at rest. In order to move the body at rest, external force has to be applied to overcome inertia of the body. Similarly, a moving body tends to move with uniform motion in a straight line. In order to change this state of motion of the body, external force is needed. The inertia of a body depends upon its mass. *The greater the mass of a body, the greater is its inertia.* Both mass and inertia are measured in the same units. Since inertia means resistance to a change in motion, mass is a measure of inertia. A 2 kg mass has twice as much inertia as 1 kg mass has. It means that resistance to any change in velocity will be twice in 2 kg mass as compared to 1 kg mass.

Note. In case of linear motion, inertia of the body is called **linear inertia**. However, in case of rotational motion, inertia of the body is called **rotational inertia** or **moment of inertia**. Note that inertia means linear inertia.

10.3. NEWTON'S FIRST LAW OF MOTION

Newton's first law of motion may be stated as under :

Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by some external force to change that state. In other words, if a body is at rest or moving with a constant velocity (*i.e.,* with constant speed in a straight line), no **net force* is acting on it.

Newton's first law of motion has two parts. The first part states that a body at rest will remain at rest unless some external force acts on it to change its state of rest. The second part states that a moving body continues to move with constant velocity unless some external force acts on the body to change its uniform motion (*i.e.,* motion with constant velocity). The first part of the law is consistent with everyday experience. It appears that second part of the law contradicts everyday experience. For example, a ball rolling on a road stops after covering some distance. The ball has stopped because external frictional force is acting on it. Had the frictional force been absent, the ball would have moved with constant velocity *i.e.,* with a constant speed in a straight line.

If a body is at rest or is moving with a constant velocity (*i.e.,* with a constant speed in a straight line), the *net force* acting on the body is zero. Now state of rest or motion with constant velocity means zero acceleration. Hence Newton's first law of motion can also be expressed as under :

If no net force acts on a body, then body's acceleration is zero (i.e., either the body is at rest or is moving with a constant velocity).

Law of **inertia.** *Newton's first law of motion is also called law of inertia.* The reason is simple. Now inertia of a body means the resistance of the body to change in its state of rest or of uniform motion in a straight line. According to Newton's first law of motion, a body at rest will remain at rest and if it is moving, it will continue to move with constant velocity unless some external force changes its state of rest or of uniform motion in a straight line. In other words, a body of its own cannot change its state of rest or of uniform motion. The statement of Newton's

* Net force means the vector sum of all the forces acting on the body.

first law of motion and definition of inertia are equivalent. Hence Newton's first law of motion is rightly called law of inertia.

10.4. TYPES OF INERTIA

Inertia of a body is of three types *viz.*, (*i*) Inertia of rest (*ii*) Inertia of motion (*iii*) Inertia of direction.

(*i*) **Inertia of rest.** The resistance of a body to change its state of rest is called inertia of rest. For example, when a train suddenly starts, the passenger sitting inside tends to fall backward. It is because the lower part of his body starts moving with the train while the upper part tends to be at rest due to inertia of rest.

(*ii*) **Inertia of motion.** The resistance of a body to change its state of motion is called inertia of motion. For example, when a bus suddenly stops, a passenger sitting in it tends to fall forward. It is because the lower part of his body comes to rest while the upper part of the body tends to remain in motion due to inertia of motion.

(*iii*) **Inertia of direction.** The resistance of a body to change its direction of motion is called inertia of direction. In other words, a body will continue to move in the same straight line unless some external force tends to change its direction. For example, the mud from the wheels of a moving vehicle flies of tangentially. As the mud leaves the wheel, there is no external force acting on it. Therefore, it follows a straight line path tangent to wheel at the point of leaving.

10.5. EXAMPLES OF LAW OF INERTIA

In everyday life, there are a large number of examples of Newton's first law of motion (or law of inertia) but only a few are given below by way of illustration :

(*i*) **When a train suddenly starts, the passengers standing in the compartment tend to fall backward.** It is because the lower part of the body of the passenger, which is in contact with the train, comes in motion but the upper part tends to be at rest due to inertia. Consequently, the passengers tend to fall backward.

(*ii*) **When a moving bus is suddenly stopped, the passengers tend to fall forward.** It is because the lower part of the body of the passenger which is in contact with the bus comes to rest but the upper part of the body tends to be in motion due to inertia. As a result, the passengers tend to fall forward.

(*iii*) **When we beat a carpet with a stick, dust particles are removed.** It is because the carpet is suddenly set into motion but the dust particles tend to remain at rest due to inertia. Therefore, dust particles get removed from the carpet.

(*iv*) **When a man jumps to get into a train, he falls backward.** It is because as soon as his feet come in contact with the train, they acquire the speed of train but the upper part of his body tends to be at rest due to inertia. Consequently, he falls backward.

(*v*) **A ball thrown upward in a train moving with constant velocity returns to the thrower.** It is because as soon as the ball is thrown, it acquires the horizontal velocity of the train which remains the same throughout the entire journey of the ball. Therefore, relative velocity between train and ball is zero during the motion of the ball. As a result, the ball comes back to the hands of the thrower.

(*vi*) **When we shake a mango tree, the mangoes fall down.** It is due to the fact that shaking brings the branches of the tree in motion while the mangoes tend to be at rest due to inertia. Consequently, the mangoes get detached.

(*vii*) **An athlete runs some distance before executing a jump.** It is because by running some distance, velocity acquired due to inertia is added to the velocity of the athlete at the time of the jump. Consequently, the athlete jumps through a longer distance.

(*viii*) **If a piece of paper under a book is suddenly withdrawn, it does not disturb the book.** It is because as the paper is suddenly withdrawn, the book tends to remain at rest due to inertia.

(*ix*) **The mud from the wheels of a moving vehicle flies of tangentially.** As the mud leaves the wheel, there is no net external force acting on it. Therefore, it follows a straight line path tangent to the wheel at the point of leaving. As a result, mud from the wheels of a moving vehicle flies off tangentially.

10.6. LINEAR MOMENTUM

The concept of linear momentum is extremely important in physics. Whenever we examine a moving object, we must consider both its mass and velocity because the quantity of motion possessed by the body depends upon these factors.

The **linear momentum** *of a body with mass m travelling with velocity* $\vec{v}$ *is defined as the product of the mass and velocity i.e.,*

Linear momentum, $\vec{p} = m\vec{v}$

Linear momentum is associated with object's linear motion. Since mass is a scalar quantity and velocity is a vector quantity, their product — linear momentum — is a vector quantity. The direction of linear momentum is in the direction of the velocity $\vec{v}$ and its magnitude is $p = mv$. The unit of linear momentum is that of mass × velocity which in SI units is kg m s^{-1} and its dimensional formula is $[MLT^{-1}]$.

Fig. 10.1 shows the graphical explanation of linear momentum of a body. If mass m of the body is constant, then $p \propto v$ and graph between p and v will be as shown in Fig. 10.1(*i*). If velocity v of the body is constant, then $p \propto m$ so that p-m graph will be as shown in Fig. 10.1(*ii*). Fig. 10.1(*iii*) shows v-m graph of two bodies having the same linear momentum but different masses. Obviously, they will also have different velocities.

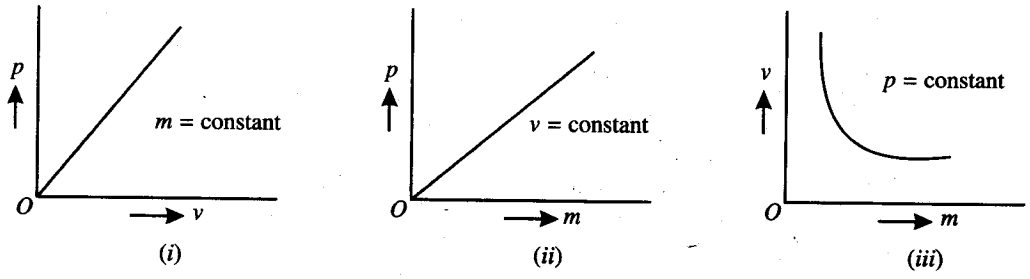


Fig. 10.1

Note. The momenum associated with rotation is called **angular momentum**. The term momentum alone always means linear momentum $\vec{p}$. When we want to speak of angular momentum, we will always specify "angular" so that there is no confusion.

10.7. NEWTON'S SECOND LAW OF MOTION

According to Newton's second law of motion, *the rate of change of linear momentum of a body with time is directly proportional to the net external force applied on it and this change takes place in the direction of the applied force i.e.,*

Applied force ∝ Time rate of change of linear momentum

Consider a body of mass m moving with a velocity $\vec{v}$ in a straight line. The linear momentum of the body is $\vec{p} = m\vec{v}$. Let $\vec{F}$ be the external force applied on the body in the direction of motion of the body. If $\vec{dp}$ is the small change in the linear momentum of the body in small time dt due to this force, then according to Newton's second law of motion,

$$\vec{F} \propto \frac{\vec{dp}}{dt} \quad \text{or} \quad F = k\frac{\vec{dp}}{dt}$$

where k is a constant of proportionality and its value depends on the choice of unit for the force. In SI units, the unit of force is so selected that $k = 1$.

$$\therefore \quad \vec{F} = \frac{\vec{dp}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{\vec{dv}}{dt}$$

But $\frac{\vec{dv}}{dt} = \vec{a}$, the acceleration produced in the body.

$$\therefore \quad \vec{F} = m\vec{a} \qquad ...(i)$$

The magnitude of the force is $F = ma$.

SI Unit of Force.

$$F = m\,a$$

The SI unit of m is 1 kg and that of a is 1 ms^{-2}.

$\therefore$ SI unit of force F = 1 kg × 1 ms^{-2} = 1 kg ms^{-2} or 1 N

Hence **one newton (1 N)** *is that force which produces an acceleration of 1m s^{-2} in a mass of 1 kg.*

Thus if a force of 40 N is applied to a body of mass 10 kg, then acceleration produced in the body is $a = F/m$ = 40/10 = 4 ms^{-2}. On the other hand, in order to produce an acceleration of 2 ms^{-2} in a mass of 5 kg, the force required is $F = ma$ = 5 × 2= 10 N. The dimensional formula of force is $[F] = [M]\,[L\,T^{-2}] = [MLT^{-2}]$.

Discussion. The following conclusions can be drawn from Newton's second law of motion:

(*i*) *Newton's second law gives us the relation between force, mass and acceleration viz.*

$$\vec{F} = m\vec{a}$$

Since m is constant, $\vec{F} \propto \vec{a}$.

Therefore, Newton's second law of motion provides us the definition of force.

A **force** *is that which produces or tends to produce acceleration (or retardation) in a body.*

(*ii*) If no net force is acting on a body, then according to Newton's second law of motion the rate of change of linear momentum with time is zero *i.e.*,

$$\frac{\vec{dp}}{dt} = \vec{F} = 0$$

or $$\vec{p} = m\vec{v} = \text{constant}$$

Thus the object's linear momentum remains constant when the net force acting on it is zero.

Since m is constant and $\vec{p}$ is also constant, it means $\vec{v}$ is constant. What does constant $\vec{v}$ mean? It means either the body is at rest or it is moving with constant velocity. Thus when net force on a body is zero, the body is either at rest or it is moving with constant velocity. This is Newton's first law of motion. Therefore, Newton's first law is contained in the second law.

(*iii*) $$F = m\,a \quad \text{or} \quad a = F/m$$

Clearly, second law tells us that mass of a body is a measure of its inertia. For given net force, an object with a larger mass will have smaller acceleration *i.e.*, smaller change in the velocity of the object. Therefore, the mass of an object is that property of the object which causes it to resist any change in its velocity. Since inertia means resistance to change, the mass of a body is a measure of its inertia.

(*iv*) $$F = \frac{dp}{dt}$$

It means that slope of linear momentum-time graph gives the magnitude of force. Referring to Fig. 10.2,

$$F = \frac{dp}{dt} = \tan\theta = \text{Slope of } p\text{-}t \text{ graph}$$

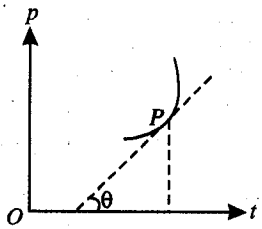


Fig. 10.2

Note. Forces acting simultaneously at the same point on a body are called **concurrent forces** and forces acting in the same plane are called **coplanar forces.** Forces acting parallel to each other are called **collinear forces.**

Example 10.1. A constant force acts for 10 seconds on a stationary body of mass 10 kg after which the force ceases to act. The body moves a distance of 50 m in next 5 seconds. Find the magnitude of force.

Solution. Initial velocity, $v_0 = 0$

When the force ceases to act, the body moves with a constant velocity. Since the body moves a distance of 50 m in 5 seconds, constant velocity $v = 50/5 = 10$ m s^{-1}. Note that this is the velocity of the body at the instant the force was removed.

$$\text{Acceleration, } a = \frac{v - v_0}{t} = \frac{10-0}{10} = 1 \text{ ms}^{-2}$$

$\therefore$ Magnitude of force, $F = m\,a = 10 \times 1 = \mathbf{10\ N}$

Example 10.2. A ship of mass 3×10^7 kg initially at rest is pulled by a force of 5×10^4 N through a distance of 3 m. Assuming the resistance due to water to be negligible, what is the speed of the ship?

Solution. Acceleration of ship, $a = \frac{F}{m} = \frac{5\times10^4}{3\times10^7} = \frac{5}{3}\times10^{-3}$ ms^{-2}

Let v be the speed of the ship. Since ship starts from rest, its initial speed $v_0 = 0$.

$\therefore$ $$v^2 - v_0^2 = 2aS \quad \text{or} \quad v^2 - 0^2 = 2\times\frac{5}{3}\times10^{-3}\times3$$

$\therefore$ $$v^2 = 10^{-2} \quad \text{or} \quad v = \sqrt{10^{-2}} = 0.1 \text{ ms}^{-1}$$

Example 10.3. A 100 kg man jumps into swimming pool from a height of 5 m. It takes 0.4 second for the water to reduce his velocity to zero. What is the average force exerted by water on the man ? Take $g = 10$ ms^{-2}.

Solution. Let v be the velocity of the man just before entering the water.

$\therefore$ $$v^2 - 0^2 = 2gh \quad \text{or} \quad v = \sqrt{2gh} = \sqrt{2\times10\times5} = 10 \text{ ms}^{-1}$$

Now $$F = \frac{mv-0}{t} = \frac{mv}{t} = \frac{100\times10}{0.4} = 2500 \text{ N}$$

Example 10.4. A girl pulls a box of mass 10.4 kg with a force of 25 N inclined to the horizontal at an angle of 37°. As a result, the box accelerates horizontally. What is the magnitude of acceleration?

Solution. Fig. 10.3 shows the conditions of the problem. The horizontal component of force accelerates the box in the horizontal direction.

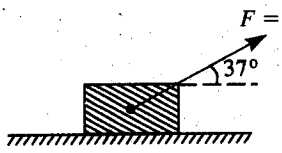


Fig. 10.3

Horizontal component of force, $F_x = F \cos 37°$

$$= 25 \cos 37° = 20 \text{ N}$$

$$\therefore \quad \text{Acceleration, } a_x = \frac{F_x}{m} = \frac{20}{10.4} = 1.92 \text{ ms}^{-2}$$

Example 10.5. **A bullet of mass 0.04 kg moving with a speed of 90 ms^{-1} enters a heavy wooden block and is stopped after a distance of 60 cm. What is the average resistive force exerted by the block?**

Solution. The initial speed of the bullet is $v_0 = 90$ m s^{-1}. As the bullet enters the wooden block, its velocity decreases to zero (*i.e.*, $v = 0$) after covering a distance of 60 cm (*i.e.*, $S = 60$ cm).

Now $$v^2 - v_0^2 = 2aS$$

or $$a = \frac{v^2 - v_0^2}{2S} = \frac{(0)^2 - (90)^2}{2 \times 0.6} = -6750 \text{ ms}^{-2}$$

$\therefore$ Average retarding force, $F = m\,a = 0.04 \times 6750 = 270$ N

Example 10.6. **Sand drops vertically at the rate of 100 gs^{-1} on to a horizontal belt conveyor moving at a steady velocity of 5 cm s^{-1}. What is the force required to keep the belt moving?**

Solution. Fig. 10.4 shows the conditions of the problem. The initial horizontal velocity of the sand is zero ($v_0 = 0$). The final horizontal velocity of sand is 5 cm s^{-1} ($v = 5$ cm s^{-1} = 0.05 m s^{-1}).

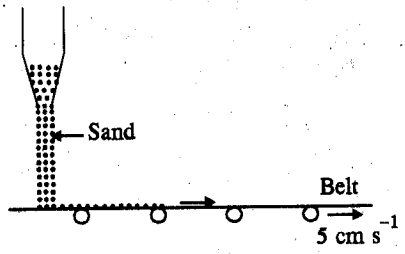


Fig. 10.4

Force required to keep belt moving, F = Rate of increase of horizontal linear momentum

$$\therefore \quad F = \frac{mv - mv_0}{t} = \frac{0.1 \times 0.05 - 0}{1} = 5 \times 10^{-3} \text{ N}$$

Example 10.7. **The mass of an elevator (lift) is 500 kg. Calculate the tension in the cable of the elevator when the elevator is (*i*) stationary (*ii*) ascending with an acceleration of 2 ms^{-2} (*iii*) descending with the same acceleration. Take $g = 9.8$ ms^{-2}.**

Solution. (*i*) When the elevator is stationary [See Fig. 10.5 (*i*)], its acceleration is zero. Therefore, net force on it is zero.

$$\text{Net force} = T - mg$$

or $$0 = T - mg$$

$\therefore$ $$T = m\,g = 500 \times 9.8 = 4900 \text{ N}$$

(*ii*) When the elevator is ascending [See Fig. 10.5 (*ii*)], the net force on it is upward. Therefore, tension T in the cable will be greater than gravity-force mg.

$$\text{Net force} = T - mg$$

or $$ma = T - mg$$

$\therefore$ $$T = mg + ma = m\,(g+a) = 500\,(9.8 + 2) = \mathbf{5900 \text{ N}}$$

Note that when the lift is ascending with an acceleration, its apparent weight is increased. This is indicated by the increased tension in the cable.

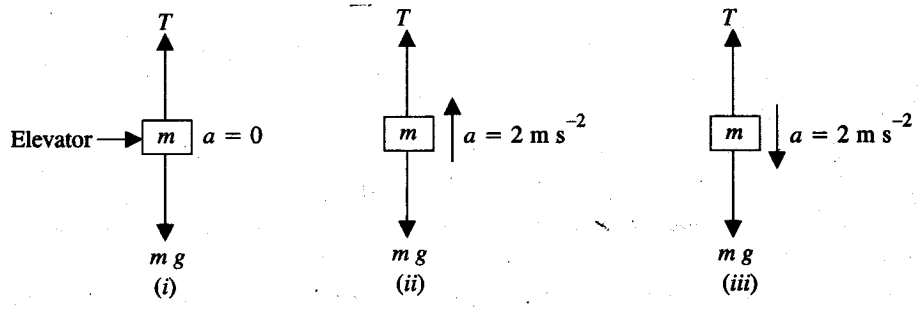


Fig. 10.5

(*iii*) When the elevator is descending [See Fig. 10.5 (*iii*)], the net force is in the downward direction. Therefore, tension T in the cable will be less than mg.

$$\text{Net force} = mg - T$$

or $$ma = mg - T$$

$$\therefore \quad T = mg - ma = m\,(g-a) = 500\,(9.8 - 2) = \mathbf{3900\ N}$$

Note that when the lift is descending with an acceleration, its apparent weight is decreased. This is indicated by the decreased tension in the cable.

Example 10.8. A hydrogen gas filled balloon having a mass of 25 g is released up in the air. The gas starts leaking from the balloon at a uniform rate with a velocity of 5 cm/s and the balloon shrinks completely in 5 s. Find the average force acting on the balloon.

Solution. Here $m = 25\ \text{g} = 0.025\ \text{kg}$; $v = 5\ \text{cm/s} = 0.05\ \text{m/s}$

$$\text{Force, } F = \frac{d}{dt}(mv) = v \times \frac{dm}{dt} \quad \text{...}v \text{ remains constant}$$

$$= 0.05 \times \frac{0.025}{5} = \mathbf{25 \times 10^{-5}\ N}$$

Example 10.9. A car starts from rest and accelerates uniformly with 2 ms^{-2}. At t = 10 s, a stone is dropped out of the window of the car 2 m high. What are the (*i*) velocity and (*ii*) acceleration of the stone at t = 10.2 s? Take g = 10 ms^{-2}.

Solution. (*i*) Initial velocity of car is $v_0 = 0$.

Velocity of car after 10 s is $v_x = v_0 + at = 0 + 2 \times 10 = 20$ m/s

Once the stone is dropped at t = 10 s, its acceleration along the horizontal direction is zero. Therefore, horizontal velocity remains unchanged *i.e.*, v_x = 20 m/s. But at t = 10 s, the stone also comes under the accelerated motion due to gravity.

$\therefore$ At t = 10.2 s : Horizontal velocity of stone = v_x = 20 m/s

Also At t = 10.2 s : Vertical downward velocity of stone is

$$v_y = 0 + gt = 0 + 10 \times 0.2 = 2\ \text{ms}^{-1} \quad (\because\ t = 10.2 - 10 = 0.2\ \text{s})$$

$\therefore$ Resultant velocity of stone at t = 10.2 s is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20)^2 + (2)^2} = \mathbf{20.1\ m/s}$$

(*ii*) Once the stone is dropped, the horizontal acceleration ceases to act on the stone and it is now only under acceleration due to gravity. Therefore, at t = 10.2 s, acceleration of stone = g = **10 m/s^2**.

PROBLEMS FOR PRACTICE

1. Find the force required to give a mass of 70 kg an acceleration of 4.5 m/s^2 (*i*) horizontally (*ii*) vertically. **[(*i*) 315 N (*ii*) 1001 N]**
2. A force acts for 1 second on a body of mass 12 kg. The force is then withdrawn and the body moves through 2 m in the next second. Find the magnitude of force. **[24 N]**
3. A cricket ball of mass 100 g is moving with a speed of 25 ms^{-1} and is brought to rest in 0.025 s. Find the average force applied. **[100 N]**
4. A bullet of mass 6 g travelling at 126 m s^{-1} penetrates into a fixed target and is brought to rest in 0.01 second. Determine (*i*) the distance of penetration in the target (*ii*) average retarding force exerted on the bullet. **[(*i*) 0.6 m (*ii*) 72 N]**
5. Sand drops vertically at the rate of 2 kg s^{-1} onto a conveyor belt moving horizontally with a velocity of 0.1 m s^{-1}. Find the force required to keep the belt moving. **[0.2 N]**
6. A man of mass 50 kg is standing in a lift. Calculate the apparent weight of man when the lift is (*i*) stationary (*ii*) moving upward with an acceleration of 2 m s^{-2} (*iii*) moving downward with an acceleration of 2 m s^{-2}. **[(*i*) 490 N (*ii*) 590 N (*iii*) 390 N]**
7. A constant force acting on a body of mass 3 kg changes its speed from 2 m s^{-1} to 3.5 ms^{-1} in 25 s. The direction of motion of the body remains unchanged. What is the magnitude and direction of the force? **[0.18 N in the direction of motion]**
8. In an X-ray machine, an electron is subjected to a force of 10^{-23} N. What is its acceleration? In how much time the electron will cover a distance of 0.1 m? Take the mass of electron as 10^{-30} kg. **[10^7 ms^{-2} ; 1.4×10^{-4} s]**
9. A body of mass 5 kg is acted upon by two perpendicular forces 8 N and 6 N. Find the magnitude and direction of the resultant force. **[10 N; 36.87° with 8 N force]**

10.8. BASIC FORCES IN NATURE

There are four basic forces that operate in nature. These are :

1. The gravitational force **2.** The electromagnetic force **3.** The strong nuclear force **4.** The weak nuclear force.

1. The gravitational force. All objects exert a force of attraction upon each other. The gravitational force of attraction between two objects is governed by Newton's law of gravitation. If two objects of masses m_1 and m_2 are held at a distance r, then according to Newton's law of gravitation, the force of attraction between the two objects is given by ;

$$F = \frac{G \times m_1 m_2}{r^2} \qquad ...(i)$$

where G (= 6.67×10^{-11} Nm^2 kg^{-2}) is called *universal gravitational constant.* The mass m_1 exerts an attractive force on mass m_2 given by eq. (*i*). Similarly, the mass m_2 exerts an attractive force on m_1 given by eq. (*i*).

Properties. The gravitational force has the following properties :

(*i*) It is always attractive*.

* Unlike electrostatic force which can be attractive or repulsive. It is because mass is of one kind whereas charge is of two kinds (positive and negative).

(*ii*) It is a central force *i.e.*, gravitational force between two objects acts along the line joining the centres of the objects.

(*iii*) It obeys inverse square law *i.e.*, gravitational force between two objects is inversely proportional to the square of distance between them.

(*iv*) Of all the four forces operating in nature, the gravitational force is the weakest.

(*v*) It is a long range force *i.e.*, it operates over very long distances.

(*vi*) It is a conservative force *i.e.*, work done by the gravitational force is recoverable.

Examples. A few examples of gravitational force in everyday life are :

(*i*) A stone dropped from a height falls because earth exerts gravitational force on the stone.

(*ii*) The rotation of earth around the sun is due to gravitational pull of sun on earth.

(*iii*) The moon revolves around the earth due to the gravitational force on it by the earth.

2. The electromagnetic force. The electromagnetic force includes both electric and magnetic forces and is relatively strong.

(*a*) The force between two static (at rest) electric charges is given by Coulomb's law in electrostatics. If two charges q_1 and q_2 are at a distance r apart, then according to Coulomb's law, the electrostatic force between them is given by ;

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \quad \text{.... in vacuum}$$

where $\varepsilon_0 \left(= 8.854\times10^{-12}\ \mathrm{C^2N^{-1}m^{-2}}\right)$ is called *absolute permittivity of free space.*

(*b*) A static charge does not produce magnetic field. However, a moving charge produces a magnetic field. When two charges are in motion, each charge experiences a force called electromagnetic force.

Properties. The electromagnetic force has the following properties :

(*i*) It may be attractive or repulsive. It is because electric charges are of two kinds (positive and negative).

(*ii*) It is a central force.

(*iii*) It obeys inverse square law.

(*iv*) It is also a long range force.

(*v*) It is also a conservative force.

(*vi*) It is about 10^{36} times stronger than the gravitational force.

Examples. A few examples of electromagnetic force in everyday life are :

(*i*) If two current carrying parallel conductors are placed near each other, then force (repulsive or attractive) acts on them. This is the electromagnetic force. If the conductors carry currents in the same direction, then they attract each other *i.e.*, electromagnetic force is attractive. If the conductors carry currents in the opposite direction, then they repel each other.

(*ii*) In atoms, the electrons are bound by the electromagnetic forces acting on them due to protons.

(*iii*) Atoms combine to form molecules due to electromagnetic forces.

3. The strong nuclear force. The strong nuclear force is the attractive force that holds the constituents of the atomic nucleus *viz.*, protons and neutrons. The stability of the nucleus suggests that attractive nuclear forces must be very strong. For example, consider the helium nucleus ($_2He^4$) consisting of two protons and two neutrons and contained in a volume with a radius of about 10^{-15} m. The protons are so close together that repulsive electric force is very large. The gravita-

tional force between them is negligible. Yet $_2He^4$ is very stable. It is because the strong attractive nuclear force acting between all the constituents of the nucleus overpowers the repulsive electric force between protons.

Properties. The strong nuclear force has the following properties :

(*i*) It is basically an attractive force.

(*ii*) It is a short-range force and operates only upto distances of the order of 10^{-14}m.

(*iii*) It does not depend upon the charge on the nucleons.

(*iv*) It is a non-central force.

(*v*) It does not obey inverse square law.

(*vi*) It is a non-conservative force.

Examples. A few examples of the strong nuclear force in everyday life are :

(*i*) It is the strong nuclear force that binds the neutrons and protons together in a nucleus.

(*ii*) The nuclear energy available from the processes of nuclear fission and fusion uses the concept of strong nuclear force.

4. The weak nuclear force. The weak nuclear force appears only in certain nuclear processes such as β-decay of a nucleus. In β-decay, electrons are emitted from a radioactive nucleus. The weak nuclear force is much stronger than the gravitational force.

10.9. IMPULSIVE FORCE AND IMPULSE

1. Impulsive forces. *The forces which act on bodies for a short time are called impulsive forces.* The examples of impulsive forces are :

(*i*) A bat hitting the ball.

(*ii*) The collision of two billiard balls.

(*iii*) The firing of a gun.

The impulsive force is not constant but varies with time. For example, when a bat hits the ball, the impulsive force jumps from zero at the moment of contact to a very large value within a very short time and then abruptly returns to zero again as shown in Fig. 10.6. Note that during the short time interval Δt ($= t_2 - t_1$), the impulsive force is varying continuously. Therefore, it is not easy to measure the impulsive force. In such cases, we measure the total effect of the impulsive force, called *impulse.*

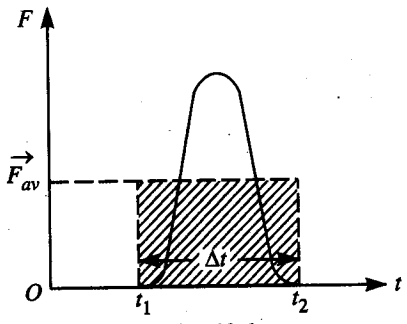


Fig. 10.6

2. Impulse. If $\vec{F}_{av}$ is the average force [See Fig. 10.6] exerted by the bat on the ball during the time interval Δt and the change in linear momentum during this time interval is $\overrightarrow{\Delta p}$, then according to Newton's second law of motion,

$$\vec{F}_{av} = \frac{\overrightarrow{\Delta p}}{\Delta t}$$

or $$\vec{F}_{av}\,\Delta t = \overrightarrow{\Delta p} = \text{Change in linear momentum}$$

The product $\vec{F}_{av}\,\Delta t$ is called *impulse* of the force.

Hence impulse *of a force is the product of average force during the impact and the time for which the impact lasts i.e.,*

$$\text{Impulse} = F_{av} \times \Delta t \quad \text{...in magnitude}$$

Mathematical analysis. If $\vec{F}$ is the force at any time t during the collision, then according to Newton's second law of motion,

$$\vec{F} = \frac{\overrightarrow{dp}}{dt}$$

or $$\overrightarrow{dp} = \vec{F}\,dt$$

If we integrate it over the time interval $\Delta t (= t_2 - t_1)$, we get the total change in linear momentum $\overrightarrow{\Delta p}$ during that time interval.

$$\therefore \quad \overrightarrow{\Delta p} = \int_{t_1}^{t_2} \vec{F}\,dt \qquad ...(i)$$

If $\vec{F}_{av}$ is the average force during the collision, then,

$$\overrightarrow{\Delta p} = \vec{F}_{av}\,\Delta t \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $\int_{t_1}^{t_2} \vec{F}\,dt = \vec{F}_{av}\,\Delta t$ = Change in linear momentum.

The quantity $\int_{t_1}^{t_2} \vec{F}\,dt$ is known as impulse of the force $\vec{F}$ during the time interval t_1 to t_2 and is equal to the change in the linear momentum of the body on which it acts (= area under the $F - t$ graph during the time interval t_1 to t_2).

Units of Impulse. The impulse has the same SI units as that of linear momentum *i.e.,* kg m/s or Ns. Therefore, the dimensional formula of impulse is also the same as that of linear momentum *i.e.,* $[MLT^{-1}]$. *Impulse is a vector quantity and its direction is the same as the direction of change in linear momentum.*

10.10. IMPULSE-LINEAR MOMENTUM THEOREM

Impulse = $F \times t$ = Change in linear momentum

Thus a given change in linear momentum can be produced by applying a larger force for a smaller time or by applying a smaller force for a larger time. This is called *impulse-linear momentum theorem.*

10.11. EXAMPLES OF IMPULSE–LINEAR MOMENTUM RELATIONSHIP

Average impulsive force, $\vec{F}_{av} = \dfrac{\text{Change in linear momentum}}{\text{Time}}$

Therefore, whenever you wish the force of impact to be small, extend the time of impact. On the other hand, if the time of impact is small, the impact of force will be large. There are a large number of practical examples where impulse-linear momentum relationship plays an important role. A few cases are being discussed below by way of illustration :

(*i*) **While catching a ball, a cricket player extends his hands forward** so that he has plenty of room to let his hands move backward after making contact with the ball. This extends the time of impact and thus reduces the force of impact.

(*ii*) **A person is better off falling on a wooden floor than a concrete floor.** The wooden floor allows for a longer time of impact and, therefore, a lesser force of impact than a concrete floor.

(*iii*) **A person jumping from an elevated position on a floor below bends his knees upon making contact.** This extends the time of impact. Therefore, the force of impact is reduced.

(*iv*) China wares are wrapped in a paper or straw before packing to avoid breakage. This increases the time of impact between various articles during jerks, thereby decreasing the force of impact on the articles.

(*v*) While catching a cricket ball, if you do not move your hands away upon contact, you may be hurt. It is because the time of impact will be small so that the impact force will be large.

(*vi*) When the car goes out of control while driving, you would prefer to hit something soft (say haystack) than something hard (say concrete wall). It is because by hitting haystack you extend your time of impact, thereby reducing the impact force.

Example 10.10. **A 3 kg hammer is used to drive a nail into a block of hard wood. If at the time of impact, hammer's speed is 5 ms^{-1} and it drives the nail 1 cm into the block, find (*i*) the acceleration (*ii*) the force exerted (*iii*) the time of impact (*iv*) impulse.**

Solution. The initial velocity of nail is $v_0 = 5$ ms^{-1} and its final velocity is $v = 0$. The distance covered by the nail is $S = 1$ cm $= 10^{-2}$ m.

(*i*) $$v^2 - v_0^2 = 2\,a\,S$$

$$\therefore \quad a = \frac{v^2 - v_0^2}{2S} = \frac{(0)^2 - (5)^2}{2\times 10^{-2}} = -1.25 \times 10^3 \text{ ms}^{-2}$$

(*ii*) Force exerted, $F = m\,a = 3 \times 1.25 \times 10^3 = 3.75 \times 10^3$ N

(*iii*) Time of impact, $t = \frac{v - v_0}{a} = \frac{0 - 5}{-1.25\times 10^3} = 4\times 10^{-3}$ s

(*iv*) Impulse $= F\,t = (3.75 \times 10^3) \times 4 \times 10^{-3} = 15$ Ns

Alternatively : Impulse $= m\,(v - v_0) = 3\,(0 - 5) = 15$ Ns .... in magnitude

Example 10.11. **A body of mass 0.25 kg moving with a velocity of 12 ms^{-1} is stopped by applying a force of 0.6 N. Calculate the time taken to stop the body. Also calculate the impulse of this force.**

Solution. Mass of body, $m = 0.25$ kg ; Retarding force, $F = -0.6$ N

$$\therefore \quad \text{Retardation, } a = \frac{F}{m} = \frac{-0.6}{0.25} = -2.4 \text{ ms}^{-2}$$

Now $$v = v_0 + at \quad \text{or} \quad 0 = 12 + (-2.4) \times t \quad \therefore \quad t = 5 \text{ s}$$

$$\text{Impulse} = F \times t = -0.6 \times 5 = -3.0 \text{ Ns}$$

Example 10.12. **A ball of mass 10 g hits a hard surface vertically with a speed of 5 m/s and rebounds with the same speed. The ball remains in contact with the surface for 0.01 second. Find the average force exerted by the surface on the ball.**

Solution. $F \times t =$ Change in linear momentum

Now change in linear momentum $= mv - (-mv) = 2\,mv$

$$\therefore \quad F \times t = 2\,mv \quad \text{or} \quad F \times 0.01 = 2\times\frac{10}{1000}\times 5 \quad \therefore \quad F = 10 \text{ N}$$

Example 10.13. **A 1200 kg car initially moving at 20 ms^{-1} strikes a tree and comes to rest in a distance of 1.5 m. Find the average stopping force the tree exerts on the car.**

Solution. $m = 1200$ kg ; $v_0 = 20$ ms^{-1} ; $v = 0$; $S = 1.5$ m

Change in car's linear momentum, $\Delta p = m\,(v - v_0) = 1200 \times (0 - 20) = -24000$ kg ms^{-1}

Average speed during impact, $\bar{v} = \frac{v_0 + v}{2} = \frac{20+0}{2} = 10 \text{ ms}^{-1}$

Time of impact, $t = \frac{S}{\bar{v}} = \frac{1.5}{10} = 0.15 \text{s}$

If $\vec{F}_{av}$ is the average force, then,

$F_{av} \times t$ = Change in linear momentum

$\therefore \quad F_{av} = \frac{\text{Change in linear momentum}}{t} = \frac{24000}{0.15} = 1.6 \times 10^5 \text{ N}$

Example 10.14. A ball moving with a momentum of 5 kg m s^{-1} strikes against a wall at an angle of 45° and is reflected at the same angle. Find the magnitude of change in momentum.

Solution. Fig. 10.7 shows the conditions of the problem. Suppose $\vec{P}_1$ is the initial linear momentum and $\vec{P}_2$ is the final linear momentum.

$\therefore$ Change in linear momentum = $\vec{p}_2 - \vec{p}_1 = \vec{p}_2 + (-\vec{p}_1)$

Therefore, in order to find the change in linear momentum, we should add $\vec{P}_2$ to $-\vec{P}_1$ as shown in Fig. 10.7. The change in linear momentum is $\overrightarrow{AB}$.

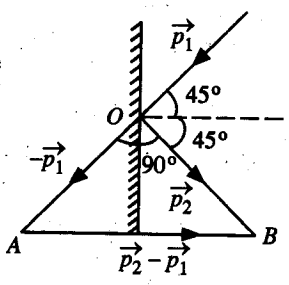


Fig. 10.7

$\therefore \quad AB = \sqrt{p_1^2 + p_2^2} = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07 \text{kg ms}^{-1}$

Note that linear momentum is a vector quantity i.e. it has magnitude as well as direction.

PROBLEMS FOR PRACTICE

1. A cricket ball of mass 100 g is moving with a speed of 25 ms^{-1} and is brought to rest in 0.025 s. Find the average force applied and the impulse. **[100 N ; 2.5 Ns]**
2. A force of 10 N acts on a body for 3 microsecond. Calculate the impulse. If the mass of the body is 5 g, what is the change in velocity? **[3 × 10^{-5} Ns ; 6 × 10^{-3} m s^{-1}]**
3. Determine the average force needed to change the velocity of 22,000 kg bus from rest to 13.6 m s^{-1} in 10.5 s. **[28500 N]**
4. A batsman deflects a ball by an angle of 45° without changing its initial speed which is equal to 54 km/h. What is the impulse imparted to the ball ? Mass of the ball is 0.15 kg. **[4.16 Ns]**
5. A 12.5 g bullet moving at 235 ms^{-1} passes through a sheet of plastic 3.4 cm thick and emerges with a speed of 125 ms^{-1}. The time taken by the bullet to pass through the sheet is 1.9 × 10^{-4} s. Find the average stopping force exerted on the bullet. **[7240 N]**

10.12. NEWTON'S THIRD LAW OF MOTION

According to Newton's third law of motion: *To every action, there is an equal and opposite reaction.* It simply means that if one body exerts a force (*action*) on the second body, then second body exerts an equal and opposite force (*reaction*) on the first body. We can restate Newton's third law of motion in more quantitative terms as under :

If body A exerts a force $\vec{F}_{AB}$ on body B (action) and body B exerts a force $\vec{F}_{BA}$ on body A (reaction), then according to Newton's third law of motion,

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Note that $\vec{F}_{AB}$ (*action*) and $-\vec{F}_{BA}$ (*reaction*) are equal in magnitude but opposite in direction. Two things are worth noting about Newton's third law of motion. First, *action and reaction act on different bodies and, therefore, they cannot cancel each other.* Thus in the above case, action ($\vec{F}_{AB}$) is on body B and reaction ($-\vec{F}_{BA}$) is on body A. Secondly, *action and reaction involve a pair of bodies.* It does not matter which force in such a pair is called the action and which the reaction.

Proof. We now prove Newton's third law of motion. If $\vec{F}_{AB}$ is the force exerted by body A on body B (action) and $\vec{F}_{BA}$ is the force exerted by body B on body A (reaction), then according to Newton's third law of motion,

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Consider an isolated system (*i.e.*, no net external force acts on the system) consisting of bodies A and B moving along the same straight line. Suppose the two bodies collide. As a result of collision, their velocities will change and hence their linear momenta will also change. Suppose the two bodies interact for time t.

Change in linear momentum of body A is

$$\overrightarrow{\Delta p_A} = \vec{F}_{BA} \times t$$

Change in linear momentum of body B is

$$\overrightarrow{\Delta p_B} = \vec{F}_{AB} \times t$$

Total change in linear momentum of bodies A and B is

$$\overrightarrow{\Delta p} = \overrightarrow{\Delta p_A} + \overrightarrow{\Delta p_B} = \vec{F}_{BA} \times t + \vec{F}_{AB} \times t$$

Since no net external force is acting on the system, according to Newton's second law of motion, the change in linear momentum is zero.

$$\therefore \quad \overrightarrow{\Delta p} = 0$$

or $$\vec{F}_{AB} \times t + \vec{F}_{BA} \times t = 0$$

or $$\vec{F}_{AB} = -\vec{F}_{BA}$$

Thus action and reaction are equal and opposite. This proves Newton's third law of motion.

10.13. EXAMPLES OF NEWTON'S THIRD LAW OF MOTION

In everyday life, there are a large number of practical examples of the applications of Newtons's third law of motion. However, we shall discuss only a few by way of illustration.

(*i*) Block resting on a table. Fig. 10.8 shows a block resting on a table. The force acting downward on the block is its weight W (= mg) due to the attraction of the earth *i.e.*, earth is exerting a force W (= mg) on the block. An equal and opposite force W' (= mg) is exerted by the block *on the earth.* Here W and W' are the action and reaction forces. Note that action and reaction act on different bodies. Therefore, they cannot cancel each other. Let us now turn to another action-reaction pair. The table is in contact with the block and exerts an upward force N on it. The block also exerts a

force N' (= N) downward on the table. The forces N and N' are also action-reaction pair. Note that N and N' act on different bodies and therefore cannot cancel each other.

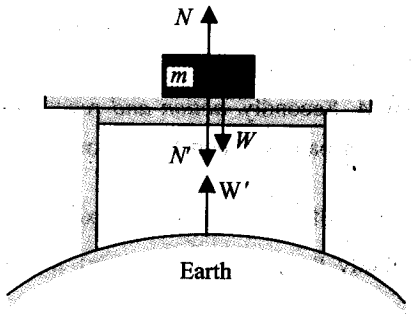


Fig. 10.8

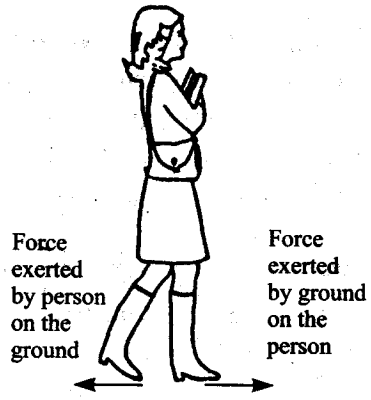


Fig. 10.9

(*ii*) **Walking.** When a person walks on the ground, he or she exerts a force on the ground and in turn ground exerts an equal force on the person. It is this force on the person which moves him or her forward. Thus is illustrated in Fig. 10.9. Similarly, when a man jumps, he exerts downward force on the ground. The ground exerts an equal upward force on the man.

(*iii*) **When a person throws a package out of the boat, the boat moves (atleast a little) in the opposite direction.** The person exerts a force on the package; the package exerts an equal and opposite force back on the person.

(*iv*) **The working of rockets is based on Newton's third law.** The rocket exerts a strong force on the gases, expelling them. The gases exert an equal and opposite force *on the rocket* and it is this force which moves the rocket forward.

(*v*) **A falling object exerts upward force on the earth as much as the earth exerts downward force on it.** This fact may seem puzzling. The downward force on the object is less puzzling because the acceleration of 9.8 m s^{-2} is quite noticeable. The same amount of force acting upward on the huge mass of the earth, however, produces an acceleration ($F = m\ a$) so small that it cannot be measured. But it is there.

(*vi*) **When a small car collides head-on with a huge truck, which is hit with a greater force?** Most people will conclude that the car is certainly hit with a greater force. Yet Newton's third law states that force exerted by the car on the truck is exactly equal in magnitude (and in the opposite direction) to the force exerted by the truck on the car. This fact seems puzzling because the car is damaged more. Note that $F = m\ a$; for the same force, larger mass will undergo less acceleration than the smaller mass. Thus the truck's velocity undergoes a relatively small change during collision, slowing down but continuing in the same direction. The much less massive car, *struck with the same force,* has a much larger change in its velocity. Thus the car is not only stopped but is compelled to move in the reverse direction. The larger acceleration puts a much greater stress on the structural frame of the car. This causes more damage to the car than the truck.

(*vii*) **If a car is accelerating forward, it is because its tyres are pushing backward on the road and the road is pushing forward on the tyres.** But if the car is moving forward and slowing down, the tyres push forward and the road pushes backward.

(*viii*) **A bird flies forward by exerting a force on the air** but it is the air pushing back on the bird's wings that moves bird forward.

(*ix*) **When a gun is fired, the force exerted on the bullet is equal to the reaction force exerted on the gun.** Since the forces are equal in magnitude, why does not gun recoil with the same speed as that of bullet. Recall that $a = F/m$. Since the gun is much more massive than the bullet, the change in its acceleration is very small as compared to that of the bullet.

(*x*) **Using Newton's third law, we can explain how a helicopter gets its lifting force.** The whirling blades of helicopter are so shaped that they exert a downward force on air. The air exerts an equal upward force on the helicopter. This upward reaction force is called *lift*. When lift equals the weight of the helicopter, it hovers in mid-air. When the lift is greater, the helicopter climbs upward.

Third law is contained in First law. Newton's third law is contained in first law of motion. As the system in Fig. 10.8 is at rest, according to Newton's first law of motion, the net force on the system must be zero. The resultant of two forces of action will be zero only when they are equal and opposite. This is third law of motion. Therefore, Newton's third law is contained in first law of motion.

10.14. NEWTON'S SECOND LAW IS THE REAL LAW OF MOTION

Newton's second law of motion is the real law of motion because Newton's first law and third law are contained in it.

(*i*) **Newton's first law is contained in second law**

According to Newton's second law of motion, the net external force F applied on a body of mass m produces an acceleration a given by ;

$$a = \frac{F}{m}$$

If the net external force is zero (*i.e.* $F = 0$), then, $a = 0/m = 0$. The zero acceleration means either the body is at rest or it is moving with constant velocity. Thus when the net external force on a body is zero, the body is either at rest or it is moving with a constant velocity. This is Newton's first law of motion. Thus Newton's first law of motion is contained in Newton's second law of motion.

(*ii*) **Newton's third law is contained in the second law**

If $\vec{F}_{AB}$ is the force exerted by body A on body B (action) and $\vec{F}_{BA}$ is the force exerted by body B on body A (reaction), then according to Newton's third law of motion,

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

We shall now prove it by using Newton's second law of motion. Consider an isolated system (*i.e,* no net external force acts on the system) consisting of bodies A and B moving along the same straight line. Suppose the two bodies collide. As a result of collision, their velocities will change and hence their linear momenta will also change. Suppose the two bodies interact for time t.

Change in linear momentum of body A is

$$\overrightarrow{\Delta p_A} = \vec{F}_{BA} \times t$$

Change in linear momentum of body B is

$$\Delta \vec{p}_B = \vec{F}_{BA} \times t$$

Total change in linear momentum of bodies A and B is

$$\Delta \vec{p} = \Delta \vec{p}_A + \Delta \vec{p}_B = \vec{F}_{BA} \times t + \vec{F}_{AB} \times t$$

Therefore, apparent weight (R) is greater than the actual weight W(= mg) of the man. The man can feel it if he walks on the floor of lift/elevator; he will need more effort to walk naturally.

(*iv*) When the lift is moving downward with acceleration a. It means that net force is acting downward [See Fig. 10.10(*iii*)]. Therefore $mg - R$ provides the downward acceleration to the man.

$$\text{Net force} = mg - R$$

or $$ma = mg - R$$

or $$R = mg - ma = m(g - a) \quad ...(i)$$

$\therefore$ $$R < W(= mg)$$

Therefore, apparent weight (R) of the man is less than the actual weight W(= mg) of the man. The man will feel lighter as he walks about on the floor of the lift/elevator.

(*v*) **For free fall.** If the lift/elevator cable breaks, then the lift/elevator will fall freely and $a = g$. Therefore, from eq. (*i*), $R = m(g - a) = m(g - g) = 0$. Hence, the apparent weight of the man will become zero.

(*vi*) When lift moves downward with $a > g$. In that case, from eq. (*i*), $R = m(g - a)$ becomes negative. *Therefore, the apparent weight of the man becomes negative.* In that event, the man will rise from the floor of the lift/elevator and stick to the ceiling of the lift/elevator.

2. Motion of cart pulled by a horse. A good example of application of Newton's laws of motion is the motion of cart pulled by a horse as shown in Fig. 10.11. The horse pulls the cart by pushing the ground backward with its foot with a force F making an angle θ with the horizontal. The ground in turn exerts a force R on the horse equal and opposite to F. The reaction R can be resolved into two rectangular components *viz.* $R \cos \theta$ and $R \sin \theta$ as shown in Fig. 10.11. Therefore, the component $R \cos \theta$ (= T) tends to move the cart to the right. Whether the cart moves or not depends upon the relative magnitudes of $R \cos \theta$ and frictional force f between the cart and the ground. If $R \cos \theta > f$, the cart will move forward *i.e.*, to the right.

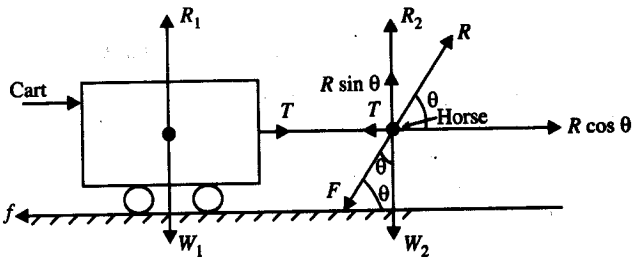


Fig. 10.11

Note that weight W_1 of the cart is balanced by the reaction R_1 of the ground on the cart. Similarly, weight W_2 of the horse is balanced by $(R_2 + R \sin \theta)$ where R_2 is the reaction of the ground on the horse.

Example 10.15. **A 62 kg diver jumps from a helicopter on an air cushion. His speed as he reaches the cushion is 24 ms^{-1}. The average force of the cushion on his body while he is being stopped is 9400 N. Calculate the distance he will sink into the cushion.**

Solution. Fig. 10.12 shows the conditions of the problem. As the diver comes in contact with the cushion, two forces act on him (*i*) gravity-force $m\,g$ (= W) vertically downward and (*ii*) upward force F of the cushion. If a_y is the vertical acceleration of the diver, then,

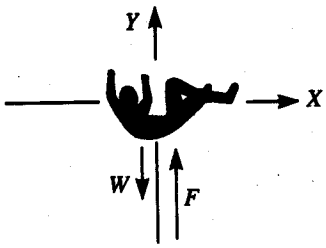


Fig. 10.12

$$\text{Net force} = F - mg$$

or $m\,a_y = F - mg$

$$\therefore \quad a_y = \frac{F-mg}{m} = \frac{9400-62\times9.8}{62} = +142\,\text{ms}^{-2}$$

Now $v^2 - v_0^2 = 2a_y S$

Here $v_0 = 24\,\text{ms}^{-1}$; $v = 0$; $a_y = 142\,\text{ms}^{-2}$; $S = ?$

$$\therefore \quad S = \frac{v^2 - v_0^2}{2a_y} = \frac{(0)^2 - (24)^2}{2\times142} = -2.\text{m}$$

The negative sign shows that his displacement while being stopped is in the negative direction.

Example 10.16. Two blocks, *A* of mass *m* and *B* of mass 3m, are side by side in contact with each other. They are pushed along a smooth floor under the action of a constant force *F* applied to *A*. Find (*i*) accelerations of the blocks (*ii*) the force exerted on *B* by *A*.

Solution. Fig. 10.13 shows the conditions of the problem.

(*i*) Since the blocks remain in contact, the acceleration of the whole system is same (say *a*). According to Newton's second law of motion,

$$F = (m+3m)a \qquad \therefore \ a = \frac{F}{4m}$$

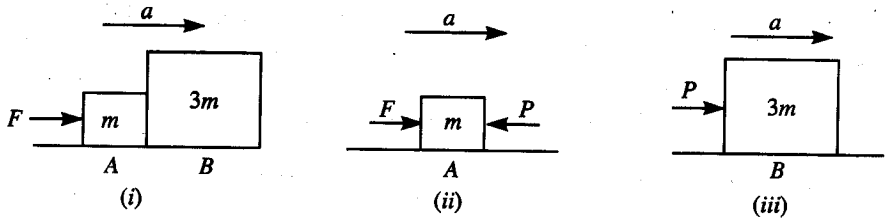


Fig. 10.13

(*ii*) Suppose the force on *B* due to *A* is *P*. According to Newton's third law, *B* will exert a force *P* on *A* but in opposite direction. This is shown in Fig. 10.13 (*iii*) and 10.13 (*ii*). Applying Newton's second law to the motion of *B* [Fig.10.13 (*iii*)],

$$P = 3\,m \times a$$

$$\therefore \quad P = 3m\times\frac{F}{4m} = \frac{3F}{4}$$

Example 10.17. Two blocks of masses 2.9 kg and 1.9 kg are suspended from a rigid support *S* by two inextensible wires each of length 1 m. The upper wire has negligible mass and the lower wire has a uniform mass of 0.2 kg/m. The whole system of blocks, wires and support have an upward acceleration of 0.2 m/s^2. Acceleration due to gravity is 9.8 ms^{-2}.

(*i*) Find the tension at the mid-point of the lower wire.

(*ii*) Find the tension at the mid-point of the upper wire.

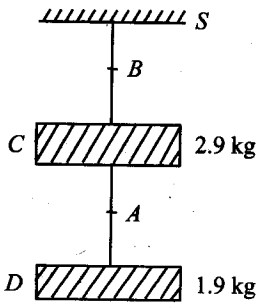


Fig. 10.14

Solution. Figure 10.14 shows the conditions of the problem.

(*i*) Suppose the tension at the mid-point *A* of the lower wire is T_A. Then,

Net upward force = $T_A - m\,g$

where m = 1.9 kg + mass of wire of length AD

$$= 1.9 + \frac{1}{2} \times 0.2 = 2 \text{ kg}$$

Since the upward acceleration is a (= 0.2 ms^{-2}),

$\therefore$ $T_A - m\,g = m\,a$

or $T_A = m(a + g) = 2(0.2 + 9.8) = \mathbf{20\ N}$

(*ii*) Suppose the tension at the mid-point B of the upper wire is T_B. Then,

Net upward force = $T_B - M\,g$

where M = 2.9 kg + 1.9 kg + mass of wire CD = 2.9 + 1.9 + 0.2 × 1 = 5 kg

Since upward acceleration is a(= 0.2 ms^{-2}),

$\therefore$ $T_B - M\,g = M\,a$

or $T_B = M(a + g) = 5\ (0.2 + 9.8) = \mathbf{50\ N}$

PROBLEMS FOR PRACTICE

1. A man standing in a lift holds a spring balance with a load of 5 kg suspended from it. What would be the reading of the balance when the lift is descending with an acceleration of 3.8 ms^{-2} ? **[30 N]**
2. An 80 kg man stands in a lift. Calculate the force he exerts on the floor of the lift when the lift is (*i*) stationary (*ii*) accelerating upward at 2 m s^{-2} (*iii*) moving with a constant velocity (upward) at 4 m s^{-1}. **[(*i*) 780 N (*ii*) 940 N (*iii*) 780 N]**
3. An elevator weighs 39200 N. When the upward tension in the supporting cable is 48000N, what is the upward acceleration? Starting from rest, how far does it rise in 3 s ? **[2.2 ms^{-2}; 9.9 m]**
4. Suppose that your physics and chemistry books are lying on your desk, with chemistry book on the top of the physics book. The chemistry and physics books weigh 14 N and 18 N respectively. Find the magnitude and direction of the force that physics book exerts on the chemistry book. **[14 N *upward*]**
5. What is the apparent weight during take off of an astronaut whose actual weight is 750 N if the resultant upward acceleration is 5 g? **[4.5 × 10^3 N]**

10.16. LAW OF CONSERVATION OF LINEAR MOMENTUM

According to this law, *if the net external force acting on a system of objects is zero, then the vector sum of linear momenta of these objects is constant (conserved), both in magnitude and direction.*

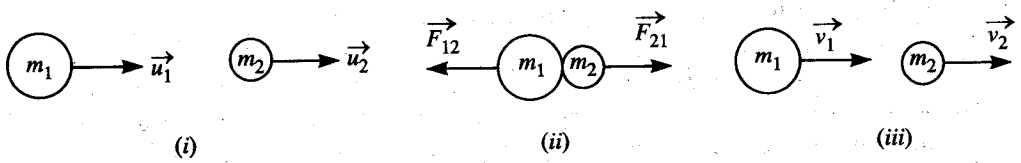


Fig. 10.15

Consider the collision of two balls of masses m_1 and m_2 as shown in Fig. 10.15 (*ii*). Before collision, their respective velocities are $\vec{u_1}$ and $\vec{u_2}$ as shown in Fig. 10.15 (*i*). After collision, the velocities of m_1 and m_2 are $\vec{v_1}$ and $\vec{v_2}$ respectively as shown in Fig. 10.15 (*iii*).

We want to relate the linear momentum of the system before collision to the linear momentum of the system after collision. This relation will be found by considering the impulse on each ball during the collision. Suppose Δt is the time of contact.

The impulse experienced by m_1 will be

$$\vec{F}_{12}\,\Delta\, t = \text{Change in linear momentum} = m_1\vec{v}_1 - m_1\vec{u}_1$$

The impulse experienced by m_2 will be

$$\vec{F}_{21}\,\Delta t = \text{Change in linear momentum} = m_2\vec{v}_2 - m_2\vec{u}_2$$

Note that Δt's are taken the same in the two cases since Δt is the time of contact of the two balls.

During collision, if m_2 exerts a force $\vec{F}_{12}$ on m_1, then m_1 will exert a force $\vec{F}_{21}$ on m_2. According to Newton's third law of motion, $\vec{F}_{12} = -\vec{F}_{21}$. Therefore, the two impulses given above are related as :

$$(m_1\vec{v}_1 - m_1\vec{u}_1) = -(m_2\vec{v}_2 - m_2\vec{u}_2)$$

or
$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \qquad ...(i)$$

Total linear momentum before collision = Total linear momentum after collision.

Thus the total linear momentum of the two balls is conserved. We can rewrite eq. (*i*) as :

$$m_1\vec{v}_1 - m_1\vec{u}_1 = m_2\vec{u}_2 - m_2\vec{v}_2$$

This shows that any linear momentum gained by one ball is lost by the other ball. The total linear momentum thus remains constant (*i.e.*, conserved).

The law of conservation of linear momentum is true for any number of objects in a system. Thus if an isolated system consists of n objects of masses $m_1, m_2, m_3, \ldots m_n$ moving with velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots \vec{v}_n$ respectively, then according to law of conservation of linear momentum,

$$m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n = \text{Constant}$$

Discussion. The following points may be noted :

(*i*) If the net external force acts on the system, the law of conservation of linear momentum will not be valid.

(*ii*) So long the interacting objects are in the isolated system, their total (vector) linear momentum is unchanged. The individual objects may change speed and direction of motion. A force may act on each object from the other object. The objects may separate after a collision or stick together. In every such collision or interaction, the total (vector) linear momentum of the isolated system remains constant.

10.17. EXAMPLES OF CONSERVATION OF LINEAR MOMENTUM

The use of the law of conservation of linear momentum is one of the most fundamental ways of describing many physical phenomena. There are several applications of this law in our everyday life. However, we shall discuss only a few examples to illustrate the application of this most important law.

(*i*) Recoil of a gun. When a gun is fired, the bullet moves with a large velocity and the gun moves with a small velocity in a direction opposite to that of the bullet. The gun and the

bullet constitute an *isolated system. Therefore, the linear momentum of the system should remain constant. Before firing, both the gun and bullet are at rest and, therefore, the total linear momentum is zero. Just after firing, the linear momentum of the system (gun and bullet) should also be zero.

Suppose M and m are the masses of the gun and bullet respectively. Just after firing, let $\vec{V}$ and $\vec{v}$ be the velocities of the gun and the bullet respectively. The conservation of linear momentum applies here, even though kinetic energy is created by the explosion. Note that linear momentum is always conserved in an isolated system. The initial linear momentum of the system was zero; the final linear momentum must be zero.

$$\therefore \quad M\vec{V} + m\vec{v} = 0$$

or
$$\vec{V} = -\frac{m\vec{v}}{M}$$

Fig. 10.16

The minus sign shows that the direction of **recoil of the gun is opposite to that of the bullet. Since the gun has a much larger mass, its velocity is much less than that of the bullet. Note that it is the *vector* sum of linear momenta that is conserved.

Note. The initial kinetic energy of the system is zero because both the gun and the bullet are at rest. However, the final kinetic energy $= \frac{1}{2}MV^2 + \frac{1}{2}mv^2$ which is greater than zero. This kinetic energy is created by the explosion of the gun powder *i.e.*, chemical energy is converted into kinetic energy.

(*ii*) **When the bullet is fired, the gun is always held close to the shoulder;** otherwise the shoulder may get hurt due to the recoil velocity of the gun. If the gun is held close to the shoulder, the total mass that recoils is equal to the sum of the masses of the man and the gun. Consequently, the recoil velocity is very much decreased and the man's shoulder will not get hurt.

(*iii*) **When a man jumps from a boat, the boat slightly moves away from the shore.** Initially, the total linear momentum of the boat and the man is zero. To conserve linear momentum, the linear momentum of the boat should be equal and opposite to that of the man. This is possible only if the boat moves in the opposite direction.

(*iv*) **Rocket Propulsion.** The motion of a rocket is based on the principle of conservation of linear momentum. The rocket and fuel form an †isolated system. Therefore, the total linear momentum of the system remains constant. Before the rocket is fired, the total linear momentum of the system is zero because the rocket is at rest. Therefore, the total linear momentum of the rocket and exhaust gases should remian zero after firing of the rocket. When the rocket is fired, fuel is burnt and very hot gases are formed. As the hot gases gain linear momentum to the rear on leaving the rocket, the rocket acquires equal linear momentum in the forward (*i.e.*, opposite) direction because linear momentum is conserved.

Example 10.18. A hunter fires a bullet of mass 10 g with a velocity of 400 ms^{-1} from a gun of mass 5 kg. What will be the recoil velocity of the gun ? If the hunter is standing on ice, then what velocity he (along with the gun) will acquire ? Mass of hunter is 75 kg.

* At the moment of firing, the forces caused by the explosion of gunpowder are much larger than the force exerted by the hand. Therefore, the system is practically isolated one.

** The backward motion of the gun is called recoil of the gun.

† At the moment of firing, the forces caused by the burning of fuels are very much larger than other forces. So the system is essentially an isolated one.

Solution. Mass of gun, $M = 5$ kg ; Mass of bullet, $m = 10$ g $= 0.01$ kg
Velocity of gun, $V = ?$; Velocity of bullet, $v = 400$ ms^{-1}

The gun and bullet form an isolated system. Therefore, the linear momentum of the system remains constant. Before firing, the linear momentum of the system is zero because both gun and bullet are at rest. Therefore, after firing, the linear momentum of the system should also be zero.

$$\therefore \quad MV + mv = 0 \quad \text{or} \quad V = -\frac{mv}{M} = -\frac{0.01 \times 400}{5} = \mathbf{-0.8\ ms^{-1}}$$

The negative sign shows that velocity of the gun will be in a direction opposite to that of the bullet.

Since the hunter is standing on ice, on firing the bullet, he himself along with the gun will move backward with a velocity V' (say).

Linear momentum of gun plus man $= (M + 75)\ V' = (5 + 75)\ V' = 80V'$

Linear momentum of bullet $= m\ v = 0.01 \times 400 = 4$

The whole system consists of man + gun + bullet.

$$\therefore \quad 80\ V' + 4 = 0 \quad \text{or} \quad V' = -\frac{4}{80} = \mathbf{-\ 0.05\ ms^{-1}}$$

Example 10.19. A 30 kg shell is flying at 48 ms^{-1}. When it explodes, its one part of 18 kg stops while the remaining part flies on. What is the velocity of the latter?

Solution. Mass of shell, $M = 30$ kg ; Mass of first part, $m_1 = 18$ kg
Velocity of shell, $V = 48$ ms^{-1} ; Mass of second part, $m_2 = 12$ kg
Velocity of first part, $v_1 = 0$; Velocity of second part, $v_2 = ?$
Initial linear momentum $= M\ V$; Final linear momentum $= m_1\ v_1 + m_2\ v_2$

According to principle of conservation of linear momentum,

$$M\ V = m_1\ v_1 + m_2\ v_2$$

or $$30 \times 48 = 18 \times 0 + 12 \times v_2 \quad \therefore \quad v_2 = \frac{30 \times 48}{12} = \mathbf{120\ ms^{-1}}$$

Example 10.20. A hunter has a machine gun that can fire 50 g bullets with a velocity of 150 m/s. A 60 kg tiger springs at him with a velocity of 10 ms^{-1}. How many bullets must the hunter fire per second into the tiger in order to stop him in his track?

Solution. Let n bullets be fired per second at the tiger to stop him in his track. Then,

Linear momentum of bullets = Linear momentum of tiger

or $$n \times (50 \times 10^{-3}) \times 150 = 60 \times 10$$

$$\therefore \quad n = \frac{60 \times 10}{(50 \times 10^{-3}) \times 150} = \mathbf{80}$$

Example 10.21. A cart of mass 500 kg is standing at rest on the rails. A man weighing 70 kg and running parallel to rail track with a velocity of 10 ms^{-1} jumps onto the cart on approaching it. Find the velocity with which the cart will start moving.

Solution. $m_1 = 500$ kg ; $v_1 = 0$; $m_2 = 70$ kg ; $v_2 = 10$ ms^{-1}

Let v be the velocity with which the cart and man move when the man jumps onto the cart.

According to principle of conservation of linear momentum,

$$(m_1 + m_2)v = m_1v_1 + m_2v_2$$

or $$(500 + 70)v = 500 \times 0 + 70 \times 10 \quad \therefore\ v = \mathbf{1.23\ ms^{-1}}$$

PROBLEMS FOR PRACTICE

1. What is the recoil velocity of 2 kg gun if it shoots a 7 g bullet out of the muzzle at a speed of 500 ms^{-1}? **[–1.75 ms^{-1}]**
2. A 40 kg shell is flying at 60 ms^{-1}. When it explodes, its one part of 10 kg stops while the remaining part flies on. Find the velocity of the latter. **[80 ms^{-1}]**
3. A 2 kg mass with a speed of 0.5 ms^{-1} collides head–on with a 1.5 kg mass moving with a speed of 0.3 ms^{-1} toward the first mass. What is the speed of the second mass after collision? **[0.37 ms^{-1}]**
4. A 3.5 kg rifle fires a 10 g bullet with a velocity of 850 ms^{-1} .What is the recoil velocity of the rifle? **[–2.5 ms^{-1}]**
5. A shell of mass 0.2 kg is fired by a gun of mass 100 kg. If the muzzle speed of the gun is 80 ms^{-1}, find the recoil speed of the gun. **[– 0.016 ms^{-1}]**

10.18. ROCKET PROPULSION – CHANGING MASS IN MOTION

So far we have considered the motion of bodies with constant mass. However, in many practical cases, the mass of the system is varying. The most common example is the motion of a rocket. In such a motion, fuel is burnt in the rocket and gases are expelled out of the back of the rocket. Therefore, the mass of the system changes with time.

Principle. *The propulsion of a rocket is based on the principle of conservation of linear momentum.* Before a rocket is fired, the total linear momentum of rocket plus fuel is zero. Since the system is essentially an isolated system, the linear momentum of the system remains the same. When rocket is fired, fuel is burnt and very hot gases are formed. These gases are expelled from the back of the rocket. Since linear momentum acquired by the gases is directed toward the rear, the rocket must acquire an equal linear momentum in the opposite direction (forward) in order to conserve linear momentum. This is how a rocket is propelled.

Theory. Consider a rocket moving vertically upward (one dimensional motion) in free space with no external forces. Thus the system (rocket plus fuel) constitutes an isolated system so that linear momentum remains the same. Suppose that at $t = 0$, the mass of the rocket (mass of container plus fuel) is m_0 and is moving with a velocity V_0 *w.r.t.* earth as shown in Fig. 10.17 (*i*). Let at any time t, the mass of the rocket be m and its velocity be V as shown in Fig. 10.17 (*ii*). Note that as the fuel is burnt, the velocity of the rocket increases and its mass decreases. Therefore, $m < m_0$ and $V > V_0$.

In time Δt, the rocket will eject a mass of Δm of the fuel (in the form of gases) with a velocity $-v$ *w.r.t the rocket.* However, the velocity of the ejected gases with respect to earth is $(*V - v)$. At the end of the time interval Δt, the mass of rocket has decreased to $(m - \Delta m)$ and its velocity has increased to $(V + \Delta V)$ [See Fig. 10.17 (*iii*)].

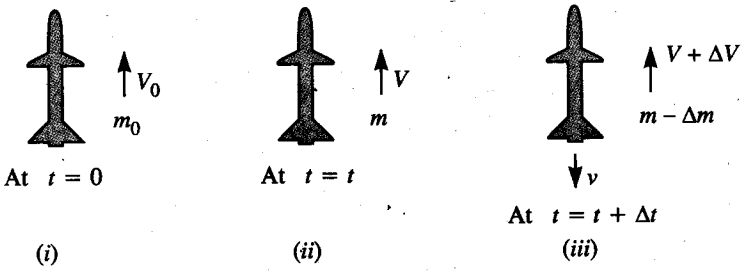


Fig. 10.17

* We are not using vector notation because it is a case of one dimensional motion.

Applying the law of conservation of linear momentum to the system (*i.e.*, rocket), the initial linear momentum (*i.e.*, at the start of time interval Δt) must be equal to the final linear momentum (*i.e.*, at the end of time interval Δt) *i.e.*,

Initial linear momentum = Final linear momentum

or $$m\,V = (m - \Delta m)\,(V + \Delta V) + \Delta m\,(V - v)$$

or $$mV = mV - V\Delta m + m\Delta V - \Delta m\,\Delta V + V\Delta m - v\Delta m$$

or $$0 = m\,\Delta V - v\,\Delta m - \Delta m\,\Delta V$$

The term $\Delta m\,\Delta V$ is the product of two small quantities and, therefore, can be neglected.

$\therefore$ $$0 = m\,\Delta V - v\,\Delta m$$

We notice that quantity of fuel ejected (Δm) is equal to the loss of mass of the rocket; $-\Delta m$. The negative sign is given to indicate a decrease in m.

$\therefore$ $$0 = m\,\Delta V - v\,(-\Delta m)$$

or $$\Delta V = -\frac{\Delta m}{m} v$$

If we pass to the limit of $\Delta t \to 0$, then ΔV and Δm can be replaced by dV and dm to yield

$$dV = -v\frac{dm}{m} \qquad ...(i)$$

At $t = 0$; $V = V_0$ and $m = m_0$

At $t = t$; $V = V$ and $m = m$

$\therefore$ $$\int_{V_0}^{V} dV = -\int_{m_0}^{m} v\frac{dm}{m}$$

Generally, the velocity of exhaust gases *w.r.t.* rocket is assumed to be constant.

$\therefore$ $$\int_{V_0}^{V} dV = -v\int_{m_0}^{m} \frac{dm}{m}$$

or $$\left|V\right|_{V_0}^{V} = -v\left|{}^{*}\log_e m\right|_{m_0}^{m}$$

or $$V - V_0 = -v\,[\log_e m - \log_e m_0] = v\,[\log_e m_0 - \log_e m]$$

$\therefore$ $$V = V_0 + v\log_e \frac{m_0}{m} \qquad ...(ii)$$

Eq. (*ii*) gives the velocity of the rocket at any time t when the mass of the rocket is m. If the initial velocity of the rocket is zero ($V_0 = 0$), then above equation becomes :

$$V = v\log_e \frac{m_0}{m} \qquad ...(iii)$$

Thus the velocity of the rocket at any instant is directly proportional to

(*i*) *exhaust speed of the ejecting gases w.r.t. rocket.*

(*ii*) *natural logarithm of the ratio of the initial mass of the rocket to its mass at that instant.*

Thrust of the rocket. An important specification for a rocket is its thrust. *It is defined to be the recoil force exerted on the rocket by the exhaust gases.* We can find it easily from eq. (*i*) above.

$$m\,dV = -v\,dm$$

---

* $\int \frac{dx}{x} = \log_e x$

or $$m\frac{dV}{dt} = -v\frac{dm}{dt}$$

Now dV/dt is the instantaneous acceleration so that $m(dV/dt)$ is the instantaneous thrust of the rocket.

$$\therefore \quad \text{Thrust of rocket, } F = -v\frac{dm}{dt}$$

Discussion. The following points may be noted :

(*i*) The thrust of rocket at any instant is equal to the product of exhaust speed of the ejecting gases *w.r.t.* rocket and the rate of combustion of the fuel at that instant.

(*ii*) The thrust of the rocket can be increased by ejecting the gases at a higher exhaust speed and by ejecting the gases at a greater rate.

(*iii*) The negative sign in the equation shows that an *increase* in speed – positive dV/dt – corresponds to a *decrease* in rocket mass – negative dm/dt.

Burnt out speed of the rocket

The speed attained by the rocket when the whole fuel of the rocket is burnt is called **burnt out speed** *of the rocket* (V_b). It is the maximum speed that can be acquired by the rocket. If m_r is the mass of the empty container, then burnt out speed V_b from eq.(*iii*) becomes :

Burnt out speed, $V_b = v \log_e \frac{m_o}{m_r}$ ..... initial velocity is zero

10.19. ANOTHER EXAMPLE OF CHANGING MASS IN MOTION

If the mass of a body is constant, then for a given force, we can easily find the acceleration and the subsequent motion of the body. However, in many practical cases, the mass is not constant. We shall give one another familiar example.

An important example of changing mass is the railroad car of Fig. 10.18. The car is in motion while taking on sand at constant rate. For simplicity, we assume that car has some initial velocity v_1 and rolls freely without friction. It is not connected to a locomotive.

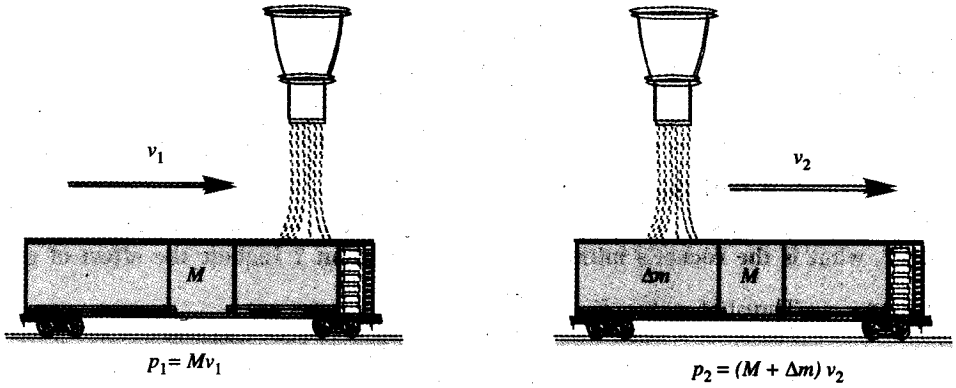


Fig. 10.18

The initial linear momentum p_1 is the product of the mass of the car M and its initial velocity v_1 *i.e.*

$$p_1 = M\, v_1$$

Although the falling sand has zero initial speed in the horizontal direction, it is accelerated horizontally when it hits the moving car. As a result, sand speeds up while slowing down the car. After time Δt, an amount of sand Δm has been added and the system (car plus sand) has a new

velocity v_2. The corresponding linear momentum of the system is then,

$$p_2 = (M + \Delta m)v_2$$

Change in linear momentum, $\Delta p = p_2 - p_1$

$$= (M + \Delta m)v_2 - Mv_1$$

or $$\Delta p = M(v_2 - v_1) + \Delta m v_2$$

or $$\Delta p = M\Delta v + \Delta m v_2 \quad (\because v_2 - v_1 = \Delta v)$$

If we divide throughout by Δt, we get system's change in linear momentum per unit time *i.e*

$$\frac{\Delta v}{\Delta t} = M\frac{\Delta v}{\Delta t} + v_2\frac{\Delta m}{\Delta t} \qquad ...(i)$$

But the change in linear momentum per unit time is equal to the net external force applied. Since there is no external force along the direction of motion, $\Delta p / \Delta t = 0$. Therefore, eq. (*i*) becomes :

$$M\frac{\Delta v}{\Delta t} = -v_2\frac{\Delta m}{\Delta t} \qquad ...(ii)$$

We see that $\Delta v / \Delta t$ is negative, as expected, the car is slowing down. If we wanted to keep the car with constant speed so that $v_1 = v_2$, then we would have to apply an external force F given by ;

$$F = v_1\frac{\Delta m}{\Delta t}$$

Example 10.22. A fully fuelled rocket has a mass of 21,000 kg of which 15000 kg is fuel. The burnt fuel is exhausted at a rate of 190 kg/s at a speed of 2800 ms^{-1} relative to the rocket. If the rocket is fired vertically upward, calculate (*i*) the thrust of the rocket (*ii*) the net force on the rocket at blast off.

Solution. (*i*) Thrust of rocket, $F = v\frac{dm}{dt}$

Here $v = 2800\,ms^{-1}$; $\frac{dm}{dt} = 190\ kg\ s^{-1}$

$\therefore$ $F = 2800 \times 190 = \mathbf{5.3 \times 10^5\ N}$

(*ii*) The rocket is acted upon by two forces *viz.*, upward thrust of rocket and a downward force mg due to gravity. At *blast off, $mg = 21{,}000 \times 9.8 = 2.1 \times 10^5$ N.

$\therefore$ Net force on rocket $= F - mg = 5.3 \times 10^5 - 2.1 \times 10^5 = \mathbf{3.2 \times 10^5\ N}$

Example 10.23. A fully fuelled rocket of mass 5000 kg is set to be fired vertically. If the rocket ejects its gases at a speed of 3×10^3 m s^{-1} *w.r.t.* the rocket and burns fuel at the rate of 50 kgs^{-1}, what is the rocket's initial upward acceleration ? Include the effect of gravity.

Solution. Thrust of rocket, $F = v\frac{dm}{dt}$

Here $v = 3 \times 10^3$ m s^{-1}; $\frac{dm}{dt} = 50\,kg\ s^{-1}$

$\therefore$ $F = 3 \times 10^3 \times 50 = 1.5 \times 10^5$ N

The rocket is acted upon by two forces *viz.*, upward thrust of the rocket and downward force mg due to gravity.

* At blast off, the mass of rocket is 21000 kg.

Net upward force initially (*i.e.*, at blast off) = $F - mg$

$$\therefore \quad \text{Initial acceleration, } a = \frac{F-mg}{m} = \frac{F}{m} - g$$

$$= \frac{1.5\times10^5}{5\times10^3} - 9.8 = \mathbf{20\ ms^{-2}}\ \textit{upward}$$

Example 10.24. A hopper drops gravel onto a conveyor belt at a rate of 75 kg s^{-1}. If the belt moves at a constant speed v = 2.20 ms^{-1}, what force is needed to keep the conveyor belt moving?

Solution. Fig. 10.19 shows the conditions of the problem. This is a case of varying mass. The belt is in motion while taking on gravel at a constant rate. We know that

$$M\frac{dv}{dt} = -v\frac{dm}{dt}$$

We see that dv/dt is negative. It means that the belt is slowing down. In order to keep the belt moving at original speed v (= 2.2 m s^{-1}), we would have to apply an external force F given by ;

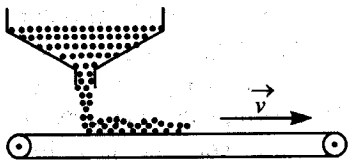


Fig. 10.19

$$F = v\frac{dm}{dt} = 2.2\times75 = \mathbf{165\ N}$$

Example 10.25. A rocket of mass 20 kg has 180 kg of fuel. The exhaust velocity of the fuel is 1.6 km/sec. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is 2 kg/sec.

Solution. The rate of consumption of fuel is minimum when the upward acceleration of the rocket is zero *i.e.*, the rocket moves with a uniform velocity. Under this condition, upthrust of rocket has to overcome the weight of rocket *i.e.*

$$v\frac{dm}{dt} = mg \quad \text{or} \quad \frac{dm}{dt} = \frac{mg}{v}$$

Here m = 20 + 180 = 200 kg ; g = 9.8 ms^{-2} ; v = 1.6 km/s = 1.6 × 10^3 m/s

$$\therefore \quad \frac{dm}{dt} = \frac{200\times9.8}{1.6\times10^3} = \mathbf{1.225\ kg/sec}$$

If the rate of consumption of fuel is 2 kg/sec., the time taken for the consumption of 180 kg fuel is

$$t = 180/2 = 90\text{s}$$

The ultimate vertical velocity attained by the rocket is given by;

$$V = v\log_e\frac{m_0}{m} - gt$$

Here v = 1.6 km/sec = 1.6× 10^3 m/sec ; m_0 = 180 + 20 = 200 kg ; m=20 kg

$$\therefore \quad V = 1.6\times10^3\log_e\frac{200}{20} - 9.8\times90 = 3680 - 882 = \mathbf{2798\ ms^{-1}}$$

If effect of gravity is neglected, then,

$$V = 1.6\times10^3\log_e\frac{200}{20} = 3680\ ms^{-1}$$

PROBLEMS FOR PRACTICE

1. What is the thrust of a rocket that burns fuel at a rate of 1.3×10^4 kgs^{-1} if the exhaust gases have a velocity of 2.5×10^3 m s^{-1} *w.r.t.* the rocket ? **[3.3×10^7 N]**
2. For a certain rocket, the velocity of the exhaust gas is 50,000 ms^{-1} and gas is expelled at the rate of 1300 kg s^{-1}. Find rocket's thrust. **[6.5×10^7 N]**
3. Sand drops vertically at the rate of 2 kg s^{-1} onto a conveyor belt moving horizontally with a velocity of 0.1 m s^{-1}. Calculate the force required to keep the belt moving. **[0.2 N]**

10.20. EQUILIBRIUM OF CONCURRENT FORCES

When forces act at the same point, they are called *concurrent forces.*

A number of concurrent forces are said to be in equilibrium if their vector sum is zero.

When the vector sum of the concurrent forces acting on a body is zero, it means that the net force on the body is zero. According to Newton's second law, the body will have zero acceleration. As a result, either the body is at rest (*i.e.*, zero velocity) or it is moving with a constant velocity (*i.e.*, with constant speed in a straight line). If the body is at rest, it is said to be in *static equilibrium.* If the body is in motion, it is said to be in *dynamic equilibrium.* Note that in either case, the acceleration of the body is zero.

Suppose three concurrent forces $\vec{A}$, $\vec{B}$ and $\vec{C}$ acting at point O are in equilibrium as shown in Fig. 10.20. Then,

$$\vec{A} + \vec{B} + \vec{C} = 0$$

or

$$\vec{A} + \vec{B} = -\vec{C}$$

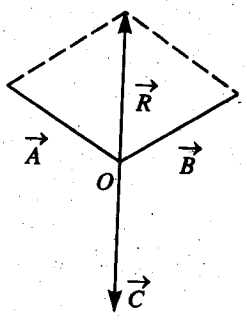


Fig. 10.20

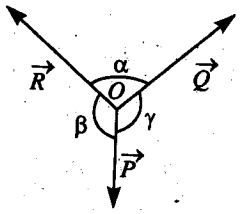


Fig. 10.21

It means that the vector sum of $\vec{A}$ and $\vec{B}$ is equal in magnitude and opposite in direction to force $\vec{C}$. Thus in Fig. 10.20, $\vec{R}$ is equal in magnitude and opposite in direction to $\vec{C}$. Therefore, point O is in equilibrium. Indeed when three concurrent forces are in equilibrium, the resultant of any two forces is equal in magnitude and opposite in direction to the third force. Lami gave his theorem regarding the three concurrent forces in equilibrium and is called Lami's theorem.

Lami's theorem. *It states that if three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the other two forces.* Thus referring to Fig. 10.21, suppose the three forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ acting at point O are in equilibrium. Let α, β and γ be the angles between forces $\vec{Q}$ and $\vec{R}$, $\vec{R}$ and $\vec{P}$ and $\vec{P}$ and $\vec{Q}$ respectively. Then according to Lami's theorem :

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

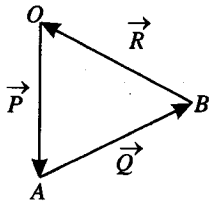


Fig. 10.22

Proof. The three forces $\vec{P}, \vec{Q}$ and $\vec{R}$ acting at point O in Fig. 10.21 are in equilibrium and can be represented by the three sides of a triangle OAB taken in the same order as shown in Fig. 10.22.

$$\therefore \quad \frac{P}{OA} = \frac{Q}{AB} = \frac{R}{BO} \qquad ...(i)$$

From triangle OAB, by sine formula, we have,

$$\frac{OA}{\sin\angle ABO} = \frac{AB}{\sin\angle BOA} = \frac{BO}{\sin\angle OAB} \qquad ...(ii)$$

From eqs. (i) and (ii), $\frac{P}{\sin\angle ABO} = \frac{Q}{\sin\angle BOA} = \frac{R}{\sin\angle OAB}$

or $$\frac{P}{\sin(180° - \alpha)} = \frac{Q}{\sin(180° - \beta)} = \frac{R}{\sin(180° - \gamma)}$$

or $$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Thus Lami's theorem stands proved.

10.21. CONNECTED MOTION

Consider two masses m_1 and m_2 tied at the ends of massless rope passing over a frictionless and massless pulley [See Fig.10.23 (i)]. Suppose $m_2 > m_1$. The heavier body (m_2) will move downward and the lighter body (m_1) will move upward. Let a be the common acceleration and T be the *tension in the rope. We shall find the values of a and T by applying Newton's second law of motion.

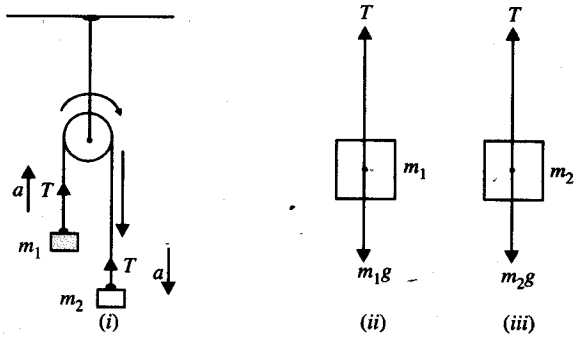


Fig. 10.23

Mass m_2. Fig. 10.23 (iii) shows the free-body diagram of m_2. The forces acting on m_2 are (i) a downward force $m_2 g$ due to gravity and (ii) upward force T provided by the rope.

$$\text{Net force on } m_2 = m_2 g - T$$

or $$m_2 a = m_2 g - T \qquad ...(i)$$

---

* Note that we assume that rope and pulley are massless so that we can neglect their inertias. Because the pulley is both massless and frictionless, the tension in the rope is the same on both sides of the pulley.

Mass m_1. Fig. 10.23 (*ii*) shows the free-body diagram of m_1.

$$\text{Net force on } m_1 = T - m_1 g$$

or $$m_1 a = T - m_1 g \quad ...(ii)$$

Adding eqs. (*i*) and (*ii*), we get,

$$(m_1 + m_2)\, a = m_2 g - m_1 g$$

$$\therefore \quad a = \frac{(m_2 - m_1)g}{m_1 + m_2} \quad ...(iii)$$

Note that $a < g$ *i.e.* acceleration a of a system of two connected masses is always less than acceleration due to gravity g.

Dividing eq. (*i*) by eq.(*ii*), we get,

$$\frac{(m_2 g - T)}{T - m_1 g} = \frac{m_2 a}{m_1 a} = \frac{m_2}{m_1}$$

or $$m_1 m_2 g - m_1 T = m_2 T - m_1 m_2 g$$

or $$(m_1 + m_2)T = 2 m_1 m_2 g$$

$$\therefore \quad T = \frac{2 m_1 m_2 g}{m_1 + m_2} \quad ...(iv)$$

Note that when $m_1 = m_2 = (= m)$, then tension T equals the weight mg and there is no acceleration.

10.22. SOLVING PROBLEMS IN MECHANICS

We now have the necessary background for applying Newton's laws of motion to a variety of situations. Before giving a number of examples, we will list a series of helpful steps for problem solving :

(*i*) Sketch the picture of the problem.

(*ii*) Isolate the body whose motion is to be analysed. Show the various forces that act on the body. Do *not* include forces that do not act directly on the body you have isolated. Such a diagram is called a *free-body diagram* or *force diagram.*

A diagram showing all the forces acting on a given body (and only at that body) is called a **free-body diagram.**

(*iii*) If necessary, resolve the forces in the free-body diagram into rectangular components.

(*iv*) Write $F = m\,a$ equations in the component form for the forces in the free body diagram.

(*v*) Solve the component equations for the unknown.

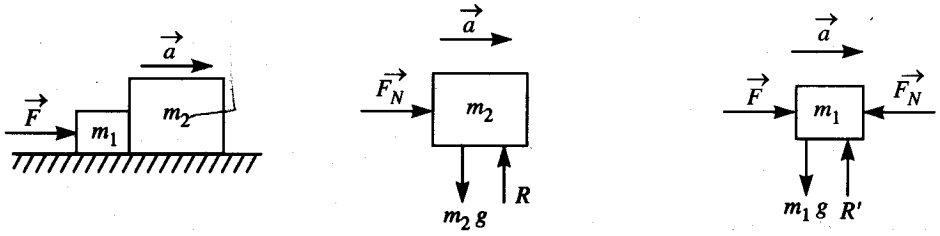
(*vi*) Check the reasonableness of the result you obtain.

Example 10.26. Two blocks having masses m_1 = 1 kg and m_2 = 2 kg are in contact with each other on a horizontal table as shown in Fig. 10.24 (*i*). A horizontal force $\vec{F}$ is applied to m_1, causing the blocks to accelerate to the right with an acceleration of 3 ms^{-2}. Calculate (*i*) the magnitude of F and (*ii*) the forces of compression between the blocks. Neglect friction.

Solution. *Fig. 10.24 (*ii*) shows the free-body diagram for m_2 whereas Fig. 10.24 (*iii*) shows the free-body diagram for m_1. Note that m_1 exerts a force $\vec{F}_N$ on m_2 and according to Newton's third law, m_2 exerts a force $\vec{F}_N$ on m_1 in the opposite direction. Since $\vec{F}$ is not applied to m_2,

* $m_2 g$ is the force with which m_2 is attracted downward due to gravity. R is the upward force which the table exerts on m_2. Since there is no vertical motion, $R = m_2 g$. Similarly, $m_1 g = R'$.

it does not appear in the free-body diagram for m_2.



(*i*) (*ii*) (*iii*)

Fig. 10.24

(*i*) The blocks move together. Therefore, they can be treated as one object of mass = m_1 + m_2 = 1 + 2 = 3 kg. Applying Newton's second law, we have,

$$F = (m_1 + m_2)\, a = (1 + 2)\, 3 = \mathbf{9\ N}$$

(*ii*) Referring to Fig. 10.24 (*ii*), there is only one horizontal force $\vec{F}_N$ that acts on m_2.

$$\therefore \qquad F_N = m_2 a = 2 \times 3 = \mathbf{6\ N}$$

Note. We can also find $\vec{F}_N$ from Fig. 10.24 (*iii*).

$$\text{Net force on } m_1 = F - F_N$$

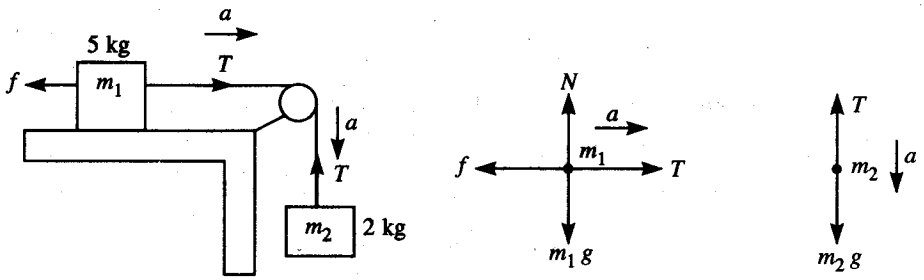
Also $$\text{Net force on } m_1 = m_1 a$$

$$\therefore \qquad F - F_N = m_1 a$$

or $$F_N = F - m_1 a = 9 - 1 \times 3 = \mathbf{6\ N}$$

This is in agreement with the result for m_2.

Example 10.27. Fig. 10.25 (*i*) shows two masses m_1 and m_2 connected by a rope. If the kinetic friction between m_1 and the table is 9.8 N, find the acceleration of the system.



(*i*) (*ii*) (*iii*)

Fig. 10.25

Solution. Fig. 10.25 (*ii*) shows the free-body diagram for m_1 whereas Fig. 10.25 (*iii*) shows the free-body diagram for m_2. Before we solve the problem, let us look at one important point. If there were no frictional force, the weight of m_2 would cause m_1 to fall no matter how large m_1 was. It is because only the frictional force f is trying to hold m_1 from moving. The weight of m_1 merely holds the mass tightly on the table top ; it has no component to left and so it cannot hold the mass from moving to the right.

Suppose a is the acceleration of the system. Applying Newton's second law to Fig. 10.25 (*ii*),

$$\text{Net force} = m_1\, a$$

or $$T - f = m_1\, a \qquad \ldots(i)$$

Applying Newton's second law to Fig. 10.25 (*iii*),

$$m_2 g - T = m_2 a \qquad ...(ii)$$

Adding eqs. (*i*) and (*ii*), we have,

$$m_2 g - f = (m_1 + m_2)a$$

$$\therefore \quad a = \frac{m_2 g - f}{m_1 + m_2} = \frac{2\times 9.8 - 9.8}{5+2} = 1.4 \text{ ms}^{-2}$$

We can also find the tension T in the rope.

$$T = f + m_1 a = 9.8 + 5 \times 1.4 = 17 \text{ N}$$

Example 10.28. **A frictionless surface of 1 m length is inclined at an angle of 40° to the horizontal as shown in Fig. 10.26. Calculate (*i*) how long it would take the object to slide down and (*ii*) how fast the object would be going at the bottom.**

Solution. An important type of one dimensional motion is motion on an inclined plane. The weight mg of the object acts vertically downward and the normal force (N) exerted by the inclined surface on the object (by definition) is perpendicular to the inclined surface. Motion is constrained to take place along the inclined surface. Therefore, it is convenient to take X-axis along the inclined surface and Y-axis perpendicular to it as shown in Fig. 10.26. Note that weight mg is resolved into two rectangular components $mg \sin\theta$ along the inclined surface and $mg \cos\theta$ perpendicular to it. Since no motion is allowed perpendicular to the inclined surface, Newton's first law requires that $N = mg \cos\theta$. Motion is along the inclined surface and is governed by Newton's second law. Suppose acceleration of the object is a.

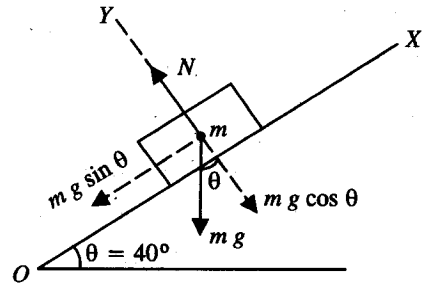


Fig. 10.26

(*ii*) Net force = ma or $mg \sin\theta = ma$

$$\therefore \quad a = \frac{mg\sin\theta}{m} = g\sin\theta = 9.8\times\sin 40° = 6.27 \text{ ms}^{-2}$$

Let us choose the positive direction along the motion of the object

$$\therefore \quad v^2 - v_0^2 = 2aS$$

or $$v^2 = v_0^2 + 2aS = (0)^2 + 2\times(6.27)\times 1 \quad \therefore \quad v = 3.55 \text{ ms}^{-1}$$

(*i*) $$x = v_0 t + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2$$

$$\therefore \quad t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{2\times 1}{6.27}} = 0.56 \text{ s}$$

Example 10.29. **A block of mass M on an inclined plane is joined to a mass m by a string passing over a pulley as shown in Fig. 10.27 (*i*). The block slides on a frictionless surface and the effects of the pulley are negligible. If the surface is inclined at 20° and $m = M/2$, find the magnitude and direction of the acceleration of the block.**

Solution. We resolve the weight Mg of the block into components parallel and perpendicular to the surface of the inclined plane as shown in Fig. 10.27 (*i*). Fig. 10.27 (*ii*) shows the free-body diagram for M whereas Fig. 10.27 (*iii*) shows free-body diagram for m.

Since no motion is allowed perpendicular to the inclined plane, Newton's first law requires that $N = Mg \cos \theta$. The motion is entirely along the plane and is governed by Newton's second law. Suppose the acceleration of the system is a. Applying Newton's second law to Fig. 10.27 (*ii*),

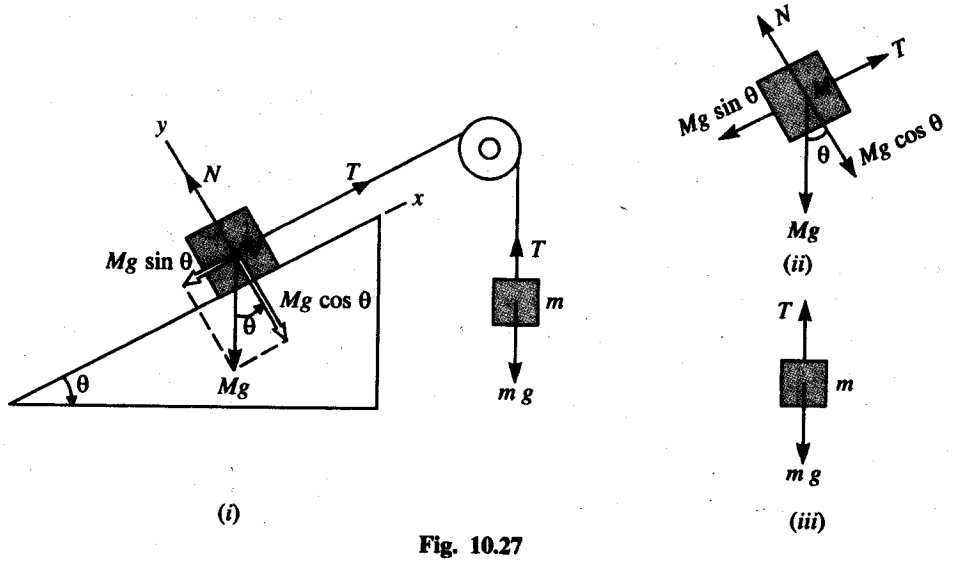


Fig. 10.27

$$T - M g \sin \theta = Ma \qquad ...(i)$$

Applying Newton's second law to Fig. 10.27 (*iii*),

$$mg - T = ma \qquad ...(ii)$$

Adding eqs. (*i*) and (*ii*), we have,

$$mg - Mg \sin \theta = (m + M)\, a$$

$$\therefore \quad \text{Acceleration, } a = \frac{(m - M \sin \theta) g}{m + M}$$

Now, $m = M/2$; $\theta = 20°$

$$\therefore \quad a = \frac{Mg\left(\frac{1}{2} - \sin 20^\circ\right)}{(3/2)M} = 1.03 \text{ ms}^{-2}$$

The direction of acceleration of the block is up the plane.

PROBLEMS FOR PRACTICE

1. In Fig. 10.28, the rope is pulling the two blocks with a force of 58 N. Find the acceleration of the blocks and the tension in the connecting rope. **[5.27 ms^{-2} ; 15.8 N]**

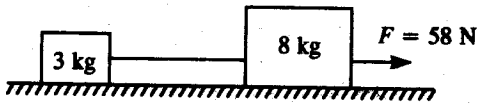


Fig. 10.28

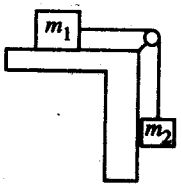


Fig. 10.29

2. In Fig. 10.29, m_1 = 3.25 kg and m_2 = 1.9 kg. Find the acceleration of the blocks and tension in the connecting cord. **[3.62 m s^{-2} ; 11.8 N]**
3. A 10 kg block is placed on a frictionless inclined plane and connected to a 5 kg block as shown in Fig. 10.30. What would the angle θ have to be for the blocks to be motionless? What would be the acceleration of the blocks if $\theta = 37°$?

[30° ; 0.65 ms^{-2} *down the plane*]

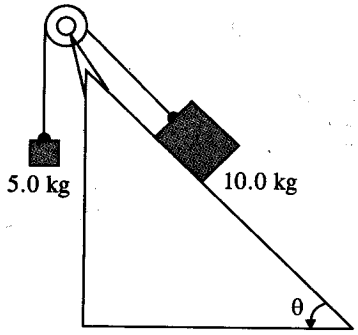


Fig. 10.30

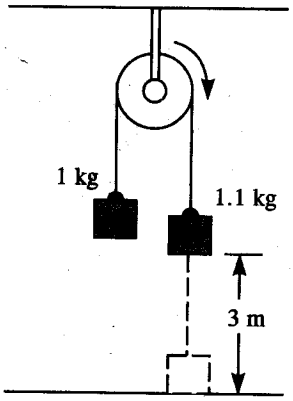


Fig. 10.31

4. In Fig. 10.31, m_1 = 1 kg and m_2 = 1.1 kg. If the larger mass descends a distance of 3 m from rest in 3.6 s, what is the acceleration due to gravity at that place? **[9.7 ms^{-2}]**
5. A 10 kg block is placed on a frictionless table and connected to a 5 kg block by a string that extends horizontally across the table over a pulley and down to the 5 kg block. What is the acceleration of the blocks? **[3.3 ms^{-2}]**

10.23. INERTIAL FRAMES OF REFERENCE

We saw earlier that a system of coordinate axes fixed to a particular body is called *a frame of reference*. Our description of motion is always in terms of motion with respect to some particular reference frame. For example, we usually describe the motion of autos, baseballs, cars and other ordinary objects with respect to the surface of the earth. Therefore, our frame of reference is always earth's surface unless stated otherwise.

An inertial frame of reference is a reference frame in which the law of inertia (Newton's first law) holds. In other words, in an inertial reference frame, an object at rest will remain at rest if no net force acts on it. Let us explain this point. Consider a passenger in an aircraft that is making its take-off run. Suppose we attach *frame of reference to the aircraft. We know that during the take-off run, the aircraft goes faster and faster, yet the passenger remains at rest relative to the frame of reference attached to the aircraft. Therefore, Newton's first law does not describe this situation; a forward force does act on the passenger but nevertheless (relative to the aircraft) he remains at rest. An aircraft gaining speed (*i.e.*, accelerating) is *not* an inertial frame of reference. Therefore, a frame of reference that has an accelerated motion is not an inertial frame of reference.

An ****inertial frame of reference*** *is one in which Newton's first law (law of inertia) holds.*

For many practical purposes, earth's surface is considered as an inertial frame of reference, even though the earth is not truly an inertial reference frame because it rotates.

* Note that the passenger is at rest relative to the aircraft but is moving faster and faster relative to the earth.

** So named because in such a frame, an object at rest remains at rest unless acted upon by a *net force*.

The effects of rotation are not large and for most purposes, Newton's laws of motion apply on earth's surface. But if a frame of reference is inertial, then a second frame moving uniformly (*i.e.*, with constant speed in a straight line, without rotation) relative to it is also inertial reference frame. It is because any body in uniform motion as seen by an observer in the first frame will also appear to an observer in the second frame to be in uniform motion. The speed and direction appear different to the two observers, but both observe that Newton's first law is obeyed. *Therefore, the reference frames that move with constant velocity are the inertial reference frames.* Many examples can be cited. Two different inertial reference frames are a train travelling at constant velocity (frame 1) relative to the ground (frame 2). An elevator rising at constant velocity (frame 1) relative to the building in which it is located (frame 2). In each case (frame 1 or frame 2), Newton's first law is obeyed.

Note. Basic laws of nature are the same in all inertial reference frames.

10.24. ACCELERATED FRAMES OF REFERENCE

A frame of reference having accelerated motion is not an inertial frame of reference *i.e.*, Newton's first law is not obeyed in such a reference frame. Let us illustrate this point. When we measure the weight of an object with a spring scale in an inertial frame (*e.g.*, on the surface of earth) and accelerated frame (*e.g.*, in an accelerating lift), the two weights will be different. Why? Because only in the inertial frame is the net force on the object is zero when it is at rest on the scale. In an accelerating frame, there must be a non-zero net force on the object for it to accelerate with the scale. Therefore, the scale reading and the gravitational force do not balance. The scale reading will be more or less than the actual weight (*i.e.*, weight in the inertial frame) depending on whether the acceleration is upward or downward.

CONCEPTUAL QUESTIONS

Q.1. A body is moving with a constant velocity. What does it mean?

Ans. It means that the body is moving with constant speed in a straight line.

Q.2. A body is moving with a constant acceleration. What about the force acting on the body ?

Ans. $F = ma$. Since mass is a constant quantity, constant acceleration means that a constant force is acting on the body.

Q.3. Can you cite a situation where no force acts on a body?

Ans. No. Atleast one force—the force of gravity—must act on the body. One can cite many examples where *net* force on the body is zero.

Q.4. What do you mean by inertia of a body?

Ans. Inertia means resistance to change. Therefore, inertia of a body means its resistance to change in motion. Now $F = m\,a$. For a given net force, an object with large mass will have a smaller acceleration *i.e.*, smaller change in velocity. This means that mass of a body is a measure of its inertia. The more mass a body has, the harder it is to change its state of motion; it is harder to start it moving from rest or to stop it when it is moving or to change its motion sideways out of straight line path.

Q.5. Why is Newton's first law of motion called law of inertia?

Ans. According to Newton's first law, in the absence of any *net* force on a body, the body is either at rest or it moves with a constant speed in a straight line. In other words, in order to change the motion (magnitude and/or direction of velocity), net external force is required. This means that everybody has a tendency to maintain its state of rest or of uniform velocity. For this reason, Newton's first law is called law of inertia.

Q.6. Can action and reaction forces cancel each other?

Ans. No. It is true that action and reaction forces are equal and opposite but they cannot cancel each other. It is because they act on different bodies. For this reason, when you are considering the forces on a body, only one of them (action or reaction) will be involved.

Q.7. What is the importance of law of conservation of linear momentum?

Ans. The law of conservation of linear momentum is a very useful tool for dealing with collision processes. For example, consider the head-on collision of two objects (*e.g.*, cars). In this situation, there is an important difference between the conservation of linear momentum and that of energy. Kinetic energy itself does not have to be conserved, because there are many other forms of energy into which it can be changed and still conserve *total* energy. But there is only one kind of linear momentum ; it cannot change into some other form. Thus conservation of linear momentum always applies in an isolated system, but you cannot say the same about kinetic energy.

Q.8. Where do we apply the concept of impulse?

Ans. Often we wish to apply the concept of impulse (*i.e.*, change in linear momentum) to situations where the applied force is not constant. For example, consider the case that a bat strikes the ball. In this case, it is fairly easy to measure the change in linear momentum. Therefore, impulse can be evaluated even though the average force and the time of contact may be extremely difficult to determine.

Q.9. What are position dependent forces?

Ans. Forces between two bodies which depend on the position of the bodies are called position dependent forces *e.g.* gravitational forces.

Q.10. Is the tension of a string on either side of a pulley the same? If so, under what condition?

Ans. The tension of a string on either side of pulley is the same if the pulley is frictionless.

Q.11. Why does a gun appear to have a greater kick when fired with the butt held loosely against the shoulder than when held tightly?

Ans. Let m = mass of bullet ; M = mass of gun

v = velocity of bullet; V = velocity of gun

When the gun is loosely held, the recoil velocity of the gun is $V = mv/M$. However, when the gun is held tightly, the man and the gun form a single rigid body. In this case, the recoil velocity of the gun = $mv/M + M'$ where M' is the mass of man. Since the velocity in the former case is larger, the man experiences a greater kick.

Q.12. A rocket moves forward by pushing the air backward. Is it true or false?

Ans. It is false. A rocket moves forward by the thrust produced by the gases ejected from the rear end.

Q.13. The linear momentum of a system of particles is always conserved. Is it true or false?

Ans. It is false. It is because linear momentum of a system is conserved only if the net external force on the system is zero.

Q.14. Can an object change speed or direction due to internal forces?

Ans. No. Changing object's state of motion requires a net external force. An object cannot change speed or direction by internal forces.

Q.15. If an object of mass m is accelerating, will its weight be mg?

Ans. An object's apparent weight will not be mg if it is accelerating. Its apparent weight is greater than mg if it is accelerating in a direction opposite to gravity and is less than mg if accelerating in the direction of gravity.

Q.16. Is weight a vector quantity?

Ans. Yes, weight is a vector quantity. The direction of weight is the direction of the gravitational force on the object.

VERY SHORT ANSWER QUESTIONS

Q.1. Several forces act simultaneously on a body. In which direction will it move?

Ans. It will move in the direction of resultant force.

Q.2. Is force needed to keep a body moving with uniform velocity?

Ans. No because $F = ma$ and in this case $a = 0$.

Q.3. If the net force on a body is zero, will the body remain necessarily in rest position?

Ans. No. The body may be moving with uniform velocity *i.e.*, with constant speed in a straight line.

Q.4. Define the SI unit of force.

Ans. The SI unit of force is newton (N). One newton (1N) is that force which produces an acceleration of 1 ms^{-2} in a mass of 1 kg.

Q.5. Several forces are acting on a body but the body is at rest. Why?

Ans. It means that vector sum of the forces (= resultant force) is zero.

Q.6. Can a force change only the direction of velocity of an object keeping its magnitude constant?

Ans. Yes. If a force is acting on a moving body perpendicular to the direction of motion, there will be change in direction of motion but no change in speed.

Q.7. What is inertia?

Ans. The tendency of a body to resist change of its state (rest or motion) is called inertia.

Q.8. What is measure of inertia?

Ans. The mass of a body is a measure of its inertia. The greater the mass of a body, the greater is its inertia and vice-versa.

Q.9. The distance travelled (x) by a moving body is directly proportional to time (t). Is any external force acting on it?

Ans. No external force is acting on the body. It is because $x = kt$ so that $d^2x/dt^2 = 0$. Since acceleration a $(= d^2x/dt^2)$ is zero, $F = ma = 0$.

Q.10. Two bodies having different masses have the same linear momentum. Which one of them will move faster?

Ans. Momentum, $p = mv$. Since p is constant, the body with smaller mass will move faster (*i.e.*, will have greater velocity).

Q.11. A ball of mass 1 kg moving with a speed of 10 ms^{-1} rebounds after striking normally a perfectly elastic wall. What is the change in linear momentum of the ball?

Ans. Change in linear momentum, Δp = Final linear momentum − Initial linear momentum

$$\therefore \quad \Delta p = -mv - (mv) = -2mv = -2 \times 1 \times 10 = -20 \text{ kg ms}^{-1}$$

Q.12. When are the two bodies said to possess equal masses?

Ans. The two bodies have equal masses if the same force produces the same acceleration in the two bodies.

Q.13. If action and reaction are equal and opposite, why do they not cancel each other?

Ans. Action and reaction act on different bodies. Therefore, they do not cancel each other.

Q.14. What is the principle of working of a rocket?

Ans. A rocket works on the principle of conservation of linear momentum.

Q.15. A bullet of mass m having velocity v gets embedded in a body of mass M. What is the velocity V of the system?

Ans. $mv = (M + m)V \quad \therefore \quad V = \dfrac{mv}{M+m}$

Q.16. An object is moving at a constant speed in a circular path. Does the object have a constant linear momentum?

Ans. No. Momentum = mv. Although speed is constant, the direction of velocity is changing continuously.

Q.17. What force will change the velocity of a body of mass 1 kg from 20 ms^{-1} to 30 ms^{-1} in 2 s?

Ans. $F \times t = mv_2 - mv_1$ or $F \times 2 = 1 \times 30 - 1 \times 20 \quad \therefore \quad F = 5$ N.

Q.18. Is the linear momentum of a ball falling freely conserved?

Ans. No. It is because freely falling ball is under the action of a net external force.

Q.19. A bomb explodes in mid-air into two equal fragments. What will be the direction of motion of the two fragments?

Ans. The two fragments will move in the opposite directions.

Q.20. What is impulse?

Ans. Impulse is equal to the change in linear momentum of the body.

Q.21. Why does an athelete run some distance before jumping?

Ans. Running provides greater linear momentum which helps in jumping through a greater distance.

Q.22. The area under force-time graph is 100 Ns for a short time interval of 1 s. What is the magnitude of impulsive force?

Ans. Impulsive force, $F = \dfrac{\text{Impulse}}{\text{Time}} = \dfrac{100\text{ Ns}}{1\text{ s}} = 100\text{ N}$

Q.23. A force of 5 N changes the velocity of a body from 20 ms^{-1} to 30 ms^{-1} in 5 s. How much force is required to bring about the same change in 2 s?

Ans. Impulse = $F \times t$ = Change in linear momentum. Here impulse is the same.

$\therefore \quad F_1 t_1 = F_2 t_2$ or $5 \times 2 = F_2 \times 2 \quad \therefore \quad F_2 = 5$ N.

Q.24. What are the different effects of a force?

Ans. A force can (*i*) cause linear motion (*ii*) cause rotation (*iii*) deform bodies.

Q.25. Two ends of a spring balance are pulled each by a force of 10 N. What will be the reading of the balance?

Ans. The reading of the balance will be 10 N.

SHORT ANSWER QUESTIONS

Q.1. While catching a ball, a cricket player lowers his hands. Explain.

Ans. Force = $\dfrac{\text{Impulse}}{\text{Time}}$.

By moving his hands backwards, a cricketer increases the time interval of impulse. The result is that the force that acts on the hand becomes much smaller.

Q.2. While firing a bullet, the gun should be held tight to the shoulder. Why?

Ans. When the bullet is fired, the gun moves in the backward direction (recoil of the gun). If there is space between the gun and the shoulder, the recoil velocity of the gun may hurt the shoulder. If the gun is held tight to the shoulder, then the gun and the body of the man become a single unit of large mass. As a result, the recoil velocity will be very small and shoulder of the man will not get hurt.

Q.3. The force acting on a body of mass 10 kg is $(2\hat{i}+\hat{j}-\hat{k})$N. If the body is initially at rest, what will be the velocity at the end of 20 s?

Ans. Acceleration, $\vec{a} = \frac{\vec{F}}{m} = \frac{2\hat{i}+\hat{j}-\hat{k}}{10}$

$\therefore$ Velocity after 20s, $\vec{v} = \vec{v_0} + \vec{a}t = 0 + \frac{2\hat{i}+\hat{j}-\hat{k}}{10} \times 20 = (4\hat{i}+2\hat{j}-2\hat{k})$

$\therefore$ $|\vec{v}| = \sqrt{(4)^2+(2)^2+(-2)^2} = 2\sqrt{6}$ ms^{-1}

Q.4. In the game of swing in a circus, the man falls on the net after leaving the swing but he is not injured. Why?

Ans. Force = Impulse/Time. When the man falls on the net, it is depressed where man falls on it and thus the contact time is increased. As a result, the force of reaction on the man is decreased to a great extent and the man is not injured.

Q.5. If the force on a rocket moving with a velocity of 300 ms^{-1} is 210 N, what is the rate of consumption of fuel?

Ans. Force on rocket, $F = v\frac{dm}{dt}$. Here F = 210 N ; v = 300 ms^{-1}.

$\therefore$ Rate of consumption of fuel, $\frac{dm}{dt} = \frac{F}{v} = \frac{210}{300}$ = 0.7 kg/s

Q.6. A stone thrown on a glass window pane smashes it to pieces but a bullet fired on it makes a clean hole in it. Why?

Ans. This is due to the fact that velocity of the stone is much less than that of the bullet. Due to low speed, the stone remains in contact with the window pane for a long time. The result is that the motion (momentum) of the stone is shared by a large part of window pane. Consequently, the window pane is smashed into pieces. However, in case of bullet, due to large speed, the bullet remains in contact with window pane for a very short time. The result is that bullet shares its motion with only a small part of the pane. Consequently, the bullet makes a clean hole in the window pane.

Q.7. A spring weighing machine inside a stationary lift reads 50 N when a man stands on it. What would happen to the scale reading if the lift is moving upward with (*i*) constant velocity (*ii*) constant acceleration?

Ans. When lift moves upward, apparent weight = $m(g + a)$ where a is the acceleration of the lift.

(*i*) Scale reading = Apparent weight = $m(g + 0) = mg$ = 50 N

(*ii*) Scale reading = Apparent weight = $m(g + a)$ = more than 50 N

Q.8 A fish of mass 2 kg is suspended on a spring balance hung vertically in a lift. What will be the reading of the balance if the lift is (*i*) falling down under acceleration due to gravity g (*ii*) going upward with g?

Ans. (*i*) Reading of balance = $m(g - a) = m(g - g) = 0$

(*ii*) Reading of balance = $m(g + a) = m(g + g) = 2\ mg = 2 \times 2 \times 9.8 = 39.2$ N

Q.9. Can a bomb, initially at rest, explode into three pieces which fly in mutually perpendicular directions?

Ans. No. The conservation of linear momentum requires that linear momentum after explosion should be zero. This is possible if the vector sum of the three momentum vectors is zero. Since in the present case, the three momentum vectors are not coplanar (*i.e.*, they are not in the same plane) such an explosion cannot take place.

Q.10. When several passengers are standing in a running bus, it is said to be dangerous. Why?

Ans. This is due to the following three reasons :

(*i*) The standing passengers raise the centre of gravity of bus. Consequently, the bus comes in unstable equilibrium and may overturn.

(*ii*) If the bus stops suddenly due to any reason, the passengers are pushed forward and standing passengers may dangerously collide with each other.

(*iii*) If sudden brakes are applied, the forward linear momentum of the overcrowded bus may push it slightly ahead. This may cause accident in congested traffic situation.

LONG ANSWER QUESTIONS

1. Explain the terms (*i*) momentum (*ii*) force and (*iii*) impulse. **[Refer to Arts. 10.6, 10.7 and 10.9]**
2. State and explain Newton's first law of motion. **[Refer to Art. 10.3]**
3. State Newton's second law of motion and derive the equation of motion: $F = ma$. **[Refer to Art. 10.7]**
4. Show that Newton's second law of motion is the real law of motion. **[Refer to Art. 10.14]**
5. State and explain with two examples Newton's third law of motion. **[Refer to Arts. 10.12 and 10.13]**
6. State and prove principle of conservation of linear momentum. **[Refer to Art. 10.16]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. When a person of mass m climbs up a rope with acceleration a, the tension T in the rope is equal to apparent weight of the person.

$$T = \text{Apparent weight of person} = m\,(g + a)$$

For downward journey with acceleration a, we have,

$$T = \text{Apparent weight of person} = m\,(g - a)$$

2. Consider two blocks of masses m_1 and m_2 in contact on a smooth horizontal surface as shown in Fig. 10.32. When a force F is applied on m_1, the two blocks move with a common acceleration 'a' given by ;

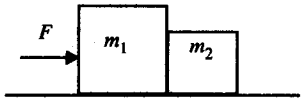


Fig. 10.32

$$\text{Common acceleration, } a = \frac{F}{m_1 + m_2}$$

$$\text{Force on } m_1,\ F_1 = F$$

$$\text{Force on } m_2,\ F_2 = m_2 a = m_2 \times \frac{F}{m_1 + m_2}$$

3. Consider three blocks of masses m_1, m_2 and m_3 in contact with one another on a smooth horizontal surface as shown in Fig. 10.33. When a force F is applied on m_1, the three blocks move with a common acceleration 'a' given by ;

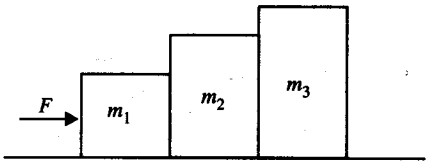


Fig. 10.33

$$\text{Common acceleration, } a = \frac{F}{m_1 + m_2 + m_3}$$

$$\text{Force on } m_1,\ F_1 = F$$

$$\text{Force on } m_2,\ F_2 = (m_2 + m_3)a = \frac{(m_2 + m_3)F}{m_1 + m_2 + m_3}$$

$$\text{Force on } m_3,\ F_3 = m_3 a = \frac{m_3 F}{m_1 + m_2 + m_3}$$

4. Consider two blocks of masses m_1 and m_2 in contact on a smooth horizontal surface as shown in Fig. 10.34 (*i*). When a force F is applied on m_1, the two blocks move with a common acceleration 'a' given by ;

$$\text{Common acceleration, } a = \frac{F}{m_1 + m_2}$$

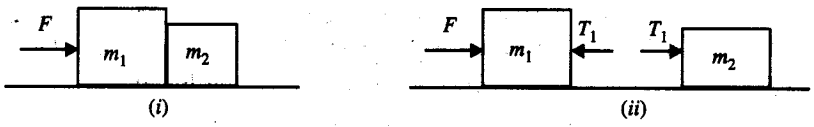


Fig. 10.34

Let us investigate the equation of motion of each body. For this purpose, refer to Fig. 10.34(*ii*) where action and reaction forces are shown.

For first body, $F - T_1 = m_1 a$; For second body, $T_1 = m_2 a$

5. Consider three blocks of masses m_1, m_2 and m_3 in contact with one another on a smooth horizontal surface as shown in Fig. 10.35 (*i*). When force F is applied on m_1, the three blocks move with a common acceleration 'a' given by ;

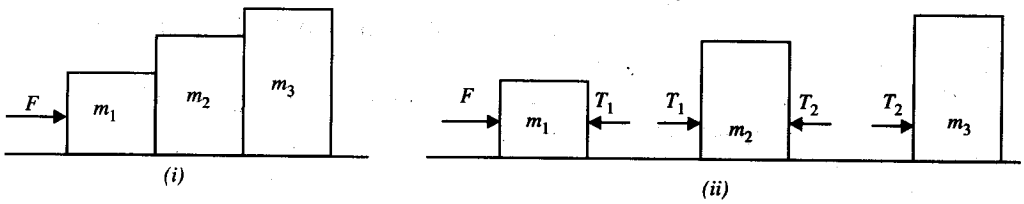


Fig. 10.35

Common acceleration, $a = \dfrac{F}{m_1 + m_2 + m_3}$

Fig. 10.35 (*ii*) shows the action-reaction forces on the three blocks. We can easily find the equation of motion for each block.

For block 1, $F - T_1 = m_1 a$; For block 2, $T_1 - T_2 = m_2 a$; For block 3, $T_2 = m_3 a$

6. Fig. 10.36 shows mass m_1 on a smooth horizontal table and connected to mass m_2 by a light string passing over a frictionless pulley of negligible mass. If 'a'is the acceleration of the system in the direction shown, then equations of motion for the two masses are :

$$T = m_1 a \ ; \ m_2 g - T = m_2 a$$

On solving, $a = \dfrac{m_2}{m_1 + m_2} \times g$; $T = \dfrac{m_1 m_2}{m_1 + m_2} \times g$

Note that T is always less than $m_2 g$ and common acceleration 'a' of the system is always less than g.

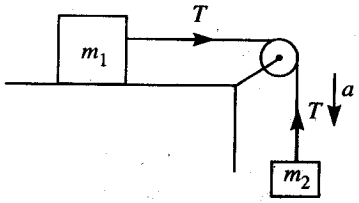


Fig. 10.36

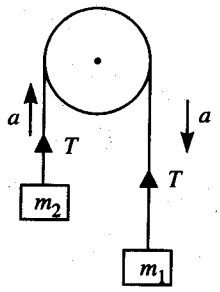


Fig. 10.37

7. Consider two masses m_1 and m_2 connected to the opposite ends of a light string passing over a frictionless and massless pulley as shown in Fig. 10.37. Suppose $m_1 > m_2$. As a result, mass m_1 moves downward while mass m_2 moves upward. Let the common acceleration of the system be 'a' . Then equations of motion for the two masses are :

$$m_1 g - T = m_1 a \quad \text{and} \quad T - m_2 g = m_2 a$$

$$\text{On solving, } a = \frac{m_1 - m_2}{m_1 + m_2} \times g \ ; \ T = \frac{2 m_1 m_2}{m_1 + m_2} \times g$$

Note that when $m_1 = m_2$, then, $a = 0$ and $T = m_1 g$.

8. Fig. 10.38 shows two masses m_1 and m_2 suspended vertically from a rigid support with the help of strings. If mass m_2 is pulled down with a force F, then tensions T_1 and T_2 are given by :

$$T_2 = F + m_2 g \quad \text{and} \quad T_1 = F + (m_1 + m_2)\, g$$

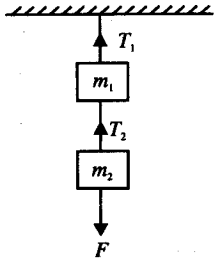


Fig. 10.38

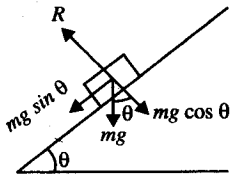


Fig. 10.39

9. Fig. 10.39 shows a block of mass m sliding down a smooth inclinded plane making an angle θ with the horizontal.

Force on block down the plane is given by ;

$$F = mg \sin\theta$$

Acceleration of the block down the plane is given by ;

$$a = \frac{F}{m} = \frac{mg\sin\theta}{m} = g\sin\theta$$

Note that gravity is the accelerating force yet the acceleration is not vertical. Further, acceleration a is independent of mass of the block. It depends only on the angle of inclination θ and g.

10. Fig. 10.40 shows two masses m_1 and m_2 attached to the opposite ends of a string passing over a frictionless pulley at the edge of a smooth inclined plane. If the system moves up the inclined plane with an acceleration 'a', then equations of motion for the two bodies are :

$$T - m_1 g\sin\theta = m_1 a \text{ and } m_2 g - T = m_2 a$$

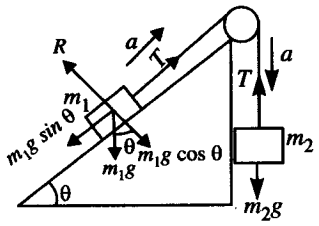


Fig. 10.40

On solving these equations, we have,

$$a = \frac{m_2 g - m_1 g\sin\theta}{m_1 + m_2} \; ; \; T = \frac{m_1 m_2 g(1+\sin\theta)}{m_1 + m_2}$$

(*i*) If m_2 is greater than $m_1 \sin\theta$, the acceleration of m_1 is up the inclinde plane. If $m_1 \sin\theta$ exceeds m_2, the acceleration of m_1 is down the inclined plane.

(*ii*) The expression for the acceleration can be interpreted as the resultant unbalanced force $(= m_2 g - m_1 g\sin\theta)$ on the system divided by the total mass of the system.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A force of 6 N acts on a body at rest and mass 1 kg. During the time the body attains a velocity of 30 m/s. The time for which the force acts on the body is **[CBSE PMT 1997]**

(*a*) 10 seconds (*b*) 8 seconds
(*c*) 7 seconds (*d*) 5 seconds

2. A body whose momentum is constant must have constant **[AIIMS 2000]**

(*a*) force (*b*) velocity
(*c*) acceleration (*d*) all of these

3. A man is standing on a spring platform. The reading of the spring balance is 60 kg wt. If

the man jumps outside the platform, then reading of the spring balance will **[AIIMS 2000]**

(*a*) increase (*b*) remain same

(*c*) become zero (*d*) first (*a*) then (*c*)

4. A body is moving with a constant speed in a straight line path. A force is not required to **[AFMC 2001]**

(*a*) increase its speed

(*b*) decrease the momentum

(*c*) change the direction

(*d*) keep it moving with uniform velocity

5. If a body is in equilibrium under a set of non-linear forces, then the minimum number of forces has to be **[Pb PMT 2001]**

(*a*) four (*b*) three

(*c*) two (*d*) five

6. A force vector applied on a mass is represented as $\vec{F} = 6\hat{i} - 8\hat{j} + 10\hat{k}$ and accelerates with $1 m/s^2$. What will be the mass of the body? **[CBSE PMT 1996]**

(*a*) $10\sqrt{2}$ kg (*b*) $2\sqrt{10}$ kg

(*c*) 10 kg (*d*) 20 kg

7. The rocket engines lift a rocket from the earth because hot gases with high velocity **[AIIMS 1998]**

(*a*) push it against the earth

(*b*) push it against the air

(*c*) react against rocket and push it up

(*d*) heat up the air which lifts the rocket

8. The motion of rocket is based on the principle of conservation of **[AFMC-Pune 2000]**

(*a*) mass (*b*) kinetic energy

(*c*) linear momentum

(*d*) angular momentum

9. If the force on a rocket moving with a velocity of 300 m/s is 210 N, then rate of combustion of fuel is **[AIIMS 1997]**

(*a*) 0.7 kg/s (*b*) 1.4 kg/s

(*c*) 7 kg/s (*d*) 10.7kg/s

10. A 5000 kg rocket is set for vertical firing. The exhaust speed is 800 m/s and acceleration (*a*) of the rocket is $20 m/s^2$. The acceleration due to gravity is $10 m/s^2$. The amount of gas ejected per second is **[Pb PMT 2001]**

(*a*) 127.5 kg/s (*b*) 137.5 kg/s

(*c*) 185.5 kg/s (*d*) 187.5 kg/s

11. A body of mass 2 kg is moving with a velocity of 8 m/s on a smooth surface. If it is to be brought to rest in 4 seconds, then force to be applied is **[Pb PMT 2000]**

(*a*) 8 N (*b*) 4 N

(*c*) 2 N (*d*) 1 N

12. The forces which meet at one point but their lines of action do not lie in one plane are called **[AIIMS 1996]**

(*a*) coplanar concurrent forces

(*b*) coplanar non-concurrent forces

(*c*) non-planar concurrent forces

(*d*) non-planar non-concurrent forces

13. A boy of mass 50 kg is standing in a lift moving down with an acceleration $9.8 m/s^2$. The apparent weight of the boy is **[Pb PMT 1999]**

(*a*) zero (*b*) 50 N

(*c*) (50×9.8)N (*d*) 50/9.8 N

14. A monkey is descending from the branch of a tree with a constant acceleration. If the breaking strength of the branch is 75% of the weight of the monkey, the minimum acceleration with which the monkey can slide down without breaking the branch is **[CBSE PMT 1993]**

(*a*) g (*b*) $3g/4$

(*c*) $g/2$ (*d*) $g/4$

15. The force acting on a body of mass 10 kg is $(2\hat{i} + \hat{j} - \hat{k})N$. If the body is initially at rest, the velocity at the end of 20 sec will be (m/s) **[Pb PMT 1997]**

(*a*) $2\sqrt{6}$ (*b*) $6\sqrt{2}$

(*c*) $3\sqrt{2}$ (*d*) $2\sqrt{3}$

16. A cricket ball of mass 250 g collides with a bat with velocity of 10 m/s and returns with same velocity within 0.01 sec. The force acted on the bat is **[CPMT-1997]**

(*a*) 25 N (*b*) 50 N

(*c*) 250 N (*d*) 500 N

17. A mass of 1 kg is thrown up with a velocity of 100 m/s. After 5 seconds, it explodes into two parts. One part of mass 400 *g* comes down with a velocity of 25 m/s. Calculate the velocity of other part. **[CBSE PMT 2000]**

(*a*) $40 m/s \uparrow$ (*b*) $40 m/s \downarrow$

(*c*) $100 m/s \uparrow$ (*d*) $60 m/s \uparrow$

18. Two bodies having same mass 40 kg are moving in opposite directions, one with a velocity of 10 m/s and other with a velocity of

7 m/s. If they collide elastically and move as one body, the velocity of combination is **[Pb PMT 2000]**

(a) 10 m/s (b) 1.5 m/s
(c) 7 m/s (d) 3 m/s

19. A person swimming in a fresh water pool obeys **[AFMC 2003]**

(a) Newton's second law
(b) Gravitational law
(c) Newton's third law
(d) Newton's first law

20. What determines the nature of the path followed by the particle? **[AFMC 2005]**

(a) Speed (b) Velocity
(c) Acceleration
(d) None of the above

21. Three blocks of masses m_1, m_2 and m_3 are connected by massless strings as shown in Fig. 10.41 on a frictionless table. If m_1 = 10 kg, m_2 = 6 kg and m_3 = 4 kg, then tension T_2 will be: **[Manipal 2004]**

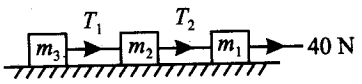


Fig. 10.41

(a) 10 N (b) 20 N
(c) 32 N (d) 40 N

22. A force of 10 N acts on a body of mass 20 kg for 10 seconds. Then change in its momentum is **[Manipal 2004]**

(a) 50 kg m/s (b) 100 kg m/s
(c) 300 kg m/s (d) 1000 kg m/s

23. A weight W is suspended from the midpoint of a rope, whose ends are at the same level. In order to make the rope perfectly horizontal, the force applied to each of its ends must be **[Manipal 2005]**

(a) less than W (b) equal to W
(c) equal to $2W$ (d) infinitely large

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (d)	**2.** (b)	**3.** (d)	**4.** (d)	**5.** (b)
6. (a)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (d)
11. (b)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (a)
16. (d)	**17.** (c)	**18.** (b)	**19.** (c)	**20.** (c)
21. (c)	**22.** (b)	**23.** (d)		

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The acceleration produced in the body is $a = \frac{F}{m} = \frac{6}{1} = 6\,\text{m/s}^2$

   Now $v = v_0 + at$ or $30 = 0 + 6 \times t$

   $\therefore \quad t = 30/6 = 5$ seconds

2. Momentum = Mass × Velocity

   Since mass is constant, constant momentum means the velocity of the body is constant.

3. When the man jumps from the platform, first the platform goes down with some force. After some ups and downs, the platform will stop. Therefore, reading of the spring balance will first increase and then becomes zero when the platform stops.

4. According to Newton's first law of motion, no force is required to keep the body in motion with uniform velocity.

5. We know that if three non-linear forces are acting on a body and the body is in equilibrium, then these forces can be represented by the sides of a triangle. Therefore, minimum number of forces is three.

6. Mass, $m = \frac{F}{a} = \frac{\sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2}}{a} = \frac{\sqrt{36+64+100}}{1} = 10\sqrt{2}$ kg

7. According to Newton's third law of motion, the force exerted by the hot gases from the rocket is equal to the thrust of the air on the rocket.Therefore, hot gases from the rocket push the rocket against air and the air pushes the rocket upward with the same force.

8. The motion of a rocket is based on the principle of conservation of linear momentum. Before the rocket is fired, the total linear momentum of the rocket plus fuel is zero. When rocket is fired, fuel is burnt and very hot gases are formed. These gases are expelled from the back of the rocket. Since the momentum acquired by the gases is directed toward the rear, the rocket must acquire an equal momentum in the opposite direction (forward) in order to conserve momentum.

9. Force on rocket, $F = v\frac{dm}{dt}$

$\therefore$ Rate of combustion of fuel is $\frac{dm}{dt} = \frac{F}{v} = \frac{210}{300} = 0.7$ kg/s

10. Force on rocket, $F = m\,(g + a) = 5000\ \ (10 + 20) = 150000$ N

Now $F = v\frac{dm}{dt}$ $\therefore$ $\frac{dm}{dt} = \frac{F}{v} = \frac{150000}{800} = 187.5$ kg/s

11. Retardation, $a = \frac{0 - v_0}{t} = \frac{0-8}{4} = -2$ m/s^2

$\therefore$ Force to be applied, $F = ma = 2 \times 2 = 4$ N

Negative sign is omitted because only magnitude is required.

12. The forces which meet at a point on a body are called concurrent forces. If their lines of action do not lie in one plane, they are called non-planar forces. Therefore, forces which meet at a point but their lines of action do not lie in one plane are called non-planar concurrent forces.

13. Apparent weight $= m(g - a) = 50(9.8 - 9.8) = 0$ N

14. Weight of monkey $= mg$; Breaking strength, $T = \frac{3}{4}mg$

If 'a' is the minimum acceleration, then equation of motion of the monkey is

$$mg - T = ma$$

or $$mg - \frac{3}{4}mg = ma \quad \therefore \quad a = g - \frac{3}{4}g = g/4$$

15. Acceleration, $\vec{a} = \frac{\vec{F}}{m} = \frac{2\hat{i} + \hat{j} - \hat{k}}{10}$

Therefore, velocity $\vec{v}$ after 20 sec will be

$$\vec{v} = \vec{v}_0 + \vec{a}t = 0 + \frac{2\hat{i} + \hat{j} - \hat{k}}{10} \times 20 = (4\hat{i} + 2\hat{j} - 2\hat{k})$$

$\therefore$ $$|\vec{v}| = \sqrt{(4)^2 + (2)^2 + (-2)^2} = 2\sqrt{6} \text{ m/s}$$

16. $$F \times t = mv - mv_0 = 0.25\ (v - v_0) = 0.25\ [10 - (-10)] = 5 \text{ Ns}$$

$\therefore$ $$F = \frac{5}{t} = \frac{5}{0.01} = 500 \text{ N}$$

17. After 5 seconds, the velocity v of 1kg mass is

$$v = v_0 - gt = 100 - 10 \times 5 = 50 \text{ m/s} \uparrow$$

According to law of conservation of linear momentum,

$$mv = m_1v_1 + m_2v_2$$

or $$1\times(-50) = 0.4\times25 + 0.6\times v_2$$

$\therefore$ $$v_2 = -100\,\text{m/s or } v_2 = \mathbf{100\ m/s}$$

18. $v_1 = 10\,\text{m/s}$; $v_2 = -7\,\text{ms}^{-1}$. If V is the velocity of the combination after collision, then according to law of conservation of linear momentum,

$$(m+m)V = mv_1 + mv_2 \text{ or } (40+40)V = 40\times10 + 40(-7)$$

or $$80V = 400 - 280 = 120 \quad \therefore \quad V = 120/80 = \mathbf{1.5\ m/s}$$

19. A person swimming in a fresh water pool obeys **Newton's third law of motion.** When the swimmer applies force (action) to push water in the backward direction, then water pushes (reaction) him in the forward direction.

20. The force applied on the particle produces acceleration in it. Therefore, **acceleration** determines the path followed by the particle.

21. The three blocks will move with a common acceleration a given by ;

$$a = \frac{F}{m_1+m_2+m_3} = \frac{40}{10+6+4} = 2\,\text{ms}^{-2}$$

$\therefore$ Tension, $T_2 = (m_1 + m_2)a = (10 + 6)2 = \mathbf{32\ N}$

22. Change in momentum $= F \times \Delta t = 10 \times 10 = \mathbf{100\ kg\ ms^{-1}}$

23. Fig. 10.42 shows the conditions of the problem. It is clear from the figure that :

$$mg = 2T\cos\theta \quad \text{or} \quad T = \frac{mg}{2\cos\theta}$$

For the string to be horizontal, θ should be 90°.

$\therefore$ $$T = \frac{mg}{2\cos90^\circ} \text{ or } T \to \infty$$

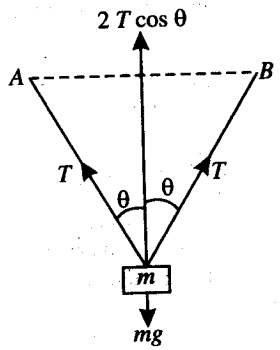


Fig. 10.42

Therefore, in order to make the rope perfectly horizontal, the applied force should be **infinitely large.**

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A cricket player catches a ball of mass 0.1 kg moving with a speed of 10 m/s in 0.1 sec. Force exerted by him is **[CET Karnataka 1999]**

(*a*) 10 N (*b*) 5 N
(*c*) 2 N (*d*) 1N

2. An impulse is applied to a moving object with a force at an angle of 20° *w.r.t.* the velocity vector. The angle between the impulse vector and the change in momentum vector is **[EAMCET 1990]**

(*a*) 0° (*b*) 60°
(*c*) 160° (*d*) 240°

3. If block A of mass m_1 is in contact with another block B of mass m_2 as shown in Fig. 10.43 and a force F is applied horizontally to block A and the surfaces are frictionless, the force exerted by A on B is given by **[CET Haryana 2000]**

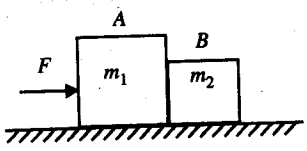


Fig. 10.43

(*a*) $\frac{m_1F}{m_1+m_2}$ (*b*) $\frac{m_2F}{m_1+m_2}$

(*c*) $\frac{m_1}{m_2}F$ (*d*) $\frac{m_2}{m_1}F$

4. A ship of mass 2×10^7 kg is at rest. On applying a force of 25×10^5 N, it is displaced through 25 m. The speed of the ship after this displacement is **[CET Karnataka 1998]**

(a) 3.5 m/s (b) 5 m/s

(c) 12.5 m/s (d) 2.5 m/s

5. A ball of mass 10 g hits a hard surface vertically with a speed of 5 m/s and rebounds with the same speed. The ball remains in contact with the surface for 0.01 sec. The average force exerted by the surface on the ball is **[Roorkee 1993]**

(a) 100 N (b) 10 N (c) 1N (d) 0.1N

6. A block of mass M is pulled along a horizontal surface by a rope of mass m by applying a force F at one end of the rope. The force which the rope exerts on the block is **[CET Haryana 2000]**

(a) $\frac{mF}{m+M}$ (b) $\frac{mF}{M-m}$

(c) $\frac{MF}{M-m}$ (d) $\frac{FM}{m+M}$

7. A light string passes over a frictionless pulley. To one of its ends, a mass of 6 kg is attached. To its other end, a mass of 10 kg is attached

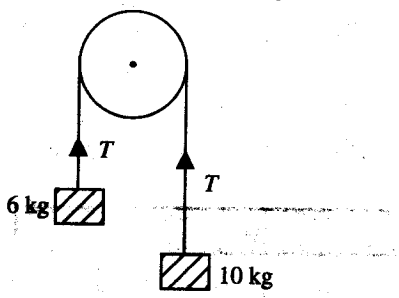


Fig. 10.44

(See Fig. (10.44). The tension in the string will be **[Rajasthan PET 1996]**

(a) 24.5 N (b) 2.45 N

(c) 79 N (d) 73.5 N

8. A mass M of 100 kg is suspended by strings A, B and C as shown in Fig. 10.45 where W is the vertical wall and R is a rigid horizontal rod. The tension in string B is **[BHU 1994]**

(a) 100g N (b) 0

(c) $100\sqrt{2}\,g$ N (d) $\frac{100}{\sqrt{2}}\,g$ N

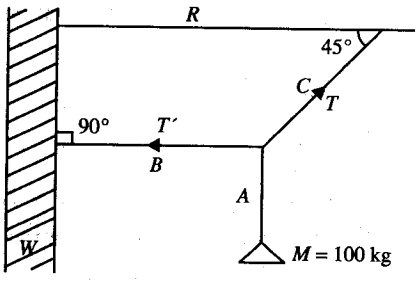


Fig. 10.45

9. A person is sitting in a travelling train and facing engine. He tosses up a coin and coin falls behind him. It can be concluded that train is **[SCRA 1994]**

(a) moving forward and gaining speed

(b) moving forward and losing speed

(c) moving forward with uniform speed

(d) moving backward with uniform speed

10. The engine of a car produces acceleration of 4 m/s^2 in car. If this car pulls another car of same mass, what will be the acceleration produced? **[Rajasthan PET 1996]**

(a) 8 m/s^2 (b) 4 m/s^2

(c) 2 m/s^2 (d) 0.5 m/s^2

11. A 100 kg man jumps into swimming pool from a height of 5 m. It takes 0.4 sec for the water to reduce his velocity to zero. The average force exerted by water on the man is **[CET 1998]**

(a) 25 N (b) 250 N

(c) 2500 N (d) 5000 N

12. A ship of mass 3×10^7 kg initially at rest is pulled by a force of 5×10^4 N through a distance of 3m. Assuming the resistance due to water to be negligible, the speed of the ship is **[I.I.T. 1990]**

(a) 0.1 m/s (b) 60 m/s

(c) 1.5 m/s (d) 5 m/s

13. A man of mass 75 kg is standing on spring balance inside a lift. If the lift falls freely downward, the reading of the spring balance will be **[CET Karnataka 1993]**

(a) 75 kgf (b) >75 kgf

(c) <75 kg f (d) zero

14. A particle of mass 0.3 kg is subjected to a force $F = -kx$ where $k = 15$ N/m. What will be its initial acceleration, if it is released from a point 20 cm away from the origin? **[AIEEE 2005]**

(a) 3 ms^{-2} (b) 15 ms^{-2}
(c) 5 ms^{-2} (d) 10 ms^{-2}

15. A machine gun fires a bullet of mass 40 g with a velocity 1200 ms^{-1}. The man holding it can exert a maximum force of 144 N on the gun. How many bullets can he fire per second at the most? **[AIEEE 2004]**
(a) One (b) Four
(c) Two (d) Three

16. A nucleus disintegrates into two nuclear parts which have their velocities in the ratio 2:1. The ratio of their nuclear sizes will be **[AIEEE 2004]**
(a) $2^{1/3} : 1$ (b) $1 : 3^{1/2}$
(c) $3^{1/2} : 1$ (d) $1 : 2^{1/3}$

17. A light spring balance hangs from the hook of the other light spring balance and a block of mass M kg hangs from the former one. Then the true statement about the scale reading is **[AIEEE 2003]**
(a) both the scales read M kg each
(b) the scale of the lower one reads M kg and of the upper one zero
(c) the reading of the two scales can be anything but the sum of the readings will be M kg
(d) both the scales read $M/2$ kg

18. Two blocks A and B of masses $2m$ and m, respectively are connected by a massless and inextensible string. The whole system is suspended by a massless spring as shown in Fig. 10.46. The magnitudes of acceleration of A and B, immediately after the string is cut, are respectively. **[IIT 2006]**

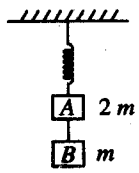


Fig. 10.46

(a) g, $g/2$ (b) $g/2$, g
(c) g, g (d) $g/2$, $g/2$

19. A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane. Initially the blocks are at rest and the spring is unstretched. Then, a constant force F starts acting on the block of mass M to pull it. Find the force on block of mass m. **[AIEEE 2007]**
(a) $\frac{mF}{M}$ (b) $\frac{(M+m)F}{M}$
(c) $\frac{mF}{(M+m)}$ (d) $\frac{MF}{(M+m)}$

20. A block is kept on a frictionless inclined surface with angle of inclination α. The incline is given an acceleration a to keep the block stationary (See Fig. 10.47). Then, a is equal to **[AIEEE 2005]**

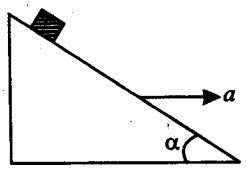


Fig. 10.47

(a) g/tan α (b) g cosec α
(c) g (d) g tan α

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (a)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (b)
6. (d)	**7.** (d)	**8.** (a)	**9.** (a)	**10.** (c)
11. (c)	**12.** (a)	**13.** (d)	**14.** (d)	**15.** (d)
16. (d)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (d)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Retardation, $a = \frac{v - v_0}{t} = \frac{0-10}{0.1} = -100 m/s^2$
Force exerted by the player is $F = ma = 0.1 \times 100 = 10$ N
Negative sign is omitted because only magnitude is required.

2. Impulse and change in linear momentum are in the same direction. Therefore, angle between the two is zero degree.

3. The two blocks will move with common acceleration 'a' given by ;

$$a = \frac{F}{m_1 + m_2}$$

$\therefore$ Force exerted by A on $B = m_2 a = \dfrac{m_2 F}{m_1 + m_2}$

4. Acceleration of the ship, $a = \dfrac{F}{m} = \dfrac{25\times10^5}{2\times10^7} = 12.5\times10^{-2}\ \text{m/s}^2$

Initial velocity, $v_0 = 0$; Final velocity, v = ? ; Displacement, S = 25 m

$\therefore \quad v^2 - v_0^2 = 2\ aS$

or $v^2 - 0^2 = 2\times(12.5\times10^{-2})\times25 = 6.25$ or $v = \sqrt{6.25} =$ **2.5 m/s**

5. $F \times t$ = Change in linear momentum

or $F\times t = m(v+v)$ or $F\times0.01 = \dfrac{10}{1000}(5+5) = \dfrac{1}{10}$

$\therefore \quad F = \dfrac{1}{10\times0.01} = \mathbf{10N}$

6. Acceleration, $a = \dfrac{F}{m+M}$ $\therefore$ Force on block $= Ma = \dfrac{FM}{m+M}$

7. Let 'a' be the acceleration of the system (The mass 10 kg will move downward while 6 kg will move upward).

$$a = \frac{\text{Resultant unbalanced force}}{\text{Total mass of system}} = \frac{(10-6)9.8}{10+6} = 2.45\ \text{m/s}^2$$

The equation of motion of 10 kg mass is

$10 \times 9.8 - T = 10a$ or $98 - T = 10 \times 2.45$ $\therefore$ $T =$ **73.5 N**

8. Since the system is stationary,

$T \sin 45° = 100$ g and $T \cos 45° = T'$

$\therefore$ Tension in B, $T' = T\cos45° = \dfrac{100g}{\sin45°}\times\cos45° = \mathbf{100g}$ **N**

9. Since the coin falls behind the person, the train must be **moving forward and gaining speed.**

10. Acceleration $= \dfrac{\text{Force}}{\text{Mass}}$. For the same force, if mass is doubled, the acceleration is halved.

$\therefore$ New acceleration = 4/2 = 2 **m/s^2**

11. Velocity v of the man just before entering water is

$v^2 - (0)^2 = 2\ gh$ or $v = \sqrt{2gh} = \sqrt{2\times10\times5} = 10\ \text{m/s}$

Now $F \times t = mv - 0$ or $F = \dfrac{mv}{t} = \dfrac{100\times10}{0.4} =$ **2500 N**

12. $$\text{Acceleration, } a = \frac{F}{m} = \frac{5\times10^4}{3\times10^7} = \frac{5}{3}\times10^{-3}\ \text{m/s}^2$$

Now $$v^2 - (0)^2 = 2as \quad \text{or} \quad v^2 = 2\times\frac{5}{3}\times10^{-3}\times3 = 10^{-2}$$

$$\therefore \quad v = \sqrt{10^{-2}} = \mathbf{0.1\ m/s}$$

13. Spring balance reading $= m(g - a) = m(g - g) = \mathbf{0}$.

14. $F = -kx$ and $F = ma$

$\therefore -kx = ma$ or $-15 \times 20 \times 10^{-2} = 0.3a \quad \therefore \quad a = \mathbf{-10\ ms^{-2}}$

15. Force exerted on the gun in firing a bullet = change in linear momentum of bullet per second $= 0.04 \times 1200 = 48$ N. The maximum force that the man can exert is 144 N. Therefore, the number of bullets fired per second $= 144/48 = \mathbf{3}$.

16. According to law of conservation of linear momentum, we have,

$$m_1v_1 = m_2v_2 \quad \text{or} \quad m_1/m_2 = v_2/v_1 \qquad ...(i)$$

Now $$m = (4/3)\,\pi\, r^3\, \rho \quad \text{or} \quad m \propto r^3$$

$\therefore$ From eq. (*i*), $$\frac{m_1}{m_2} = \frac{r_1^3}{r_2^3} = \frac{v_2}{v_1} = \frac{1}{2}$$

$$\therefore \quad \frac{r_1}{r_2} = \left(\frac{1}{2}\right)^{1/3} \quad \text{or} \quad r_1 : r_2 = \mathbf{1 : 2^{1/3}}$$

17. Both scales read M kg. It is because tension in the series-connected spring balances is the same.

18. Just before the string is cut, the force on the spring pulling up = 3 *mg*. After the string is cut, the free–body diagram for block *A* is shown in Fig. 10.48(*i*). Net force on block *A* = 3 *mg* − 2 *mg* = *mg* so that magnitude of acceleration of block *A* is $a_A = mg/2m = g/2$. After the string is cut, the free–body diagram for block *B* is shown in Fig. 10.48(*ii*). It is clear that $a_B = mg/m = g$.

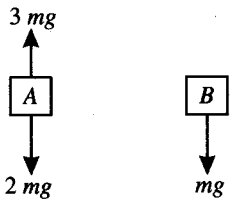


Fig. 10.48

19. Acceleration of the system, $a = \dfrac{\text{Applied force}}{\text{Total mass}} = \dfrac{F}{M+m}$

$\therefore$ Force on block of mass $m = ma = m\left(\dfrac{F}{M+m}\right) = \dfrac{mF}{M+m}$

20. When the incline is given an acceleration *a* towards the right, the block receives a reaction *ma* towards the left. The block will remain stationary if the reaction component *ma* cos α and weight component *mg* sin α cancel each other *i.e.*,

$$ma\cos\alpha = mg\sin\alpha \quad \text{or} \quad a = g\tan\alpha$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. What should be the tension in a rope pulling vertically upward a mass of 100 kg so that starting from rest it acquires a velocity of 4 ms^{-1} in 1.5 seconds?

Hint. Acceleration, $a = \dfrac{v - v_0}{t} = \dfrac{4-0}{1.5} = 2.67\ \text{ms}^{-2}$

If T is the tension in the rope, then,

Net upward force $= T - m\,g$

or $m\,a = T - mg$

or $T = m\,(a+ g) = 100\,(2.67 + 9.8) = 1247$ N

2. **The total mass of an elevator, with 80 kg man in it, is 1000 kg. This elevator moving upward with a speed of 8 ms^{-1} is brought to rest over a distance of 16 m. Calculate (*i*) the tension *T* in the cable supporting the elevator (*ii*) the force exerted on the man by the elevator floor.**

Hint. Acceleration *a* of the elevator is given by ;

$$v^2 - v_0^{\,2} = 2\,as \quad \text{or } a = \frac{v^2 - v_0^{\,2}}{2s} = \frac{(0)^2 - (8)^2}{2\times 16} = -2\,ms^{-2}$$

(*i*) If *T* is the tension in the cable and *M* is the total mass of the elevator, then,

Net upward force = *M a*

or $T - Mg = Ma \quad \therefore\ T = M\,(a + g) = 1000\,(-2 + 9.8) = 7800$ N

(*ii*) Let *R* be the force exerted on the man by the elevator floor. If *m* is the mass of the man, then,

Net upward force on man = *m a*

or $R - mg = ma$

$\therefore\quad R = m\,(a + g) = 80\,(-2 + 9.8) = 624$ N

3. **A block of metal of mass 2 kg is resting on a frictionless floor. It is struck by a jet releasing water at the rate of 1 kg/sec and at a speed of 5 ms^{-1}. What will be the initial acceleration of the block?**

Hint. Force exerted on the block by jet of water is

$$F = \frac{d}{dt}(m\,v) = v\,\frac{dm}{dt} \qquad (\because v \text{ is constant})$$

Here, $v = 5\ ms^{-1}\ ;\ \frac{dm}{dt} = 1$ kg/sec

$\therefore\quad F = 5 \times 1 = 5$ N

Now $F = m\,a$

$\therefore\quad a = \frac{F}{m} = \frac{5}{2} = 2.5\ ms^{-2}$

4. **A certain force applied to a mass m_1 gives it an acceleration of 15 m/sec^2. The same force applied to a mass m_2 gives it an acceleration of 10 m/sec^2. If the two masses are fixed together and the same force is applied to the combination, find the acceleration.**

Hint. Let the applied force be *F* newton.

$F = m_1 \times 15 = m_2 \times 10 \quad \therefore \quad m_1 = F/15$ and $m_2 = F/10$

If *a* is the acceleration for the combined masses, then,

$$a = \frac{F}{m_1 + m_2} = \frac{F}{(F/15) + (F/10)} = \frac{15\times 10}{15+10} = 6\ ms^{-2}$$

5. **A body of mass 10 kg falls vertically on a muddy ground with a velocity of 20 ms^{-1} and penetrates through a vertical depth of 0.3 m. Calculate the average force exerted by the muddy ground. Take g =10 ms^{-2}.**

Hint. $m = 10$ kg ; $v_0 = 20\ ms^{-1}$; $v = 0$; $s = 0.3$ m

$\therefore\quad a = \frac{v^2 - v_0^{\,2}}{2s} = \frac{0 - (20)^2}{2\times 0.3} = -\,666.7\,ms^{-2}$

Let F be the average force exerted by the muddy ground. Then,

$$mg - F = ma \text{ or } F = mg - ma = 10\,[10 - (-666.7)] = \mathbf{6767\ N}$$

6. Masses of 50 g and 40 g are connected by a string passing over a smooth pulley. If the system travels 2.18 m in the first two seconds, find the value of g.

Hint. Fig. 10.49 shows the conditions of the problem. The acceleration a of the masses is given by ;

$$S = v_0 t + \frac{1}{2}at^2 \text{ or } 2.18 = 0 + \frac{1}{2} \times a \times (2)^2$$

$$\therefore \quad a = 1.09\ \text{ms}^{-2} \qquad ...(i)$$

$$\text{Net force} = (m_2\, g - m_1 g) = g(m_2 - m_1)$$

$$= g\ (0.05 - 0.04) = 0.01\ g \text{ newton}$$

$$\text{Acceleration, } a = \frac{\text{Net force}}{m_1 + m_2} = \frac{0.01g}{0.04 + 0.05} = \frac{0.01g}{0.09} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $1.09 = \dfrac{0.01g}{0.09}$

$$\therefore \quad g = 1.09 \times \frac{0.09}{0.01} = \mathbf{9.81 ms^{-2}}$$

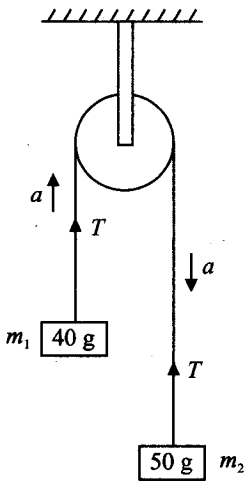


Fig. 10.49

7. A lift is going up. The total mass of the lift and the passengers is 1500 kg. The variation of the speed of the lift is given by graph shown in Fig. 10.50.

(*i*) What will be the tension in the rope at t = 1 sec; at t = 6 sec and at t = 11 sec?

(*ii*) What is the height to which the lift takes the passengers?

(*iii*) What will be the average velocity and average acceleration during the course of entire motion ?

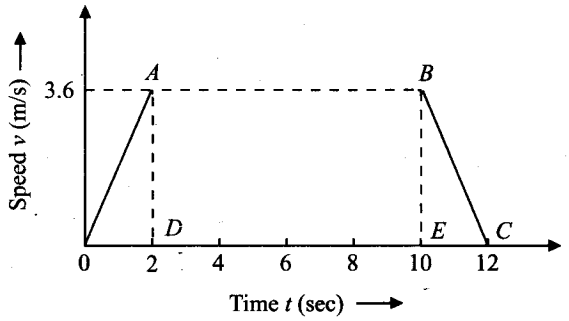


Fig. 10.50

Hint. From 0 – 2 sec, the lift has uniform acceleration given by ;

$$a = \frac{\Delta v}{\Delta t} = \frac{3.6 - 0}{2} = 1.8\ \text{ms}^{-2}$$

From t= 2 sec to t = 10 sec, the speed of the lift is constant so that during this period, a = 0. From t = 10 sec to t = 12 sec, the lift has uniform retardation given by ;

$$a = \frac{0 - 3.6}{12 - 10} = -1.8\,\text{ms}^{-2}$$

(*i*) Suppose m is the mass of lift and passengers. If a is the upward acceleration and T is the tension in the rope, then,

$$T - mg = ma \text{ or } T = m(g + a)$$

At t = 1 sec, a = 1.8 ms^{-2} $\quad \therefore\ T = 1500(9.8 + 1.8) = 17400$ N

At t = 6 sec, a = 0 $\quad \therefore\ T = 1500(9.8 + 0) = 14700$ N

At t = 11 sec, a = –1.8 ms^{-2} $\quad \therefore\ T = 1500(9.8 - 1.8) = 12000$ N

(*ii*) Height reached by the lift is

h = Area under speed-time graph

= Area of trapezium $OABC$

$$= \frac{(AB+OC)}{2} \times AD = \frac{(8+12)}{2} \times 3.6 = 36 \text{ m}$$

(*iii*) $$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{36}{12} = 3 \text{ ms}^{-1}$$

$$\text{Average acceleration} = \frac{\text{Total change in velocity}}{\text{Total time}} = \frac{0}{12} = 0$$

8. A train is moving along a straight horizontal track. A pendulum suspended from the roof makes an angle of 5° with the vertical. Find the acceleration of the train. Take g = 10 ms^{-2}.

Hint. Fig. 10.51 shows the conditions of the problem. The forces acting on the bob are :

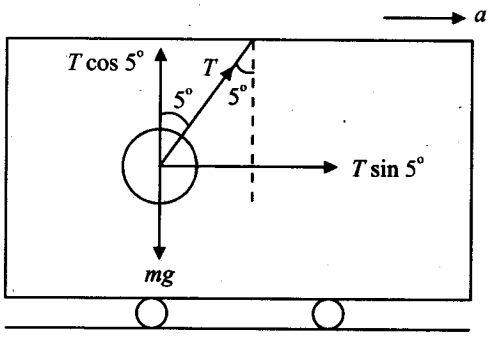


Fig. 10.51

(*i*) Weight mg of the bob.

(*ii*) Tension T developed in the string. Resolving T along vertical and horizontal directions, we have, T cos 5° and T sin 5°.

$\therefore\ T \cos 5° = mg$ ... (*i*)

Now T sin 5° is the unbalanced force and causes the acceleration a in the pendulum.

$\therefore\ T \sin 5° = ma$ ...(*ii*)

From eqs. (*i*) and (*ii*), we have, $\dfrac{T \sin 5°}{T \cos 5°} = \dfrac{ma}{mg}$

or $\tan 5° = \dfrac{a}{g}$ $\therefore\ a = g \tan 5° = 10 \tan 5° = 0.87$ ms^{-2}

9. The masses m_1, m_2 and m_3 of the three bodies shown in Fig. 10.52 are 5 kg, 2 kg and 3 kg respectively. Calculate the values of tensions T_1, T_2 and T_3 when (*i*) the whole system is going upward with an acceleration of 2 ms^{-2} (*ii*) the whole system is stationary.

Hint. (*i*) In this case, all the bodies are moving upward together. The upward force on the system is T_1 and the downward force is $(m_1 + m_2 + m_3)\ g$.

$\therefore$ Net upward force = $T_1 - (m_1 + m_2 + m_3)\ g$

If a is the upward acceleration in the system, then,

$$(m_1 + m_2 + m_3)\ a = T_1 - (m_1 + m_2 + m_3)\ g \quad \text{...(i)}$$

or $$T_1 = (m_1 + m_2 + m_3)\ (a + g) = (5 + 2 + 3)\ (2 + 9.8) = 118 \text{ N}$$

The force pulling m_2 and m_3 in the upward direction is T_2 and the downward force on them is $(m_2 + m_3)g$. Hence, the net upward force = $T_2 - (m_2 + m_3)\ g$.

$\therefore \quad (m_2 + m_3)\, a = T_2 - (m_2 + m_3)\, g$ ...(*ii*)

or $T_2 = (m_2 + m_3)\,(a + g) = (2 + 3)\,(2 + 9.8) = \mathbf{59\ N}$

The net upward force on $m_3 = T_3 - m_3 g$

$\therefore \quad m_3\, a = T_3 - m_3\, g$ ...(*iii*)

or $T_3 = m_3\,(a + g) = 3\,(2 + 9.8) = \mathbf{35.4\ N}$

(*ii*) When the whole system is stationary or moving with a constant velocity, $a = 0$. Putting $a = 0$ in eqs. (*i*), (*ii*) and (*iii*), we get,

$T_1 = (m_1 + m_2 + m_3)\, g = (5 + 2 + 3)\, 9.8 = \mathbf{98\ N}$

$T_2 = (m_2 + m_3)\, g = (2 + 3)\, 9.8 = \mathbf{49\ N}$

$T_3 = m_3 g = 3 \times 9.8 = \mathbf{29.4\ N}$

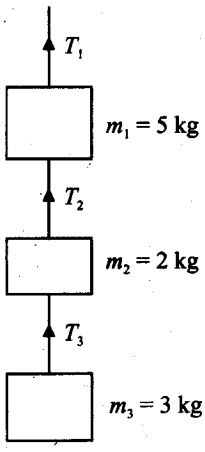


Fig. 10.52

10. Four blocks of the same mass *m* connected by cords are pulled by a force *F* on a smooth horizontal surface as shown in Fig. 10.53. Determine the tensions T_1, T_2 and T_3 in the cords.

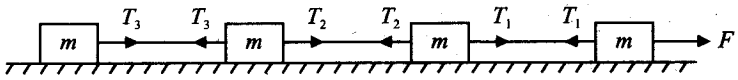


Fig. 10.53

Hint. Acceleration of the system, $a = \dfrac{F}{4m}$

Considering each block separately, net force = mass × acceleration.

$\therefore \quad F - T_1 = m\, a$ or $F - T_1 = m \times \dfrac{F}{4m}$ $\quad \therefore T_1 = \dfrac{3}{4}F$

$T_1 - T_2 = m\, a$ or $\dfrac{3}{4}F - T_2 = m \times \dfrac{F}{4m}$ $\quad \therefore T_2 = \dfrac{1}{2}F$

$T_2 - T_3 = m\, a$ or $\dfrac{1}{2}F - T_3 = m \times \dfrac{F}{4m}$ $\quad \therefore T_3 = \dfrac{1}{4}F$

11. Two masses 10 kg and 20 kg are connected by a massless spring as shown in Fig. 10.54. A force of 200 N acts on the 20 kg mass. At the instant shown, the 10 kg mass has an acceleration of 12 ms^{-2}. What is the acceleration of 20 kg mass ?

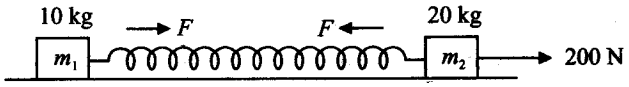


Fig. 10.54

Hint. For mass m_1; $F = m_1\, a_1 = 10 \times 12 = 120$ N

For mass m_2; $200 - F = m_2\, a_2$

or $200 - 120 = 20 \times a_2 \quad \therefore a_2 = 80/20 = 4\ ms^{-2}$

12. Three blocks are connected by strings as shown in Fig. 10.55. They are pulled by a force *F* on a frictionless horizontal surface. The tension *P* in the first string is 16 N. Find (*i*) acceleration of blocks (*ii*) Force *F* (*iii*) tension *Q* in the second string.

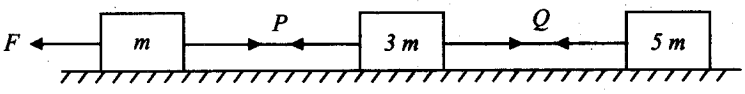


Fig. 10.55

Hint. The blocks will have the same acceleration a. The force F is pulling all the three blocks; force P pulling blocks 3 m and 5 m; force Q pulling only block 5 m.

(*i*) $\therefore \quad P = (3\ m + 5\ m)\ a = 8\ m\ a$

or $\quad a = \dfrac{P}{8m} = \dfrac{16}{8m} = \dfrac{2}{m}$ metre/sec^2

(*ii*) $\quad F = (m + 3m + 5m)a = 9\ ma = 9m \times \dfrac{2}{m} = 18$ N

(*iii*) $\quad Q = 5ma = 5m \times \dfrac{2}{m} = 10$ N

13. **A massless string passes over a frictionless pulley and carries a mass m_1 hanging at one end and mass m_2 connected by another massless string to mass m_3 at the other end as shown in Fig. 10.56. Calculate the tension in the string joining masses m_2 and m_3.**

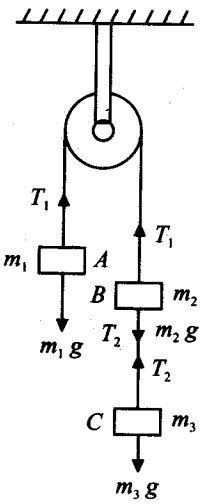


Fig. 10.56

Hint. Let a be the acceleration of the masses. The net force causing this acceleration $= (m_2 + m_3 - m_1)\ g$.

$\therefore \quad$ Acceleration, $a = \dfrac{\text{Net force}}{\text{Total mass}} = \dfrac{(m_2 + m_3 - m_1)\,g}{m_1 + m_2 + m_3}$

Applying Newton's second law to mass m_3, we have,

$$m_3\ g - T_2 = m_3\ a$$

or $\quad m_3 g - T_2 = \dfrac{m_3(m_2 + m_3 - m_1)\,g}{m_1 + m_2 + m_3}$

$\therefore \quad T_2 = m_3 g - \dfrac{m_3(m_2 + m_3 - m_1)\,g}{m_1 + m_2 + m_3}$

or $\quad T_2 = \dfrac{2m_1 m_3 g}{m_1 + m_2 + m_3}$

14. **A metal plate of mass 200 g is balanced in mid-air by throwing 40 balls per second, each of mass 2g vertically upwards from below. The balls get rebounded with the same speed with which they strike the plate. Find the speed with which the balls strike the plate.**

Hint. Mass of plate, $M = 200$ g $= 200 \times 10^{-3}$ kg

Mass of ball, $m = 2$ g $= 2 \times 10^{-3}$ kg

In order that the plate remains balanced in air, the weight Mg of the plate must be balanced by the upward impulsive force imparted by the balls to the plate. Let the required striking speed of the balls be v metres/sec. It is given that the ball rebounds with the same speed after striking the plate.

$\therefore$ Change of linear momentum of ball in one collision

$$= -\ m\ v - (m\ v) = -2\ m\ v$$

Change of linear momentum of plate per collision of each ball is

$$\Delta p = 2\ mv$$

Time for one collision, $\Delta t = 1/40$ sec

$\therefore$ Rate of change of linear momentum of plate is

$$\frac{\Delta p}{\Delta t} = \frac{2mv}{1/40} = 80\ mv$$

This is the force imparted to the plate by the balls.

For the equilibrium of the plate, we have,

$$M g = 80\, mv$$

or $$v = \frac{Mg}{80m} = \frac{(200\times10^{-3})\times9.8}{80\times2\times10^{-3}} = 12.25 \text{ ms}^{-1}$$

15. A rifle man, who together with his rifle has a mass of 100 kg, stands on a smooth surface and fires 10 shots horizontally. Each bullet has a mass of 10 g with a muzzle velocity of 800 ms^{-1}.

(*i*) What velocity does the rifle man acquire at the end of 10 shots ?

(*ii*) If the shots were fired in 5 s, what was the average force exerted on him ?

Hint. Mass of bullet, $m = 10 \times 10^{-3}$ kg ; Mass of rifle and man, M= 100 kg

Velocity of bullet, $v = 800$ ms^{-1}; Velocity of rifle and man, V= ?

(*i*) Initially the linear momentum of the system is zero because the bullet and the rifle man are at rest. Therefore, the linear momentum of the system remains constant after firing the bullet *i.e.*,

$$mv + MV = 0$$

or $$V = -\frac{mv}{M} = -\frac{(10\times10^{-3})\times800}{100} = -0.08 \text{ ms}^{-1}$$

$\therefore$ Velocity acquired after 10 shots = $10 \times (-0.08) = -0.8$ ms^{-1}

Note that negative sign shows that velocity of rifle man is in a direction opposite to that of bullet.

(*ii*) The linear momentum acquired by the rifle man is $\Delta p = MV = 100 \times 0.8 = 80$ kg ms^{-1}.

This change in linear momentum occurs in 5 sec. Therefore, average force exerted on rifle man is

$$F = \frac{\Delta p}{\Delta t} = \frac{80}{5} = 16 \text{ N}$$

16. An 8000 kg rocket is set for a vertical firing. If the exhaust speed is 800 ms^{-1}, how much gas must be ejected per second to supply thrust needed (*i*) to overcome the weight of the rocket (*ii*) to give the rocket an initial upward acceleration of 19.6 ms^{-2}?

Hint. The upward thrust F of the rocket is given by ;

$$F = v\frac{dm}{dt}$$

where v = velocity of exhaust = 800 ms^{-1}

dm/dt = time rate of mass of gas ejected

(*i*) The weight of the rocket acting downward = mg.

The upthrust needed to overcome the weight of rocket (*i.e.*, acceleration of rocket = 0),

$$v\frac{dm}{dt} = mg \quad \therefore \quad \frac{dm}{dt} = \frac{mg}{v} = \frac{8000\times9.8}{800} = 98 \text{ kg/sec}$$

(*ii*) Net force on rocket = Upthrust – Weight of rocket

$$= v\frac{dm}{dt} - mg$$

If a is the upward acceleration of the rocket, then,

$$v\frac{dm}{dt} - mg = ma \quad \text{or} \quad v\frac{dm}{dt} = (ma + mg) = m(a+g)$$

$$\therefore \quad \frac{dm}{dt} = \frac{m(a+g)}{v} = \frac{8000(19.6+9.8)}{800} = 294 \text{ kg/sec}$$

11

Friction and Dynamics of Circular Motion

INTRODUCTION

When an object (*e.g.*, wooden block) moves over the surface of another object (*e.g.*, floor), its motion is always opposed by a retarding force that resists motion. This retarding force is called *friction* and always acts in a direction opposite to motion. For example, an object sliding down an inclined plane experiences a frictional force directed up the inclined plane. Similarly, an object that moves to the right on a level floor experiences the frictional force to the left. Friction can operate between two solids, between a solid and a fluid or between two layers of a fluid. However, in this chapter, we shall discuss only the friction between solid surfaces and its role in motion of vehicles on circular roads.

11.1. FRICTION

When a solid object moves over the surface of another solid object, its motion is always opposed by a retarding force, called friction.

Hence friction is an opposing force that comes into play when one body moves or attempts to move over the surface of another body. The frictional force is always parallel to the surfaces in contact.

Let us discuss *how frictional force behaves.* Consider a wooden block of mass m placed on a horizontal surface as shown in Fig. 11.1. The block pushes down on the horizontal surface with a force equal to block's weight (= mg). The supporting horizontal surface pushes back with an equal and opposite force R so that $R = mg$. The force R is perpendicular to the surface and is called **normal reaction** (normal means perpendicular).

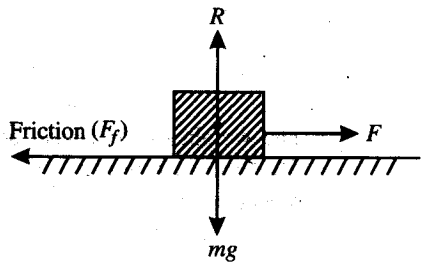


Fig. 11.1

Suppose a small horizontal force F is applied on the block to the right but the block does not move. It means that a force has come into play to oppose the motion of the block. This is the force of friction and is called **static friction** F_S because the block remains at rest or static even though we have applied force. As we increase the applied force, the static friction F_S also increases till a stage is reached when the block is about to move. The value of static friction F_S is now maximum and is called **limiting friction** or **critical friction** F_C. Once the limiting friction (*i.e.*, maximum

static friction) is overcome, the block begins to move and frictional force decreases. The friction is now called **kinetic friction** F_k. Note that the word *kinetic* means that the block is moving. Thus the behaviour of friction can be summed up as under :

(*i*) Before the body begins to move, the static friction plays the role ; the static friction increases as the applied force increases upto the value of limiting friction (*i.e.,* maximum static friction).

(*ii*) Once limiting friction is overcome by the applied force, the role of static friction is over. Now smaller frictional force F_k (= kinetic friction) opposes the motion.

(*iii*) Both limiting friction (F_C) and kinetic friction (F_k) are important. The limiting friction must be overcome to start the body moving and then smaller frictional force F_k to oppose the motion.

11.2. CAUSE OF FRICTION

Friction is a complicated phenomenon. Many aspects of friction even today are not completely understood. We now give the old view and modern view about the cause of friction.

Old view. According to the old view, friction comes into play due to the irregularities in the surfaces in contact. These irregularities are in the form of projections and depressions in the surfaces as shown in Fig. 11.2. When two surfaces come in contact, these irregularities get interlocked. This interlocking opposes the sliding of one surface over the other. Once sliding has begun, the surfaces do not have time to "settle down" onto each other completely. Therefore, the friction decreases. This explains why kinetic friction is less than limiting friction (maximum static friction).

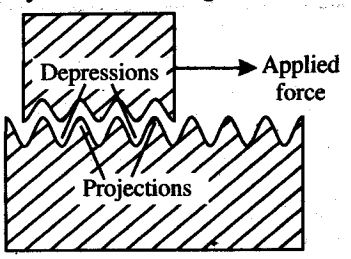


Fig. 11.2

Drawbacks. The old view has the following drawbacks :

(*i*) According to old view, the greater the area of contact between the surfaces, the greater should be the force of friction between them. However, experiments show that force of friction is independent of area of contact of the two surfaces.

(*ii*) According to the old view, when the surfaces are made smooth, the friction decreases because the size of projections and depressions in the surfaces is decreased. However, experiments show that when surfaces are highly polished, the force of friction becomes very large.

Modern view. According to modern view, *the friction is due to the atomic and molecular forces of attraction between the two surfaces at the point of their actual contact.*

On a microscopic level, even the most polished surface is far from plain *i.e.,* every surface has roughness. Therefore, when two surfaces are put together, the actual area of contact is very much less than the apparent area of contact (See Fig. 11.3).

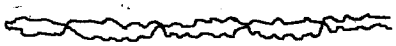


Fig. 11.3

For example, it has been found that for steel on steel, the actual contact area can be as little as one ten-thousandth of the apparent area. The pressures at the contact points are very high and the molecules are pushed into such close proximity that the attractive forces between them weld the surfaces together at contact points. In other words, the contact points are *cold-welded* together. When we try to move one body over another body, these tiny welds oppose motion. This gives rise to the opposing force of friction.

Explanation of friction on modern view. We now explain many experimentally observed facts about friction on the basis of modern view.

(*i*) The tiny welds formed have to be broken before one surface can move over the other. Therefore, no matter in which direction the motion occurs, there is a force which opposes it. *This explains why force of friction always acts in a direction opposite to motion.*

(*ii*) When the apparent area of contact of the body is *decreased, the number of contact points is decreased. Since the weight of the body is not changed, there is increased pressure at the contact points. As a result, contact points flatten so that total contact area and pressure return to their original values. *This explains why force of friction is independent of the area of contact of the surfaces.*

(*iii*) The actual area of contact between two surfaces depends upon the force with which the surfaces are pressed together. In other words, actual area of contact will depend upon the normal force exerted by the supporting surface on the body. *This explains why force of friction (static or kinetic) is directly proportional to the normal reaction.*

11.3. TYPES OF FRICTION

There are two common types of friction *viz.* **1.** Static friction **2.** Kinetic friction.

1. Static friction. *The opposing force that comes into play when an object does not move over the surface of another even though force is applied to make it move is called* **static friction.**

Consider a wooden block lying on a horizontal surface. If we apply a small horizontal force (say 1 N) on the block and the block does not move, it means that a force (= 1 N) has come into play to oppose the motion. This is the force of friction and is called *static friction* F_S. As the applied force is increased, the static friction also increases (being always equal and opposite to the applied force) till a stage is reached when the block is about to move. The value of static friction is now maximum and is called *limiting friction* or *critical friction* F_C. Note that static friction always adjusts itself equal to the applied force upto the value of limiting friction.

Thus **limiting friction** *is the maximum static friction that comes into play when one body is just at the verge of moving over the surface of another body.*

Once the limiting friction is overcome by the applied force, the block begins to move and limiting friction F_C is converted into kinetic friction F_k. The subsequent motion of the block is determined by the applied force (F) and kinetic friction F_k. It is reminded that $F_k < F_C$.

Discussion. The following points may be noted :

(*i*) Static friction comes into play *only* when the applied force tends to move a body on the surface of another body. If the applied force is zero, static friction is also zero.

(*ii*) Static friction is a *self-adjusting force.* As the applied force increases, static friction also increases (being always equal and opposite to the applied force) upto limiting friction.

2. Kinetic Friction. *Kinetic friction is the opposing force that comes into play when one body moves over the surface of another body.* It is sometimes called *dynamic friction.*

Once the limiting friction is overcome by the applied force, the body begins to move and the limiting friction is converted into kinetic friction. Fig. 11.4 shows the graph between the applied force and force of friction. The part *OA* of the graph is a straight line showing that static friction increases as the applied force increases upto limiting friction F_C. The body will remain at rest till the applied force does not exceed *OC*. Once the applied force exceeds *OC*, the body starts moving and friction decreases to F_k (kinetic friction). Thus the limiting friction must be overcome to start the body moving and then a smaller kinetic frictional force F_k acts on the body to oppose the motion of the body.

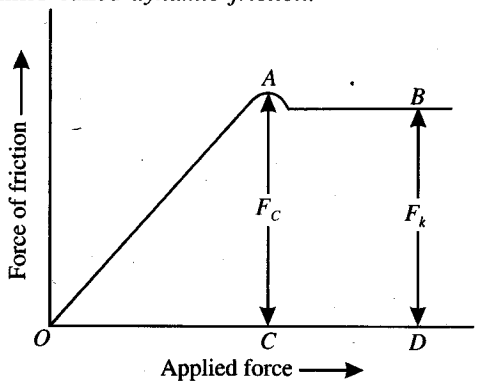


Fig. 11.4

* This can be done by turning the body so that it rests on one of its smaller faces.

Types of kinetic friction. Kinetic friction is of two types *viz.* (*a*) **Sliding friction** (*b*) **Rolling friction**.

When one body slides over the surface of another body, then opposing force that comes into play is called the *sliding friction.* For example, when a rectangular wooden block is moved on a level floor, the opposing force is the sliding friction.

When one body rolls over the surface of another body, then opposing force that comes into play is called the *rolling friction.* Thus when a ball rolls over a surface, the opposing force is the rolling friction. We shall discuss the cause of rolling friction in Art. 11.10.

11.4. LAWS OF FRICTION

Static friction as well as kinetic friction is a complicated phenomenon. Many aspects of friction even today are not completely understood. However, experiments show that to a good approximation, the force of friction obeys the following laws :

(*i*) *The frictional force (static as well as kinetic) depends upon the nature of the two surfaces in contact and their state of roughness.*

(*ii*) *The frictional force (static as well as kinetic) always acts parallel to the surfaces in contact and its direction is opposite to motion or attempted motion.*

(*iii*) *The frictional force (static as well as kinetic) is independent of the area of contact of the two surfaces.*

(*iv*) ***The magnitude of frictional force (kinetic and static) is directly proportional to the perpendicular force (normal reaction R) between the two surfaces in contact.***

(*v*) *The frictional force is independent of the speed (applicable to kinetic friction only) of one surface relative to the other surface.*

11.5. MAGNITUDE OF LIMITING FRICTION

Consider a block of mass m lying on a horizontal surface as shown in Fig. 11.5. The normal reaction R is the perpendicular force exerted by the surface on the block. Clearly, $R = mg$. If F_C is the limiting friction (*i.e.,* maximum static friction), then according to laws of friction, F_C is directly proportional to the normal reaction R *i.e.,*

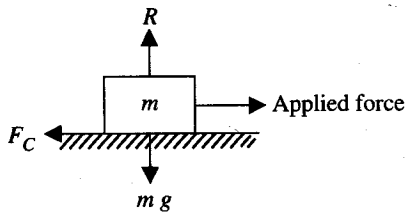


Fig. 11.5

$$F_C \propto R$$

or $$F_C = \mu_S R$$

where μ_S is a constant of proportionality and is called *coefficient of static friction.* Its value depends upon the nature and condition of the surfaces in contact.

$$\mu_S = \frac{F_C}{R} = \frac{\text{Limiting friction}}{\text{Normal reaction}}$$

Therefore, ***co-efficient of static friction*** *between two surfaces in contact is equal to the ratio of limiting friction and the normal reaction.* Since μ_S is a ratio of two forces, it has no units *i.e.* it is a pure number.

Illustration. Let us find the magnitude of limiting friction in the three cases shown in Fig. 11.6. Suppose that co-efficient of static friction between the block and the surface is μ_S in

each case. The normal reaction R is the perpendicular force that the surface exerts on the block. Its value is different in the three cases shown in Fig. 11.6.

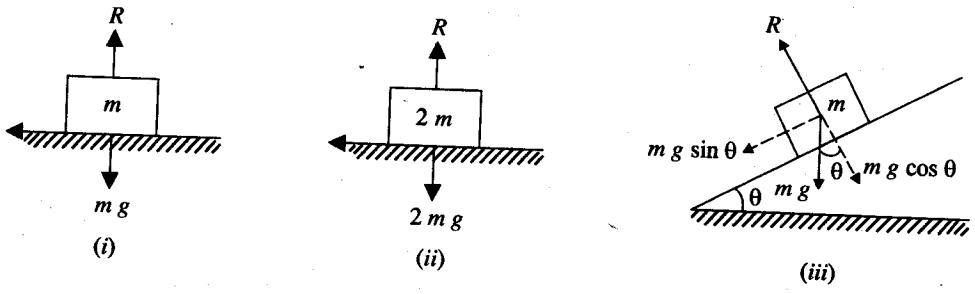


Fig. 11.6

(*i*) For the first case, $R = m\,g$ [See Fig. 11.6 (*i*)]. Therefore, $F_C = \mu_S R = \mu_S\, mg$. This is the value of maximum static friction that the surface can exert on the block.

(*ii*) For the second case [See Fig. 11.6 (*ii*)], $R = 2\,m\,g$ so that $F_C = \mu_S R = 2\mu_S\, mg$. Note that now normal force is double its previous value. Therefore, the magnitude of limiting friction becomes two times the previous value.

(*iii*) For the third case [See Fig. 11.6 (*iii*)], $R = m\,g\cos\theta$ so that $F_C = \mu_S R = \mu_S mg\cos\theta$.

11.6. MAGNITUDE OF KINETIC FRICTION

Like static friction, the magnitude of kinetic friction is directly proportional to the perpendicular force between the surfaces in contact (i.e. normal reaction between the surfaces). Consider a block of mass m moving with a constant velocity $\vec{v}$ on a horizontal surface as shown in Fig. 11.7. The normal reaction R is the perpendicular force exerted by the surface on the block. Clearly, $R = m\,g$. If F_k is the kinetic friction exerted by the surface on the block, then,

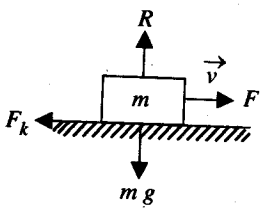


Fig. 11.7

$$F_k \propto R$$

or

$$F_k = \mu_k R$$

where μ_k is a constant of proportionality and is called *co-efficient of kinetic friction.* Its value depends upon the nature and condition of the surfaces in contact. To a good approximation, value of F_k (or μ_k) is independent of relative speed between the surfaces and their area of contact.

$$\mu_k = \frac{F_k}{R} = \frac{\text{Kinetic friction}}{\text{Normal reaction}}$$

Therefore, ***co-efficient of kinetic friction*** *between two surfaces in contact is equal to the ratio of kinetic friction and the normal reaction.* Since μ_k is a ratio of two forces, it has no units; it is a pure number. *Note that for a given pair of surfaces,* $\mu_k < \mu_S$. Therefore, we require a larger force to set an object in motion than to keep it in motion with constant velocity.

Illustration. Suppose a box of mass 3 kg is being pulled horizontally on a level surface by a force of 20 N [See Fig. 11.8]. If the co-efficient of kinetic friction between the box and surface is $\mu_k = 0.5$ and $g = 10\ \text{ms}^{-2}$, we are to find the acceleration of the box. The normal reaction R is the perpendicular force exerted by the surface on the box. Clearly, $R = mg$.

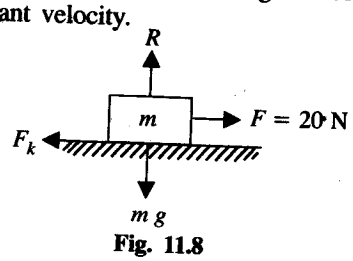


Fig. 11.8

Kinetic friction, $F_k = \mu_k R = \mu_k\, mg = 0.5 \times 3 \times 10 = 15$ N

Net force $= F - F_k = 20 - 15 = 5$ N

According to Newton's second law, the acceleration a of the box is given by ;

$$a = \frac{\text{Net force}}{\text{Mass}} = \frac{5}{3} = 1.67\,\text{ms}^{-2}$$

Note that role of limiting friction is only confined to set the body moving. Once the body is in motion, its acceleration is determined by the applied force and kinetic friction between the surfaces in contact.

Example 11.1. A block of 10 kg is pulled at a constant speed on a rough horizontal surface by a force of 20 N. Find the co-efficient of kinetic friction. Given that $g = 10\ \text{ms}^{-2}$.

Solution. Since the block is moving with a constant speed on the horizontal surface, kinetic friction exerted by the surface on the block is equal and opposite to the applied force.

$$F_k = 20\text{ N};\ R = mg = 10 \times 10 = 100\text{ N};\ \mu_k = ?$$

$$\therefore \quad \mu_k = \frac{F_k}{R} = \frac{20}{100} = 0.2$$

Example 11.2. A box is lying on the floor of a moving train and the coefficient of friction between the floor and the box is 0.15. Find the maximum acceleration of the train for which the box will lie stationary on the floor. Given that $g = 10\ \text{ms}^{-2}$.

Solution. Suppose m is the mass of the box. Let a_{max} be the maximum acceleration of the train at which the box can remain stationary on the floor.

The limiting friction between floor and box is

$$F_C = m\, a_{max}$$

Now $$\mu_S = \frac{F_C}{R} = \frac{m\, a_{max}}{mg} = \frac{a_{max}}{g}$$

$$\therefore \quad a_{max} = \mu_S\, g = 0.15 \times 10 = 1.5\ \text{ms}^{-2}$$

Example 11.3. Find the distance travelled by a body before coming to rest if it is moving with a velocity of 15 ms^{-1} and the coefficient of friction between the ground and the body is 0.45. Take $g = 10\ \text{ms}^{-2}$.

Solution. Let a be the retardation of the body.

Now $$\mu_S = \frac{F_C}{R} = \frac{ma}{mg} = \frac{a}{g}$$

$$\therefore \quad \text{Retardation, } a = \mu_S \times g = 0.45 \times 10 = 4.5\ \text{ms}^{-2}$$

Now $$v^2 - v_0^2 = 2\,aS$$

or $$0^2 - 15^2 = 2 \times (-4.5) \times S \quad \therefore\ S = 25\text{ m}$$

Example 11.4. A 3 kg box rests on a horizontal table. The coefficients of static and kinetic friction between the box and table are $\mu_S = 0.6$ and $\mu_k = 0.5$. The box is pulled by a string at an angle of 30° as shown in Fig. 11.9. Find the frictional force and the acceleration of the box if the tension in the string is 10 N.

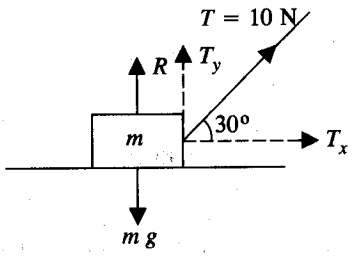


Fig. 11.9

Solution. Fig. 11.9 shows the conditions of the problem. The horizontal and vertical components of tension T are :

$$m\,v = (M + m)\,V$$

or $$v = \frac{(M+m)V}{m} = \frac{4+0.01}{0.01} \times 9.9 = 3969.9 \text{ ms}^{-1}$$

Example 11.7. Referring to Fig. 11.10, what is the least amount of force F you must exert on the block to keep it in place ? The block has a mass of 2.2 kg and the co-efficient of static friction between the wall and the block is $\mu_S = 0.65$. Take $g = 10$ ms^{-2}.

Solution. The block is being pressed against the wall by the horizontal force F [See Fig. 11.10]. The wall exerts a normal force R on the block so that $R = F$. You can hold the block in place if you push hard enough horizontally. It is because friction between the block and wall depends upon R and the value of R depends upon the value of F. If the block is to remain stationary, there must be enough static friction upward to balance the downward force $m\,g$.

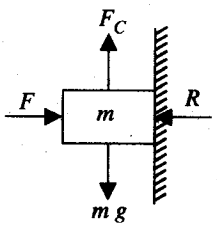


Fig. 11.10

Weight of block = $m\,g = 2.2 \times 10 = 22$ N

The static friction must be equal to 22 N.

Limiting friction, $F_C = \mu_S R$

or $$R = \frac{F_C}{\mu_S} = \frac{22}{0.65} = 34 \text{ N}$$

$\therefore$ $$F = R = 34 \text{ N}$$

Therefore, the least force F that will keep the block in place is 34 N.

Example 11.8. A 4 kg block A is placed at the top of 8 kg block B, which rests on smooth table. The block A just slips on B when a force of 12 N is applied on block B. What is the maximum horizontal force F required to make both A and B move together?

Solution. Fig. 11.11 shows the conditions of the problem. The maximum force of static friction is $F_S = 12$ N. For A to move together with B, the force on A should not exceed 12 N. If a is the corresponding acceleration, then,

$$12 = M_A \times a$$

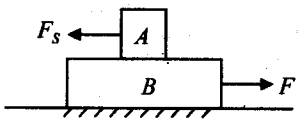


Fig. 11.11

or $$a = \frac{12}{M_A} = \frac{12}{4} = 3 \text{ ms}^{-2}$$

The equation of motion for body B is

$$F - F_S = M_B \times a$$

or $$F = F_S + M_B \times a = 12 + 8 \times 3 = 36 \text{ N}$$

PROBLEMS FOR PRACTICE

1. In Fig. 11.12, find the normal reaction. **[19.4 N]**

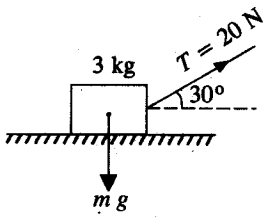


Fig. 11.12

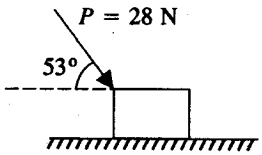


Fig. 11.13

2. In Fig. 11.13, the weight of the block is 47 N. Find the normal reaction. **[69.4 N]**

3. If the co-efficient of friction between a car's tyres and a roadway is 0.62, what is the least distance in which the car can accelerate from rest to 20.7 ms^{-1} ? **[35.3 m]**
4. A 50 kg block rests on a horizontal surface. If this block is pulled as shown in Fig. 11.14, it will not move unless the tension exceeds 350 N. What is the co-efficient of static friction between the block and the surface ? **[0.714]**

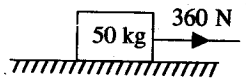


Fig. 11.14

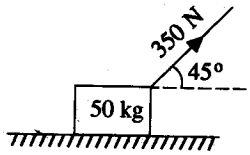


Fig. 11.15

5. Will the block shown in Fig. 11.15 move ? **[Yes]**
6. A 50 kg stone falls into a mud pot from a height of 6 m. It sinks 0.5 m below the mud surface before coming to rest. Assuming that a constant frictional force between the mud and stone brought the stone to rest, determine the magnitude of that force. **[7.84 × 10^3 N]**
7. The coefficient of static friction between the floor of a truck and a box resting on it is 0.3. The truck is travelling at 80 km/h. What is the least distance in which the truck can stop if the box is not to slide? **[83.9 m]**
8. How long will it take for a wooden block of mass 2 kg to slide from rest a distance of 3 m down a rough plane inclined at 30° to the horizontal ? Given $\mu = 0.20$. **[1.37 s]**
9. The co-efficient of friction between a rubber tyre and a wet concrete road is 0.5. Find the minimum time in which a car whose initial velocity is 15 ms^{-1} can come to a stop on such a road. **[3 s]**

11.7. ANGLE OF FRICTION

The angle which the resultant of the limiting friction and the normal reaction makes with the normal reaction is called the **angle of friction.**

Consider a block of mass m resting on a horizontal surface as shown in Fig. 11.16. The weight mg of the block is balanced by the normal reaction R *(i.e.* $R = mg$*).* The magnitude of applied force is equal to the limiting friction F_C between the surfaces. Under such condition, the block is at the point of sliding. The resultant of R and F_C makes an angle α with R. This angle is the angle of friction. Referring to Fig. 11.16,

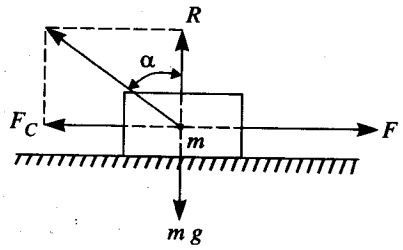


Fig. 11.16

$$\tan\alpha = \frac{F_C}{R} = \frac{\mu_S R}{R} = \mu_S$$

$$\therefore \quad \mu_S = \tan\alpha$$

Thus the coefficient of static friction is equal to the tangent of the angle of friction.

11.8. MEASUREMENT OF μ_S AND μ_k

We can measure μ_s and μ_k for the two surfaces simply by placing a block on a plane surface and incline the plane until the block begins to slide.

(*i*) Measurement of μ_S. Consider a block of mass m placed on a plane surface. If the plane surface is raised gradually above the horizontal, eventually a point is reached when the block just begins to slide. Let θ_C be the critical angle at which sliding starts. In this condition, the block is under the action of the following forces (See Fig. 11.17) :

(*a*) The weight *mg* of the block acting vertically downward.

(*b*) The limiting friction F_C acting parallel to the surfaces in contact but opposite to the attempted motion. In this case, limiting friction is directed upward along the inclined plane.

We choose *X*-axis parallel to the plane and *Y*-axis perpendicular to the plane. The weight *mg* of the block is resolved into an *x*-component $mg \sin\theta_C$ down the inclined plane and a *y*-component $mg \cos \theta_C$ perpendicular to the inclined plane. The incline exerts a normal force *R* on the block. Since the block is just at the point of sliding, the *x* and *y* components of the net force are zero *i.e.*,

$$R - mg \cos\theta_C = 0 \quad \text{or} \quad R = mg \cos\theta_C \qquad ...(i)$$

and $$mg \sin\theta_C - F_C = 0 \quad \text{or} \quad F_C = mg \sin\theta_C \qquad ...(ii)$$

Dividing eq. (*ii*) by (*i*), we have,

$$\frac{F_C}{R} = \frac{mg \sin\theta_C}{mg \cos\theta_C} = \tan\theta_C \qquad (\because F_C = \mu_S R)$$

or $$\frac{\mu_S R}{R} = \tan\theta_C$$

$\therefore$ $$\mu_S = \tan\theta_C$$

Thus the coefficient of static friction (μ_S) *is equal to the tangent of the angle of inclination* (θ_C) *at which the block just begins to slide.*

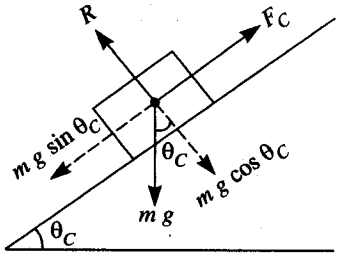


Fig. 11.17

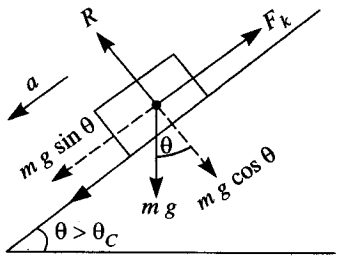


Fig. 11.18

The angle θ_C is called the *angle of repose*. Thus if the angle of repose for a block on a plank is 38°, then co-efficient of static friction between the block and plank is $\mu_S = \tan\theta_C = \tan 38° = 0.78$.

Angle of repose. *The angle of the inclined plane at which a body placed on it just begins to slide is called the* ***angle of repose***. It is also called *angle of sliding.*

Now $\mu_S = \tan\theta_C$ ...θ_C is angle of repose

Also $\mu_S = \tan\alpha$ ... α is angle of friction

$\therefore$ $\alpha = \theta_C$

Hence angle of friction (α) *is equal to the angle of repose* (θ_C).

(*ii*) Measurement of μ_k. At an angle greater than θ_C (See Fig. 11.18), the block begins to slide down the inclined plane with an acceleration *a*. Note that static friction changes to kinetic friction F_k($= \mu_k R$). The net force down the inclined plane is

Net force = $mg \sin\theta - F_k = mg \sin\theta - \mu_k R = mg \sin\theta - \mu_k mg \cos\theta$

$\therefore$ Acceleration of block, $a = \dfrac{\text{Net force}}{\text{Mass}} = \dfrac{mg \sin\theta - \mu_k mg \cos\theta}{m}$

or $$a = g(\sin\theta - \mu_k \cos\theta)$$

We can measure the acceleration *a* of the block. Thus we can determine μ_k for the two surfaces. Note that mass of the block does not enter into the expression. This means that all masses will have the same acceleration as long as the co-efficients of friction are the same.

11.9. ACCELERATION OF A BODY DOWN ROUGH INCLINED PLANE

An important type of one dimensional motion is the motion of an object on an inclined plane. Fig. 11.19 shows a block placed on a rough inclined plane; the angle of inclination being θ. If θ is greater than the angle of repose, the static friction changes to kinetic friction and the block slides down with an acceleration a. Referring to Fig. 11.19,

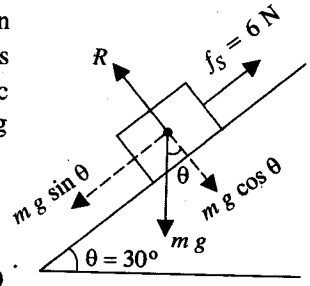


Fig. 11.19

$$R = m\ g \cos\theta$$

Net force on block down the plane = $mg \sin\theta - F_k$

or $\quad m\ a = mg \sin\theta - \mu_k\ mg \cos\theta \quad (\because F_k = \mu_k\ mg \cos\theta)$

or $\quad a = g\ (\sin\theta - \mu_k \cos\theta)$

It is clear that $a < g$. *We arrive at a very important conclusion that acceleration of a body down the inclined plane is always less than the acceleration due to gravity.*

Note. If $\theta < \theta_C$ (angle of repose), the body will not slide down and force is required to move the body down the plane.

Minimum force required to move the body down the inclined plane

$$= F_k - mg \sin\theta$$

$$= mg\ \mu_k \cos\theta - mg \sin\theta$$

$$= mg\ (\mu_k \cos\theta - \sin\theta)$$

Minimum force required to move the body up the inclined plane

$$= mg \sin\theta + F_k$$

$$= mg \sin\theta + mg\ \mu_k \cos\theta$$

$$= mg\ (\sin\theta + \mu_k \cos\theta)$$

Example 11.9. A box of mass 4 kg is placed on a wooden plank of length 1.5 m which is lying on the ground. The plank is lifted from one end along its length so that it becomes inclined. It is noted that when the vertical height of the top end of the plank from the ground becomes 0.5 m, the box begins to slide. Find the coefficient of friction between the box and the plank.

Solution. Fig. 11.20 shows the conditions of the problem. The angle of the inclined plane at which the box placed on it just begins to slide is called the angle of repose. Therefore, $\angle AOB = \theta_C$ = Angle of repose.

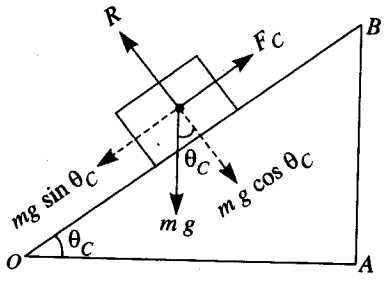


Fig. 11.20

Here $BA = 0.5$ m ; $OB = 1.5$ m

$$\therefore\ OA = \sqrt{OB^2 - BA^2} = \sqrt{(1.5)^2 - (0.5)^2} = \sqrt{2}$$

Therefore, co-efficient of friction μ is given by ;

$$\mu = \tan\theta_C = \frac{BA}{OA} = \frac{0.5}{\sqrt{2}} = 0.35$$

Example 11.10. A book that weighs 12 N is supported by an inclined plane. The coefficient of static friction between the book and the inclined plane is $\mu_S = 0.7$. What is the frictional force that acts on the book if the angle of inclination of the plane is 30° with the horizontal?

Solution. Fig. 11.21 shows the conditions of the problem. The weight mg (= 12 N) of the book acts vertically downward. The normal force R is exerted by the surface on the book.

$$R = mg \cos\theta = 12 \cos 30° = 10.4 \text{ N}$$

The component of weight down the plane

$$= mg \sin\theta = 12 \sin 30° = 6 \text{ N}$$

The force of static friction can be as great as $\mu_S R$.

$\therefore$ Limiting friction, $F_C = \mu_S R = \mu_S mg \cos\theta$

$$= 0.7 \times 10.4 = 7.28 \text{ N}$$

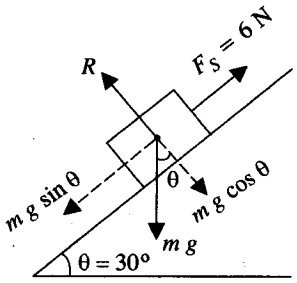


Fig. 11.21

However, the force along incline due to book's weight is only 6 N. Therefore, the force of static friction that acts on the book is also 6 N and acts upward along the incline.

$\therefore$ Frictional force on book = **6 N**

Example 11.11. **A block of wood is supported on an inclined plane that makes an angle of 37° with the horizontal. The block is given a push so that it slides up with an initial speed of 5 ms^{-1}. The coefficients of static and kinetic friction between the block and inclined plane are μ_S = 0.8 and μ_k= 0.5.**

(*i*) How far along the inclined plane will the block move before it comes to rest ?

(*ii*) Will the block slide back down the plane ?

Solution. Fig. 11.22 (*i*) shows the conditions of the problem.

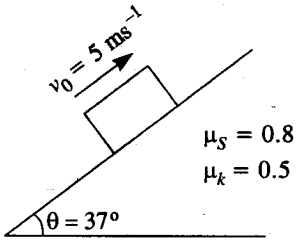
(*i*) Fig. 11.22 (*ii*) shows the conditions of the problem while the block is sliding upward along the plane. Let us take the positive x direction upward along the plane. When the block is moving up along the plane, the component $mg \sin\theta$ as well as kinetic friction F_k act down along the plane. Therefore, net force on the block is

$$\text{Net force on block} = (-mg \sin\theta) + (-F_k)$$

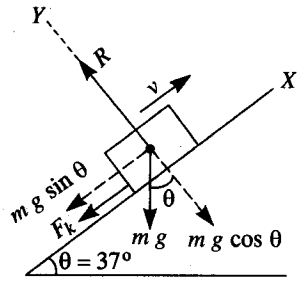
$$= -mg \sin\theta + (-\mu_k mg \cos\theta) \qquad (\because F_k = \mu_k R)$$

$$= -mg (\sin 37° + 0.5 \cos 37°)$$

$$\text{Acceleration of block, } a = \frac{-mg (\sin 37° + 0.5 \cos 37°)}{m} = -g$$



(*i*)



(*ii*)

Fig. 11.22

Now $v^2 - v_0^2 = 2aS$

Here $v_0 = 5$ ms^{-1}; $v = 0$; $a = -g$

$\therefore$ $S = \dfrac{0^2 - 5^2}{2\times(-9.8)} = 1.28$ m

(*ii*) When the block has reached the highest point, it is momentarily at rest and it is then the static friction that determines whether the block will remain at rest or slide down the plane. We know that maximum angle (θ_C) that an inclined plane can have without causing motion of an object resting on the plane is given by ;

$$\mu_S = \tan\theta_C$$

or $$\theta_C = \tan^{-1}\mu_S = \tan^{-1} 0.8 = 38.7°$$

Since the angle of inclined plane is 37° (*i.e.*, less than θ_C), the block will not slide down the plane but will remain at the point where it comes to rest.

Example 11.12. An object of weight *W* is resting on an inclined plane at an angle θ to the horizontal. The coefficient of static friction is μ. Find the horizontal force needed to just push the object up the plane.

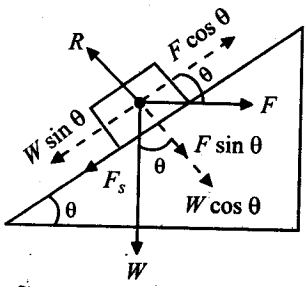


Fig. 11.23

Solution. Figure 11.23 shows the conditions of the problem. Let F be the horizontal force needed to just push the object up the inclined plane. The force F and weight W are resolved into rectangular components along the plane and perpendicular to the plane.

Frictional force, $F_s = \mu R = \mu(W\cos\theta + F\sin\theta)$

Now $$F\cos\theta = W\sin\theta + F_s$$

or $$F\cos\theta = W\sin\theta + \mu(W\cos\theta + F\sin\theta)$$

or $$F\cos\theta - \mu F\sin\theta = W\sin\theta + \mu W\cos\theta$$

or $$F(\cos\theta - \mu\sin\theta) = W(\sin\theta + \mu\cos\theta)$$

$$\therefore \quad F = \frac{W(\sin\theta + \mu\cos\theta)}{(\cos\theta - \mu\sin\theta)}$$

Example 11.13. Find the force required to move a train of 2000 quintals up an incline of 1 in 50 with an acceleration of 2 ms^{-2}, the force of friction being 0.5 N/quintal.

Solution. Here m = 2000 quintals = 2000 × 100 kg ; $\sin\theta$ = 1/50

Force of friction, F_f = 0.5 × 2000 = 1000 N

In moving up the inclined plane, the force required against gravity is

$$F_G = mg\sin\theta = (2000 \times 100) \times 9.8 \times 1/50 = 39200 \text{ N}$$

Force required to produce acceleration a (= 2 ms^{-2}) is

$$F = ma = (2000 \times 100) \times 2 = 400{,}000 \text{ N}$$

$\therefore$ Total force required = $F_f + F_G + F$ = 1000 + 39200 + 400,000 = 440,200 N

Example 11.14. A block of mass 50 kg is placed on a rough inclined plane of angle 30°. If the coefficient of friction between the surface of the plane and the block is $1/\sqrt{3}$, find the values of maximum and minimum force along the plane to keep the mass in equilibrium.

Solution. Here m = 50 kg ; θ = 30° ; $\mu = 1/\sqrt{3}$

Force of friction, $F = \mu R = \mu\, mg\cos\theta$ ($\because R = mg\cos\theta$)

$$= (1/\sqrt{3}) \times 50 \times 9.8 \times \cos 30° = 245 \text{ N}$$

The component of weight acting down the plane is

$$P = mg\sin\theta = 50 \times 9.8 \times \sin 30° = 245 \text{ N}$$

To keep the block in equilibrium, the force required will be *maximum* when the block is moved up the plane and *minimum* when it is moved down the plane.

$\therefore$ Maximum force = $P + F$ = 245 + 245 = 490 N

Minimum force = $P - F$ = 245 − 245 = 0 N

PROBLEMS FOR PRACTICE

1. A block rests on a rough incline whose angle θ with the horizontal can be varied. A trial shows that the block does not begin sliding down the incline until $\theta = 30°$. What is the co-efficient of static friction between the block and the surface of the incline ? **[0.58]**
2. Show that if a block slides at constant velocity down a plank that makes an angle θ_k with the horizontal, then the coefficient of friction between the block and the plank is
$$\mu_k = \tan\theta_k$$
3. When an automobile moving with a speed of 36 km/h reaches an upward inclined road of angle 30°, its engine is switched off. If the co-efficient of friction involved is 0.1, how much distance will the automobile move before coming to rest? Take $g = 10\ ms^{-2}$. **[8.52 m]**
4. A body placed on a rough inclined plane just begins to slide when the slope of the plane is 1 in 4. What is the value of coefficient of static friction? **[0.26]**
5. What is the coefficient of friction between a block and a plane inclined at 30° from the horizontal if the block just slides without accelerating when given an initial push? **[0.58]**

11.10. ROLLING FRICTION

When one body rolls over another, the resistance offered to motion is called rolling friction. It is generally much less than the sliding friction. Rolling friction results from the deformation produced where a wheel pushes against the surface on which it rolls. The deformation of an automobile tyre in contact with the pavement is readily visible. Even in case of a steel wheel rolling on a steel rail, there is some deformation of the surfaces. The deformation of the two surfaces produces rolling friction. The force of rolling friction varies inversely as the radius of the roller and it is less, the more rigid the surface.

Cause. When a wheel rolls on a level surface, the area of contact is very small and the pressure (weight/area) becomes very large. As a result, there is depression in the surface below as shown in Fig. 11.24. In climbing up the depression *BC*, the wheel encounters some opposition to motion. Further, the wheel also experiences opposition to motion in detaching itself from portion *AB* against the forces of adhesion. This causes the rolling friction. Experiments show that rolling friction is directly proportional to the normal reaction (R) and inversely proportional to the radius (r) of the rolling wheel.

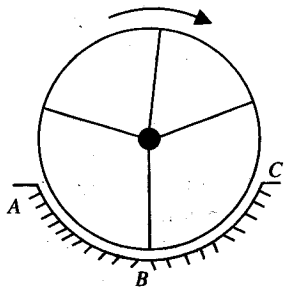


Fig. 11.24

$$\text{Rolling friction, } F_r = \frac{\mu_r R}{r}$$

where μ_r = coefficient of rolling friction

11.11. ADVANTAGES AND DISADVANTAGES OF FRICTION

Friction always opposes motion. It has advantages as well as disadvantages.

Advantages. Although we often view friction as an undesirable evil and go to great pains to reduce it, we would really be in deep trouble without it. The following examples illustrate this point.

(*i*) We would not be able to walk if there had been no friction between the soles of our shoes and the ground. While walking, we push on the ground and the force of friction acts in the opposite direction. In the absence of friction, for example, on icy surface, the person's shoes would slip when placed on the ground. The frictional force always opposes the motion of the shoes.

(*ii*) The frictional forces are very beneficial for cars and other moving vehicles. Without friction between tyres and the road, the car could not stop, start or turn corners.

(*iii*) The brakes of a car or any moving device depend upon friction. When we apply brakes, the car is stopped due to the force of friction between the brake lining and the drum on the wheel.

(*iv*) Friction helps (*a*) to transmit power (*b*) hold screws in wood (*c*) drive nails in wood etc.

Disadvantages. Friction is associated with many disadvantages and much of our efforts are directed to reduce it. The following examples illustrate this point.

(*i*) Much of energy is wasted in overcoming frictional forces. For example, as much as 20% of the gasoline burnt in a car is used to overcome friction in the engine.

(*ii*) The energy used to overcome the frictional forces is converted into heat. This raises the temperature. In almost every machinery, the generation of heat causes several problems.

(*iii*) Since energy due to friction is converted into heat, it lowers the efficiency of every machinery.

11.12. METHODS OF REDUCING FRICTION

The following methods are used to reduce friction :

(*i*) By use of lubricants. When a machinery is working, many moving solid parts come in direct contact. By using a lubricant (*e.g.* oil or grease), a thin layer of the lubricant is formed between the solid parts. Since friction between two liquid layers is much smaller than that between two solid surfaces, the use of lubricants reduces the friction considerably.

(*ii*) **Polishing.** Polishing means to deposit a fine layer of the material over the surface of the body. This layer covers the irregularities of the surfaces. Consequently, friction is greatly reduced. The same result can be achieved by rubbing the surfaces with sand paper or emery powder.

(*iii*) Use of ball bearings. In rotating machinery, the shafts are mounted on ball bearings to reduce friction. It is because rolling friction is very small as compared to sliding friction. The free wheels of cycles, the axles of motor cars and shafts of motors are provided with ball bearings.

(*iv*) **Proper selection of materials.** The friction depends upon the nature of the material. We normally use materials which form low friction combinations. For example, machine bearings are made of one metal (*e.g.*, bronze) while rotating shaft is made from other metal (*e.g.*, steel). Similarly, tyres are made of rubber because rubber-road forms a low friction combination.

(*v*) **Streamlining.** In order to reduce frictional forces of air, moving objects are given special shapes called streamline shapes. In a streamline shape, the cross-section of the body is gradually tapered toward the rear with no sharp corners for edges anywhere. By streamlining, the power necessary to drive the aeroplane or automobile is reduced to a great extent.

11.13. CENTRIPETAL FORCE

We have already seen (Art. 9.13) that an object moving in a circular path of radius r with a constant speed v (See Fig. 11.25) has centripetal acceleration directed radially towards the centre of the circle. The magnitude of this acceleration is given by ;

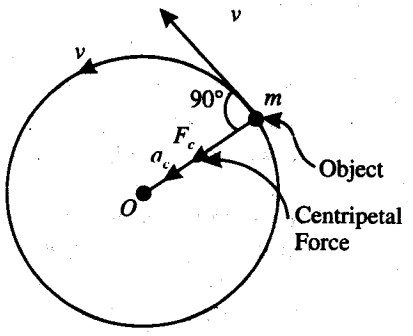


Fig. 11.25

Centripetal acceleration, $a_c = \frac{v^2}{r} = \omega^2 r \quad (\because v = \omega r)$

In order to produce acceleration in a body, the force must act in the direction of the acceleration (Newton's second law of motion). Since centripetal acceleration (a_c) is directed radially towards the centre of the circle, the centripetal force (F_c) must act on the object radially towards the centre. If m is the mass of the object, then magnitude of centripetal force is given by ;

Centripetal force, $F_c = m \times a_c = m \times \frac{v^2}{r}$

$$\therefore \quad F_c = \frac{mv^2}{r} = mr\,\omega^2 \qquad (\because v = \omega r)$$

Hence the force required to move an object with a constant speed in a circle is called ***centripetal force.*** Without this centripetal force, the object would move in a straight line and not in a circle.

Centripetal force is required in many practical situations. For example, a car moving along a circular level road must be acted on by a force in order to execute circular rather than straight line motion. This centripetal force is provided by the friction between the tyres and the road.

11.14. CENTRIFUGAL FORCE

In order to move a body in a circular path, a centripetal force F_c ($= mv^2/r$) is required. This force acts along the radius towards the centre of the circle. *The reaction of this centripetal force is the* ***centrifugal force.*** Both these forces are equal in magnitude but opposite in direction and act on different bodies. For example, consider the case of a stone tied at one end and rotated in a circle as shown in Fig. 11.26. Note that the centripetal force F_1 is applied on the stone by the hand and acts towards the centre of the circle. The centrifugal force F_2 acts on the hand and pulls it away from the centre of the circle. Centrifugal means "centre-fleeing" or away from the centre. Therefore, the outward force in circular motion is called centrifugal force. The magnitude of centrifugal force is the same as that of the centripetal force *i.e.*, mv^2/r. However, its direction is opposite to that of the centripetal force.

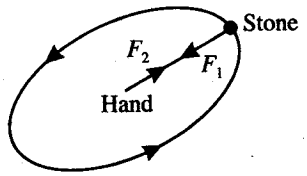


Fig. 11.26

Example 11.15. A car is moving with a speed of 30 ms^{-1} on a circular track of radius 500 m. Its speed is increasing at the rate of 2 ms^{-1}. Determine the value of its acceleration.

Solution. Since the *speed* of the car moving on a circular track is increasing, the car has centripetal (a_c) as well as tangential acceleration (a_T). The two accelerations are at right angles to each other.

Now, $\quad a_c = \frac{v^2}{r} = \frac{(30)^2}{500} = 1.8 \text{ ms}^{-2}$ and $a_T = 2 \text{ ms}^{-2}$

$\therefore$ Resultant linear acceleration a is given by :

$$a = \sqrt{a_c^2 + a_T^2} = \sqrt{(1.8)^2 + (2)^2} = 2.7 \text{ ms}^{-2}$$

Example 11.16. A small mass of 1 kg is attached to the lower end of a string 1 m long whose upper end is fixed. The mass is made to rotate in a horizontal circle of radius 0.6 m. If the circular speed of the mass is constant, find (*i*) tension in the string (*ii*) the period of motion.

Solution. Fig. 11.27 shows the conditions of the problem. Here A is the mass and AB is the string. Suppose θ is the angle of inclination of AB to the horizontal. The tension T in the string can be resolved in horizontal ($T \cos\theta$) and vertical ($T \sin\theta$) components. As the mass moves

with a constant speed v in a circle of radius r (= 0.6 m), there must be a centripetal force mv^2/r acting toward the centre O. The horizontal component $T\cos\theta$ of T provides this force along AO.

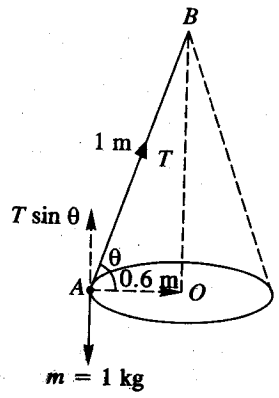


Fig. 11.27

(*i*) $\therefore \quad T\cos\theta = \dfrac{mv^2}{r} = \dfrac{mv^2}{0.6}$ ...(*i*)

Since mass does not move in the vertical direction,

$\therefore \quad T\sin\theta = mg$ ...(*ii*)

Now $\cos\theta = \dfrac{3}{5}$ and hence $\sin\theta = \dfrac{4}{5}$

From eq. (*ii*), $T = \dfrac{mg}{\sin\theta} = \dfrac{1\times 9.8}{4/5} = \mathbf{12.25\ N}$

(*ii*) From eq. (*i*), $v = \sqrt{\dfrac{0.6T\cos\theta}{m}} = \sqrt{\dfrac{0.6\times 12.25\times 3}{1\times 5}} = 2.1\ \text{ms}^{-1}$

$\therefore$ Angular velocity, $\omega = \dfrac{v}{r} = \dfrac{2.1}{0.6} = \dfrac{7}{2}\ \text{rad s}^{-1}$

$\therefore$ Time period, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{7/2} = \mathbf{1.8\ s}$

Example 11.17. A circular disc is rotating at 90 r.p.m. and a coin of mass 20 g is placed at a distance of 10 cm from its centre. What is the centrifugal force acting on the coin?

Solution. Here m = 20 g = 0.02 kg ; f = 90 r.p.m. = 90/60 r.p.s. = 1.5 r.p.s.; r = 10 cm = 0.1 m.

Angular frequency, $\omega = 2\pi f = 2\pi \times 1.5 = 3\pi\ \text{rad s}^{-1}$

Centrifugal force, $F = mr\omega^2 = 0.02 \times 0.1 \times (3\pi)^2 = \mathbf{0.177\ N}$

PROBLEMS FOR PRACTICE

1. A body of mass 0.1 kg is being rotated in a circular path of diameter 1 m on a frictionless horizontal plane by means of a string. It performs 10 revolutions in 31.4 s. Calculate the tension in the string. **[0.2 N]**
2. An object of mass 0.5 kg is rotated in a horizontal circle by a string 1 m long. The maximum tension in the string before it breaks is 50 N. What is the greatest number of revolutions per second of the object? **[1.6 r.p.s.]**
3. A certain string breaks when a weight of 24.5 N acts on it. A mass of 0.5 kg attached to one end of the string 1 m long and is rotated in a horizontal circle. Find the greatest number of revolutions per minute which can be made without breaking the string. **[66.88 r.p.m.]**

11.15. MOTION OF A VEHICLE ON A LEVEL CIRCULAR ROAD

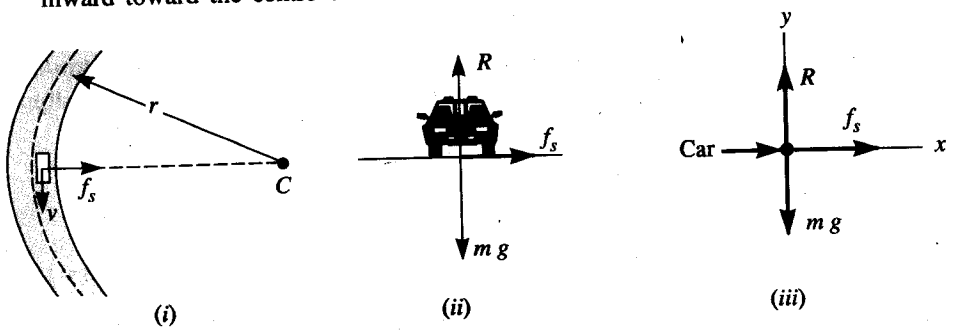
When a vehicle goes around a curve, a centripetal force must act on the vehicle to keep it in the circular path. On level circular road; this force is provided by the **static friction** between the tyres and the road. If the friction is not great enough, the vehicle fails to make a curve and the tyres slide sideways ; we say the vehicle skids.

Consider a car of mass m going around a curve of radius r with constant speed v on a level circular road as shown in Fig. 11.28 (*i*). The car's free-body diagram is shown in Fig. 11.28 (*iii*). The various forces acting on the car are :

(*i*) Weight mg of the car acting vertically downward.

(*ii*) Normal reaction R acting vertically upward (*i.e.*, perpendicular to road).

(*iii*) The *static friction f_s (rather than kinetic friction f_k), provides the centripetal force because car's tyres do not slide on the road. Note that f_s acts inwards, pointing toward the centre C. The reason is simple. As the car negotiates the curve, the tyres tend to go away from the centre of the curve. Consequently, the force of friction acts inward toward the centre of the curve.



(*i*) (*ii*) (*iii*)

Fig. 11.28

Since there is no vertical motion of the car,

$$\therefore \quad R = m g$$

The centripetal force required by the car to go around the curve is mv^2/r and this is provided by the static friction f_s between the tyres and the road.

$$\therefore \quad f_s = \frac{mv^2}{r} \qquad ...(i)$$

As the speed v of the car increases, f_s required also increases. But f_s cannot exceed the maximum value, $f_{s(max)}$. The static friction can have a maximum value of

$$f_{s\ (max)} = \mu_s R = \mu_s m g$$

Here μ_s is the coefficient of static friction.

$\therefore$ The maximum speed v_m corresponding to $f_{s\ (max)}$ is

$$f_{s\ (max)} = \frac{mv_m^2}{r} \quad \text{or} \quad \mu_s m g = \frac{mv_m^2}{r}$$

$$\therefore \quad v_m = \sqrt{\mu_s g r}$$

Should the car exceed this speed, sliding will begin and the car will no longer travel in a circle. Note that v_m is independent of the mass of the car but depends upon the coefficient of static friction and the radius of the curve. When $\mu_s = 1.2$ and $r = 150$ m, then,

$$v_m = \sqrt{\mu_s g r} = \sqrt{1.2 \times 9.8 \times 150} = 42 \text{ m s}^{-1}$$

Example 11.18. **A car of mass 1500 kg is moving along a circular level road of radius 150 m with a maximum speed of 15 ms^{-1} without skidding. Find (*i*) force of friction and (*ii*) coefficient of friction between the tyres and the road.**

Solution. Here m = 1500 kg ; v = 15 ms^{-1} ; r = 150 m.

* Note that friction is static and not kinetic. Under normal conditions, the portion of the tyre that contacts the road is temporarily at rest. Therefore, static friction plays the role.

(*i*) Since it is circular level road, the force of friction between the tyres and road provides the necessary centripetal force.

$\therefore$ Force of friction, $F = \frac{mv^2}{r} = \frac{1500\times(15)^2}{150} = 2250$ N

(*ii*) Normal reaction, $R = mg = 1500 \times 9.8$ N

$\therefore$ Coefficient of friction, $\mu = \frac{F}{R} = \frac{2250}{1500\times9.8} = 0.153$

Example 11.19. A car takes a circular turn of radius 20 m at a speed of 54 km/h. What is the least coefficient of friction between tyres and road that can prevent slide slipping ?

Solution. To prevent the car from slipping,

$$\frac{mv^2}{r} = f_{s(max)} \text{ or } \frac{mv^2}{r} = \mu_s mg$$

$\therefore$ $\mu_s = \frac{v^2}{rg} = \frac{(15)^2}{20\times9.8} = 1.15$ ($\because$ 54 km/h = 15 ms^{-1})

Example 11.20. A bend on a level road has a radius of 20 metres. Find the maximum speed which a car turning this bend may have without skidding. The coefficient of static friction between tyres and the road is 0.8.

Solution. Let v_m be the maximum speed. The centripetal force required to keep the car in the circular track at this speed is mv_m^2/r. This force is to be provided by the maximum static friction between the tyres and the road *i.e.*, $f_{s\,(max)} = \mu_s R = \mu_s mg$ where m is the mass of the car.

$\therefore$ $\frac{mv_m^2}{r} = \mu_s mg$ or $v_m = \sqrt{\mu_s rg} = \sqrt{0.8\times20\times9.8} = 12.5$ ms^{-1}

Example 11.21. A car whose wheels are 1.5 m apart laterally and whose centre of gravity is 1.5 m above the ground rounds a curve of radius 250 m. Find the maximum speed at which the car can travel without toppling.

Solution. Fig. 11.29 shows the conditions of the problem.

Here, $d = 1.5$ m ; $h = 1.5$ m.

$R_1 + R_2 = mg$

$F = \frac{mv^2}{r}$

The frictional force F between the tyres and the road provides the necessary centripetal force. At the instant of toppling, the left-hand wheel leaves contact with the ground so that $R_1 = 0$.

$\therefore$ $R_2 = mg$

Let v_m be the maximum allowed speed.

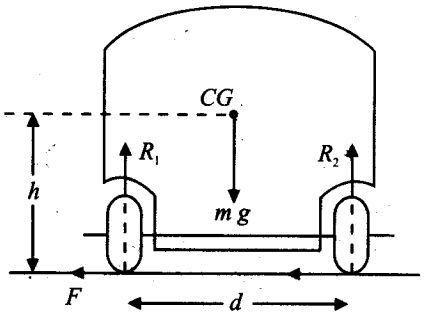


Fig. 11.29

Moment of frictional force about C.G. $= F \times h = \frac{mv_m^2}{r} \times h$

Moment of R_2 about C.G. $= R_2 \times \frac{d}{2} = \frac{mgd}{2}$

$\therefore$ $\frac{mv_m^2}{r}\times h = \frac{mgd}{2}$ or $v_m = \sqrt{\frac{gdr}{2h}} = \sqrt{\frac{9.8\times1.5\times250}{2\times1.5}} = 35$ ms^{-1}

PROBLEMS FOR PRACTICE

1. A car of mass 1000 kg rounds a curve on a flat road of radius 50 m at a speed of 50 km/h. Will the car make the turn if the static coefficient of friction between the tyres and the road is 0.2? **[No; car will skid]**
2. How large must the coefficient of friction be between the tyres and the road if a car is to round a level curve of radius 62 m at a speed of 55 km/h? **[0.39]**
3. A certain car of mass m has a maximum frictional force of 0.7 $m\,g$ between the tyres and the road as it rounds a curve on a flat road. How fast the car be moving if it is to successfully negotiate a curve of radius 5 m? **[5.86 ms^{-1}]**

11.16. BANKING OF ROADS

When an automobile goes around a curve, the road must exert an inward centripetal force on the automobile in order to move it in the circular path. This force is provided by the friction between the tyres and the road. Generally, this friction is inadequate so that the automobile has the tendency to skid *i.e.*, automobile has the tendency to leave the circular path. In order to avoid this, road curves are banked *i.e.*, outer edge of the road is raised by some angle θ above the level of the inner edge as shown in Fig. 11.30 (*i*). The result of the banked road is that the horizontal component of normal reaction R is directed towards the centre of the curve and provides the necessary centripetal force.

The process of raising the outer edge of a curved road above the level of the inner edge is called ***banking of road.***

For each banking angle (θ), there is one particular speed for which the horizontal component of normal reaction R provides the necessary centripetal force for turning the car without skidding even if there is no friction between the tyres and the road.

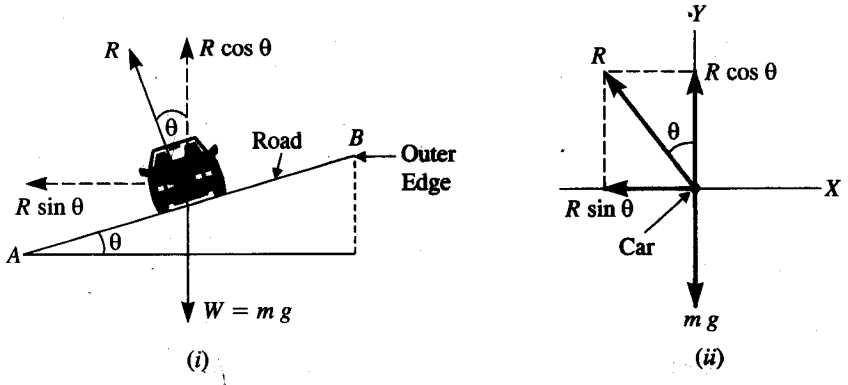


Fig. 11.30

Expression for banking angle. Fig. 11.30(*i*) shows the curved road AB of radius r banked at an angle θ. Suppose a car goes around this curve with a speed v. Fig. 11.30(*ii*) shows the free-body diagram of the car. The various forces acting on the car are :

(*i*) The weight $m\,g$ of the car acting vertically downward.

(*ii*) The normal reaction R acting normal to the road AB. Note that R is at an angle θ to the vertical. It has a horizontal component $R \sin\theta$ and a vertical component $R \cos\theta$. Since there is no vertical motion of the car,

$$\therefore \qquad R \cos\theta = mg \qquad \ldots(i)$$

The horizontal component $R \sin\theta$ acts inward (i.e., toward the centre of the curve) and provides the necessary centripetal force $m\,v^2/r$ to keep the car in the circular track.

$$\therefore \quad R\sin\theta = \frac{mv^2}{r} \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we have,

$$\tan\theta = \frac{v^2}{rg} \qquad ...(iii)$$

Eq. (*iii*) gives the banking angle θ for a curve of given radius r to be negotiated at speed v without skidding. Note that the tangent of the banking angle is directly proportional to the square of speed and inversely proportional to the radius of the curve. For a given radius, no one angle is correct for all speeds. Hence, θ is adjusted for the average speed of the traffic on the road.

Note. In deriving the above relation, friction between the tyres and the road is neglected *i.e.*, it is assumed that friction is too small to provide any appreciable centripetal force. If friction between road and tyre is considered, then the free-body diagram will be as shown in Fig. 11.31. Note that direction of the frictional force f_s is such that the force resists the tendency of the car to skid to the outisde of the curve. The horizontal component $f_s \cos\theta$ is added to the $R\sin\theta$ to provide a larger centripetal force. This added force will allow the car to go around the curve at higher speeds. Referring to Fig. 11.31,

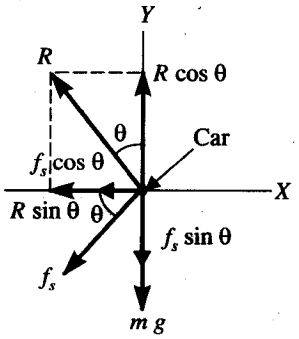


Fig. 11.31

$$R\cos\theta = mg + f_s\sin\theta$$

or $$R\cos\theta - f_s\sin\theta = mg$$

or $$R(\cos\theta - \mu_s\sin\theta) = mg \qquad ...(i) \quad (\because f_s = \mu_s R)$$

Also $$R\sin\theta + \mu_s R\cos\theta = \frac{mv_m^2}{r} \qquad ...(ii) \quad (\because {}^*f_s = \mu_s R)$$

Putting the value of R from eq. (*i*) into eq (*ii*), we get,

$$\frac{mg(\sin\theta + \mu_s\cos\theta)}{\cos\theta - \mu_s\sin\theta} = \frac{mv_m^2}{r}$$

$$\therefore \quad v_m = \sqrt{\frac{gr(\sin\theta + \mu_s\cos\theta)}{\cos\theta - \mu_s\sin\theta}}$$

or $$v_m = \sqrt{\frac{gr(\tan\theta + \mu_s)}{1 - \mu_s\tan\theta}}$$

If friction is neglected (*i.e.*, $\mu_s = 0$), then,

$$v_m = \sqrt{gr\tan\theta}$$

Example 11.22. **At what angle must a track with a bend of radius 200 m be banked for safe running of trains at a speed of 36 km/h?**

Solution.

Radius of the bend, r = 200 m ; Speed of the train, v = 36 km/h $= \frac{36\times1000}{60\times60} = 10\,\text{m s}^{-1}$

Let θ be the angle of banking. Then,

$$\tan\theta = \frac{v^2}{rg} = \frac{(10)^2}{200\times9.8} = 0.051 \quad \therefore \quad \theta = \tan^{-1}0.051 = 2.92°$$

* Maximum value of static friction = $\mu_s R$.

Example 11.23. A curve on a highway forms an arc of radius 100 m. If the road is 10 m wide and its outer edge is 1 m higher than the inner edge, for what speed the road is banked?

Solution. Fig. 11.32 shows the conditions of the problem.

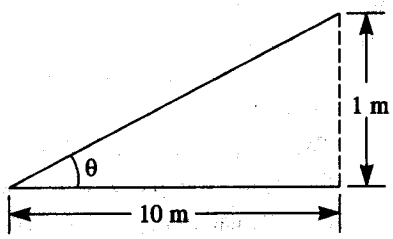


Fig. 11.32

Radius of path, r = 100 m

Referring to Fig. 11.32, $\tan\theta = \frac{1}{10} = 0.1$

Now $\tan\theta = \frac{v^2}{rg}$ or $0.1 = \frac{v^2}{100\times 9.8}$

or $v^2 = 100\times 9.8\times 0.1 = 98$

$\therefore$ $v = \sqrt{98} = 9.9 \text{ ms}^{-1} = \frac{9.9\times 3600}{1000}$ = **35.64 km/h**

Example 11.24. A curve in a road has 60 m radius. The angle of bank of the road is 47°. Find the maximum speed a car can have without skidding if the coefficient of static friction between tyres and road is 0.8.

Solution. $v_m = \sqrt{\frac{gr(\tan\theta+\mu_s)}{1-\mu_s\tan\theta}}$

Here θ = 47° ; $g = 9.8 \text{ ms}^{-2}$; $\mu_s = 0.8$; $r = 60$ m

$\therefore$ $v_m = \sqrt{\frac{9.8\times 60(\tan 47°+0.8)}{1-0.8\times\tan 47°}} = 88 \text{ ms}^{-1}$ = **317 km/hr**

PROBLEMS FOR PRACTICE

1. At what angle must a track with a bend of radius 300 m be banked for safe running of trains at a speed of 72 km/h? **[7.75°]**
2. A road curve is banked so that there is no side thrust on the passengers in a car which takes the curve at 72 km/h speed. Find the angle of banking for a circular curve making a 90° turn in a distance of 157 m along the road. **[22°14′]**
3. A train has to negotiate a curve of 400 m radius. By how much should the outer rail be raised *w.r.t.* the inner rail for a speed of 48 km/h? The distance between the rails is 1 m. **[0.0454 m]**
4. A car whose wheels are 1.4 m apart laterally and whose centre of gravity is 0.5 m above the ground rounds a curve of radius 60 m. Assuming no slipping of the wheels on the road, find the greatest speed at which the car can round the curve without slipping over. **[28.69 ms^{-1}]**
5. A race track designed for an average speed of 240 km/h is to have a turn with a radius of 975 m. To what angle must the track be banked so that cars travelling at 240 km/h have no tendency to slip sideways? **[25°]**

11.17. BENDING OF A CYCLIST

A cyclist going round a curved path has to bend toward the centre of the curve. By this action, the cyclist provides the necessary centripetal force required to keep him in the circular track. Consider a cyclist taking a turn of radus r with a speed v as shown in Fig. 11.33. In order to

provide the necessary centripetal force, suppose the cyclist leans inward making an angle θ with the vertical. The various forces acting on the system are :

(*i*) The weight $m\,g$ acting vertically downward at the centre of gravity of the system. Note that m is the mass of the cyclist plus the cycle.

(*ii*) The normal reaction R of the ground on the cyclist and the cycle. Note that R is at angle θ to the vertical. It has a horizontal component $R \sin\theta$ and a vertical component $R\cos\theta$. Since there is no vertical motion of the cycle and the cyclist,

$$\therefore \qquad R\cos\theta = m\,g \qquad ...(i)$$

The horizontal component $R\sin\theta$ acts inward (*i.e.*, toward the centre of the curve) and provides the necessary centripetal force mv^2/r to keep the cycle in the circular track.

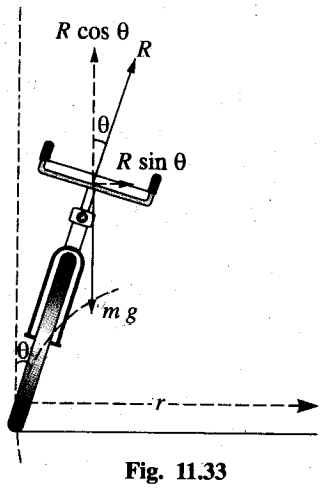


Fig. 11.33

$$\therefore \qquad R\sin\theta = \frac{mv^2}{r} \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we have,

$$\tan\theta = \frac{v^2}{rg} \qquad ...(iii)$$

Eq. (*iii*) gives the angle θ through which the cyclist leans with the vertical while going round a curve of radius r at a speed v. Actually, θ is slightly less than the value calculated above because we have not taken friction into account.

Note. For safe turn, θ should be small. This means that v should be small and r should be large.

Example 11.25. Calculate the angle through which a cyclist bends with the vertical when he crosses a circular path 34.3 m in circumference in $\sqrt{22}$ seconds.

Solution. Circumference of path, $2\pi r = 34.3$ $\quad\therefore\quad$ Radius of path, $r = \dfrac{34.3}{2\pi}$

Speed of cycle, $v = \dfrac{34.3}{\sqrt{22}}$

If the cyclist leans toward the centre of the curve making an angle θ with the vertical, then,

$$\tan\theta = \frac{v^2}{rg} = v^2 \times \frac{1}{rg} = \left(\frac{34.3}{\sqrt{22}}\right)^2 \times \frac{2\pi}{34.3\times 9.8} = 1 \qquad \therefore\ \theta = 45^\circ$$

Example 11.26. A cyclist speeding at 18 km/h on a level road takes a sharp circular turn of radius 3 m without reducing the speed. The coefficient of static friction between the tyres and the road is 0.1. Will the cyclist slip while taking the turn ?

Solution. The condition for the cyclist not to slip is

$$v^2 \le \mu_s rg \le 0.1 \times 3 \times 9.8 \le 2.94\ \text{m}^2\text{s}^{-2}$$

Therefore, for the cyclist not to slip, v^2 should be less than 2.94 m^2s^{-2}. But it is given that

$$v = 18\ \text{km/h} = \frac{18\times 1000}{60\times 60} = 5\ \text{ms}^{-1} \qquad \therefore\ v^2 = 25\ \text{m}^2\text{s}^{-2}$$

It is clear that the required condition is not obeyed. Hence, the cyclist will slip.

11.18. MOTION IN A VERTICAL CIRCLE

Consider a body of mass m tied at one end of a string and whirled in a vertical circle of radius r as shown in Fig. 11.34. The motion, while circular, is not uniform since the speed of the body is different at different points on the circular path. As the body moves from the lowest point L of the vertical circle to the highest point H, the speed of the body goes on decreasing because gravity opposes motion. On the other hand, as the body moves from the highest point H to the lowest point L, the speed of the body increases because gravity helps the motion. Therefore, the speed of the body is minimum at the highest point H while the speed of the body is maximum at the lowest point L. Note that linear velocity is always along the tangent to the circular path at every instant.

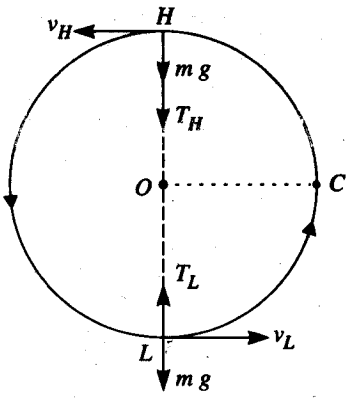


Fig. 11.34

(*i*) **Tension in the string at points L and H.** As the body moves in a vertical circle, the tension in the string varies. It will be maximum when the body is at the lowest point L and minimum when the body is at the highest point H.

Suppose v_L are v_H are the velocities of the body and T_L and T_H are the tensions in the string at the lowest point L and at the highest point H respectively. At the lowest point L of the motion, the weight mg of the body acts in *opposite* direction to the tension T_L in the string. Therefore, $T_L - mg$ provides the necessary centripetal force $m v_L^2 / r$ *i.e.*,

$$T_L - mg = \frac{m v_L^2}{r} \text{ or } T_L = \frac{m v_L^2}{r} + mg \qquad ...(i)$$

At the highest point H of the motion, both the tension T_H and the weight mg of the body act downwards towards the centre O of the circle. Therefore, $T_H + mg$ provides the necessary centripetal force $m v_H^2 / r$ *i.e.*,

$$T_H + mg = \frac{m v_H^2}{r} \text{ or } T_H = \frac{m v_H^2}{r} - mg \qquad ...(ii)$$

Note that *tension in the string is maximum when the body is at the lowest point L.* It is here that the string is most likely to break. However, *tension in the string is minimum when the body is at the highest point H.*

$$\therefore \qquad T_{max} = T_L \text{ and } T_{min} = T_H$$

(*ii*) **Minimum velocity at highest point H to complete circle.** Now let us find the minimum velocity at the highest point H so that body completes the circle. At the highest point,

$$T_H = \frac{m v_H^2}{r} - mg = m\left(\frac{v_H^2}{r} - g\right) \qquad ...(iii)$$

It is clear from eq. (*iii*) that velocity at the highest point H will be minimum when T_H is minimum *i.e.*, $T_H = 0$ ($\because$ T_H cannot be negative).

$$\therefore \qquad 0 = m\left(\frac{v_H^2}{r} - g\right) \quad \text{or} \quad v_H = \sqrt{rg}$$

Thus in order that the body loops the vertical circle, the minimum velocity of the body at the highest point H should be $\sqrt{rg}$. If velocity at point H is less than $\sqrt{rg}$, the body will fail to loop the circle.

(*iii*) **Minimum velocity at the lowest point L to complete circle.** In order that the body loops the vertical circle, it must possess certain minimum velocity at the lowest point L. This minimum velocity should be such that when the body reaches the highest point H, its velocity should be $\sqrt{rg}$. According to principle of conservation of energy,

K.E. of body at point L = (P.E. + K.E.) of body at point H

or $$\frac{1}{2}mv_L^2 = mg(2r) + \frac{1}{2}mv_H^2$$

or $$v_L^2 = 4gr + v_H^2$$

For the limiting case, $v_H = \sqrt{rg}$ or $v_H^2 = rg$.

$\therefore$ $$v_L^2 = 4gr + rg = 5rg$$

or $$v_L = \sqrt{5rg}$$

Thus in order that the body loops the vertical circle, the minimum velocity of the body at the lowest point L should be $\sqrt{5rg}$. If velocity at point L is less than $\sqrt{5rg}$, the body will fail to loop the vertical circle.

(*iv*) **Minimum velocity when the string is horizontal.** In order that the body loops the vertical circle, it must have minimum velocity v_C when the string becomes horizontal at point C (See Fig. 11.34).

Total energy at C = Total energy at L

or $$\frac{1}{2}mv_C^2 + mgr = \frac{1}{2}mv_L^2$$

But minimum velocity at L should be $v_L = \sqrt{5gr}$.

$\therefore$ $$\frac{1}{2}mv_C^2 + mgr = \frac{1}{2}m \times 5gr$$

or $$v_C = \sqrt{3\,gr}$$

Thus in order that the body loops the vertical circle, the minimum velocity of the body at point C (i.e., when the string is horizontal) should be $\sqrt{3\,gr}$.

(*v*) **To find $T_L - T_H$.** The tension T_L at the lowest point L is

$$T_L = \frac{mv_L^2}{r} + mg$$

If $v_L = \sqrt{5rg}$ and $v_H = \sqrt{rg}$, then $T_H = 0$.

$\therefore$ $$T_L = \frac{5\,mgr}{r} + mg = 6\,mg$$

$\therefore$ $$T_L - T_H = 6\,mg - 0 = 6\,mg$$

Therefore, the tension in the string at the lowest point L is greater than the tension at the highest point H by six times the weight of the body.

Summary. The above discussion about the motion of a body in a vertical circle can be summed up as under :

(*a*) For the body to loop the vertical circle, its velocity at points L, C and H should satisfy the following conditions :

$$v_L \geq \sqrt{5gr}\ ;\ v_C \geq \sqrt{3gr}\ ;\ v_H \geq \sqrt{gr}$$

(*b*) $T_L - T_H = 6\,mg$ = Six times the weight of the body

(*c*) During the motion of the body in the vertical circle, the tension in the string is maximum when the body is at the lowest point *L* and minimum when the body is at the highest point *H*.

11.19. PRACTICAL EXAMPLES OF MOTION IN A VERTICAL CIRCLE

(*i*) If a bucket containing water is rotated in a vertical circle such that its velocity at the lowest point of motion is equal to or greater than $\sqrt{5\,g\,r}$, the water will not fall out even if the bucket is at the highest point with its open top pointing downwards. It is because in this case the weight $m\,g$ of the water is less than the required centripetal force mv_H^2 / r towards the centre. So the water stays in the bucket.

(*ii*) Similarly, when an aeroplane loops a vertical circle such that its velocity at the lowest point is equal to or greater than $\sqrt{5\,g\,r}$, the pilot does not fall down even when he is not tied down to his seat in the cockpit.

(*iii*) For the same reason, a motor cyclist is able to perform the feat of driving the motor cycle in a vertical circle inside a spherical cage.

Example 11.27. **Describe the motion of a body in a vertical circle taking any point *P* on the circle.**

Solution. Consider a body of mass m tied at one end of a string and whirled in a vertical circle of radius r as shown in Fig. 11.35. Let at any time, the body be at point P at an angular position θ. Suppose at point P, T is the tension in the string and v is the linear velocity of the body. The various forces acting on the body are :

(*i*) Weight mg of the body acting vertically downwards.

(*ii*) Tension T in the string acting along PO.

The weight mg can be resolved into two rectangular components *viz.* $mg\cos\theta$ opposite to T and $mg\sin\theta$ along tangent to the circle at P. The net force on the body at P is $(T - mg\cos\theta)$ and provides the necessary centripetal force (mv^2/r) required by the body.

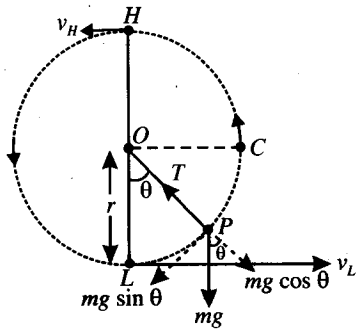


Fig. 11.35

$$\therefore \qquad T - mg\cos\theta = \frac{mv^2}{r} \qquad \ldots(i)$$

Eq. (*i*) is the general equation describing the motion of the body in a vertical circle.

At the lowest point *L*. At the lowest point L, $T = T_L$, $v = v_L$ and $\theta = 0°$ so that $\cos\theta = 1$.

$$\therefore \qquad T_L - mg = \frac{mv_L^2}{r}$$

At the highest point *H*. At the highest point H, $T = T_H$, $v = v_H$ and $\theta = 180°$ so that $\cos\theta = -1$.

$$\therefore \qquad T_H + mg = \frac{mv_H^2}{r}$$

When string is horizontal. The string is horizontal at point C where velocity is $v_C = \sqrt{3rg}$ and $\theta = 90°$ so that $\cos\theta = 0$. Therefore, from eq. (*i*),

$$T_C - mg\cos 90° = \frac{m(\sqrt{3rg})^2}{r} \quad \text{or} \quad T_C = 3\,mg$$

Example 11.28. **A string 0.5 m long is used to whirl a 1 kg stone in a vertical circle at uniform speed of 5 ms^{-1}. Determine the tension in the string when the stone is (*i*) at the top of the circle (*ii*) at the bottom of the circle.**

Solution. m = 1 kg ; r = 0.5 m ; v = 5 ms^{-1}

The centripetal force required to keep the stone moving at 5 ms^{-1} is

$$F = \frac{mv^2}{r} = \frac{1\times(5)^2}{0.5} = 50 \text{ N}$$

Weight of stone = $m\,g$ = 1 × 9.8 = 9.8 N

(*i*) When the stone is at the top of the circle, the tension T_H in the string and the weight $m\,g$ of the stone provide the necessary centripetal force.

$$\therefore \quad T_H + mg = \frac{mv^2}{r}$$

or $$T_H = \frac{mv^2}{r} - mg = 50 - 9.8 = \mathbf{40.2\ N}$$

(*ii*) When the stone is at the lowest point of the circle, the weight $m\,g$ of the stone acts in opposite direction to the tension T_L in the string. Therefore, $T_L - m\,g$ provides the necessary centripetal force.

$$\therefore \quad T_L - mg = \frac{mv^2}{r} \text{ or } T_L = \frac{mv^2}{r} + mg = 50 + 9.8 = \mathbf{59.8\ N}$$

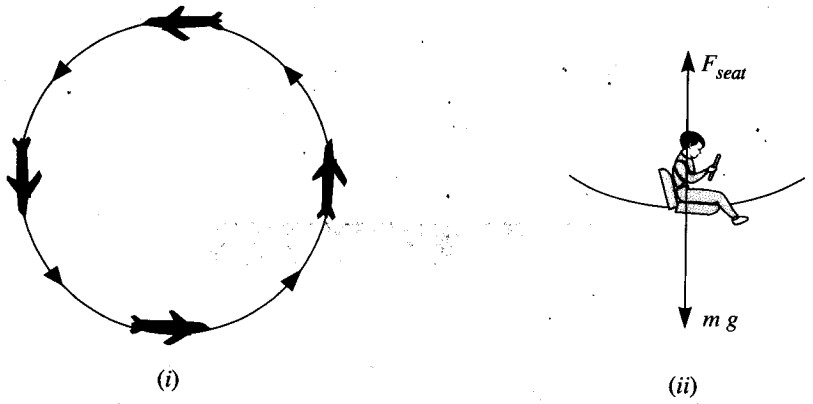
Example 11.29. A bucket containing water is tied to one end of a rope 1.25 m long and rotated about the other end in a vertical circle. What should be the minimum velocity at the highest and lowest points so that water in the bucket may not spill?

Solution. In order that water in the bucket may not spill, the minimum velocity at the highest point should be $\sqrt{rg}$ and that at the lowest point $\sqrt{5rg}$.

$\therefore$ Minimum velocity at the highest point = $\sqrt{rg} = \sqrt{1.25\times9.8} = \mathbf{3.5\ ms^{-1}}$

Minimum velocity at the lowest point = $\sqrt{5rg} = \sqrt{5\times1.25\times9.8} = \mathbf{7.83\ ms^{-1}}$

Example 11.30. An airplane moves at 64 ms^{-1} in a vertical loop of radius 120 m as shown in Fig. 11.36 (*i*). Calculate the force of the plane's seat on a 72 kg pilot while passing through the bottom part of the loop.



(*i*) (*ii*)

Fig. 11.36

Solution. Fig. 11.36 (*ii*) shows the free-body diagram of the pilot at the bottom of the loop. The two forces acting on the pilot are :

(*i*) Weight $m\,g$ of the pilot acting vertically downward.

(*ii*) The upward force F_{seat} of the airplane's seat.

At the bottom of the loop, the weight $m\,g$ of the pilot acts in the opposite direction to F_{seat}. Therefore, $F_{seat} - mg$ provides the necessary centripetal force $m\,v^2/r$ *i.e.*,

$$F_{seat} - m\,g = \frac{mv^2}{r} \quad \text{or} \quad F_{seat} = \frac{mv^2}{r} + mg$$

Here $m = 72\text{kg}$; $v = 64\,\text{m s}^{-1}$; $r = 120\,\text{m}$

$$\therefore \quad F_{seat} = \frac{72\times(64)^2}{120} + 72\times 9.8 = \mathbf{3163.2\ N}$$

Note that the pilot will feel very heavy. To keep him in circular path, the seat must push upon the pilot with a force 3163.2 N *i.e.*, 4.5 times his normal weight. He is said to experience an acceleration of 4.5 g *i.e.*, 4.5 times the acceleration due to gravity.

Example 11.31. The roadway bridge over a canal is in the form of an arc of a circle of radius 20 m. What is the maximum speed with which a car can cross the bridge without leaving contact with the ground at the highest point?

Solution. At the highest point of the roadway bridge, the centripetal force is provided by the weight of the car. If v_{max} is the maximum speed with which the car can cross the bridge without leaving contact with the ground, then,

$$\frac{mv_{max}^2}{r} = mg \quad \text{or} \quad v_{max} = \sqrt{gr}$$

Here $g = 9.8\ \text{ms}^{-2}$; $r = 20\ \text{m}$

$$\therefore \quad v_{max} = \sqrt{9.8\times 20} = \mathbf{14\ ms^{-1}}$$

Example 11.32. A stone of mass 0.5 kg tied to a rope of length 0.5 m revolves along a circular path in a vertical plane. The tension in the rope at the bottom point of the circle is 45 N. To what height the stone will rise if the rope breaks the moment the velocity is directed upwards ? Take g =10 ms^{-2}.

Solution. Let v be the velocity of the stone at the bottom point of the vertical circle.

$$\therefore \quad T - mg = m\frac{v^2}{r},$$

$$\text{or} \quad 45 - 0.5\times 10 = \frac{0.5\times v^2}{0.5} \quad \text{or} \quad v^2 = 40$$

When the rope breaks, the stone rises to a height h until its entire K.E. is converted into P.E.

$$\therefore \quad \frac{1}{2}mv^2 = mgh \quad \text{or} \quad h = \frac{v^2}{2g} = \frac{40}{2\times 10} = \mathbf{2\ m}$$

PROBLEMS FOR PRACTICE

1. An object of mass 8 kg is whirled round in a vertical circle of radius 2 m with a constant speed of 6 ms^{-1}. Calculate the maximum and minimum tensions in the string. Take g = 10 ms^{-2}. **[224 N; 64 N]**
2. A mass of 0.4 kg is rotated at a constant speed v in a vertical circle of radius 1 m. If the minimum tension in the string is 3 N, calculate (*i*) v (*ii*) the maximum tension (*iii*) the tension when the string is just horizontal. Take g = 10 ms^{-2}.
[(*i*) 4.2 ms^{-1} (*ii*) 11 N (*iii*) 7 N]
3. An old trick is to swing a pail of water in a vertical circle. If the rotation rate is large enough,water will not fall out of the pail when the pail is upside down at the top of the

At the bottom of the loop, the weight $m\,g$ of the pilot acts in the opposite direction to F_{seat}. Therefore, $F_{seat} - mg$ provides the necessary centripetal force $m\,v^2/r$ *i.e.*,

$$F_{seat} - m\,g = \frac{m v^2}{r} \quad \text{or} \quad F_{seat} = \frac{m v^2}{r} + m\,g$$

Here $m = 72\,\text{kg}$; $v = 64\,\text{m s}^{-1}$; $r = 120\,\text{m}$

$$\therefore \quad F_{seat} = \frac{72\times(64)^2}{120} + 72\times 9.8 = \mathbf{3163.2\ N}$$

Note that the pilot will feel very heavy. To keep him in circular path, the seat must push upon the pilot with a force 3163.2 N *i.e.*, 4.5 times his normal weight. He is said to experience an acceleration of 4.5 g *i.e.*, 4.5 times the acceleration due to gravity.

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$$\frac{m v_{max}^2}{r} = mg \quad \text{or} \quad v_{max} = \sqrt{gr}$$

Here $g = 9.8\ \text{ms}^{-2}$; $r = 20\ \text{m}$

$$\therefore \quad v_{max} = \sqrt{9.8\times 20} = \mathbf{14\ ms^{-1}}$$

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Solution. Let v be the velocity of the stone at the bottom point of the vertical circle.

$$\therefore \quad T - mg = m\frac{v^2}{r},$$

$$\text{or} \quad 45 - 0.5 \times 10 = \frac{0.5\times v^2}{0.5} \quad \text{or} \quad v^2 = 40$$

When the rope breaks, the stone rises to a height h until its entire K.E. is converted into P.E.

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PROBLEMS FOR PRACTICE

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[(*i*) 4.2 ms^{-1} (*ii*) 11 N (*iii*) 7 N]
3. An old trick is to swing a pail of water in a vertical circle. If the rotation rate is large enough,water will not fall out of the pail when the pail is upside down at the top of the

path. What is the minimum speed your hand must have at the top of the circle if the trick is to succeed ? Assume your arm is 0.72 m long. **[2.66 ms^{-1}]**

4. A ball of mass 0.3 kg is tied to one end of a string 0.8 m long and rotated in a vertical circle. At what speed of the ball will the tension in the string be zero at the highest point of the circle? What would be the tension at the lowest point in this situation? **[2.8 ms^{-1}; 17.6 N]**

5. A stone is tied to a weightless string and revolved in a vertical circle of radius 5 m. (*i*) What should be the minimum speed of the stone at the highest point of the circle so that the string does not slack? (*ii*) What should be the speed of the stone at the lowest point in this situation ? **[(*i*) 7 ms^{-1} (*ii*) $7\sqrt{5}$ ms^{-1}]**

CONCEPTUAL QUESTIONS

Q.1. Why is kinetic friction less than the limiting friction ?

Ans. The maximum static friction must be overcome to cause the object to start moving. Before this stage is reached, the interlocking between the irregularities of the surfaces resist sliding. However, once sliding has begun, the surfaces do not have time to "settle down" onto each other completely. As a result, less force is required to keep the object moving than to start its motion.

Q.2. Why does friction (static as well as kinetic) depend upon normal force between the surfaces in contact ?

Ans. When normal force between the surfaces increases, interlocking between the irregularities of the surfaces becomes more intense. Therefore, the more forcefully the two surfaces are pressed together, the greater is the friction between them. This explains why force of friction is directly proportional to the normal reaction.

Q.3. Friction between two surfaces is independent of area of contact. Why are tyres of a car made wider ?

Ans. The friction between the tyre and the road is the same whether the tyre is wide or narrow. The wider tyre simply spreads the weight of the car over more surface area to reduce heating and wear.

Q.4. Does friction occur in liquids and gases?

Ans. Friction also occurs in liquids and gases, collectively called **fluids**. But unlike the friction between solid surfaces, fluid friction does depend upon speed and area of contact.

Q.5. Why do smoother surfaces produce less friction?

Ans. When the surfaces are smooth, the size of the irregularities in them is less. As a result; the area of actual contact decreases. Therefore, the force of molecular attraction and hence the friction decreases.

Q.6. What are the magnitudes and direction of limiting friction and kinetic friction ?

Ans. The maximum value of static friction is called limiting friction.

Limiting friction, $F_C = \mu_s R$

Kinetic friction, $F_k = \mu_k R$

μ_S = coefficient of static friction; μ_k = coefficent of kinetic friction; R = normal force on the surface

F_C or F_k opposes motion and their direction is parallel to the surfaces.

Q.7. Is it possible to expect the coefficient of friction to be more than 1?

Ans. Yes. The coefficient of friction $\mu = \tan \alpha$, where α is the angle of friction. Since tangent of an angle can vary from zero to infinity, it is possible to expect μ to be greater than 1.

Q.8. Out of coefficient of static friction, kinetic friction and rolling friction, which is the least ?

Ans. The coefficient of rolling friction (μ_r) is the least.

$$\mu_s > \mu_k > \mu_r$$

Q.9. A block of mass 2 kg rests on a rough inclined plane making an angle of 30° with the horizontal. The coefficient of static friction for the plane is 0.7. What will be the frictional force on the block ?

Ans. Normal reaction, $R = mg\cos\theta = 2 \times 9.8 \times \cos 30° = 16.97$ N

Limiting friction, $F_C = \mu_S R = 0.7 \times 16.97 = 11.88$ N

Force down the plane, $F = mg\sin\theta = 2 \times 9.8 \times \sin 30° = 9.8$ N

Since the force down the plane (= 9.8 N) is less than the limiting friction (= 11.88 N), the force of static friction = 9.8 N.

Q.10. Why is force of limiting friction (F_C) greater than force of kinetic friction (F_k)?

Ans. Once the motion starts, the inertia of rest is overcome. Further, once motion takes place, the irregularities of one surface have little time to get locked into the irregularities of the other surface.

Q.11. Why are passengers of a car rounding a curve thrown outward ?

Ans. When the car rounds a curve, the passengers in the car are thrown outward. It is not because of some outward force but because there is no centripetal force to hold the passengers in circular motion.

Q.12. For uniform circular motion, does the direction of centripetal force depend on the sense of rotation (*i.e.*, clockwise or anticlockwise) ?

Ans. The direction of centripetal force does not depend upon the sense of rotation. The centripetal force always acts along the radius and towards the centre.

Q. 13. What is a centripetal force ?

Ans. Any force that is directed at right angles to the path of a moving object and produces circular motion is called a centripetal force. Note that centripetal force is not a new kind of force but is simply the name given to a force that keeps an object in circular path.

Q.14. If both the speed of a body and the radius of its circular path are doubled, what will happen to the centripetal force ?

Ans. Centripetal force, $F_c = \dfrac{mv^2}{r}$

If both speed and radius of the circular path are doubled, the centripetal force will be doubled.

Q.15. In rain, generally the scooter slips at the turning of a road. Why ?

Ans. On the wet road, the friction between the tyres and road is reduced. Therefore, necessary centripetal force is not provided.

Q.16. A car is moving in a circular track of radius 10 m with a constant speed of 10 m/s. A bob is suspended from the roof of the car by a light rod of length 1m. What is the angle made by the rod with the track ? (g = 10 ms^{-2}).

Ans. $\tan\theta = \dfrac{v^2}{rg} = \dfrac{(10)^2}{10\times10} = 1 \qquad \therefore\ \theta = 45°$

VERY SHORT ANSWER QUESTIONS

Q.1. A table is lying on the floor. Is force of friction acting on it?

Ans. No. When applied force is zero, force of friction is also zero.

Q.2. A book is lying on an inclined plane. Is some force of friction acting on the book?

Ans. Yes. It is because the component $mg \sin\theta$ of the weight ($= mg$) of the book is acting down the inclined plane. Since book is at rest, static friction is also $mg \sin\theta$.

Q.3. Why do we slip on a rainy day?

Ans. Due to wet ground, the coefficient of friction (μ) between ground and our feet is very much reduced. This reduces the force of friction F ($= \mu R$) between ground and our feet. It may cause us to slip.

Q.4. Why are wheels made circular?

Ans. The rolling friction is less than the sliding friction. The wheels are made circular to convert sliding friction into rolling friction.

Q.5. Why are tyres made of rubber and not of iron?

Ans. It is because the coefficient of friction between rubber tyre and road is less than that between iron and road.

Q.6. Automobile tyres have generally irregular projections over their surface. Why?

Ans. To increase friction between the tyres and ground for better road grip.

Q.7. Is static friction a self-adjusting force?

Ans. Yes. When the applied force is zero, static friction is zero. As the applied force is increased, force of static friction also increases and is equal to the applied force. This happens till the body does not start moving. For this reason, static friction is called self-adjusting force.

Q.8. When does a body start moving?

Ans. When the applied force exceeds the limiting friction (*i.e.*, maximum static friction).

Q.9. Which is greatest out of dynamic friction, static friction and limiting friction?

Ans. Limiting friction is the greatest of all types of friction.

Q.10. How does air friction affect the maximum height of a projectile?

Ans. The maximum height attained by a projectile is reduced.

Q.11. Why are fast moving vehicles given streamline shape?

Ans. To reduce force of friction due to air.

Q.12. What is the relation between angle of repose and angle of friction?

Ans. Angle of friction is equal to angle of repose.

Q.13. On the horizontal surface of a truck, a block of mass 1 kg is placed ($\mu_S = 0.6$) and the truck is moving with an acceleration of 5 ms^{-2}. What is the frictional force on the block?

Ans. Forward force on the block $= ma = 1 \times 5 = 5$ N. Therefore, frictional force on the block is 5 N.

Q.14. Why is friction a non-conservative force?

Ans. It is because work done against friction over a closed path is non-zero.

Q.15. Sand is thrown on the tracks covered with snow. Why?

Ans. The sand increases the friction between tyres and snow covered tracks.

Q.16. Why are ball bearings used in the moving parts of machines?

Ans. The ball bearings have rolling friction which has least value out of all types of friction. As a result, energy is saved and better motion of the moving parts is ensured.

Q.17. What are the factors on which coefficient of friction depends?

Ans. The coefficient of friction between two surfaces depends on (*i*) material of the surfaces and (*ii*) nature of the surfaces.

Q.18. Why is it difficult to move a cycle with its brakes on?

Ans. With the brakes of a cycle on, its wheel cannot rotate but can only skid. Since sliding friction is greater than the rolling friction, it becomes difficult to move the cycle with brakes on.

Q.19. The coefficient of friction between two surfaces in contact is $1/\sqrt{3}$. What is the angle of friction?

Ans. If θ is angle of friction, $\mu = \tan\theta$ or $1/\sqrt{3} = \tan\theta \therefore \theta = 30°$.

Q.20. What is the relation between coefficient of friction (μ) and the angle of repose (θ_C)?

Ans. The relation between the two is : $\mu = \tan\theta_C$.

Q.21. What provides the centripetal force for electrons to go round the nucleus?

Ans. The electrostatic force of attraction on the electron due to nucleus.

Q.22. What is the maximum speed (v_m) with which a car can negotiate a turn of radius r safely if the coefficient of friction between the tyres and the road is μ?

Ans. The maximum speed is given by : $v_m = \sqrt{\mu rg}$.

Q.23. Does the angle of banking depend on the mass of vehicle?

Ans. No, because banking angle θ is given by ; $\tan\theta = v^2/rg$.

Q.24. A stone is tied to one end of a string and rotated in a vertical circle. What is the difference in the tensions of the string at the lowest and highest point of the circle?

Ans. $T_L - T_H = 6\ mg = 6 \times$ weight of the stone.

Q.25. Are larger brakes on a bicycle wheel more effective than the smaller ones?

Ans. No. It is because the limiting friction is independent of the area of contact of the surfaces as long as the normal reaction remains the same.

SHORT ANSWER QUESTIONS

Q.1. What are the factors upon which rolling friction depends?

Ans. It has been found experimentally that force of rolling friction (F_r) is (*i*) directly proportional to the normal reaction (R) and (*ii*) inversely proportional to the radius (r) of the rolling wheel *i.e.*, $F_r = \mu_r \times R/r$.

Q.2. How do we save petrol when the tyres of a vehicle are fully inflated?

Ans. Under the weight of the vehicle, the tyres are somewhat deformed. When the tyres are fully inflated, the deformation of tyres will be small. As a result, the rolling friction is reduced. This means that the vehicle will cover more distance for a given quantity of petrol; thus saving petrol.

Q.3. Explain how does friction help in walking?

Ans. When we walk, we exert a force on the ground (action) and in turn ground exerts an equal force on us (reaction). If the friction between foot and ground is not sufficient, the reaction force will not be equal to action force and we may slip.

Q.4. When two surfaces in contact are polished beyond a certain limit, frictional force is increased. Why?

Ans. When surfaces are polished beyond a certain limit, the area of contact between them increases. As a result, the number of atoms/molecules of the surfaces in contact increases and the distance between them decreases. The cohesive forces between atoms/molecules become large so that friction increases.

Q.5. A horse has to apply more force to start a cart than to keep it moving. Why?

Ans. When the cart is to start from rest, it has to overcome the limiting friction (*i.e.*, maximum static friction). Once the cart starts moving it has to overcome kinetic friction. Since kinetic friction is less than static friction, a horse has to apply more force to start than to keep it moving.

Q.6. A wheel is rolling on a level road. What is the direction of frictional force between the wheel and the road?

Ans. As the wheel is moving on the road, the portion of the tyre that contacts the road moves backward. Since friction opposes motion, the force of friction acts in the forward direction.

Q.7. When a car is taking a turn on a level road, which friction (static or kinetic) provides the centripetal force?

Ans. When the car is taking a turn on the level road, static friction (rather than kinetic friction) provides the centripetal force. It is because the portion of the tyre that contacts the road is temporarily at rest. Therefore, static friction plays the role.

Q.8. Why is it difficult to climb a greasy pole?

Ans. For climbing a pole, the climber presses the pole with his feet which in turn pushes the feet and the person climbs up. When grease is applied to the pole, the friction is greatly reduced and the person is not able to press it hard. Since action is small, the reaction of the pole is also small. Therefore, it becomes difficult for a person to climb a greasy pole.

Q.9. Why does a pilot not fall when his aeroplane loops a vertical loop?

Ans. It is because at the highest point of the vertical loop, the weight of the pilot is used up in providing the centripetal force. As a result, the pilot does not fall.

Q.10. Why is the outer rail of a curved railway track generally raised over the inner?

Ans. When the outer rail of a curved railway track is raised, the weight of the train provides a force component along the radius of the curved track towards the centre. This component of weight provides the necessary centripetal force to the train to enable it to move along the curved path.

LONG ANSWER QUESTIONS

1. What is friction? What are the factors on which force of friction depends? **[Refer to Arts. 11.1 and 11.4]**
2. State and explain the laws of friction. **[Refer to Art. 11.4]**
3. Give some advantages and disadvantages of friction. **[Refer to Art. 11.11]**
4. Derive an expression for acceleration of a body down a rough inclined plane. **[Refer to Art. 11.9]**
5. Explain coefficient of static friction and angle of friction. Give the relation between the two. **[Refer to Arts. 11.5 and 11.9]**
6. Derive an expression for centripetal force. **[Refer to Art. 11.13]**
7. Explain with a diagram how banking provides the centripetal force necessary for a car moving along a circular path. **[Refer to Art. 11.16]**
8. Discuss the motion of a body in a vertical circle. **[Refer to Art. 11.18]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Fig. 11.37 shows a block of mass m lying on a horizontal surface. The normal reaction R is the perpendicular force exerted by the surface on the block.

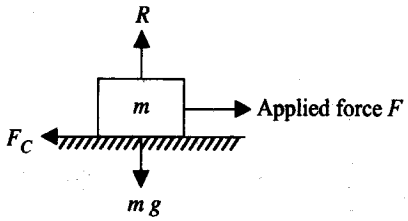


Fig. 11.37

2. When the applied horizontal force is zero, the frictional force is also zero. If a small horizontal force is applied to the right, the frictional force acts to the left parallel to the surfaces in contact and the block remains stationary. It means *static friction force* ($\because$ block is stationary) is equal and opposite to the applied force. As the applied force increases, static frictional force also increases.

   $\therefore$ Static friction, $f_s \leq \mu_S R$ ... (*i*)

   where μ_S = coefficient of static friction

   The equality of eq (*i*) holds when the block is on the *verge* of moving *i.e.* when static friction is maximum. This maximum static friction is called critical or limiting friction.

   Maximum static friction, $F_C = \mu_S R$

   The inequality in eq (*i*) holds when the applied force is less than $\mu_S R$.

3. If the applied force exceeds F_C ($= \mu_S R$), the block starts moving. It is found that frictional force decreases. This new frictional force is called *sliding* or *kinetic friction*. Its value is given by :

   $$F_k = \mu_k R$$

   where μ_k = coefficient of kinetic friction

4. The magnitudes of F_C and F_k depend upon the normal reaction R.
5. Both F_C and F_k act opposite to the direction of motion and parallel to the surfaces in contact.
6. F_k is usually independent of the speed.
7. Both F_C and F_k do not depend upon the area of contact between the surfaces.
8. The value of μ_k is generally less than μ_S.
9. Fig. 11.38 shows a block of mass m lying on a rough inclined plane.

   $$R = mg\cos\theta$$

   Remember that frictional force always acts parallel to the surfaces in contact and opposite to the motion.

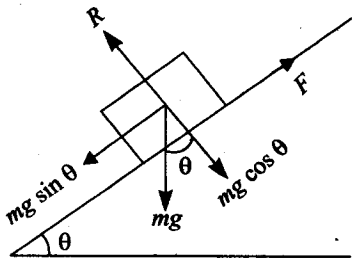


Fig. 11.38

   (*i*) If $mg\sin\theta < \mu_S R$ ($= \mu_S mg\cos\theta$), the block will remain stationary.

   (*ii*) If $mg\sin\theta > \mu_S R$, the block will move down the plane with acceleration 'a'.

   The equation of motion for the block is

   $mg\sin\theta - \mu_k mg\cos\theta = ma$

   Acceleration, $a = g(\sin\theta - \mu_k\cos\theta)$

   It is clear that acceleration 'a' is less than g.

   For smooth inclined plane, $\mu_k = 0$. Therefore, $a = g\sin\theta$.

10. Fig. 11.39 shows a block of mass m on a rough inclined plane being pulled up the incline by a force F making an angle α with the incline. If 'a' is the acceleration produced, then the equation of motion of the block is

$$F\cos\alpha - mg\sin\theta - F_k = ma$$

Note that force of kinetic friction in this case is given by ;

$$F_k = \mu_k (mg\cos\theta - F\sin\alpha)$$

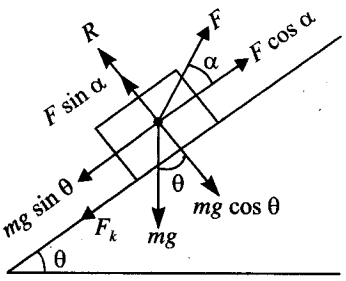


Fig. 11.39

11. Fig. 11.40 shows a block of wood being pressed against a wall by a horizontal force F. For the block to remain stationary ($R = F$),

$$F_C = mg$$

or $$\mu_S F = mg$$

This is an example to show that normal reaction R need not always be vertical. Its direction depends upon the orientation of the surfaces.

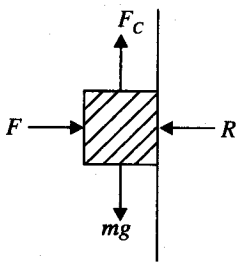


Fig. 11.40

12. On a level circular road, the maximum speed that the car can have to negotiate the turn without skidding is given by ;

$$v_m = \sqrt{\mu_S\, gr}\ ; \quad r = \text{radius of the curve}$$

13. The banking angle θ for a curve of given radius r to be negotiated at speed v without skidding is given by ;

$$\tan\theta = \frac{v^2}{rg}$$

Note that in the above expression, friction between tyres and road is neglected.

14. When there is a friction on the banked road, the horizontal component ($F_S\cos\theta$) of the frictional force is added to the horizontal component $R\sin\theta$ of the normal reaction R. In that case,

$$R\sin\theta + F_S\cos\theta = \frac{mv^2}{r}$$

The added friction component allows the car to go around the curve at higher speeds.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. On the horizontal surface of a truck, a block of mass 1 kg is placed ($\mu_S = 0.6$) and truck is moving with an acceleration of 5m/sec^2. Then frictional force on the block will be **[CBSE PMT 2001]**
   (*a*) 5 N (*b*) 6 N
   (*c*) 5.88 N (*d*) 8 N
2. When two surfaces are coated with lubricant, then they **[AIIMS 2001]**
   (*a*) stick to each other
   (*b*) slide upon each other
   (*c*) roll upon each other
   (*d*) none of these
3. A body is placed on an inclined plane and has to be pushed down. The angle made by the normal reaction with the vertical will be **[Pb PMT 2001]**
   (*a*) equal to the angle of repose
   (*b*) equal to the angle of friction
   (*c*) less than the angle of repose
   (*d*) more than the angle of friction
4. If a 100 N force is applied to a 10 kg block as shown in Fig. 11.41, then acceleration produced in the slab is **[CBSE – PMT 1999]**

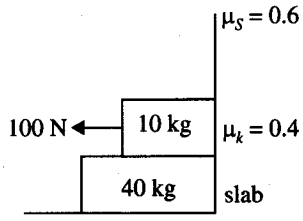


Fig. 11.41

(*a*) 1.65 m/s^2 (*b*) 0.98 m/s^2
(*c*) 0.5 m/s^2 (*d*) 0.25 m/s^2

5. An army vehicle of mass 1000 kg is moving with a velocity of 10 m/s and is acted upon by a forward force of 1000 N due to engine and a retarding force of 500 N due to friction. What will be its velocity after 10 sec? **[Pb PMT 2000]**

(*a*) 5 m/s (*b*) 10 m/s
(*c*) 15 m/s (*d*) 20 m/s

6. The upper half of an inclined plane of inclination θ is perfectly smooth while the lower is rough. A body starting from rest at the top comes back to rest at the bottom, if the coefficient of friction for the lower half is given by **[Pb PMT 2000]**

(*a*) $\mu = \sin\theta$ (*b*) $\mu = \cot\theta$
(*c*) $\mu = 2\cos\theta$ (*d*) $\mu = 2\tan\theta$

7. A block has been placed on an inclined plane. The slope angle θ of the plane is such that the block slides down with a constant speed. The coefficient of kinetic friction is equal to **[CBSE PMT 1993]**

(*a*) $\sin\theta$ (*b*) $\tan\theta$
(*c*) $\cos\theta$ (*d*) $\cot\theta$

8. If a ladder weighing 250N is placed against a smooth vertical wall and the coefficient of friction between the ladder and floor is 0.3, then what is the maximum force of friction available at the point of contact between the ladder and floor? **[AFMC-Pune 1996]**

(*a*) 75 N (*b*) 50 N
(*c*) 35 N (*d*) 25 N

9. Consider a car moving along a straight horizontal road with a speed of 72 km/h. If the coefficient of static friction between the tyres and the road is 0.5, the shortest distance in which the car can be stopped is (g = 10 m/s^2) **[CBSE PMT 1992]**

(*a*) 30 m (*b*) 40 m
(*c*) 72 m (*d*) 20 m

10. A block of mass 2 kg is placed on the floor. The coefficient of static friction is 0.4. If a force of 2.8 N is applied on the block parallel to the floor, the force of friction between the block and floor (g = 10 m/s^2) is **[CPMT 1993]**

(*a*) 2.8 N (*b*) 8 N
(*c*) 2 N (*d*) zero

11. A body of mass m is placed on a rough surface with coefficient of friction (μ) inclined at θ with the horizontal. If the mass is in equilibrium, then **[AIIMS 1996]**

(*a*) $\theta = \tan^{-1}(\mu)$ (*b*) $\theta = \tan^{-1}\left(\frac{\mu}{2}\right)$
(*c*) $\theta = \tan^{-1}\left(\frac{m}{\mu}\right)$ (*d*) $\theta = \tan^{-1}\left(\frac{\mu}{m}\right)$

12. A body of mass 10 kg is sliding down a rough inclined plane which makes an angle of 30° with the horizontal. If the coefficient of friction is 0.25, what is the acceleration of the body? **[CPMT 1993]**

(*a*) 1.39 m/s^2 (*b*) 2.78 m/s^2
(*c*) 3.65 m/s^2 (*d*) 5.56 m/s^2

13. As shown in Fig. 11.42, a mass is performing vertical circular motion. If the average velocity of the mass is increased, then at which point the string will break ? **[CBSE PMT 2000]**

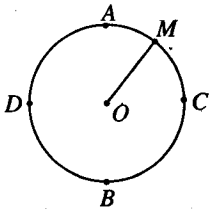


Fig. 11.42

(*a*) *A* (*b*) *B*
(*c*) *C* (*d*) *D*

14. If a cyclist moving with a speed of 4.9 m/s on a level road can take a sharp circular turn of radius 4 m, then co-efficient of friction between the cycle tyres and road is **[AFMC-Pune 2001]**

(*a*) 0.41 (*b*) 0.51
(*c*) 0.61 (*d*) 0.71

15. A body of mass 0.4 kg is whirled in a vertical circle making two rev/sec. If the radius of the circle is 1.2 m, then tension in the string when the body is at the top of circle is **[AFMC–Pune 1998]**

(*a*) 71.88 N (*b*) 89.88 N
(*c*) 109.88 N (*d*) 115.88 N

16. If a cycle wheel of radius 4 m completes one revolution in two seconds, then acceleration of the cycle is **[Pb PMT 1995]**

(a) π m/s^2 (b) 4π m/s^2

(c) $2\pi^2$ m/s^2 (d) $4\pi^2$ m/s^2

17. The coefficient of static friction μ_S between block A of mass 2 kg and the table as shown in Fig. 11.43 is 0.2. What would be the maximum mass value of block B so that the two blocks do not move? The string and the pulley are assumed to be smooth and massless (g = 10 ms^{-2}). **[CBSE 2004]**

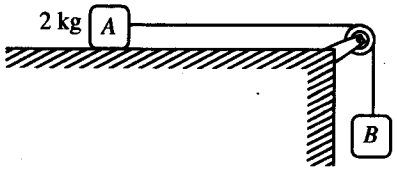


Fig. 11.43

(a) 2.0 kg (b) 4.0 kg

(c) 0.2 kg (d) 0.4 kg

18. If a ladder is not in equilibrium against a smooth vertical wall, then it can be made in equilibrium by **[CBSE 1998]**

(a) increasing the angle of inclination

(b) decreasing the angle of inclination

(c) increasing the length of the ladder

(d) decreasing the length of the ladder

19. A body of mass 2 kg is placed on rough horizontal plane. The coefficient of friction between body and plane is 0.2. Then, **[Manipal 2005]**

(a) body will move in forward direction if F = 5 N

(b) body will move in backward direction with acceleration 0.5 ms^{-2} if force F = 3 N

(c) If F = 3 N, then body will be in rest condition.

(d) both (a) and (c) are correct.

20. A cyclist moving at a speed of 20 m/s takes a turn, if he doubles his speed then chance of overturn **[AFMC 2003]**

(a) is doubled (b) is halved

(c) becomes 4 times (d) becomes 1/4 times

21. A bucket full of water is revolved in a vertical circle of radius 2 m. What should be the maximum time-period of revolution so that the water doesn't fall out of the bucket? **[AFMC 2004]**

(a) 1 s (b) 2 s

(c) 3 s (d) 4 s

22. A stone is tied to a string of length l and is whirled in a vertical circle with the other end of the string as the centre. At a certain instant of time, the stone is at its lowest position and has a speed u. The magnitude of the change in velocity as it reaches a position where the string is horizontal (g being acceleration due to gravity) is **[CBSE 2004]**

(a) $\sqrt{2(u^2 - gl)}$ (b) $\sqrt{u^2 - gl}$

(c) $u - \sqrt{u^2 - 2gl}$ (d) $\sqrt{2gl}$

23. A given shaped glass tube having uniform cross-section is filled with water and is mounted on a rotatable shaft as shown in Fig. 11.44. If the tube is rotated with a constant angular velocity ω, then, **[AIIMS 2005]**

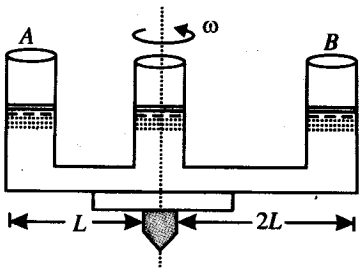


Fig. 11.44

(a) water levels in both sections A and B go up

(b) water levels in both sections A and B remain same

(c) water level in section A goes up and that in B comes down

(d) water level in section A comes down and that in B goes up

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (a)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (c)
6. (d)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (a)
11. (a)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (a)
16. (d)	**17.** (d)	**18.** (a)	**19.** (d)	**20.** (a)
21. (a)	**22.** (a)	**23.** (a)		

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Forward force on the block = ma = 1 × 5 = 5 N. Therefore, frictional force on the block is 5 N.
2. When two surfaces are coated with a lubricant, the friction between the surfaces decreases. Therefore, they can **slide upon each other**.
3. When angle of inclination is equal to angle of repose, the body slides down the inclined plane without applying external force. Since the body has to be pushed down, the angle made by normal reaction with the vertical will be **less than the angle of repose**.
4. Limiting friction, $F_C = \mu_S\, mg = 0.6 \times 10 \times 9.8 = 58.8$ N

   Since applied force of 100 N is greater than F_C, the kinetic friction is

   $$F_k = \mu_k\, mg = 0.4 \times 10 \times 9.8$$

   $\therefore$ Acceleration for the slab, $a = \dfrac{F_k}{M} = \dfrac{0.4 \times 10 \times 9.8}{40} = \mathbf{0.98\ m/s^2}$
5. Net forward force, $F = 1000 - 500 = 500$ N

   $$\text{Acceleration, } a = \frac{F}{m} = \frac{500}{1000} = 0.5\ \text{m/s}^2$$

   The velocity v of the vehicle after 10 sec is

   $$v = v_0 + at = 10 + 0.5 \times 10 = 15\ \text{m/s}$$
6. The acceleration down a perfectly smooth inclinded plane = $g \sin\theta$ and acceleration down a rough inclined plane = $g(\sin\theta - \mu\cos\theta)$. Since the body starting from the top comes to rest at the bottom, acceleration in the first half is equal to retardation in the second half

   $\therefore$ $\quad g\sin\theta = -g(\sin\theta - \mu\cos\theta)$ or $2\sin\theta = \mu\cos\theta$

   $\therefore$ $\quad \mu = 2\dfrac{\sin\theta}{\cos\theta} = \mathbf{2\tan\theta}$
7. Co-efficient of kinetic friction, $\mu_k = \dfrac{F_k}{R}$

   Now $R = mg\cos\theta$ and $F_k = mg\sin\theta$ $\quad\therefore\ \mu_k = \dfrac{mg\sin\theta}{mg\cos\theta} = \tan\theta$
8. Fig. 11.45 shows a 250 N ladder standing against a smooth vertical wall. The various forces acting on the ladder are shown. At the floor, friction can exert a horizontal force H to the right and the floor supplies a vertical force R. The wall, being frictionless, can only exert a horizontal push to the left. Since the ladder is in equilibrium,

   $$H = P \text{ and } R = 250\ \text{N}$$

   $\therefore$ Maximum friction available

   $$= \mu R = 0.3 \times 250 = 75\ \text{N}$$

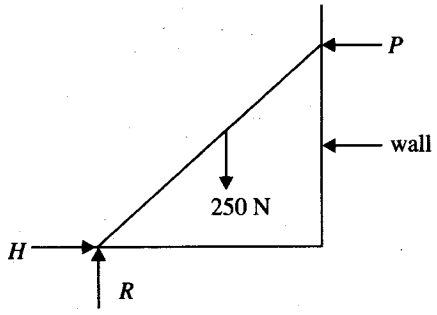


Fig. 11.45

9. Here $v_0 = 72$ km/h $= 20$ m/s ; $v = 0$; $a = -\mu g = -0.5 \times 10 = -5\ \text{m/s}^2$

   Now $\quad v^2 - v_0^2 = 2aS$

$$\therefore \quad S = \frac{v^2 - v_0^2}{2a} = \frac{(0)^2 - (20)^2}{2\times(-5)} = 40 \text{ m}$$

10. Limiting friction is given by ;

$$F_C = \mu R = \mu mg \quad \therefore \quad F_C = 0.4 \times 2 \times 10 = 8 \text{ N}$$

Since the applied force is less than F_C, the block will not move. The force of friction will, therefore, be **2.8 N.**

11. Coefficient of friction, $\mu = \frac{F_k}{R}$

$$\therefore \quad F_k = mg \sin\theta \text{ and } R = mg \cos\theta$$

$$\therefore \quad \mu = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta \quad \text{or } \theta = \tan^{-1}(\mu)$$

12. Acceleration, $a = g(\sin\theta - \mu\cos\theta) = 9.8(\sin 30° - 0.25 \cos 30°) = 2.78 \text{ m/s}^2$

13. Let tensions at points A and B be T_H and T_L respectively (See Fig. 11.46). Then,

$$T_L - mg = \frac{mv_L^2}{r}$$

or $$T_L = mg + \frac{mv_L^2}{r}$$

If $$v_L = \sqrt{5rg}, \; v_H = \sqrt{rg} \quad \text{and} \quad T_H = 0, \text{ then,}$$

$$T_L = mg + \frac{5mgr}{r} = 6mg$$

$$\therefore \quad T_L - T_H = 6mg$$

Therefore, T_L is greater than T_H by 6 times the weight of the mass. The string will break when mass is at point B.

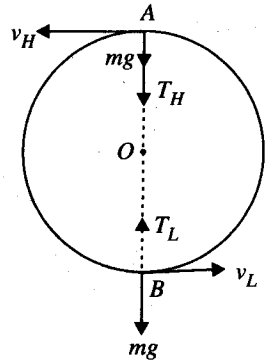


Fig. 11.46

14. The frictional force ($= \mu mg$) provides the necessary centripetal force (mv^2/r).

$$\therefore \quad \frac{mv^2}{r} = \mu mg$$

$$\therefore \quad \mu = \frac{v^2}{rg} = \frac{(4.9)^2}{4\times9.8} = 0.61$$

15. Tension T in the string when the body is at the top of the circle is

$$T = \frac{mv^2}{r} - mg$$

Here $m = 0.4$ kg ; $r = 1.2$ m ; $v = 2\pi rn = 2\pi \times 1.2 \times 2 = 15.08$ m/s

$$\therefore \quad T = \frac{0.4\times(15.08)^2}{1.2} - 0.4\times9.8 = 71.88 \text{ N}$$

16. Frequency, $f = \frac{1}{2}$ rev / sec

Angular velocity, $\omega = 2\pi f = 2\pi\times\frac{1}{2} = \pi$ rad / sec

Acceleration, $a = r\omega^2 = 4\times\pi^2 = 4\pi^2 \text{ m/s}^2$

17. Fig. 11.47 shows the conditions of the problem. Since the two blocks do not move, $T - Mg = 0$ or $T = Mg$.

Now $T = \mu_S\, mg$.

$\therefore \quad Mg = \mu_S\, mg$

or $\quad M = \mu_S\, m$

$= 0.2 \times 2 = \mathbf{0.4}$ kg

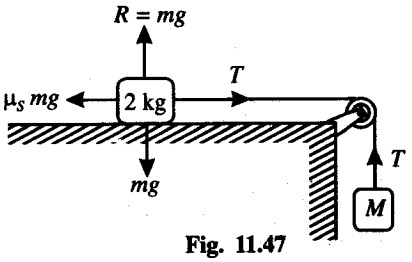


Fig. 11.47

18. The ladder can be made in equilibrium by increasing the angle of inclination.

19. Limiting friction, $F_C = \mu_S R = \mu_S mg = 0.2 \times 2 \times 9.8 = 3.92$ N. For moving the body, the applied force should be equal to or greater than F_C (= 3.92 N). It is clear that option (*d*) is correct.

20. When the cyclist takes a turn, the centripetal force acting on the cyclist is

$$F_c = \frac{mv^2}{r} \quad \text{or} \quad F_c \propto v^2$$

If the speed is doubled, the centripetal force becomes four times. Therefore, the chance of overturning is 1/4 times.

21. The minimum speed required at the lowest point of motion to complete the circle is

$$v = \sqrt{5\,gr} \quad \text{or} \quad r\omega = \sqrt{5\,gr} \quad \text{or} \quad \frac{2\pi}{T_{max}}\, r = \sqrt{5\,gr}$$

$$\therefore \qquad T_{max} = 2\pi\sqrt{\frac{r}{5g}} = 2\pi\sqrt{\frac{2}{5\times 10}} = 1.256 \text{ s}$$

Therefore, time period of revolution $\simeq$ 1s.

22. Let v be the speed of the stone when the string becomes horizontal (point A in Fig. 11.48). At point A, the stone rises through a height l and its P.E. = mgl. Applying the law of conservation of energy at points B and A, we have,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgl$$

$$\therefore \qquad v = \sqrt{u^2 - 2gl}$$

Now $\quad |\Delta \vec{v}| = |\vec{v} - \vec{u}| = \sqrt{v^2 + u^2 - 2uv\cos 90°}$

$$= \sqrt{v^2 + u^2} = \sqrt{2(u^2 - gl)}$$

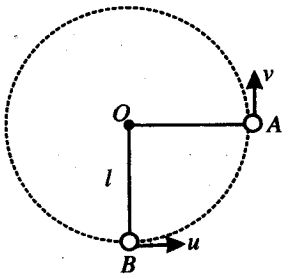


Fig. 11.48

23. Due to rotation of the shaft, the centrifugal force on liquid columns will push the liquid in both sections A and B with a fall in the central column. Therefore, water levels in both the sections *A* and *B* go up.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A block of mass 0.1 kg is held against a wall by applying a horizontal force of 5N on the block. If the coefficient of friction between the block and wall is 0.5, the magnitude of frictional force is **[IIT 1994]**

(*a*) 0.98 N (*b*) 4.9 N

(*c*) 2.5 N (*d*) 0.49 N

2. Pushing force is making an angle θ to the horizontal on a block of weight W on a horizontal table. If the angle of friction is α, then

magnitude of force required to move the body is equal to **[CET Haryana 1997]**

(a) $\frac{W\cos\alpha}{\cos(\theta-\alpha)}$ (b) $\frac{W\sin\alpha}{g\tan(\theta-\alpha)}$

(c) $\frac{W\tan\alpha}{\sin(\theta-\alpha)}$ (d) $\frac{W\sin\alpha}{\cos(\theta-\alpha)}$

3. A body of mass 60 kg is dragged with just enough force to start moving on a rough surface with coefficients of static and kinetic friction 0.5 and 0.4 respectively. On applying the same force, what is the acceleration? **[EAMCET 1993]**

(a) 0.98 m/s^2 (b) 9.8 m/s^2

(c) 0.54 m/s^2 (d) 5.292 m/s^2

4. A block of mass m lying on a rough horizontal surface, is acted upon by a horizontal force P and another force Q at an angle θ to the vertical [See Fig. 11.49]. The block will remain in equilibrium if coefficient of friction between it and surface is **[CET Haryana 1996]**

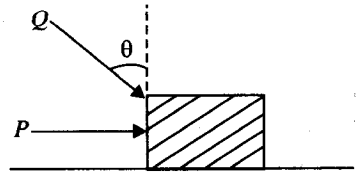


Fig. 11.49

(a) $(P + Q\sin\theta)/(mg + Q\cos\theta)$

(b) $(P\cos\theta + Q)/(mg - Q\sin\theta)$

(c) $(P + Q\cos\theta)/(mg + Q\sin\theta)$

(d) $(P\sin\theta - Q)/(mg - Q\cos\theta)$

5. A body of mass 100 g is sliding from an inclined plane of inclination 30°. What is the frictional force experienced if $\mu = 1.7$? **[BHU 1998]**

(a) $1.7\times\sqrt{2}\times\frac{1}{\sqrt{3}}$N (b) $1.7\times\sqrt{3}\times\frac{1}{2}$N

(c) $1.7\times\sqrt{3}$N (d) $1.7\times\sqrt{2}\times\frac{1}{\sqrt{3}}$N

6. A 20 kg block is initially at rest. A 75 N force is required to set the block in motion. After the motion, a force of 60 N is applied to keep the block moving with constant speed. The coefficient of friction is **[AMU 1999]**

(a) 0.6 (b) 0.52

(c) 0.44 (d) 0.35

7. A train of mass 2000 quintals is moved up an incline of 1 in 50 with an acceleration of 2 m/s^2. The force of friction is 0.5 N/quintal. The magnitude of applied force is **[Rajasthan CET 1996]**

(a) 3.526×10^6 N (b) 4.402×10^5 N

(c) 2.862×10^4 N (d) none of above

8. A 10 kg block slides down with acceleration down a rough inclined plane making an angle of 20° with the horizontal. Calculate the acceleration when the inclination of the plane is increased to 30°. **[CET Karnataka 1997]**

(a) 1.81 m/s^2 (b) 2.53 m/s^2

(c) 3.41 m/s^2 (d) 1.45 m/s^2

9. A cricket ball rolling on ice with a velocity of 5.6 m/s comes to rest after travelling 8 m. What is the coefficient of friction ? **[MP PET 1996]**

(a) 0.5 (b) 0.2

(c) 0.6 (d) 0.3

10. A marble block of mass 2 kg lying on ice when given a velocity of 6 m/s is stopped by friction in 10 s. Then the coefficient of friction is **[AIEEE 2003]**

(a) 0.02 (b) 0.03

(c) 0.06 (d) 0.01

11. A coin is placed on a gramophone record rotating at a speed of 45 r.p.m. It flies away when the rotational speed is 50 r.p.m. If two such coins are placed one over the other on the same record, both of them will fly away when rotational speed is **[MP PET 1996]**

(a) 100 r.p.m. (b) 25 r.p.m.

(c) 12.5 r.p.m. (d) 50 r.p.m.

12. A stone of mass 1 kg tied to one end of a string of length 1m is whirled in a horizontal circle with a constant speed of 2 rad/sec. Tension of the string is **[CET Haryana 1998]**

(a) 0.5 N (b) 1 N

(c) 2 N (d) 4 N

13. A body is revolving with a uniform speed v in a circle of radius r. The tangential acceleration is **[CET Haryana 1999]**

(a) zero (b) v^2/r

(c) v/r (d) v/r^2

14. A stone of mass 16 kg is attached to a string 144 m long and is whirled in a horizontal circle. The maximum tension the string can stand is 16 N. The maximum speed that can be given to the stone without breaking it will be **[CET Haryana 1994]**

(*a*) 12 m/s (*b*) 16 m/s

(*c*) 20 m/s (*d*) 14 m/s

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*a*)	**2.** (*d*)	**3.** (*a*)	**4.** (*a*)	**5.** (*b*)
6. (*d*)	**7.** (*b*)	**8.** (*a*)	**9.** (*b*)	**10.** (*c*)
11. (*d*)	**12.** (*d*)	**13.** (*a*)	**14.** (*a*)	

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Fig. 11.50 shows the conditions of the problem. In the equilibrium position, R = 5 N. Frictional force, $F_C = mg = 0.1 \times 9.8 = 0.98$ N

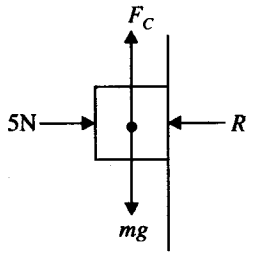


Fig. 11.50

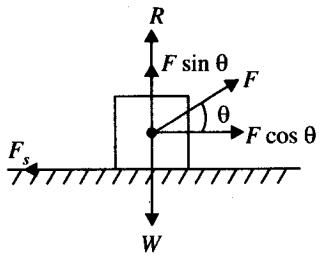


Fig. 11.51

2. Fig. 11.51 shows the conditions of the problem.

$$\tan\alpha = \frac{F_s}{R} = \mu \; ; \; R = W - F\sin\theta$$

$$\therefore \quad F_s = \mu R = \mu(W - F\sin\theta)$$

Now in order to move the body, $F\cos\theta$ should be equal to F_s.

$$\therefore \quad F\cos\theta = \mu(W - F\sin\theta) \quad \text{or} \quad F(\cos\theta + \mu\sin\theta) = \mu W$$

or $$F(\cos\theta + \tan\alpha \sin\theta) = \tan\alpha \; W \qquad (\because \mu = \tan\alpha)$$

or $$F = \frac{\tan\alpha \, W}{\cos\theta + \tan\alpha\sin\theta} = \frac{W\sin\alpha}{\cos(\theta - \alpha)}$$

3. Limiting friction, $F_C = \mu_S \, mg = 0.5 \times 60 \times 9.8$ N

Kinetic friction, $F_k = \mu_k \, mg = 0.4 \times 60 \times 9.8$ N

Accelerating force, $F = F_C - F_k = 60 \times 9.8(0.5 - 0.4) = 60 \times 9.8 \times 0.1$ N

$$\therefore \quad \text{Acceleration, } a = \frac{F}{m} = \frac{60 \times 9.8 \times 0.1}{60} = 0.98 \text{ m/s}^2$$

4. Applied force in horizontal direction = $(P + Q\sin\theta)$ ...(*i*)

Normal reaction, $R = (mg + Q\cos\theta)$

Frictional force = $\mu R = \mu(mg + Q\cos\theta)$ ... (*ii*)

For the block to be in equilibrium, eq. (*i*) = eq. (*ii*).

$\therefore \quad \mu(mg + Q\cos\theta) = (P + Q\sin\theta)$ or $\mu = P + Q\sin\theta/mg + Q\cos\theta$

5. Frictional force $= \mu R = \mu mg\cos\theta$

$$= 1.7 \times \left(\frac{100}{1000}\right) \times 10\cos 30° = 1.7 \times \frac{\sqrt{3}}{2} \text{ N}$$

6. Since the block is moving with constant speed,

Applied force = Force of friction = 60 N

$$\therefore \quad \mu = \frac{60}{R} = \frac{60}{mg} = \frac{60}{20 \times 9.8} = 0.35$$

7. The magnitude of applied force is

$F = mg\sin\theta + ma +$ force of friction

$$= 2000 \times 100 \times 9.8 \times \frac{1}{50} + 2000 \times 100 \times 2 + 0.5 \times 2000$$

$$= 39200 + 400000 + 1000 = 440200 \text{ N} = 4.402 \times 10^5 \text{ N}$$

8. Coefficient of friction, $\mu = \tan 20° = 0.365$

Acceleration, $a = g(\sin\theta - \mu\cos\theta)$

$= 9.8 (\sin 30° - 0.365 \times \cos 30°) = 1.81 \text{ m/s}^2$

9. $v^2 - v_0^2 = 2as$

or $$a = \frac{v^2 - v_0^2}{2s} = \frac{(0)^2 - (5.6)^2}{2 \times 8} = 1.96 \text{ m/s}^2$$

Negative sign is omitted because only magnitude is required.

Force of friction $= ma$

$$\therefore \quad \mu = \frac{ma}{R} = \frac{ma}{mg} = \frac{a}{g} = \frac{1.96}{9.8} = 0.2$$

10. $F = ma$ or $\mu mg = ma \quad \therefore \quad \mu = a/g$

Now $v = v_0 + at$ or $0 = 6 + 10a \quad \therefore a = -0.6 \text{ ms}^{-2}$

Therefore, coefficient of friction is $\mu = a/g = 0.6/10 = 0.06$

11. A coin flies off when centrifugal force just exceeds the force of friction *i.e.*,

$$mr\omega^2 \geq \mu mg \quad \text{or} \quad \omega \geq \sqrt{\frac{\mu g}{r}}$$

Since ω is independent of mass, ω will remain the same.

12. The tension T in the string is equal to the centripetal force.

$$\therefore \quad T = \frac{mv^2}{r} = m\omega^2 r = 1 \times (2)^2 \times 1 = 4 \text{ N}$$

13. Since speed is constant, there is no tangential acceleration. Only centripetal acceleration will be there.

14. Let v_m be the maximum speed that can be given to the stone without breaking the string.

$$\therefore \quad T = \frac{m v_m^2}{r} \text{ or } 16 = \frac{16 \times v_m^2}{144} \quad \therefore v_m = \sqrt{144} = 12 \text{ m/s}$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **An aeroplane requires for take off a speed of 80 km/h, the run on the ground being 100 m. The mass of the plane is 10,000 kg and the coefficient of friction between the plane and the ground is 0.2. Assume that plane accelerates uniformly during the take off. What is the minimum force required by the engine of the plane for take off ?** **[I.I.T.]**

Hint. Take off speed, v= 80 km/h $= \frac{200}{9}$ ms^{-1} ; Mass of plane, M =10,000 kg

The acceleration a required to attain the take off speed is

$$a = \frac{v^2 - v_0^2}{2S} = \frac{(200/9)^2 - (0)^2}{2\times 100} = \frac{200}{81} \text{ ms}^{-2}$$

Force required to give this acceleration (without friction) is

$$F_1 = M\,a = 10{,}000 \times \frac{200}{81} = 2.47 \times 10^4 \text{ N}$$

Frictional force, $F_2 = \mu R = \mu Mg = 0.2 \times 10{,}000 \times 9.8 = 1.96 \times 10^4$ N

$\therefore$ Total force required, $F = F_1 + F_2 = (2.47 + 1.96)10^4 = \mathbf{4.43 \times 10^4}$ N

2. **A heavy uniform chain lies on a horizontal table top. If the coefficient of friction between the chain and the table surface is 0.25, what is the maximum fraction of length that can hang over one edge ?**

Hint. Let L be the total length of the chain and l that of the hanging part. Let m be the mass per unit length of the chain. For equilibrium,

Weight of hanging part = Frictional force on chain remaining on table

or $$mlg = \mu m(L - l)g$$

or $$\frac{l}{L} = \frac{\mu}{\mu + 1} = \frac{0.25}{0.25 + 1} = 0.2$$

3. **Calculate the acceleration of the block-trolley system shown in Fig. 11.52 if the coefficient of kinetic friction between the trolley and the surface is 0.04. What is the tension in the string? Take g = 10 ms^{-2} and neglect the mass of the string.**

Hint. Let the acceleration produced in the block-trolley system be a.

Frictional force on trolley is

$$F_k = \mu_k \times (\text{mass of trolley}) \times g$$
$$= 0.04 \times 30 \times 10 = 12 \text{ N}$$

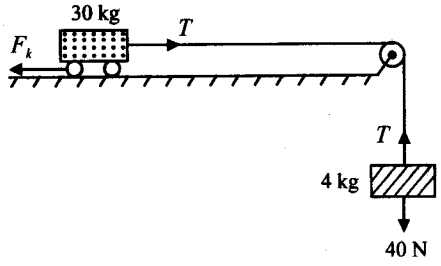


Fig. 11.52

Considering the motion of 4 kg mass, we have,

$$40 - T = 4a \qquad ...(i)$$

Considering the motion of trolley, we have,

$$T - F_k = 30\,a$$

or $$T - 12 = 30\,a \qquad ...(ii)$$

Adding eqs. (i) and (ii), we get,

$$28 = 34\,a \text{ or } a = 28/34 = \mathbf{0.82} \text{ ms}^{-2}$$

Putting the value of a in eq. (i), we have,

$$T = 40 - 4a = 40 - (4 \times 0.82) = \mathbf{36.72} \text{ N}$$

4. The block A in Fig. 11.53 weighs 100 N. The coefficient of static friction between block A and table is 0.25. Find the maximum weight of block *B* for which the system is in equilibrium.

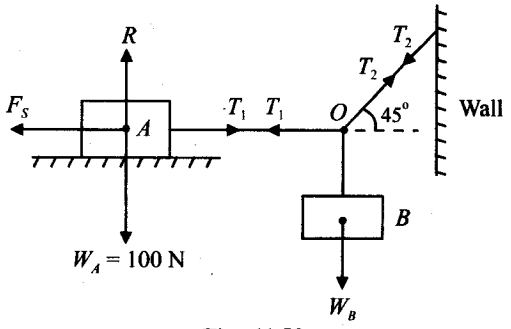


Fig. 11.53

Hint. The various forces acting on the block *A* are shown in Fig. 11.53. The limiting force (maximum) of static friction on block *A* is given by ;

$$F_C = \mu_S R = \mu_S W_A = 0.25 \times 100 = 25 \text{ N}$$

For equilibrium of block *A*, $T_1 = F_C = 25$ N

For equilibrium of block *B*, we have,

$$T_1 = T_2 \cos 45° \text{ and } W_B = T_2 \sin 45°$$

$$\therefore \quad W_B = \left(\frac{T_1}{\cos 45°}\right) \sin 45° = T_1 \tan 45° = T_1 = 25 \text{ N}$$

5. A motor car running at the rate of 7 m/s can be stopped by its brakes in 10 m. Prove that the total resistance to the motion, when the brakes are on, is one-fourth the weight of the car.

Hint. Initial velocity, $v_0 = 7$ m s^{-1} ; Final velocity, $v = 0$; $s = 10$ m

Now $v^2 - v_0^2 = 2\,a\,s$

$$\therefore \quad a = \frac{v^2 - v_0^2}{2s} = \frac{(0)^2 - (7)^2}{2 \times 10} = \frac{-49}{20} \text{ms}^{-2}$$

The negative sign indicates retardation.

$\therefore$ Force of friction = $ma = m \times (49/20)$

Weight of car = $m\,g$

$$\therefore \quad \text{Ratio} = \frac{\text{Force of friction}}{\text{Weight of car}} = \frac{m \times (49/20)}{m \times 9.8} = \frac{49}{20 \times 9.8} = \frac{1}{4}$$

6. A block of metal of mass 0.05 kg when placed on an inclined plane at an angle of 15° slides down without acceleration. Find the acceleration if the angle is increased by 15°.

Hint. Angle of repose = Angle of inclination when body just slides = 15°

$$\therefore \quad \mu = \tan 15° = 0.268$$

As the angle is further increased by 15°, $\theta = 15° + 15° = 30°$.

The acceleration of the body moving along the inclined plane downward is given by;

$$a = g\,(\sin\theta - \mu\cos\theta) = 9.8\,(\sin 30° - 0.268 \times \cos 30°) = 2.63 \text{ ms}^{-2}$$

7. A block of mass 5 kg lies on rough horizontal surface. When a horizontal force of 24.5 N is applied to the block, it just starts sliding while 19.6N is enough to keep it sliding at uniform velocity. Find (*i*) coefficient of static friction (*ii*) coefficient of

sliding friction. Find also the acceleration of the block when at rest is acted by a force of 24 N and a negligible impulse is imparted.

Hint. Limiting friction, F_C= 24.5 N ; Kinetic friction, F_k = 19.6 N

(*i*) $\mu_S = \frac{F_C}{mg} = \frac{24.5}{5\times9.8} = 0.5$

(*ii*) $\mu_k = \frac{F_k}{mg} = \frac{19.6}{5\times9.8} = 0.4$

When the body is given slight impulse, the relative motion is initiated and the static friction changes to kinetic friction.

Net force on the block = $F - F_k = F - \mu_k m g$

$$\therefore \quad \text{Acceleration} = \frac{\text{Net Force}}{\text{mass}} = \frac{F - \mu_k mg}{m}$$

$$= \frac{24-(0.4\times5\times9.8)}{5} = 0.88 \text{ ms}^{-2}$$

8. A 40 kg slab rests on a frictionless floor. A 10 kg block rests on the top of the slab [See Fig. 11.54]. The static coefficient of friction between the block and the slab is 0.6 while the coefficient of kinetic friction is 0.4. The 10 kg block is acted upon by a horizontal force *F*. What is the resulting acceleration of the slab if (*i*) *F* = 40 N (*ii*) *F* = 100 N ?

Hint. The maximum force of static friction on block M_2 is

$$F_C = \mu_S M_2 g = 0.6 \times 10 \times 9.8 = 58.8 \text{ N}$$

The action force F is on the block M_2 and the reaction force is on slab M_1.

(*i*) When the applied force F is 40 N, the limiting friction (F_C) is not overcome. Therefore, both the block and slab move together.

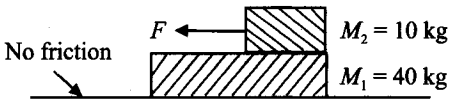


Fig. 11.54

$$\therefore \quad F = (M_1 + M_2)\, a$$

$$\text{or} \quad a = \frac{F}{M_1+M_2} = \frac{40}{40+10} = 0.8 \text{ ms}^{-2}$$

(*ii*) When the applied force F is 100 N, the limiting friction (F_C) is overcome. Therefore, block M_2 moves over the slab M_1. Now the frictional force on M_2 is kinetic.

Kinetic frictional force on the block M_2 is

$$F_k = \mu_k M_2 g = 0.4 \times 10 \times 9.8 = 39.2 \text{ N}$$

The slab M_1 moves due to the reaction of F_k.

$$\therefore \quad F_k = M_1 a$$

$$\text{or} \quad a = \frac{F_k}{M_1} = \frac{39.2}{40} = 0.98 \text{ ms}^{-2}$$

9. A body of mass *m* rests on a horizontal surface with which it has a coefficient of static friction μ. The applied force makes an angle θ with the horizontal. What should be the minimum force that can be applied to move the body ? [I.I.T.]

Hint. Figure 11.55 shows the conditions of the problem. The various forces acting on the body are shown.

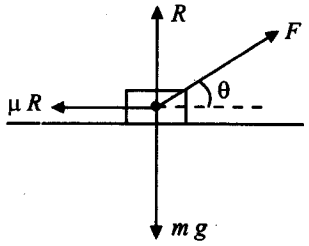


Fig. 11.55

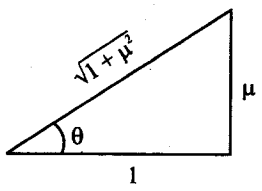


Fig. 11.56

For vertical equilibrium : $R + F \sin\theta = m g$

or $$R = mg - F \sin\theta \quad ...(i)$$

For horizontal equilibrium : $F \cos\theta = \mu R$ ...(ii)

Putting the value of R from eq. (*ii*) into eq. (*i*), we get,

$$\frac{F\cos\theta}{\mu} = mg - F\sin\theta$$

or $$F = \frac{\mu mg}{\cos\theta + \mu\sin\theta} \quad ...(iii)$$

For the force F to be minimum, the denominator [*i.e.*, $\cos\theta + \mu\sin\theta$] of eq. (*iii*) should be maximum *i.e.*,

$$\frac{d}{d\theta}[\cos\theta + \mu\sin\theta] = 0 \quad \text{or} \quad -\sin\theta + \mu\cos\theta = 0 \quad \therefore \mu = \tan\theta$$

If $\mu = \tan\theta$, then $\sin\theta$ and $\cos\theta$ (See Fig. 11.56) are given by ;

$$\sin\theta = \frac{\mu}{\sqrt{1+\mu^2}}\ ;\ \cos\theta = \frac{1}{\sqrt{1+\mu^2}}$$

Putting the values of $\sin\theta$ and $\cos\theta$ in eq. (*iii*), we get,

$$F_{min} = \frac{\mu mg}{\frac{1}{\sqrt{1+\mu^2}} + \frac{\mu\mu}{\sqrt{1+\mu^2}}} = \frac{\mu m g}{\frac{1+\mu^2}{\sqrt{1+\mu^2}}} = \frac{\mu mg}{\sqrt{1+\mu^2}}$$

10. **A block of mass 2 kg slides on an inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{3/2}$.**
   (*i*) What force should be applied to the block along the incline so that the block moves down without any acceleration?
   (*ii*) What force should be applied to the block along the incline so that the block moves up without any acceleration ? **[I.I.T.]**

Hint. Here m = 2 kg ; θ = 30° ; $\mu = \sqrt{3/2}$

(*i*) Fig. 11.57 (*i*) shows the conditions for the first case. Let P_1 be the required force. The various forces acting on the block are shown. Since the block moves without any acceleration downward along the plane, the net force on the block is zero *i.e.*,

$$P_1 + m g \sin\theta - F_k = 0$$

Now $m g \sin\theta = 2 \times 9.8 \times \sin 30° = 9.8$ N

and $F_k = \mu R = \mu (m\ g\ \cos\theta) = \sqrt{3/2}\ (2 \times 9.8 \times \cos 30°) = 20.8$ N

$\therefore \quad P_1 + 9.8 - 20.8 = 0$

or $P_1 = 20.8 - 9.8 = 11$N

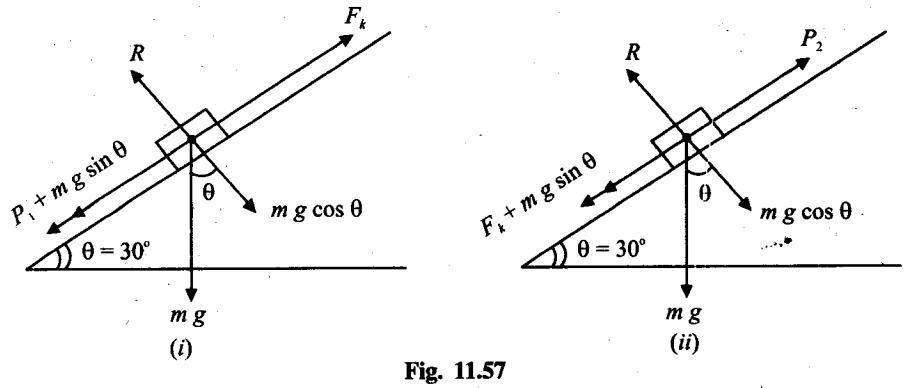


Fig. 11.57

(*ii*) Figure 11.57 (*ii*) shows the conditions for the second case. Let P_2 be the required force. The various forces acting on the block are shown. Since the block moves without any acceleration upward along the plane, the net force on the block is zero

i.e., $P_2 - (mg \sin\theta + F_k) = 0$

As already calculated, $mg \sin\theta = 9.8$ N ; $F_k = 20.8$ N

$\therefore \quad P_2 - (9.8 + 20.8) = 0$

or $P_2 = 9.8 + 20.8 = 30.6$ N

11. Two blocks *A* and *B* of masses M_A and M_B respectively are connected with a string passing over a frictionless pulley fixed at the corner of a table (See Fig. 11.58). Calculate the minimum mass of M_C of a third block *C* that may be placed on *A* to prevent *A* from moving. The coefficient of static friction between *A* and the table is μ_S. If $M_A = 10$ kg, $M_B = 5$ kg and $\mu_S = 0.2$, find M_C.

Hint. The maximum force of static friction on *A* is given by ;

$$F_C = \mu_S R_A = \mu_S (M_A + M_C)\ g$$

Since *A* and *B* are stationary, net force on each is zero.

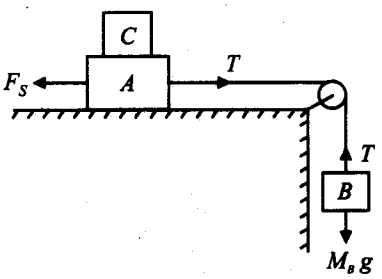


Fig. 11.58

For *A*. $T = F_C$

or $T = \mu_S (M_A + M_C) g \quad ...(i)$

For *B*. $T = M_B\ g$

Putting this value of T in eq. (*i*), we have,

$$M_B\ g = \mu_S (M_A + M_C)\ g$$

or $$M_C = \frac{M_B - \mu_S M_A}{\mu_S}$$

or $$M_C = \frac{M_B}{\mu_S} - M_A$$

For the given values of M_A, M_B and μ_S,

$$M_C = \frac{5}{0.2} - 10 = 25 - 10 = 15 \text{ kg}$$

12. A truck tows a trailer of mass 1200 kg at a speed of 10 ms^{-1} on a level road. The tension in the coupling is 1000 N. Find the tension in the coupling when the truck ascends a road having an inclination of 1 in 6. Assume that frictional resistance on the incline is the same as that on the level road.

Hint. The speed of truck-trailer is constant (= 10 ms^{-1}) on the level road. It means the frictional force on the trailer is 1000 N. It will also be 1000 N when the trailer ascends the inclined plane. Now the truck has to overcome a frictional force of 1000 N plus the downward component of weight $mg \sin\theta$ [See Fig. 11.59].

$\therefore$ Forward force (or tension in the coupling) is

$$F = 1000 + m\,g\,\sin\theta$$

Here m = 1200 kg ; g = 9.8 ms^{-2}; $\sin\theta = 1/6$

$$\therefore \quad F = 1000 + 1200 \times 9.8 \times \frac{1}{6}$$

$$= 1000 + 1960 = 2960 \text{ N}$$

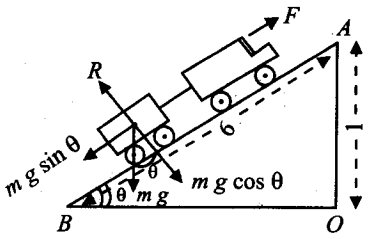


Fig. 11.59

13. Two blocks *A* and *B* are connected through a string and a spring. The string passes over a frictionless pulley as shown in Fig. 11.60 (*i*). The block *B* slides over the horizontal top surface of a stationary block *C* and the block *A* slides along the vertical surface of *C*. Both the blocks move with the same uniform speed. The coefficient of friction between the surfaces of the blocks is 0.2 and the force constant of the spring is 1960 N/m. If the mass of block *A* is 2 kg, calculate the mass of block *B* and the energy stored in the spring. [I.I.T.]

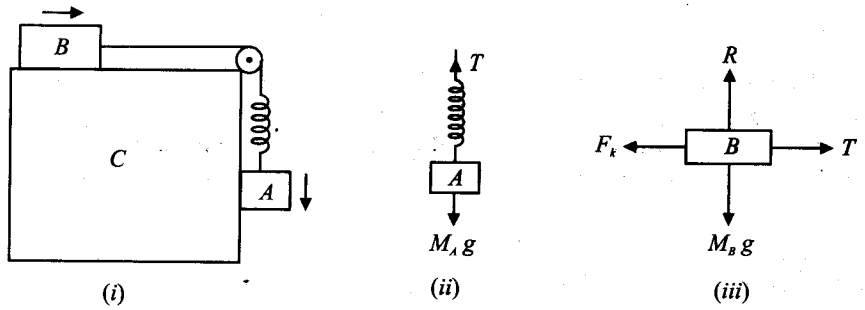


Fig. 11.60

Hint. Let M_A and M_B be the masses of blocks *A* and *B* respectively and *T* the tension in the string. The free-body diagrams for blocks *A* and *B* are shown in Figs. 11.60 (*ii*) and 11.60 (*iii*) respectively.

Motion of block A. Since block *A* moves with constant speed (zero acceleration), the net force on it is zero. Therefore, referring to Fig. 11.60 (*ii*), we have,

$$T = M_A\,g \qquad ...(i)$$

Note the weight $M_A\,g$ moves parallel to the surface of *C* and, therefore, there is no normal reaction on *A* due to C.

Motion of block *B*. Since block *B* moves with constant speed (zero acceleration), the net force on it is zero. For horizontal equilibrium [See Fig. 11.60 (*iii*)], we have,

$$T = F_k$$

or

$$T = \mu_k M_B\,g \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $M_A g = \mu_k M_B g$

$$\therefore \quad M_B = \frac{M_A}{\mu_k} = \frac{2}{0.2} = 10 \text{ kg}$$

Also $T = M_A g = 2 \times 9.8 = 19.6$ N

Let x be the extension of the spring due to the tension T. If k is the force constant of the spring, then,

$$x = \frac{\text{Tension}}{\text{Force constant}} = \frac{T}{k} = \frac{19.6}{1960} = 0.01 \text{m}$$

Energy stored in the spring $= \frac{1}{2}kx^2 = \frac{1}{2} \times 1960 \times (0.01)^2 = 0.098$ J

14. Fig. 11.61 (*i*) shows the system in equilibrium.

(*i*) What is the maximum value W can have if the frictional force on 40 N block cannot exceed 12 N ?

(*ii*) What is the coefficient of static friction between the block and the table top ?

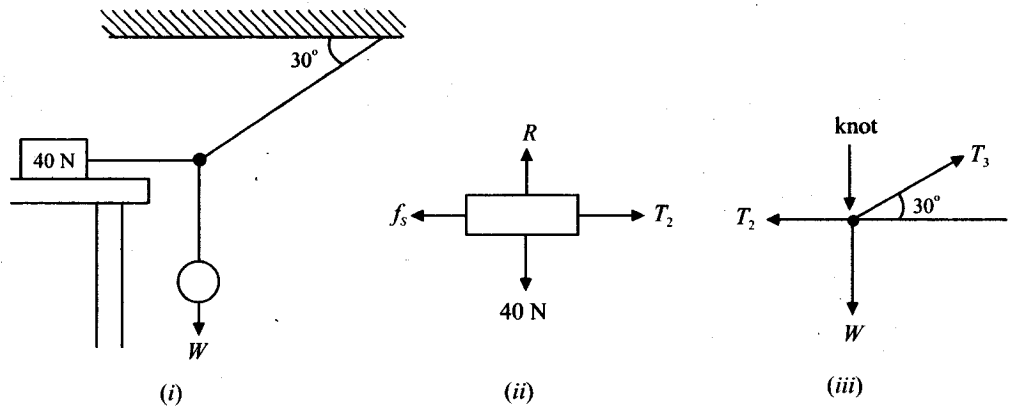


Fig. 11.61

Hint. The free-body diagrams for the block and knot are shown in Fig. 11.61 (*ii*) and Fig. 11.61 (*iii*) respectively.

For block : $f_{s(max)} = 12$ N $\quad \therefore \quad T_{2\,(max)} = 12$ N

For knot : $T_2 = T_3 \cos 30°$ and $W = T_3 \sin 30°$

$$\therefore \quad \frac{W}{T_2} = \frac{T_3 \sin 30°}{T_3 \cos 30°} = \tan 30°$$

or $W = T_2 \tan 30° = 0.577\ T_2$

(*i*) $W_{(max)} = 0.577\ T_{2\,(max)} = 0.577 \times 12 = \mathbf{6.92}$ **N**

(*ii*) $\mu_s = \frac{f_{s(max)}}{R} = \frac{12}{40} = \mathbf{0.3}$

15. A block is kept on a horizontal table. The table is undergoing simple harmonic motion of frequency 3Hz in a horizontal plane. The coefficient of static friction between the block and the table surface is 0.72. Find the maximum amplitude of the table at which the block does not slip on the surface (g = 10 ms^{-2}). [Roorkee]

Hint. When the block does not slip, it performs simple harmonic motion along with the table. Let A be the required maximum amplitude of the table. It means for this amplitude of the table motion, the force on the block is equal to static friction force on the block. The equation of S.H.M. is

$$x = A \sin \omega t$$

The instantaneous acceleration of the block is

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t$$

The magnitude of maximum acceleration $= \omega^2 A$

Maximum force on block, f_{max} = mass × maximum acceleration $= m\omega^2 A$

Frictional force on block = $\mu_S R = \mu_S m g$

Since the block is at rest *w.r.t.* the table,

$\therefore$ $m\omega^2 A = \mu_S mg$

or $A = \frac{\mu_S g}{\omega^2} = \frac{0.72 \times 10}{4\pi^2 f^2} = \frac{0.72 \times 10}{4\pi^2 \times (3)^2} = \mathbf{0.02\ m}$

16. If the rotation of the earth were such that $g = 0$ at the equator, what would be the length of the day? Take $g = 10\ ms^{-2}$ at the equator and take $\pi = 3$. The radius of earth = 6400 km.

Hint. The apparent weight of a body on the surface of earth is the vector difference between the gravitational pull ($= m g$) and the centripetal force ($= m v^2/R = m \omega^2 R$). If the speed of rotation of earth were so large that $g = 0$, it means that centripetal force is equal to the gravitational pull *i.e.*,

$$mg = m \omega^2 R$$

or $\omega = \sqrt{\frac{g}{R}}$

or $\frac{2\pi}{T} = \sqrt{\frac{g}{R}}$ $\therefore$ $T = 2\pi \times \sqrt{\frac{R}{g}} = 2 \times 3 \times \sqrt{\frac{6400 \times 1000}{10}} = 4800\ \text{s} =$ **4/3 hours**

17. The mass of a cycle rider alongwith the bicycle is 100 kg. He wants to cross over a circular road of radius 100 m with a speed of 10 ms^{-1}. If the coefficient of friction between the tyres and the road is $\mu = 0.6$, will the rider cross the turn ? Take $g = 10\ ms^{-2}$.

Hint. The necessary centripetal force F_c is provided by the friction between the tyres and the road.

Required centripetal force, $F_c = \frac{mv^2}{r} = \frac{100 \times (10)^2}{100} = 100\,\text{N}$

$\therefore$ Frictional force = $\mu R = \mu m g = 0.6 \times 100 \times 10 = 600$ N

Since the required centripetal force (100 N) is less than provided by the frictional force, the **rider will cross over the turn.**

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 5 : LAWS OF MOTION

Q.1. Give the magnitude and direction of the net force acting on

(*a*) a drop of rain falling down with a constant speed

(*b*) a cork of mass 10g floating on water

(*c*) a kite skillfully held stationary in the sky

(*d*) a car moving with a constant velocity of 30 km/h on a rough road

(*e*) a high-speed electron in space far from all material objects and free of electric and magnetic fields.

Ans. (*a*) Since the rain drop is falling with a constant speed, its acceleration $a = 0$. Therefore, net force on the rain drop is $F = ma = 0$.

(*b*) Since the cork is floating on water, its weight is balanced by the upthrust (= weight of water displaced). Therefore, the net force on the cork is zero.

(*c*) The kite is held stationary in the sky. Therefore, according to Newton's first law of motion, the net force on the kite is zero.

(*d*) Since the car is moving with a constant velocity, its acceleration $a = 0$. Therefore, the net force on the car is $F = ma = 0$.

(*e*) Since no field (gravitational, electric or magnetic) is acting on the electron, the net force on it is zero.

Q.2. A pebble of mass 0.05 kg is thrown vertically upwards. Give the magnitude and direction of net force on the pebble (*a*) during its upward motion (*b*) during its downward motion (*c*) at the highest point, where it is momentarily at rest. Do your answers change if the pebble were thrown at an angle of, say 45° with the horizontal direction?

Ans. The acceleration due to gravity g (= 9.8 ms^{-2}) is always *downward* whether the body is rising or falling. Therefore, the net force on the pebble in all the three cases is vertically downwards. Now m = 0.05 kg and g = 9.8 ms^{-2}. Therefore, $F = mg = 0.05 \times 9.8 = 0.49$ N vertically downward. If the pebble were thrown at an angle of 45° with the horizontal, it will have horizontal and vertical components of velocity. However, these components do not affect the force on the pebble ($F = ma = mg$ and both m and g are constant). Therefore, our answers will not change. However in case of (*c*), the pebble will not be at rest but will have horizontal component of velocity at the highest point.

Q.3. Give the magnitude and direction of the net force acting on :

(*a*) a stone of mass 0.1 kg just after it is dropped from the window of a stationary train.

(*b*) the same stone as above just after it is dropped from the window of a train running at a constant velocity of 36 km/hr.

(*c*) the same stone as above just after it is dropped from the window of a train accelerating with 1 ms^{-2}.

(*d*) the same stone as above lying on the floor of a train, which is accelerating with 1 ms^{-2}, the stone being at rest relative to the train.

Neglect the resistance of air throughout and take $g = 9.8$ m/s^2.

Ans. (*a*) In this case, the only force acting on the stone is that due to gravity.
$\therefore$ Net force = $ma = mg = 0.1 \times 9.8 =$ 0.98 N (vertically downwards)

(*b*) When the train is running at constant velocity in the horizontal direction, its acceleration, $a = 0$. Therefore, no force acts on the stone due to this motion. The only force acting on the stone is that due to gravity.

$\therefore$ Net force = $ma = mg = 0.1 \times 9.8 =$ 0.98 N (vertically downwards)

(*c*) When the train is accelerating with 1 ms^{-2}, a force $F = ma = 0.1 \times 1 = 0.1$ N acts on the stone in the horizontal direction. But once the stone is dropped from the train, F becomes zero and the net force is F_g due to gravity.

$\therefore$ Net force, $F_g = ma = mg = 0.1 \times 9.8 =$ 0.98 N (vertically downwards).

(*d*) In this case, the weight of the stone is balanced by the normal reaction. As the stone is lying on the floor, its acceleration is the same as that of the train.

$\therefore$ Net force on stone = ma = 0.1 × 1 = 0.1 N

The force is along the horizontal direction of motion of the train.

Q.4. One end of a string of length r is connected to a particle of mass m and the other end to a small peg on a smooth horizontal surface. If the particle moves in a circle with speed v, the net force on the particle (directed towards centre) is (*a*) T (*b*) $T - \frac{mv^2}{r}$ (*c*) $T + \frac{mv^2}{r}$ (*d*) 0. Choose correct alternative. T is tension in the string.

Ans. The net force on the particle is T [alternative (*a*)] and is directed towards the centre of the circle. The tension T provides the necessary centripetal force to the particle moving in the circle.

Q.5. A constant retarding force of 50 N is applied to a body of mass 20 kg moving initially with a speed of 15 ms^{-1}. How long does the body take to stop ?

Ans. The initial speed of the body is v_0 = 15 ms^{-1}. When a constant retarding force is applied, the speed of the body decreases uniformly to zero (*i.e.*, v = 0) in t seconds.

Now, $F = m\,a$

Here F = – 50 N ; m = 20 kg

$\therefore$ Retardation, $a = \frac{F}{m} = \frac{-50}{20} = -2.5$ ms^{-2}

Now, $v = v_0 + at$ or $0 = 15 - 2.5\,t$ $\therefore$ t = 15/2.5 = 6 s

Q.6. A constant force acting on a body of mass 3 kg changes its speed from 2 ms^{-1} to 3.5 ms^{-1} in 25 s. The direction of the motion of the body remains unchanged. What is the magnitude and direction of the force?

Ans. Here m = 3 kg ; v_0 = 2 ms^{-1} ; v = 3.5 ms^{-1} ; t = 25 s

$\therefore$ Force, $F = \frac{m(v-v_0)}{t} = \frac{3(3.5-2)}{25}$ = 0.18 N

Since the applied force increases the speed of the body, it acts in the direction of motion.

Q.7. A body of mass 5 kg is acted upon by two perpendicular forces of 8 N and 6 N. Find the magnitude and direction of the acceleration of the body.

Ans. Fig. 5.01 shows the situation. Here F_1 = 8 N ; F_2 = 6 N ; m = 5 kg

Resultant force, $F = \sqrt{F_1^2 + F_2^2} = \sqrt{(8)^2 + (6)^2}$ = 10 N

Suppose the resultant force F makes an angle α with F_1 (= 8N).

Then, $\alpha = \tan^{-1}\frac{F_2}{F_1} = \tan^{-1}\frac{6}{8} = 37°$ with 8 N force

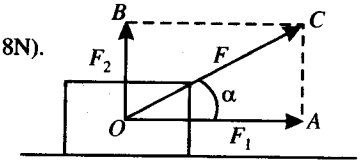


Fig. 5.01

The acceleration produced in the body is given by ;

$$a = \frac{F}{m} = \frac{10}{5} = 2 \text{ ms}^{-2}$$

The acceleration is along the direction of the resultant force *i.e.*, it makes an angle of 37° with the direction of 8 N force.

Q.8. The driver of a three-wheeler moving with a speed of 36 km/h sees a child standing in the middle of the road and brings his vehicle to rest in 4 s just in time to save the child. What is the average retarding force on the vehicle? The mass of the three-wheeler is 400 kg and the mass of the driver is 65 kg.

Ans. Here v_0 = 36 km/h = 10 ms^{-1} ; v = 0; t = 4 s

Now $v = v_0 + at$ or $0 = 10 + a \times 4$ $\therefore$ a = –2.5 ms^{-2}

Total mass = Mass of three-wheeler + Mass of driver = 400 + 65 = 465 kg

$\therefore$ Retarding force, $F = ma$ = 465 × 2.5 = **1162.5 N**

Q.9. A rocket with a lift off mass 20,000 kg is blasted upward with a net initial acceleration of 5 ms^{-2}. Calculate the initial thrust (force) of the blast.

Ans. The rocket moves up with an acceleration a (= 5 ms^{-2}). Therefore, total acceleration $a' = a + g$ = 5 + 9.8 = 14.8 ms^{-2}.

$\therefore$ Initial thrust of blast, $F = ma' = m(a + g) = 20{,}000 \times 14.8 = 296000$ N

Q.10. A particle of mass 0.4 kg moving initially with a constant speed of 10 ms^{-1} to the north is subjected to a constant force of 8 N directed towards the south for 30 s. Take the instant the force is applied to be $t = 0$, the position of the particle at that time to be $x = 0$, and predict its position at $t = -5$ s, 25 s, 100 s.

Ans. Let us take the north direction to be positive. F = 8 N due south = – 8 N due north.

Mass of particle, m = 0.4 kg

$\therefore$ Acceleration = $\frac{F}{m} = \frac{-8}{0.4} = -20\ ms^{-2}$ (towards north)

The position of the particle at any time t is given by ;

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

The position of the particle at $t = 0$ is

$$x = 0 + v_0 t + \frac{1}{2}at^2$$

or $$x = v_0 t + \frac{1}{2}at^2 \quad \ldots(i)$$

Position of particle at t = –5 s. Here no force is applied because force is applied at $t = 0$.

$\therefore$ $v_0 = 10\ ms^{-1}$; $t = -5$s ; $a = 0$

Putting these values in eq. (*i*), we get,

$$x = 10\times(-5) + \frac{1}{2}\times 0\times(-5)^2 = -50 \text{ m}$$

Position of the particle at t = 25 s

Here v_0 = 10 ms^{-1} ; t = 25 s ; $a = -20\ ms^{-2}$

Putting these values in eq. (*i*), we get,

$$x = 10\times 25 + \frac{1}{2}\times(-20)\times(25)^2 = 250 - 6250 = -6000 \text{ m} = -6 \text{ km}$$

Position of the particle at t = 100 s. Since the force acts for 30 s, the particle will be subjected to this force for 30 s and acquires a certain velocity and thereafter will move with constant velocity for 70 s.

For first 30 s, v_0 = 10 ms^{-1} ; $a = -20\ ms^{-2}$; t = 30 s

Putting these values in eq. (*i*), we get the distance x_1 travelled by the particle during first 30 s.

$\therefore$ $$x_1 = 10\times 30 + \frac{1}{2}\times(-20)\times(30)^2 = 300 - 9000 = -8700 \text{ m}$$

If v_1 is the velocity of the particle at t = 30s, then,

$$v_1 = v_0 + at = 10 + (-20)\ (30) = -590\ ms^{-1}$$

Distance x_2 travelled by the particle during next 70s with constant velocity v (= –590 ms^{-1}) is

$$x_2 = v_1\ t = -590 \times 70 = -41300 \text{ m}$$

$\therefore$ Position of the particle at t = 100 s is

$$x = x_1 + x_2 = (-8700) + (-41300) = -50{,}000 \text{ m} = -50 \text{ km}$$

Q.11. A truck starts from rest and accelerates uniformly at 2 ms^{-2}. At t = 10 s, a stone is dropped by a person standing on the top of the truck (6m high from the ground). What are the (*a*) velocity and (*b*) acceleration of the stone at *t* = 11s? (Neglect air resistance and take *g* = 9.8 ms^{-2}).

Ans. Let us first find the velocity v of the truck after 10 s.

$$v = v_0 + at = 0 + 2 \times 10 = 20 \text{ ms}^{-1}$$

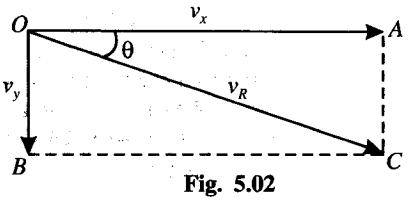


Fig. 5.02

(*a*) When the stone is dropped, the horizontal velocity of the truck is $v_x = v = 20$ ms^{-1}. Since air resistance is neglected, v_x remains constant.

When the stone is dropped, the conditions of motion along vertical are :

$$v_0 = 0;\ a = g = 9.8 \text{ ms}^{-2};\ t = 11 - 10 = 1\text{s}$$

$\therefore$ After one second, $v_y = v_0 + gt = 0 + 9.8 \times 1 = 9.8$ ms^{-1}

As shown in Fig. 5.02, the resultant velocity v_R is given by ;

$$v_R = \sqrt{v_x^2 + v_y^2} = \sqrt{(20)^2 + (9.8)^2} = 22.3 \text{ ms}^{-1}$$

If θ is the angle which the resultant makes with the horizontal, then,

$$\tan\theta = \frac{v_y}{v_x} = \frac{9.8}{20} = 0.49 \quad \therefore \quad \theta = 26.1°$$

(*b*) As soon as the stone is dropped from the truck, it no longer shares the accelerated horizontal motion of the truck. The only acceleration of the stone is that due to gravity *i.e.*, $a = g = 9.8$ ms^{-2} vertically downward. Therefore, at $t = 11$s, the acceleration of stone is $g = 9.8$ ms^{-2} vertically downward. It may be noted that path followed by the stone will be parabolic.

Q.12. A bob of mass 0.1 kg hung from the ceiling of a room by a string 2 m long is set into oscillation. The speed of the bob at its mean position is 1 ms^{-1}. What is the trajectory of the bob if the string is cut when the bob is (*a*) at one of its extreme positions (*b*) at its mean position?

Ans. (*a*) When the bob is at the extreme position, its velocity is zero. Therefore, if the string is cut when the bob is at one of the extreme positions, the bob will fall vertically downward under the action of gravity.

(*b*) When the bob is at its mean position, its velocity is 1 ms^{-1} along tangent to the circular path *i.e.*, along horizontal direction. Therefore, if the string is cut when the bob is at mean position, the bob will behave as a projectile thrown horizontally with a velocity of 1 ms^{-1}. Consequently, the bob will follow a parabolic path.

Q.13. A man of mass 70 kg stands on a weighing scale in a lift which is moving (*a*) upwards with a uniform speed of 10 ms^{-1} (*b*) downwards with a uniform acceleration of 5 ms^{-2} (*c*) upwards with a uniform acceleration of 5 ms^{-2}. What would be the readings on the scale in each case? (*d*) What would be the reading if the lift mechanism failed and it came down freely under gravity? Take *g* = 9.8 ms^{-2}.

Ans. Suppose the mass of the man is m. The gravity-force mg acts on the man in the vertically downward direction. The man exerts on the floor a force vertically downward which we call weight W of the man. According to Newton's third law, the floor exerts an equal force R on the man in the upward direction. Therefore $R = W$. Note that only two forces are acting on the man *viz.*, mg acting vertically downward and R acting vertically upward. The difference of the two forces will determine in which direction the net force acts.

(*a*) When the lift moves upward with a uniform speed (= 10 ms^{-1}), the acceleration is zero so that net force on the lift is zero.

$\therefore$ Net force = $R - mg$ or $0 = R - mg$ $\therefore$ $R = mg = 70 \times 9.8 =$ **686 N**

(*b*) When the lift moves downwards with acceleration $a = 5$ ms^{-2}, the net force is acting downwards.

$\therefore$ Net force = $mg - R$ or $ma = mg - R$ $\therefore$ $R = m(g - a) = 70(9.8 - 5) =$ **336 N**

(*c*) When the lift moves upwards with acceleration $a = 5\ ms^{-2}$, the net force is acting upward.
$\therefore$ Net force $= R - mg$ or $ma = R - mg$ $\therefore$ $R = m(g + a) = 70(9.8 + 5) = 1036$ N

(*d*) If the lift were to come down freely under gravity, then $a = g$ so that $R = m(g - a) = m(g - g) = 0$.

Q.14. Figure 5.03 shows the position-time graph of a particle of mass 4 kg. What is the (*a*) force on the particle for $t < 0$, $t > 4s$, $0 < t < 4s$? (*b*) Impulse at $t = 0$ and $t = 4s$? (Consider one dimensional motion only).

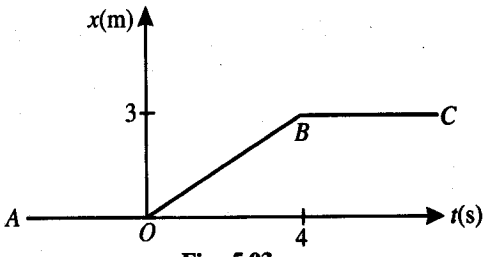


Fig. 5.03

Ans. (*a*) When $t < 0$, the position-time graph is *AO* so that displacement is zero *i.e.*, particle is at rest. Hence force on the particle is zero for $t < 0$. When $t > 4s$, the position-time graph is *BC i.e.*, displacement of the particle remains 3m as the time changes. This means that the particle is at rest. Hence the force on the particle is zero for $t > 4s$. When $0 < t < 4s$, the position-time graph is *OB* which has constant slope. Therefore, the velocity of the particle is constant during this time interval *i.e.*, acceleration is zero. Hence the force on the particle is zero for $0 < t < 4s$.

(*b*) **Impulse at $t = 0$.** At $t = 0$, the particle is at rest and after $t = 0$, the particle has a constant velocity = slope of graph *OB* $= 3/4 = 0.75\ ms^{-1}$.

$\therefore$ Impulse = Final momentum − Initial momentum
$= 4 \times 0.75 - 4 \times 0 = 3$ kg ms^{-1}

Impulse at $t = 4$ s. At $t = 4s$, the particle has constant velocity of $0.75\ ms^{-1}$ and after $t = 4s$, the velocity of the particle is zero.

$\therefore$ Impulse $= 4 \times 0 - 4 \times 0.75 = -3$ kg ms^{-1}

Q.15. Two bodies of masses 10 kg and 20 kg respectively kept on a smooth horizontal surface are tied to the ends of a light string. A horizontal force $F = 600$ N is applied to (*i*) *A*, (*ii*) *B* along the direction of the string. What is the tension in the string in each case?

Ans. Here, $F = 600$ N ; $m_1 = 10$ kg ; $m_2 = 20$ kg

(*i*) When force is applied on m_1 (= 10 kg). Fig. 5.04(*i*) shows the conditions when force F (= 600 N) is applied to body of mass m_1(= 10 kg). Fig. 5.04(*ii*) shows the free-body diagram for m_1 whereas Fig. 5.04(*iii*) shows the free-body diagram for m_2.

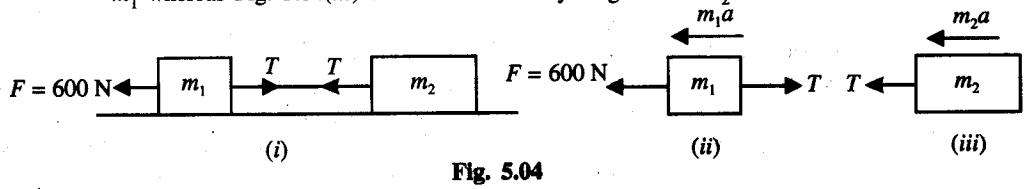


Fig. 5.04

Acceleration of the system, $a = \dfrac{F}{m_1 + m_2} = \dfrac{600}{10 + 20} = 20\ ms^{-2}$

Referring to Fig. 5.04 (*iii*), $T = m_2a = 20 \times 20 = 400$ N

We can also find the value of T from Fig. 5.04(*ii*). Referring to F ig. 5.04(*ii*), $F - T = m_1a$ or $600 - T = 10 \times 20$ $\therefore$ $T = 400$ N

(*ii*) **When force is applied on m_2 (= 20 kg).** Fig. 5.05(*i*) shows the conditions when force F(= 600 N) is applied to body of mass m_2(= 20 kg). Fig. 5.05(*ii*) shows the free-body diagram for m_1 whereas Fig. 5.05(*iii*) shows the free-body diagram for m_2.

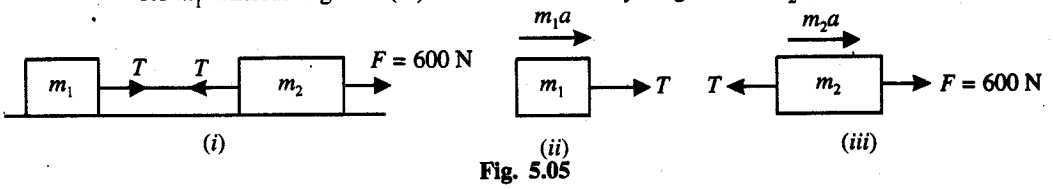


Fig. 5.05

Acceleration of the system, $a = \frac{F}{m_1+m_2} = \frac{600}{10+20} = 20\ \text{ms}^{-2}$ .....same as before

Referring to Fig. 5.05(*ii*), $T = m_1 a = 10 \times 20 =$ **200 N**

We can also find the value of T from Fig. 5.05(*iii*). Referring to Fig. 5.05(*iii*), $F - T = m_2 a$ or $600 - T = 20 \times 20 \therefore T = 200$ N.

Clearly, tension in the string depends on which mass the force is applied.

Q.16. Two masses 8 kg and 12 kg are connected at the two ends of a light inextensible string that goes over a frictionless pulley. Find the acceleration of the masses and the tension in the string when masses are released.

Ans. Fig. 5.06 shows the conditions of the problem. Here $m_1 = 8$ kg and $m_2 = 12$ kg.

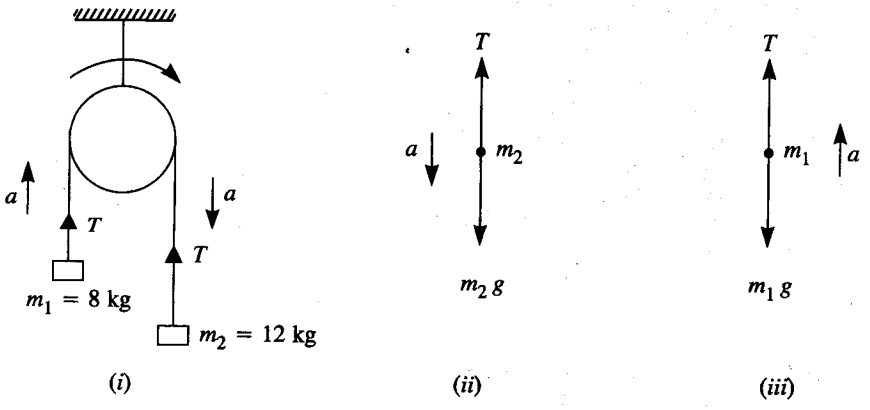


Fig. 5.06

The acceleration a of the masses (See Art. 10.21) is

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g = \left(\frac{12-8}{12+8}\right) \times 9.8 = \mathbf{1.96\ ms^{-2}}$$

The tension T in the string (See Art. 10.21) is

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2}\right) g = \left(\frac{2\times 8\times 12}{8+12}\right) \times 9.8 = \mathbf{94.1\ N}$$

Q.17. A nucleus is at rest in the laboratory frame of reference. Show that if it disintegrates into two smaller nuclei, the products must be emitted in the opposite directions.

Ans. The nucleus forms an isolated system. Therefore, its linear momentum should remain the same. Initially, the nucleus is at rest and, therefore, its linear momentum is zero. After disintegrating into two parts, the total linear momentum of the two parts will also be zero. Let the nucleus of mass M disintegrate into two smaller parts of masses m_1 and m_2 which move with velocities v_1 and v_2.

$$\therefore \qquad m_1 v_1 + m_2 v_2 = 0 \quad \text{or} \quad m_1 v_1 = -m_2 v_2$$

Since masses cannot be negative, the relation shows that the two parts must be emitted in the opposite directions.

Q.18. Two billiard balls each of mass 0.05 kg moving in opposite directions with speed 6 ms^{-1} collide and rebound with the same speed. What is the impulse imparted to each ball due to the other?

Ans. Let the two billiard balls be A and B.

Initial linear momentum of $A = 0.05 \times 6 = 0.3$ kg ms^{-1}; Final linear momentum of $A = -0.05 \times 6 = -0.3$ kg ms^{-1}

Impulse received by A = Final linear momentum − Initial linear momentum

$$= (-0.3) - (0.3) = \mathbf{-0.6\ kg\ ms^{-1}}$$

The body B will also receive an equal impulse (*i.e.*, −0.6 kg ms^{-1}).

Q.19. A shell of mass 0.02 kg is fired by a gun of mass 100 kg. If the muzzle speed of the shell is 80 ms^{-1}, what is the recoil speed of the gun?

Ans. Mass of shell, m = 0.02 kg ; Mass of gun, M = 100 kg ; Muzzle speed of shell, V = 80 ms^{-1}; Recoil speed of gun, v = ?

According to principle of conservation of linear momentum,

$$mV + Mv = 0 \quad \text{or} \quad v = -\frac{mV}{M} = -\frac{0.02\times80}{100} = \mathbf{-0.016\ ms^{-1}}$$

The negative sign shows that the recoil speed of the gun will be in a direction opposite to that of the shell.

Q.20. A batsman deflects a ball by an angle of 45° without changing its initial speed which is equal to 54 km/h. What is the impulse imparted to the ball? (Mass of the ball is 0.15 kg).

Ans. Fig. 5.07 shows the conditions of the problem. Here point O represents the position of the bat. The ball of mass m initially moves along the path AO with speed v_0 and is deflected by the batsman along OB (with speed v_0) such that $\angle AOB = 45°$. ON is normal to the horizontal line CD drawn through O. Clearly, $\theta = \angle AON = \angle BON = 45°/2 = 22.5°$. Note that magnitude v_0 of the velocity does not change; only the direction of velocity changes.

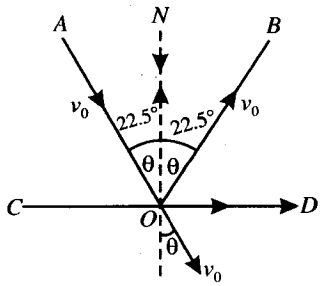


Fig. 5.07

The initial velocity along AO can be resolved into two rectangular components *viz.* (*i*) component $v_0 \sin\theta$ along OD and (*ii*) component $v_0 \cos\theta$ along NO produced. Similarly, the final-velocity along OB can be resolved into two rectangular components *viz.* (*i*) component $v_0 \sin\theta$ along OD and (*ii*) component $v_0 \cos\theta$ along ON. Note that component ($v_0 \sin\theta$) of the velocity does not change but the component $v_0 \cos\theta$ of the velocity is reversed.

$\therefore$ Change in velocity = $v_0 \cos\theta - (-v_0 \cos\theta) = 2v_0 \cos\theta$

$\therefore$ Impulse imparted to ball = Change in momentum = $m(2v_0 \cos\theta) = 2m\, v_0 \cos\theta$

Here m = 0.15 kg; v_0 = 54 kmh^{-1} = 15 ms^{-1}; $\theta = 22.5°$

$\therefore$ Impulse = $2 \times 0.15 \times 15 \times \cos 22.5° = \mathbf{4.16\ kg\ ms^{-1}}$

Q.21. A stone of mass 0.25 kg tied to the end of a string is whirled round in a circle of radius 1.5 m with a speed of 40 rev/min in a horizontal plane. What is the tension in the string? What is the maximum speed with which the stone can be whirled around if the string can withstand a maximum tension of 200 N?

Ans. Angular speed, ω = 40 revolutions/minute = $\frac{40}{60}$ r.p.s. = $\frac{40}{60}\times 2\pi$ rad s^{-1} = $\frac{4\pi}{3}$ rad s^{-1}

Mass of stone, m = 0.25 kg ; Radius of circular path, r = 1.5 m

The tension T in the string provides the necessary centripetal force which keeps the stone in the circular path.

$\therefore$ $$T = mr\omega^2 = 0.25\times1.5\times\left(\frac{4\pi}{3}\right)^2 = \mathbf{6.6\ N}$$

The string can withstand a maximum tension of 200 N *i.e.*, T_m = 200 N. If v_m is the maximum corresponding speed, then,

$$\frac{mv_m^2}{r} = T_m \text{ or } v_m = \sqrt{\frac{T_m \times r}{m}} = \sqrt{\frac{200\times1.5}{0.25}} = \mathbf{34.6\ ms^{-1}}$$

Q.22. If in question 21, the speed of the stone is increased beyond the maximum permissible value and the string breaks suddenly, which of the following correctly describes the trajectory of the stone after the string breaks: (*a*) the stone moves radially outwards (*b*) the stone flies off tangentially from the instant the string breaks (*c*) the stone flies off at an angle with the tangent whose magnitude depends on the speed of the stone?

Ans. When the stone is whirled in a circle, the instantaneous velocity of the stone is always tangent to the circle. When the string breaks, the centripetal force ceases to act on the stone. According to Newton's first law of motion, the stone continues to move along the straight line in the direction of the instantaneous velocity. This is why it flies off tangentially.

Q.23. Explain why (*a*) a horse cannot pull a cart and run in empty space (*b*) passengers are thrown forward from their seats when a speeding bus stops suddenly (*c*) it is easier to pull a lawn mower than to push it (*d*) a cricketer moves his hands backwards while holding a catch.

Ans. (*a*) To pull the cart, a horse pushes the ground in the backward direction. The ground offers equal reaction on the feet of the horse in the forward direction. This causes the cart to move. In empty space, the horse cannot push in backward direction and get a forward reaction. For this reason, a horse cannot pull a cart and run in an empty space.

(*b*) When a speeding bus is suddenly stopped, the passengers tend to fall forward. It is because the lower part of the body of the passenger which is in contact with the bus comes to rest but the upper part of the body tends to be in motion due to inertia. As a result, the passengers tend to fall forward.

(*c*) Fig. 5.08 (*i*) shows the pulling of the lawn mower while Fig. 5.08 (*ii*) shows the pushing of the lawn mower. In case of pulling, normal reaction = $m\,g - F\sin\theta$ while in case of pushing, normal reaction = $m\,g + F\sin\theta$.
Since frictional force exerted by the ground on the lawn mower is directly proportional to the normal reaction, it is easier to pull the lawn mower than to push it.

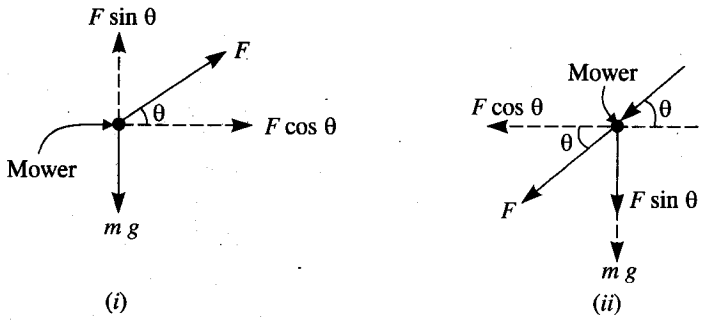


Fig. 5.08

(*d*) While catching a ball, a cricket player extends his hands forward so that he has plenty of room to let his hands move backward after making contact with the ball. This extends the time of impact and thus reduces the force of impact.

Q.24. Fig. 5.09 shows the position-time graph of a particle of mass 0.04 kg. Suggest a suitable physical context for this motion. What is the time between two consecutive impulses received by the particle? What is the magnitude of each impulse?

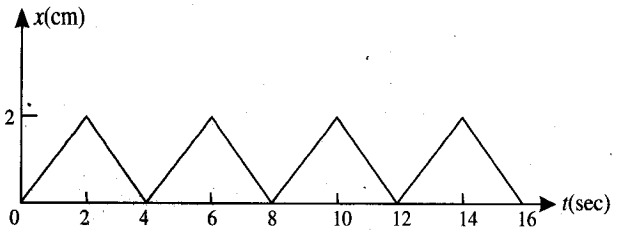


Fig. 5.09

Ans. The $x - t$ graph shows that the particle moves from position $x = 0$ to $x = 2$ cm in 2 s with a constant speed and then returns with same speed to position $x = 0$ in 2 s and so on. Therefore, it may be the $x - t$ graph of an object (*e.g.* a ball) getting rebounded again and again between two parallel walls 2 cm apart.

Constant speed of particle $= 2/2 = 1\text{cm/s} = 0.01\ \text{ms}^{-1}$

Initial momentum of the particle $= mv_0 = 0.04 \times 0.01 = 4 \times 10^{-4}\text{kg ms}^{-1}$

After 2 seconds, the speed of the particle is $-0.01\ \text{ms}^{-1}$ because the particle now moves in the opposite direction.

Momentum of particle after 2 s = $mv = 0.04 \times (-0.01) = -4 \times 10^{-4}\ \text{kg ms}^{-1}$.

Magnitude of impulse = Total change in momentum of the particle

$= 4 \times 10^{-4} - (-4 \times 10^{-4}) =$ **8×10^{-4} kg ms^{-1}**

Time between two consecutive impulses = **2 s**

Q.25. Fig. 5.010 shows a man standing stationary *w.r.t.* a horizontal conveyor belt that is accelerating with 1 ms^{-2}. What is the net force on the man ? If the coefficient of static friction between the man's shoes and the belt is 0.2, upto what acceleration of the belt can the man continue to be stationary relative to the belt? (mass of man = 65 kg).

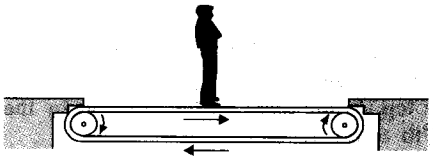


Fig. 5.010

Ans. Since the man is stationary w.r.t. horizontal conveyor belt, acceleration of man = acceleration of belt.

$\therefore$ Acceleration of man, $a = 1\ \text{ms}^{-2}$; Mass of man, $m = 65$ kg

$\therefore$ Net force on man, $F = ma = 65 \times 1 =$ **65 N**

The limiting friction (*i.e.*, maximum static friction) between the shoes of the man and belt is given by ;

$$F_C = \mu_S\, mg = 0.2 \times 65 \times 9.8$$

The man will not fall if the accelerating force does not exceed the force of limiting friction (F_C). Let a' be the acceleration upto which man remains stationary on the belt. Then,

$$F_C = ma' \text{ or } a' = \frac{F_C}{m} = \frac{0.2 \times 65 \times 9.8}{65} = \mathbf{1.96\ ms^{-2}}$$

Q.26. A stone of mass *m* tied to one end of a string revolves in a vertical circle of radius *R*. The net forces at the lowest and highest points of the circle directed vertically downwards are: [Choose the correct alternative]

Lowest Point	**Highest Point**
(*a*) $mg - T_1$	$mg + T_2$
(*b*) $mg + T_1$	$mg - T_2$
(*c*) $mg + T_1 - (mv_1^2)/R$	$mg - T_2 + (mv_1^2)/R$
(*d*) $mg - T_1 - (mv_1^2)/R$	$mg + T_2 + (m\,v_1^2)/R$

Ans. At the lowest point, mg acts downwards and T_1 upwards so that net force = $(mg - T_1)$. At the highest point, both mg and T_2 act downward so that the net force = $(mg + T_2)$. Therefore, alternative (*a*) is correct.

Q.27. A helicopter of mass 1000 kg rises with a vertical acceleration of 15 ms^{-2}. The crew and passengers weigh 300 kg. Give the magnitude and direction of (*a*) force on the floor by the crew and passengers (*b*) action of the rotor of the helicopter on surrounding air (*c*) force on the helicopter due to the surrounding air. Take $g = 10\ \text{ms}^{-2}$.

Ans. $m_1 = 1000$ kg ; $m_2 = 300$ kg ; $a = 15\ \text{ms}^{-2}$ (upward) ; $g = 10\ \text{ms}^{-2}$

(*a*) The force exerted by the crew and passengers is equal to their apparent weight *W*. Since the helicopter rises vertically with an acceleration a (= 15 ms^{-2}),

$$W = m_2\,(g + a) = 300\,(10 + 15) = \mathbf{7500\ N}\ \textit{vertically downward}$$

(*b*) The rotor of the helicopter exerts a vertically downward force F on the surrounding air.

$$F = (m_1 + m_2)(a + g) = (1000 + 300)(15 + 10) = \mathbf{32500\ N}\ \textit{downward}$$

(*c*) According to Newton's third law, the surrounding air will exert force equal to F (= 32500 N) on the helicopter in the vertically upward direction.

$\therefore$ Force on helicopter = **32500 N** *vertically upward*

Q.28. A stream of water flowing horizontally with a speed of 15 ms^{-1} gushes out of a tube of cross-sectional area 10^{-2} m^2 and hits a vertical wall nearby. What is the force exerted on the wall by the impact of water, assuming it does not rebound?

Ans. Volume of water striking the wall per second

$$= \text{Area} \times \text{Velocity} = (10^{-2}) \times 15 = 15 \times 10^{-2}\ \text{m}^3/\text{sec}$$

Mass of water striking the wall per sec. is

$$m = \text{volume} \times \text{density} = 15 \times 10^{-2} \times 10^3 = 150\ \text{kg/sec}$$

$\therefore$ Force on wall, $F = \dfrac{\text{Change in momentum}}{t} = \dfrac{150 \times (15 - 0)}{1} = \mathbf{2250\ N}$

Q.29. Ten one-rupee coins are put on top of each other on a table. Each coin has a mass m kg. Give the magnitude and direction of (*a*) the force on the 7th coin (counted from the bottom) due to all the coins on its top (*b*) the force on the 7th coin by the eighth coin (*c*) the reaction of the 6th coin on the 7th coin.

Ans. (*a*) Force on the 7th coin is due to the weight of three coins lying above it.

$\therefore$ Force on 7th coin = **3 mg N**

The force acts vertically downward.

(*b*) The 8th coin is under the weight of two coins above and also has his own weight. Therefore, force F on the 7th coin due to the 8th coin will be equal to the sum of the weight of the eighth coin and the two coins above it

$$F = (2m + m)g = \mathbf{3\ mg\ N}$$

The force acts vertically downward.

(*c*) The 6th coin is under the weight of four coins above it. Therefore, reaction due to the 6th coin on the seventh coin, $R = \mathbf{-4\ mg\ N}$.

The negative sign indicates that the reaction acts vertically upward (*i.e.*, opposite to weight).

Q.30. An aircraft executes a horizontal loop at a speed of 720 km/h with its wings banked at 15°. What is the radius of the loop?

Ans. When an aeroplane has to negotiate a circular path, it has to raise the outer wing *w.r.t.* the inner wing as shown in Fig. 5.011. This is called aircraft banking. Consider an aircraft of mass m with a lifting force L. If θ is the banking angle, then horizontal and vertical components of L are $L \sin\theta$ and $L \cos\theta$. Since there is no vertical motion of the aircraft,

$\therefore$ $$L \cos\theta = mg \qquad ...(i)$$

The horizontal component $L \sin\theta$ provides the necessary centripetal force mv^2/r required to keep the aircraft in the circular track.

$\therefore$ $$L\sin\theta = \frac{mv^2}{r} \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we have, $\tan\theta = \dfrac{v^2}{rg}$

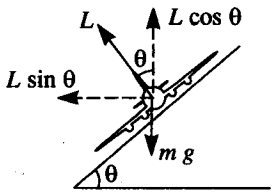


Fig. 5.011

Here $v = 720\ \text{km/h} = 200\ \text{ms}^{-1}$; $\theta = 15°$; $r = ?$

$\therefore$ $$r = \frac{v^2}{g\tan\theta} = \frac{(200)^2}{9.8 \times \tan 15°} = 15.23 \times 10^3\ \text{m} = \mathbf{15.23\ km}$$

Q.31. A train rounds an unbanked circular bend of radius 30 m at a speed of 54 km/h. The mass of the train is 10^6 kg. What provides the centripetal force required for this purpose? The engine or the rails? The outer or the inner rail? Which rail would wear out faster, the outer or the inner rail? What is the angle of banking required to prevent the wearing out of rails?

Ans. The centripetal force required is provided by the lateral thrust of the *outer rail* on the flanges of the wheel. By Newton's third law of motion, the train exerts an equal and opposite thrust on the outer rail. Therefore, the outer rail will wear out faster than the inner rail.

Radius of bend, r = 30 m ; Speed of train, $v = 54 \text{ km/h} = \frac{54 \times 1000}{60 \times 60} = 15 \text{ ms}^{-1}$

Let θ be the angle of banking required to prevent the wearing out of the rails.

$$\tan\theta = \frac{v^2}{rg} = \frac{(15)^2}{30 \times 9.8} = 0.7653 \quad \therefore \ \theta = \tan^{-1} 0.7653 = 37.4^\circ$$

Q.32. A block of mass 25 kg is raised by a 50 kg man in two different ways as shown in Fig. 5.012. What is the action on the floor by the man in the two cases? If the floor yields to a normal force of 700 N, which mode the man should adopt to lift the block without the floor yielding?

Ans. Mass of block, m = 25 kg ; Mass of man, M = 50 kg

Force applied to lift the block, $F = mg = 25 \times 9.8 = 245$ N

Weight of man, $W = Mg = 50 \times 9.8 = 490$ N

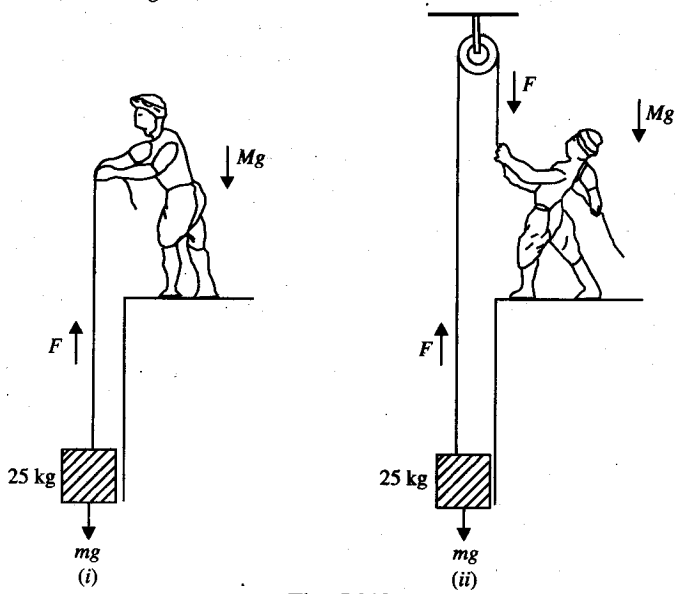


Fig. 5.012

In the *first case [See Fig. 5.012 (*i*)], the action on the floor by the man = $Mg + F$ = 490 + 245 = **735 N.** In the second case [See Fig. 5.012 (*ii*)], the action on the floor by the man = $Mg - F$ = 490 – 245 = **245 N.** Since the floor yields to a normal force of 700N, the man should adopt mode (*ii*) to lift the block without the floor yielding.

Q.33. A monkey of mass 40 kg climbs on a rope which can stand a maximum tension of 600 N. In which of the following cases the rope will break?

(*a*) If monkey climbs up with an acceleration of 6 ms^{-2}.

(*b*) If monkey climbs down with an acceleration of 4 ms^{-2}.

(*c*) If monkey climbs up with a uniform speed of 5 ms^{-1}.

(*d*) If monkey falls down the rope nearly freely under gravity. Ignore the mass of the rope and take g = 10 ms^{-2}.

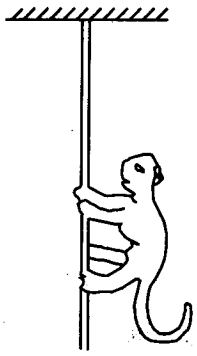


Fig. 5.013

Ans. The tension in the rope will be equal to apparent weight of the monkey.

(*a*) If the monkey climbs up with an acceleration a (= 6 ms^{-2}), its apparent weight W is

* In the first case, reaction on the floor will be downwards and in the second case, it will be upwards.

$W = m\,(g + a) = 40\,(10 + 6) = 640$ N

Since the tension in the rope is greater than the permissible tension of 600 N, *the rope will break* in this case.

(*b*) When $a = 4$ ms^{-2} (downward)

$W' = m\,(g - a) = 40\,(10 - 4) = 240$ N

In this case, *the rope will not break.*

(*c*) Here, $a = 0$ so that $W'' = m\,g = 40 \times 10 = 400$ N

In this case, *the rope will not break.*

(*d*) Here $a = g$ (downward)

$W''' = m\,(g - a) = m\,(g - g) = 0$

In this case, *the rope will not break.*

Q.34. Two bodies *A* and *B* of masses 5 kg and 10 kg in contact with each other rest on a table against a rigid partition. The co-efficient of friction between the bodies and the table is 0.15. A force of 200 N is applied horizontally at *A*. What are (*a*) the reaction of the partition (*b*) the action and reaction forces between *A* and *B*? What happens when the partition is removed? Does the answer to (*b*) change when bodies are in motion? Ignore difference between μ_s and μ_k.

Ans. Mass of body *A*, $m_A = 5$ kg ; Mass of body *B*, $m_B = 10$ kg ; $\mu = 0.15$

(*a*) Fig. 5.014 (*i*) shows the conditions of the problem. If F_C is the limiting friction acting on the combination of *A* and *B*, then,

$$F_C = \mu R = \mu(m_A + m_B)\ g = 0.15\ \ (5 + 10)\ 9.8 = 22.05 \text{ N}$$

Net force on the partition $= F - F_C = 200 - 22.05 = 177.95$ N (toward right)

$\therefore$ Reaction of partition = 177.95 N (toward left)

(*b*) Force of limiting friction on body *A* is

$$F_{CA} = \mu\, m_A\, g = 0.15 \times 5 \times 9.8 = 7.35 \text{ N (toward left)}$$

Net force exerted by body *A* on body *B* is

$$F_1 = F - F_{CA} = 200 - 7.35 = \mathbf{192.65\ N}$$

According to Newton's third law, body *B*-exerts a force of 192.65 N on body *A* in the opposite direction.

When the partition is removed, the situation becomes as shown in Fig. 5.014 (*ii*). The system behaves as one block of mass $(m_A + m_B)$.

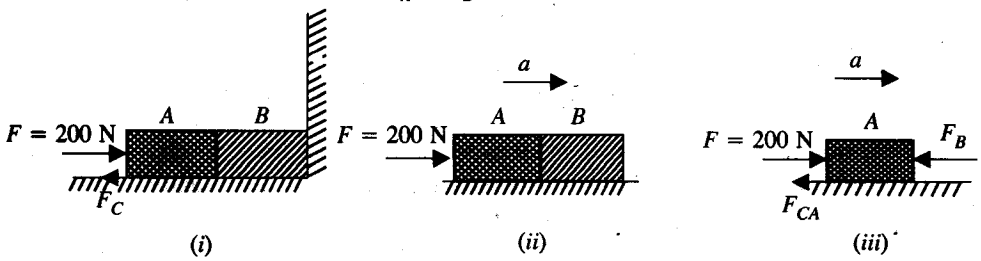


Fig. 5.014

$\therefore$ Net force $= F - F_C = 200 - 22.05 = 177.95$ N

$\therefore$ Acceleration, $a = \dfrac{\text{Net force}}{\text{Total mass}} = \dfrac{177.95}{m_A + m_B} = \dfrac{177.95}{15} = 11.86$ ms^{-2}

Thus the two blocks move with an acceleration of 11.86 ms^{-2}.

Refer to Fig. 5.014 *(iii)*. Net force on body *A* is $F - F_{CA} - F_B = 200 - 7.35 - F_B$

$\therefore \quad 200 - 7.35 - F_B = m_A a$

or $\quad F_B = 200 - 7.35 - 5 \times 11.86$

$\therefore \quad F_B = 133.35$ N

According to Newton's third law, body A also exerts a force of 133.35 N on body B but in opposite direction. Therefore, the answer changes.

Q.35. A block of mass 15 kg is placed on a long trolley. The coefficient of static friction between the block and the trolley is 0.18. The trolley accelerates from rest with 0.5 ms^{-2} for 20s and then moves with a uniform velocity. Discuss the motion of the block as viewed by (*a*) a stationary observer on the ground (*b*) an observer moving with the trolley.

Ans. Here $m = 15$ kg ; $\mu_S = 0.18$; $a = 0.5$ ms^{-2} ; $t = 20$ s

Force on block due to motion of trolley, $F = ma = 15 \times 0.5 = 7.5$ N

Force of limiting friction on block, $F_C = \mu_S R = \mu_S mg = 0.18 \times 15 \times 9.8 = 26.46$ N

(*a*) The forward force F(= 7.5 N) is less than limiting friction F_C (= 26.46 N) so that the static friction will adjust itself to 7.5 N. Therefore, to a stationary observer, the block will appear to be at rest *w.r.t.* trolley. When the trolley moves with uniform velocity, acceleration is zero so that forward force is zero. Again, the block will continue to be stationary.

(*b*) The motion of the trolley is accelerated motion. Therefore, trolley is a non-inertial frame of reference and the observer moving with the trolley is in a non-inertial frame. The law of inertia is no longer valid in such a frame.

Q.36. The rear side of a truck is open and a box of 40 kg mass is placed 5m away from the open end as shown in Fig. 5.015. The co-efficient of friction between the box and the surface below it is 0.15. On a straight road, the truck starts from rest and accelerates with 2 m s^{-2}. At what distance from the starting point does the box fall off the truck? (Ignore the size of the box)

Ans. Mass of box, $m = 40$ kg ; Acceleration of truck, $a = 2$ ms^{-2} ; Distance of box from open end, $S = 5$ m.

The box is accelerating along with the truck. Therefore, force on the box is $F = m\,a = 40 \times 2 = 80$ N. Since the truck is accelerating in the forward direction, the reaction of the box is equal to F (= 80 N) but acts in the backward direction (See Fig. 5.016). The backward motion of the box is opposed by the limiting friction (F_C) between the box and the surface of the truck.

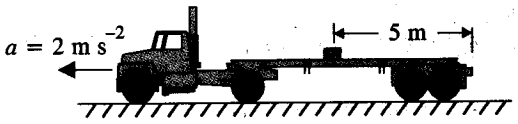


Fig. 5.015

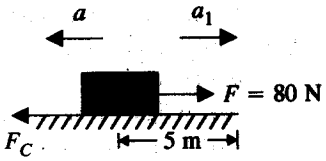


Fig. 5.016

Therefore, net force F_1 on the box in the backward direction is

$$F_1 = F - F_C = F - \mu\, m\, g$$
$$= 80 - 0.15 \times 40 \times 9.8 = 21.2 \text{ N}$$

Acceleration a_1 of the box in the backward direction is

$$a_1 = \frac{F_1}{m} = \frac{21.2}{40} = 0.53 \text{ ms}^{-2}$$

Suppose t is the time in which the box falls off the truck.

Now $\quad S_1 = ut + \frac{1}{2} a_1 t^2 \quad$ or $5 = 0 + \frac{1}{2} \times 0.53\, t^2 \quad \therefore \quad t^2 = \frac{10}{0.53}$

During the time t, the truck will move a distance S given by ;

$$S = ut + \frac{1}{2} a t^2 = 0 + \frac{1}{2} \times 2 \times \frac{10}{0.53} = 18.9 \text{ m}$$

Q.37. A record revolves with a speed of $33\frac{1}{3}$ revolutions per minute and has a radius of 15 cm. Two coins are placed at 4 cm and 14 cm away from the centre of the record. If the coefficient of friction between the coins and the record is 0.15, which of the two coins will revolve with the record?

Ans. As the record revolves, the coins tend to go away from the centre of the record. Therefore, friction between the coins and the record acts inward toward the centre of the record. In order that a coin revolves with the record, the frictional force must be large enough to provide the necessary centripetal force. If m is the mass of each coin, then maximum static friction between the coin and the record is

$$f_{S(max)} = \mu_s R = \mu_s m g = 0.15 \times m \times 9.8 = 1.47 m\text{N}$$

Angular speed of record, $\omega = 33\frac{1}{3}\text{r.p.m} = \frac{100}{3\times 60}\text{r.p.s.} = \frac{2\pi\times 100}{3\times 60} = \frac{10\pi}{9}\text{rad s}^{-1}$

The centripetal force required by the coin to move in a circular path of radius 4 cm (= 0.04 m)

$$= mr\omega^2 = m\times 0.04\times\left(\frac{10\pi}{9}\right)^2 = 0.48m\text{N}$$

Since the centripetal force required is less than the limiting friction, the coin placed 4 cm from the centre of the record will revolve with the record.

The centripetal force required by the coin to move in a circular path of radius 14 cm (= 0.14 m)

$$= mr\omega^2 = m\times 0.14\times\left(\frac{10\pi}{9}\right)^2 = 1.706m\text{N}$$

Since the centripetal force required is more than the limiting friction, the coin placed 14 cm from the centre of the record will not revolve with the record.

Q.38. You may have seen in a circus, a motorcyclist driving in vertical loops inside a 'death-well' (a hollow spherical chamber with holes so the spectators can watch from outside). Explain clearly why the motorcyclist does not drop down when he is at the uppermost point of death well with no support from below. What is the minimum speed required at the uppermost position to perform a vertical loop if the radius of the chamber is 25 m?

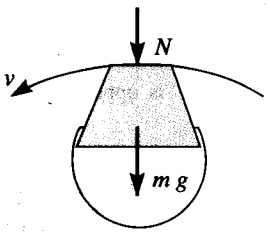


Fig. 5.017

Ans. At the top of the vertical circle (See Fig. 5.017), the forces acting on the motorcyclist are :

(*i*) Weight of motorcyclist (motorcycle + man) acting vertically downward.

(*ii*) Normal reaction N on the motorcyclist by the ceiling of the chamber.

Note that both these forces act downward. The acceleration which is toward the centre of the circle is also downward at this point. Therefore, according to Newton's second law,

$$N + mg = \frac{mv^2}{r} \qquad \therefore\ N = \frac{mv^2}{r} - mg = m\left(\frac{v^2}{r} - g\right)$$

The motorcyclist will remain in the circular path if $\frac{v^2}{r} \geq g$. The minimum speed of the motorcyclist at the highest point of the vertical circle is

$$v_H = \sqrt{gr} = \sqrt{9.8\times 25} = 15.65\ \text{ms}^{-1}$$

Q.39. A 70 kg man stands in contact against the inner wall of a hollow cylindrical drum of radius 3 m rotating about its vertical axis with 200 rev/min. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed?

Ans. Here m = 70 kg ; r = 3 m ; μ = 0.15 ; $\omega = 2\pi N/60 = 2\pi \times 200/60 = 20.94$ rad s^{-1}

When the hollow cylindrical drum rotates about its vertical axis, the man standing in contact against its wall moves along a horizontal circular path. The man is under the following forces : The normal reaction R by the wall on the man provides the necessary centripetal force ($= mv^2/r$) *i.e.,*

$$R = \frac{mv^2}{r} = \frac{m(\omega r)^2}{r} = mr\omega^2$$

The weight ($= mg$) of the man acts vertically downward. Since the man stands in contact with the wall, the frictional force F_r ($= \mu R$) acts vertically upward *i.e.,*

$$F_r = \mu R = \mu\, mr\, \omega^2 \qquad ...(i)$$

When the floor is suddenly removed, the man will remain stuck to the wall of the drum if frictional force is *atleast* equal to weight of the man *i.e.,*

$$F_r = mg \qquad (ii)$$

From eqs. (*i*) and (*ii*), $\mu\, mr\, \omega^2 = mg$

$$\therefore \quad \omega_{min} = \sqrt{\frac{g}{\mu r}} = \sqrt{\frac{9.8}{0.15 \times 3}} = 4.67 \text{ rad } s^{-1}$$

The angular speed ω_{min} is the minimum speed that will make $F_r = mg$, thus enabling the man to remain stuck to the wall of the drum. Since ω (= 20.94 rad s^{-1}) is greater than ω_{min} (= 4.67 rad s^{-1}), the man will remain stuck to the wall of drum after the removal of the floor.

Q.40. A thin circular loop of radius R rotates about its vertical diameter with an angular frequency ω. Show that a small bead on the wire loop remains at its lowermost point for $\omega \leq \sqrt{g/R}$. What is the angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{2g/R}$? Neglect friction.

Ans. Fig. 5.018 shows the conditions of the problem. The circular loop of radius R is rotating about a vertical diameter with an angular velocity ω. Suppose at any instant, the bead is at point P and the radius vector joining centre O of the loop to the bead makes an angle θ with the vertical as shown in Fig. 5.018. When the loop rotates about the vertical diameter, the bead moves along a circular path whose centre is point C and radius is $r = CP = R \sin\theta$.

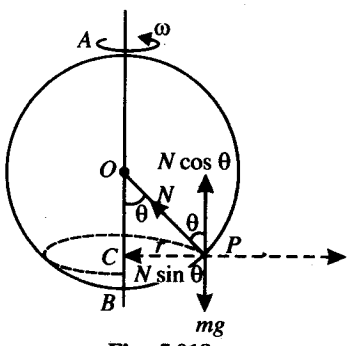


Fig. 5.018

Let us now discuss the dynamics of circular motion of the bead. The weight mg of the bead acts vertically downward. The normal reaction N of the wire loop on the bead acts along PO and can be resolved into two rectangular components $N \cos\theta$ and $N \sin\theta$ as shown in Fig. 5.018. The component $N \cos\theta$ balances the weight mg of the bead while component $N \sin\theta$ provides the necessary centripetal force ($= m\, r\, \omega^2$) for the circular motion of the bead with a circle of radius r.

$$\therefore \quad mg = N\cos\theta \quad ...(i) \qquad m\, r\, \omega^2 = N \sin\theta \quad ...(ii)$$

Since $r = R \sin\theta$, eq. (*ii*) becomes :

$$m(R \sin\theta)\omega^2 = N \sin\theta \quad \text{or} \quad mR\omega^2 = N \qquad ...(iii)$$

From eqs. (*i*) and (*iii*), $\cos\theta = \dfrac{g}{R\omega^2}$ ...(*iv*)

Now $\cos\theta \leq 1$. Therefore, the bead will remain at its lowest point B if

$$\frac{g}{R\omega^2} \geq 1 \text{ or } \frac{g}{R} \geq \omega^2 \text{ or } \omega \leq \sqrt{\frac{g}{R}}$$

If $\omega > \sqrt{g/R}$, the bead will rise above lower most point B. When $\omega = \sqrt{2g/R}$, from eq. (*iv*),

$\cos\theta = \dfrac{g}{R} \times \dfrac{R}{2g} = \dfrac{1}{2}$. $\therefore \quad \theta = 60°$

Unit V

WORK, ENERGY AND POWER

- Work, Energy and Power
- N.C.E.R.T. Textbook Excercises

12

Work, Energy and Power

INTRODUCTION

In everyday language, the word *work* is used to describe any activity in which muscular or mental effort is exerted. In physics, the word work has a special meaning. Work is only done when the force acting on a body produces motion in it in the direction of force (or in the direction of component of force). Thus a boy pushing the wall is doing no work from physics' point of view. It is because the force exerted by the boy is not producing motion of the wall. The speed at which work can be done is an indication of the power of the body doing work. For example, a boy may carry a suitcase upstairs in 3 minutes while a man may do it in 1 minute. Obviously, the power of the man is more than the power of the boy. Thus, time factor is important for power. A body which has the capacity to do work is said to possess energy. The greater the capacity of a body to do work, the greater the energy it has. Thus work, energy and power are related to each other. In this chapter, we shall deal with these three important concepts of physics.

12.1. WORK

In everyday language, the word work is used to describe any activity in which muscular or mental effort is exerted. In physics, the word work has a special meaning.

Work *is only done when the force acting on an object produces displacement in it in the direction of force or in the direction of component of the force.*

Therefore, for work to be done by a force on an object, the following two conditions must be met :

(*i*) *There must be displacement of the object.*

(*ii*) *There must be a component of force in the direction of the displacement.*

Let us apply these conditions to some practical situations.

(*a*) Suppose you are pushing the wall very hard but the wall does not move. In this case, you are doing zero work on the wall because its displacement is zero. You may feel tired after pressing hard against the wall but from physics' point of view, work done is zero.

(*b*) Suppose a man holding bucket of water is walking on a horizontal road. According to the definition of work, the man is doing no work. In this case, the man is applying an upward force $\vec{F}$ equal to the weight of the bucket. The direction of the force $\vec{F}$ he applies is perpendicular to the horizontal motion of the bucket (object). Therefore, there is no component of force $\vec{F}$ in the direction of motion (displacement). Hence work done by man on the bucket is zero.

From eq. (*i*), two points are very clear. First, work is the scalar product of two vectors ($\vec{F}$ and $\vec{S}$). Therefore, work is a scalar quantity. Secondly, vector notation expresses the work done in a compact way that accounts for both direction and magnitude of the vectors involved. This is a good example of the application of scalar product in physics.

Note. We can also express work done in terms of rectangular components of $\vec{F}$ and $\vec{S}$.

$$\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z \; ; \; \vec{S} = \hat{i}x + \hat{j}y + \hat{k}z$$

$$\therefore \quad W = \vec{F}.\vec{S} = (\hat{i}F_x + \hat{j}F_y + \hat{k}F_z).(\hat{i}x + \hat{j}y + \hat{k}z)$$

or $$W = x F_x + y F_y + z F_z$$

Note that work done is equal to the product of component of force in the direction of displacement and magnitude of displacement. Since work is a scalar quantity, total work done is equal to the algebraic sum of the works done.

12.4. NATURE OF WORK DONE

Work done, $W = FS \cos\theta$

Note that the presence of the term $\cos\theta$ in the expression for work implies that the work done can be **positive**, **negative** or **zero** depending upon the value of θ.

1. **Positive work done.** If $\theta < 90°$, $\cos\theta$ is positive so that work done is positive. Under this condition, there will be a component of force in the direction of object's displacement.

   Examples. Some examples of positive work done are :

   (*i*) When a body falls freely under gravity, $\theta = 0°$ so that $\cos\theta = +1$. Therefore, work done by the gravitational force on the body is positive.

   (*ii*) When a spring is stretched, the displacement and force are in the same direction so that $\theta = 0°$. Therefore, the work done by the stretching force is positive.

   (*iii*) When a horse pulls a cart on a level road, the work done by the horse is positive.

2. **Negative work done.** If $\theta > 90°$, $\cos\theta$ is negative so that work done is negative. Under this condition, there is a component of force opposite to the direction of object's displacement.

   Examples. Some examples of negative work done are :

   (*i*) When a body is moved over a rough horizontal surface, the work done by the frictional force is negative. It is because frictional force opposes motion so that $\theta = 180°$ and $\cos\theta = -1$. Note that work done by the applied force is not negative.

   (*ii*) When a positive charge is moved towards another positive charge, the work done by the electrostatic force between them is negative. It is because in this case $\theta = 180°$ so that $\cos\theta = -1$. Again work done by the applied force is not negative.

   (*iii*) When a body is thrown up, the work done by the gravitational force on the body moving upwards is negative. In this case, the gravitational force is vertically downwards while the displacement is vertically upwards so that $\theta = 180°$ and $\cos\theta = -1$.

3. **Zero work done.** If $\theta = 90°$, $\cos\theta = 0$ and no work is done by the force on the body. Also work done is zero when either $\vec{F}$ or $\vec{S}$ or both are zero.

   Examples. Some examples of zero work done are :

   (*i*) When a man carrying some load on his head moves on a level road, work done by the man is zero. In this case, the load force is vertically downward and displacement is horizontal so that $\theta = 90°$ and $\cos\theta = 0$.

(*ii*) A man pushing the wall is doing no work. It is because there is no displacement of the wall (*i.e.*, $\vec{S} = 0$).

(*iii*) When a body is moved in a circle with the help of a string, the work done by the tension in the string is zero. It is because angle θ is 90° (*i.e.*, angle between $\vec{T}$ and $\vec{S}$ is 90°) and $\cos\theta = 0$.

12.5. WORK DONE AGAINST FRICTION

We shall find the work done against friction in the following three cases :

(*i*) **Along a horizontal surface.** Consider a block of mass m lying on a rough horizontal surface as shown in Fig. 12.2. If μ_k is the coefficient of kinetic friction between the two surfaces, then kinetic friction F_k is given by ;

$$F_k = \mu_k R = \mu_k m g$$

In order to move the block with constant velocity (*i.e.* zero acceleration), the force F required is equal to the force of kinetic friction F_k *i.e.*

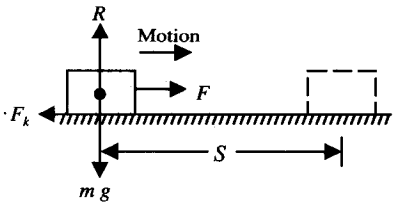


Fig. 12.2

$$F = F_k = \mu_k m g$$

If S is the displacement of the block, then work done W is

$$W = FS \cos 0° = \mu_k m g S$$

(*ii*) **When block is moved up the inclined plane.** Suppose a block of mass m is to be moved up a rough inclined plane AB inclined at an angle θ to the horizontal as shown in Fig. 12.3. The weight (= mg) of the block can be resolved into two rectangular components :

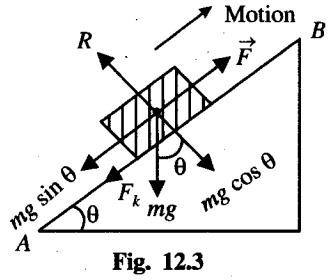


Fig. 12.3

(*a*) $mg \sin\theta$ parallel to the inclined plane

(*b*) $mg \cos\theta$ perpendicular to the inclined plane

$\therefore$ Normal reaction, $R = mg \cos\theta$

When the block is moved up the inclined plane, the force of kinetic friction F_k is

$$F_k = \mu_k R = \mu_k mg \cos\theta$$

The force of friction F_k acts parallel to the inclined plane and in downward direction. In order to move the block up the inclined plane with constant velocity (*i.e.* zero acceleration), force F required is

$$F = mg\sin\theta + F_k = mg\sin\theta + \mu_k mg\cos\theta$$

or $$F = mg(\sin\theta + \mu_k \cos\theta)$$

If the block is moved through a displacement S up the inclined plane, then work done W is

$$W = F \times S = mg\,(\sin\theta + \mu_k \cos\theta) S$$

or $$W = mgS\,(\sin\theta + \mu_k \cos\theta)$$

(*iii*) **When the block is moved down the inclined plane.**

Suppose the angle of inclination θ is less than the angle of repose (See Fig. 12.4). When the block is moved down the inclined plane, the force of kinetic friction F_k is

$$F_k = \mu_k R = \mu_k mg\cos\theta$$

The force of friction F_k acts parallel to the inclined plane and in the upward direction. In order to move the block down the inclined plane with constant velocity (*i.e.* zero acceleration), force F required is

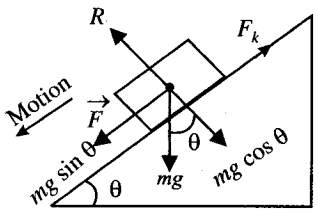


Fig. 12.4

$$F = F_k - mg\sin\theta = \mu_k mg\cos\theta - mg\sin\theta$$

$$\therefore \quad F = mg\,(\mu_k\cos\theta - \sin\theta)$$

$\therefore$ Work done W to move the block down the inclined plane through displacement S is

$$W = F\times S = mg\,(\mu_k\cos\theta - \sin\theta)\cdot S$$

or $$W = mgS\,(\mu_k\cos\theta - \sin\theta)$$

Example 12.1. Find the work done by a force of 10 N applied to a lawn roller, when the force acts making an angle of 60° with the horizontal, moving the roller through a horizontal distance of 10 m.

Solution. Work done, $W = FS\cos\theta$

Here, $F = 10$ N ; $S = 10$ m ; $\theta = 60°$

$\therefore$ $W = 10 \times 10 \times \cos 60° = \mathbf{50\ J}$

Example 12.2. A man cycles up a hill whose slope is 1 in 20. If the mass of the man and bicycle is 150 kg and the man covers a distance of 100 m along the incline, calculate the work done by man.

Solution. Fig. 12.5 shows the conditions of the problem. The cycle goes along OB. The magnitude of the force applied by the man $F = mg\sin\theta$.

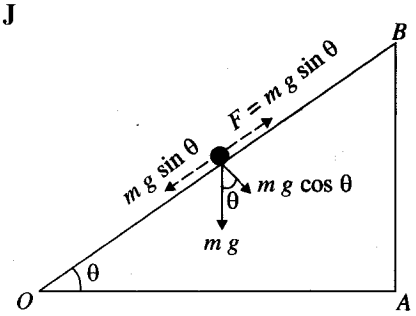


Fig. 12.5

Work done by man, $W = FS\cos\phi$

Here $F = mg\sin\theta$; $S = OB = 100$ m ; $\phi = 0°$

$\therefore$ $W = (mg\sin\theta)S\cos 0°$

$$= \left(150\times 9.8\times\frac{1}{20}\right)\times 100\times 1 = \mathbf{7350\ J}$$

Example 12.3. The displacement x of a particle moving in one dimension under the action of a constant force is related to time by the equation :

$$t = \sqrt{x} + 3$$

where x is in metres and t is in seconds.

(*i*)Find the displacement of the particle when its velocity is zero.

(*ii*)Find work done by the force in first 6 seconds.

Solution. $t = \sqrt{x}+3$ or $x = t^2 - 6t + 9$ ...(*i*)

Velocity, $v = \dfrac{dx}{dt} = \dfrac{d}{dt}(t^2 - 6t + 9) = 2t - 6$ ... (*ii*)

$$\text{Acceleration, } a = \frac{d^2x}{dt^2} = \frac{d}{dt}(2t-6) = 2 \qquad \ldots (iii)$$

(*i*) When $dx/dt = 0$, then from eq. (*ii*), we have,

$$0 = 2t - 6 \text{ or } t = 3 \text{ s}$$

$$\text{From eq. (}i\text{)}, \quad x = (3)^2 - 6\times3 + 9 = 0$$

Hence, displacement of the particle is zero when its velocity is zero.

(*ii*) Work done = Force × distance $(\because \cos\theta = 1)$

Let m be the mass of the particle.

$$\text{For } t = 6\text{s}; \; x = (6)^2 - (6\times6) + 9 = 9 \text{ m}$$

$$\therefore \quad \text{Work done} = \left(m\frac{d^2x}{dt^2}\right)x = (m\times2)\times9 = 18 \text{ m joules}$$

Example 12.4. A gardner moves a lawn roller through a distance of 50 m with a force of 50 N inclined at an angle of 60° to the direction of motion. Calculate his wages if he is to be paid 20 paise for doing 25 J of work.

Solution. Work done, $W = FS\cos\theta$

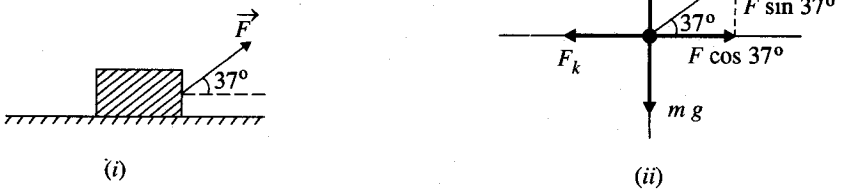
Here $F = 50$ N ; $S = 50$ m ; $\theta = 60°$

$\therefore$ $W = 50 \times 50 \times \cos 60° = 1250$ J

$$\text{Wages} = \text{Rs } \frac{0.2\times1250}{25} = \text{Rs } 10$$

Example 12.5. The box in Fig. 12.6 (*i*) is being pulled along the floor at constant speed by a force $\vec{F}$. The force of friction opposing the motion is 20 N and the box has a mass of 30 kg. Find (*i*) the magnitude of $\vec{F}$ and (*ii*) work done by $\vec{F}$ as the box is moved 5 m.

Solution. Fig. 12.6 (*i*) shows the conditions of the problem whereas Fig. 12.6 (*ii*) shows free-body diagram.



(*i*) (*ii*)

Fig. 12.6

(*i*) Referring to Fig. 12.6 (*ii*), the box is under the action of two opposing horizontal forces *viz.* frictional force F_k and the horizontal component F_x (= $F\cos 37°$) of force $\vec{F}$. Since the box moves with constant speed, the net force on it is zero *i.e.*,

$$F\cos 37° = F_k$$

$$\therefore \text{ Magnitude of } \vec{F}, \; F = \frac{F_k}{\cos 37°} = \frac{20}{0.8} = 25 \text{ N}$$

(*ii*) Work done by $\vec{F}$ on the box is

$$W = FS\cos\theta. \quad \text{Here } F = 25 \text{ N}; S = 5 \text{ m}; \theta = 37°$$

$$\therefore \quad W = 25 \times 5 \times \cos 37° = 100 \text{ J}$$

Example 12.6. A particle moves from position $\vec{r_1} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ to position $\vec{r_2} = 14\hat{i} + 13\hat{j} - 9\hat{k}$ under the action of a force $(4\hat{i} + \hat{j} + 3\hat{k})$ newton. Calculate the work done.

Solution. Displacement, $\vec{r_{12}} = \vec{r_2} - \vec{r_1} = (14\hat{i} + 13\hat{j} - 9\hat{k}) - (3\hat{i} + 2\hat{j} - 6\hat{k}) = (11\hat{i} + 11\hat{j} - 3\hat{k})$

$\therefore$ Work done, $W = \vec{F} \cdot \vec{r_{12}} = (4\hat{i} + \hat{j} + 3\hat{k}) \cdot (11\hat{i} + 11\hat{j} - 3\hat{k})$

$= 44 + 11 - 9 = 46$ J

Example 12.7. A uniform chain of length l and mass m is lying on a smooth table and one-third of its length is hanging vertically down over the edge of the table. If g is the acceleration due to gravity, calculate the work required to pull the hanging part on the table.

Solution. Weight of hanging chain $= \frac{1}{3} \times mg = \frac{mg}{3}$

The centre of gravity of the hanging part of the chain lies at its centre *i.e.* $l/6$ below the edge of the table.

$\therefore$ Work required to pull the hanging part of chain is

$$W = \text{Force} \times \text{Displacement} = \frac{mg}{3} \times \frac{l}{6} = \frac{mgl}{18}$$

Example 12.8. When an automobile moving with a speed of 36 km/h reaches the foot of an upward inclined road of angle 30°, its engine is switched off. The coefficient of kinetic friction involved is 0.1. How much distance will automobile move before coming to rest (assume $g = 10$ ms^{-2})?

Solution. Fig. 12.7 shows the conditions of the problem.

$v_0 = 36$ km/h $= 10$ ms^{-1}; $v = 0$

$\theta = 30°$; $\mu_k = 0.1$; $g = 10$ ms^{-2}

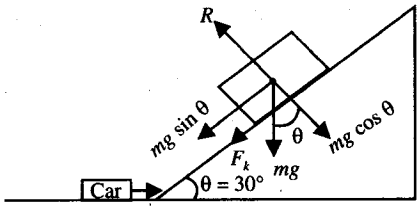


Fig. 12.7

Downward force acting on automobile on the inclined plane is

$$F = mg(\sin\theta + \mu_k \cos\theta)$$

Retardation, $a = \frac{F}{m} = g(\sin\theta + \mu_k \cos\theta) = 10(\sin 30° + 0.1 \times \cos 30°) = 5.866$ ms^{-2}

Now $v^2 - v_0^2 = 2aS$ or $(0)^2 - (10)^2 = 2 \times (-5.866) \times S$ $\therefore$ $S = 8.52$ m

PROBLEMS FOR PRACTICE

1. How much work is done in pulling a box 2 m across a table top with a horizontal force of 35 N? **[70 N]**
2. A woman pushes a lawn roller with a force of 180 N at an angle of 24° downward from the horizontal. How much work does she do as she pushes the roller a horizontal distance of 50 m? **[8200 J]**
3. A worker does 300 J of work against a frictional retarding force of 15 N in pushing a box across a floor in 3 s. If the box moves with a constant speed, how fast is it going? **[6.7 ms^{-1}]**

4. Find the work done in moving 50 kg block through a horizontal distance of 10 m by applying a force of 100 N which makes an angle of 60° with the horizontal. **[500 J]**

5. The coefficient of kinetic friction between a 20 kg box and the floor is 0.4. How much work does a pulling force do on the box in pulling it 8 m across the floor at constant speed? The pulling force is directed 37° above the horizontal. **[482 J]**

6. A coin of mass m slides a distance D along a table top. If the coefficient of friction between the coin and the table is μ, find the work done on the coin by friction.

[$-\mu m g D$]

7. A 52 kg box rests on the floor. How much work is required to move it at constant speed (*i*) 12 m along the floor against a frictional force of 150 N and (*ii*) 12 m vertically upward? **[(*i*) 1.8×10^3 J (*ii*) 6.9×10^3 J]**

8. How much work is needed to push a 1250 kg car 11.5 m up a 13.5° incline at constant velocity? (*i*) Ignore friction (*ii*) Assume the coefficient of friction is 0.09.

[(*i*) 3.29×10^5 J (*ii*) 4.5×10^5 J]

9. A particle moves from position $3\hat{i}+2\hat{j}-6\hat{k}$ to position $(14\hat{i}+13\hat{j}+9\hat{k})$ in metres and a constant force $(4\hat{i}+2\hat{j}+3\hat{k})$ N acts on it. Calculate the work done by the force.

[111 N m]

10. A child pulls a toy 2 m across the floor by a string, applying a force of constant magnitude 0.8 N. During the first metre, the string is parallel to the floor. During the second metre, the string makes an angle of 30° with the horizontal. What is the total work done by the child on the toy? **[1.5 J]**

12.6. WORK DONE BY A VARIABLE FORCE

So far we have considered the work done by constant forces. In many cases, the forces we encounter are not constant. For example, as a rocket moves away from the earth, work is done against the force of gravity which varies as the inverse square of the distance from the earth's centre.

Consider a *variable force acting on an object that moves along a straight line, X-axis, as shown in Fig. 12.8 (*i*). Suppose the x-component of force needed to pull it from $x = a$ to $x = b$ varies as shown in Fig. 12.8. (*ii*). It is desired to find the work done by the variable force when the object moves from the initial position $x = a$ to the final position $x = b$.

Fig. 12.8 (*ii*) shows how F_x varies with the position x of the object. To find the work done by this force, we divide the total displacement into a large number of very short segments (say N segments) such as $\Delta x_1, \Delta x_2, \Delta x_3 ..... \Delta x_N$. In each segment, the force is constant or approximately constant. Thus when the object moves through a very small [illegible]

Since the motion of the object from a to b is divided into N very small movements such as Δx_3, the total work done will, therefore, be

$$W = F_{x_1} \Delta x_1 + F_{x_2} \Delta x_2 + F_{x_3} \Delta x_3 + ... + F_{x_N} \Delta x_N \quad ...(i)$$

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* The variation in the force could easily arise if dirt on the surface made the object harder to pull at some points than at the others.

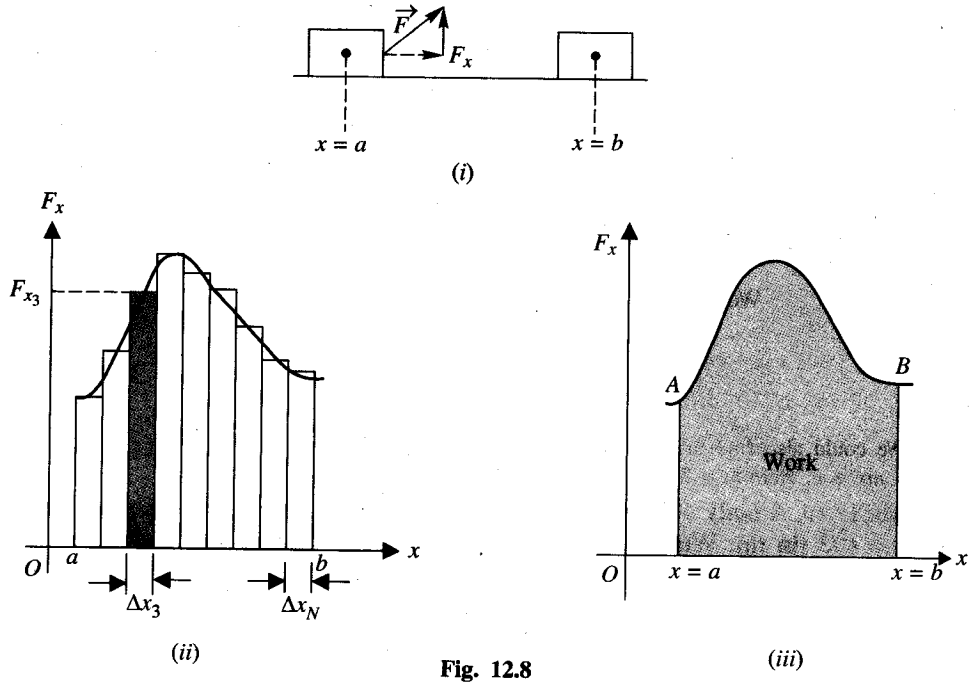

Fig. 12.8

But the products $F_{x_1}\Delta x_1$, $F_{x_2}\Delta x_2$ ... etc., are the areas of the various rectangles [See Fig. 12.8 (*ii*)]. Thus the estimate of the work done given by eq. (*i*) is equal to the sum of the areas of all the rectangles from $x = a$ to $x = b$. Of course, this expression for the work done is approximate because the force during each Δx is not precisely constant. But in the limit $\Delta x \to 0$ and $N \to \infty$, the sum of the areas of these rectangles will be exactly equal to the area under the curve [See Fig. 12.8 (*ii*)].

Thus the work done by a variable force in moving an object between two points is equal to the area under $F_x - x$ graph between those two points.

If a typical term in eq. (*i*) is represented by $F_{x_i}\Delta x_i$, we can write the equation as :

$$W = \sum_{i=1}^{N} F_{x_i}\Delta x_i$$

In the limit $\Delta x \to 0$ and $N \to \infty$, we have,

$$W = \int_a^b F_x dx = \int_a^b F\cos\theta dx$$

As $\Delta x \to 0$, the infinitesimal distance dx equals the magnitude of the infinitesimal displacement vector $\overrightarrow{dx}$. The direction of $\overrightarrow{dx}$ is along the tangent to the curve at that point so that θ is the angle between $\vec{F}$ and $\overrightarrow{dx}$ at any point. Thus, we can write W using dot product notation as :

$$W = \int_a^b F\cos\theta dx = \int_a^b \vec{F}_x . \overrightarrow{dx}$$

In general, the work done for the case of variable force can be written as :

$$W = \int_a^b \vec{F} . \overrightarrow{dS} = \text{Area } aABb \text{ [See Fig. 12.8 (}iii\text{)]} \qquad ...(ii)$$

This is the *most general definition of work.* The integral in eq. (*ii*) is called the **line integral** since it is the integral of $F \cos\theta$ along the line that represents the path of the object. Note that work done by a constant force is a special case of eq. (*ii*).

Example 12.9. **A force which pushes the object in the x direction is given by:**

$$F_x = (3x + 2)\text{N}$$

Find the work done by this force as it pushes the object from $x = 0$ to $x = 4$m.

Solution. Since the force and displacement are in the same direction, $\theta = 0°$.

$\therefore$ Work done, $W = \int_0^4 (3x+2)\,dx$. Here dx replaces dS.

$\therefore$ $$W = \left|\frac{3x^2}{2}\right|_0^4 + |2x|_0^4 = (24 - 0) + (8 - 0) = 32 \text{ J}$$

Note. We could also find the solution graphically. Now $F_x = 3x + 2$. Compare it with the standard equation $y = mx + c$. Here $m = 3$ and $c = 2$.

Example 12.10. **A body of mass 3 kg is under a force which causes a displacement in it given by $S = t^2/3$ (in m). What is the work done by the force in 2 seconds?**

Solution.

$$S = \frac{t^2}{3}; \frac{dS}{dt} = \frac{2t}{3}; \frac{d^2S}{dt^2} = \frac{2}{3}$$

$$\text{Work done, } W = \int_0^2 F dS = \int_0^2 m\frac{d^2S}{dt^2}dS = \int_0^2 m\frac{d^2S}{dt^2}\frac{dS}{dt}dt$$

$$= \int_0^2 3\times\frac{2}{3}\times\frac{2t}{3}dt = \frac{4}{3}\left[\frac{t^2}{2}\right]_0^2 = 2.6 \text{ J}$$

Example 12.11. **A body moves under the action of a force that varies in magnitude as shown in Fig. 12.9. If the force is expressed in newtons and displacement in metres, find the work done.**

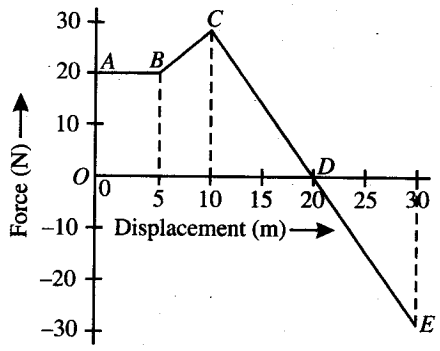

Fig. 12.9

Solution. Work done is equal to the *net* area under the force-displacement ($F - S$) graph.

$\therefore$ Work done = (Area under AB) + (Area under BC) + (Area under CD) − (Area under DE)

$$= (20\times5) + \left[\frac{1}{2}\times(20+30)\times5\right] + \left(\frac{1}{2}\times30\times10\right) - \left(\frac{1}{2}\times30\times10\right)$$

$$= 100 + 125 + 150 - 150 = \mathbf{225 \text{ J}}$$

12.7. POWER

The rate at which an agent (e.g., electric motor) can do work is called its power i.e.,

$$\text{Power, } P = \frac{\text{Work done}}{\text{Time taken}} = \frac{W}{t}$$

The power of an agent indicates how fast it can do work. The faster a given amount of work is done by an agent, the greater is its power and vice-versa.

We can also obtain another useful relation for power. Suppose a force $\vec{F}$ acts on a body so that it moves with velocity $\vec{v}$.

Now $$P = \frac{\text{Work}}{\text{Time}} = \frac{\vec{F} \cdot \vec{S}}{t} = \vec{F} \cdot \vec{v} \qquad \left(\because \vec{v} = \frac{\vec{S}}{t} \right)$$

or $$P = \vec{F} \cdot \vec{v}$$

Thus power transferred to a body is the dot product of the force on the body and the velocity of the body.

Now $$P = \vec{F} \cdot \vec{v} = Fv \cos \theta$$

Here θ is the angle between $\vec{F}$ and $\vec{v}$. If $\theta = 0°$, $P = Fv \cos 0° = Fv$. However, if $\theta = 90°$, $P = Fv \cos 90° = 0$. Since power is the dot product of $\vec{F}$ and $\vec{v}$, it is a scalar quantity *i.e.,* power has magnitude but no direction. The dimensional formula of power is

$$P = \frac{W}{t} = \frac{ML^2 T^{-2}}{T} = [ML^2 T^{-3}]$$

Units of Power. $$\text{Power} = \frac{\text{Work done}}{\text{Time interval taken}}$$

The SI unit of work is 1J and that of time is 1s. Therefore, the SI unit of power is 1 J/s or 1 watt.

Hence, the power of an agent is 1 W if it is doing 1J of work in 1s.

Sometimes bigger units of power *viz.* kilowatt (kW), megawatt (MW) or horse power (h.p.) are used.

$$1 \text{ kW} = 1000 \text{ W} \; ; \; 1 \text{ MW} = 10^6 \text{ W} = 10^3 \text{ kW} \; ; \; 1 \text{ h.p.} = 746 \text{ W}$$

Notes. (*i*) The slope of work-time graph (work along *Y*-axis and time along *X*-axis) gives the instantaneous power.

(*ii*) The area under power-time graph gives the total work done during the considered time.

Example 12.12. Oil is pumped from the tanks of a ship to a storage tank on land 45 m higher in elevation. What is the power required to pump 20,000 litres of oil per hour? Given that 1 litre of oil has a mass of 0.8 kg.

Solution. Work done $= mgh$ $\therefore$ Power, $P = \dfrac{mgh}{t}$

Here, $m = 20{,}000 \times 0.8 = 16{,}000$ kg ; $g = 9.8$ ms^{-2} ; $h = 45$ m ; $t = 1$ hour $= 60 \times 60$ s.

$\therefore$ $$P = \frac{16000 \times 9.8 \times 45}{60 \times 60} = \mathbf{1960 \text{ W}}$$

Example 12.13. An elevator cable pulls a fully loaded elevator upward at a constant speed of 0.75 ms^{-1}. The power delivered by the cable is 23 kW. What is the tension in the cable?

Solution. $P = \vec{F}.\vec{v} = Fv\cos\theta$

Here $P = 23 \text{ kW} = 23 \times 10^3 \text{ W}$; $v = 0.75 \text{ ms}^{-1}$; $\theta = 0°$; $F = ?$

$\therefore$ $23\times10^3 = F\times0.75\times\cos0°$ or $F = \dfrac{23\times10^3}{0.75\times\cos0°} = 31 \times 10^3 \text{ N}$

Therefore, tension in the elevator cable is 31 kN.

Example 12.14. A man cycles up a hill whose slope is 1 in 20 at the rate of 12 km/hr. The mass of the man and bicycle is 150 kg. Find the power of man.

Solution. Fig. 12.10 shows the conditions of the problem. The weight mg of man and bicycle can be resolved into two rectangular components *viz* $mg\sin\theta$ parallel to hill and $mg\cos\theta$ perpendicular to hill. To move at a steady speed up the hill, the man must exert force equal to $mg\sin\theta$ *i.e.*, $F = mg\sin\theta$.

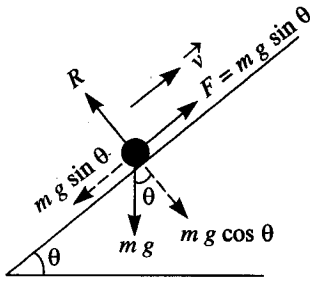

Fig. 12.10

$\therefore$ Force exerted by man, $F = mg\sin\theta = 150\times9.8\times\dfrac{1}{20} = 73.5 \text{ N}$

Velocity of man, $v = 12 \text{ km/h} = \dfrac{12\times1000}{3600} = \dfrac{10}{3}\text{ms}^{-1}$

$\therefore$ Power of man, $P = \vec{F}.\vec{v} = Fv\cos0° = Fv = 73.5\times10/3 = 245 \text{ W}$

Example 12.15. The human heart discharges 75 ml of blood at each beat against a pressure of 0.1m of Hg. Calculate the power of the heart assuming that the pulse frequency is 80 beats per minute. Given that density of mercury = 13.6×10^3 kg/m^3.

Solution. Volume of blood discharged/beat is

$\Delta V = 75 \text{ ml} = 75 \times 10^{-6} \text{ m}^3$

Pressure, $p = 0.1 \text{ m of Hg} = 0.1\times(13.6\times10^3)\times9.8 \text{ Nm}^{-2}$ $(\because p = h\rho g)$

$= 13.33 \times 10^3 \text{ Nm}^{-2}$

Work done by heart/beat $= p\times\Delta V = 13.33\times10^3\times75\times10^{-6} = 999.6\times10^{-3}\text{J}$

$\therefore$ Power of heart $= \dfrac{\text{Work done in 80 beats}}{\text{Time}} = \dfrac{999.6\times10^{-3}\times80}{60} \text{ W} = 1.33 \text{ W}$

Example 12.16. A car of mass 2000 kg is lifted up a distance of 30 m by a crane in 1 minute. A second crane does the same job in 2 minutes. Do the cranes consume the same or different amounts of fuel. What is the power supplied by each machine? Neglect power dissipation due to friction.

Solution. Mass of car, $m = 2000$ kg ; Lifting height, $h = 30$ m

For first crane. Work done, $W = mgh = 2000 \times 9.8 \times 30 = 5.88 \times 10^5$ J

$\therefore$ Power, $P_1 = \dfrac{W}{t_1} = \dfrac{5.88\times10^5}{1\times60} = 9800 \text{ W}$

For second crane. Work done, $W = mgh = 2000 \times 9.8 \times 30 = 5.88 \times 10^5$ J

$\therefore$ Power, $P_2 = \dfrac{W}{t_2} = \dfrac{5.88\times10^5}{2\times60} = 4900 \text{ W}$

Since work done by each crane is the same, the fuel consumed by each crane will also be same.

PROBLEMS FOR PRACTICE

1. A 70 kg man runs up a long flight stairs in 4s. The vertical height of stairs is 4.5 m. Calculate the power output of man **[770 W]**
2. A 1 h.p. electric motor (1 h.p. = 746 W) operates a pump continuously. How much work is performed by the motor in one day ? **[18 kWh]**
3. A man cycles up a 6° hill at a steady speed of 7 km/h. The mass of the man and the bicycle is 75 kg. Find the power output of man. **[300 W]**
4. Show that 1 kW = 3.6 MJ.
5. A log is pulled across a level floor at a constant speed of 2.3 ms^{-1} by a horizontal cable. If the power delivered by the cable is 940 W, what is the tension in the cable? **[410 N]**

12.8. ENERGY

If a body can do work, we say that the body has energy.

Thus energy of a body is defined as the ability (or capacity) of the body to do work.

Whenever work is done on a body, it gains energy. If 100 J of work is done, the body gains 100 J of energy. It means that the energy of a body is the work stored in it. The following points may be noted about energy :

(*i*) Since energy of a body is the stored work, energy is a scalar quantity ($\because$ Work is a scalar quantity).

(*ii*) Energy is measured in the same units as work *i.e.,* joules.

(*iii*) The dimensional formula of energy is $[ML^2 T^{-2}]$ *i.e.,* the same as for work.

(*iv*) Energy may appear in various forms such as mechanical energy, heat energy, sound energy, light energy *etc.*

12.9. MECHANICAL ENERGY

The energy associated with the motion and position/configuration of mechanical systems is called mechanical energy. The mechanical energy is of two types *viz.*

1. Kinetic energy **2.** Potential energy.

1. **Kinetic energy.** *The energy possessed by a body by virtue of its motion is called kinetic energy of the body.* For example,

   (*i*) The kinetic energy of a hammer drives a nail into a piece of wood.

   (*ii*) The kinetic energy of a cannon ball strikes down a brick wall.

   (*iii*) A bullet fired from a gun can pierce a target due to its kinetic energy.

   (*iv*) The kinetic energy of air is used to run wind mills.

2. **Potential energy.** *The energy possessed by a body by virtue of its position or configuration (shape or size) is called potential energy of the body.*

   A body may store energy because of its position or configuration. This is called potential energy because in the stored state, the body has the potential to do work. For example, when you do work to wind the spring of a toy car, the work done is stored in the spring in the form of potential energy. The spring now has the ability to move the car when it unwinds and therefore to do work. Two important types of potential energy are: (*i*) gravitational potential energy (*ii*) elastic potential energy.

   (*i*) **Gravitational potential energy.** The work done against the gravitational force is stored in the form of gravitational potential energy. When a heavy brick is lifted from the ground to some height, work is done against the gravitational force. This work done is stored in the brick in the form of gravitational potential energy.

(*ii*) **Elastic potential energy.** The potential energy associated with elastic materials is called elastic potential energy. For example, when a spring is compressed or stretched, work will have to be done against the spring force. This work done is stored in the spring in the form of elastic potential energy.

12.10. EXPRESSION FOR KINETIC ENERGY

Consider a body of mass m moving in a straight line with velocity v. The kinetic energy of the body must be equal to the work done in imparting it a velocity v. Suppose initially the body is at rest (*i.e.*, $v_0 = 0$). Let a constant horizontal force $\vec{F}$ be applied to the body till it attains the velocity v (See Fig. 12.11). If $\vec{S}$ is the displacement during this time, then work done by the force on the body is

$$W = \vec{F}.\vec{S} = FS \qquad (\because \theta = 0°)$$

If a is the constant acceleration produced, then according to Newton's second law, $F = ma$.

$\therefore$ $$W = maS$$

Now, $$v^2 - v_0^2 = 2aS \quad \text{or} \quad v^2 - 0^2 = 2\,a\,S \quad \therefore \quad S = \frac{v^2}{2a}$$

$\therefore$ $$W = ma\frac{v^2}{2a} = \frac{1}{2}m\,v^2$$

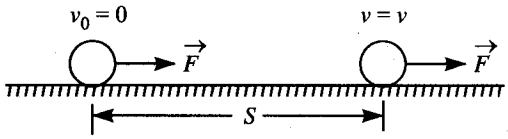

Fig. 12.11

Since W = Kinetic energy gained by the body,

$\therefore$ Kinetic energy, $K = \frac{1}{2}mv^2$

Note that kinetic energy of the moving body is directly proportional to the mass and square of the velocity of the body.

Calculus method. We can also obtain expression for the kinetic energy of a moving body by the method of calculus.

Suppose a body of mass m is initially at rest (*i.e.*, $v = 0$). If a force $\vec{F}$ applied on the body increases its velocity from zero to v (magnitude), then total work done W is given by ;

$$W = \int_0^v \vec{F}.\overrightarrow{dS} = \int_0^v F\,dS \qquad (\because \theta^* = 0°)$$

According to Newton's second law, $F = m\frac{dv}{dt}$

$\therefore$ $$W = \int_0^v m\frac{dv}{dt}dS = \int_0^v mv\,dv \qquad \left(\because v = \frac{dS}{dt}\right)$$

$$= \left|\frac{mv^2}{2}\right|_0^v = \frac{1}{2}mv^2$$

$\therefore$ Kinetic energy, $K = \frac{1}{2}mv^2$

---

* Note that a force (or component of a force) acting perpendicular to velocity vector does no work.

Discussion. The following points may be noted :

(*i*) The kinetic energy of a body at rest is zero ($\because v = 0$).

(*ii*) Since m and v^2 are both positive, *kinetic energy (K.E.) of a body is always positive.*

(*iii*) Kinetic energy, like work, is a scalar quantity. The kinetic energy of a moving body depends on its speed (*magnitude* of velocity) but not on the *direction* in which it is moving.

12.11. KINETIC ENERGY AND LINEAR MOMENTUM

The kinetic energy (K) of a moving body may be expressed in terms of the magnitude of its linear momentum ($p = mv$).

Now $$K = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m}$$

$$\therefore \quad K = \frac{p^2}{2m} \qquad ...(i)$$

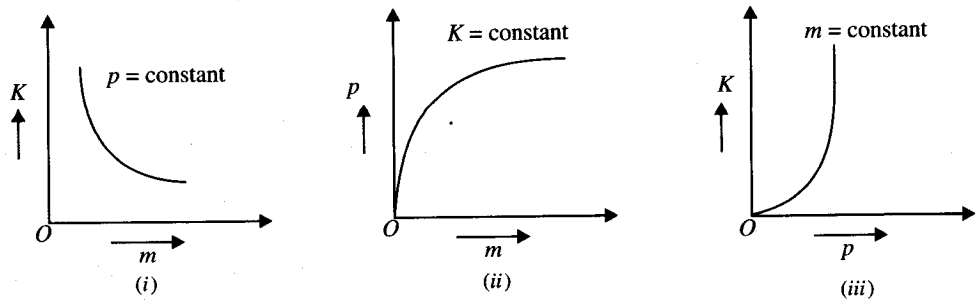

Fig. 12.12

(*i*) If linear momentum p of the body is constant, then its kinetic energy $K \propto 1/m$ [See eq. (*i*)]. The graph between K and m for this situation is shown in Fig. 12.12(*i*).

(*ii*) If kinetic energy K of the body is constant, then its linear momentum $p \propto \sqrt{m}$ [See eq. (*i*)]. The graph between p and m for this situation is shown in Fig. 12.12(*ii*).

(*iii*) If mass m of the body is constant, then its linear momentum and kinetic energy are related as: $p \propto \sqrt{K}$ [See eq. (*i*)]. The graph between K and p for this situation is shown in Fig. 12.12(*iii*).

12.12. WORK-ENERGY THEOREM

According to this theorem, *the work done by the net force acting on a body is equal to the change in the kinetic energy of the body.*

This theorem has a simple meaning. When a force is applied on a body in the direction of motion of the body, the speed and hence kinetic energy of the body increases. According to work-energy theorem, the increase in kinetic energy of the body is equal to the work done by the force on the body. If a force is applied on a body in a direction opposite to its motion, the kinetic energy of the body decreases. The decrease in the kinetic energy of the body is equal to the work done by the body against the retarding force. In either case, the change in kinetic energy of the body is equal to the work done (positive or negative work done).

Proof. We now prove work-energy theorem.

Let m = Mass of the body

v_0 = Initial velocity of the body (magnitude)

$\vec{F}$ = Force applied on the body in the direction of motion

v = Final velocity of the body (magnitude)

The small amount of work dW done by the force in causing small displacement $\overrightarrow{dS}$ in the direction of $\vec{F}$ is

$$dW = \vec{F} \cdot \overrightarrow{dS} = F\,dS \qquad (\because \theta = 0°)$$

or
$$dW = ma\,dS \qquad (\because F = m\,a)$$

$$= m\left(\frac{dv}{dt}\right)dS \qquad \left(\because a = \frac{dv}{dt}\right)$$

$$= m\frac{dS}{dt}dv$$

$$= mv\,dv \qquad \left(\because v = \frac{dS}{dt}\right)$$

The total work done by the force in increasing the velocity of the body from v_0 to v is

$$W = \int_{v_0}^{v} mv\,dv = \left|\frac{mv^2}{2}\right|_{v_0}^{v} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$\therefore$
$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

i.e., **Net work done = Change in kinetic energy**

Thus work-energy theorem stands proved.

Discussion. The following points may be noted :

(*i*) If net force does positive work (*i.e.,* $\vec{F} \cdot \vec{S}$ is positive) on a body, the K.E. of the body increases.

(*ii*) If net force does negative work (*i.e.,* $\vec{F} \cdot \vec{S}$ is negative) on a body, the K.E. of the body decreases.

(*iii*) If work done on a body is zero, K.E. of the body remains the same *i.e.,* it does not change.

12.13. GRAVITATIONAL POTENTIAL ENERGY (G.P.E.)

Consider a brick lying on the surface of the earth. If the brick is raised through a certain height, work is done against gravity. This work done is stored in the brick in the form of energy. We say that the brick has a potential energy or more specifically gravitational potential energy because work is done against gravity. Note that brick has gravitational potential energy due to its *position* above the surface of the earth and not due to motion. If the brick is allowed to fall, it can do the amount of work equal to its gravitational potential energy.

The energy possessed by a body by virtue of its position above the surface of earth is called gravitational potential energy of the body.

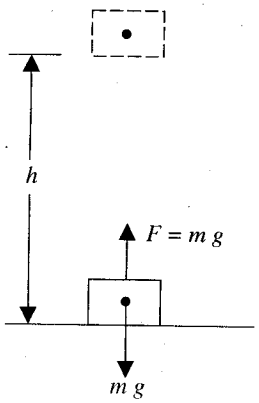

Fig. 12.13

Expression for G.P.E. Consider a body of mass m lying at rest on the surface of earth as shown in Fig. 12.13. The gravitational force on the body is mg acting vertically downward. In order to lift the body vertically upward at a constant velocity (*i.e.,* zero acceleration), we require an upward force $F = mg$. If the body is raised through height h, the work done by the lifting force is

$$W = \vec{F} \cdot \vec{S} = FS\cos 0° = F\,h = m\,g\,h \qquad (\because S = h)$$

There is no change in the kinetic energy of the body because its velocity has been the same throughout. Therefore, the work done on the body has been used to increase its gravitational potential energy. If the gravitational potential energy of the body is taken to be zero on earth's surface, then its gravitational potential energy at height h is mgh.

$\therefore$ Gravitational potential energy = mgh

Suppose a mass of 4 kg is raised through a height 5 m above the ground; then its gravitational potential energy = $mgh = 4 \times 9.8 \times 5 = 196$ J.

Note. We have seen that in order to lift a body of mass m vertically upward through a distance h, work done is mgh. Even if the body is lifted through height h following any path (not necessarily vertical), the work done will still be mgh.

12.14. CONSERVATIVE AND NON-CONSERVATIVE FORCES

1. Conservative Force. *A force is said to be* ***conservative*** *if the amount of work done in moving an object against that force is independent of how the object moves from the initial position to the final position.*

One important example of conservative force is the gravitational force. It means that amount of work done in moving a body against gravity from location A to location B is the same whichever path you may follow in going from A to B. This is illustrated in Fig. 12.14. The mass m is located at position A on the surface of the earth. In order to move the mass m from location A to location B, there can be several paths; only three paths 1, 2 and 3 are shown for clarity of the diagram. Whatever path you may follow in moving the mass from A to B, work done against gravity is mgh. Thus work done against gravity is path independent; it only depends upon the mass of the body and on the initial and final positions of the body.

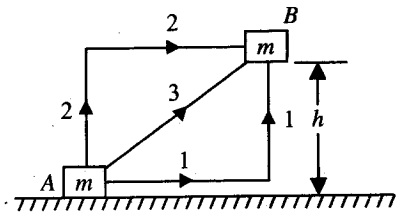

Fig. 12.14

Fig. 12.14 illustrates another interesting feature of a conservative force. In moving the body from A to B, the work done by gravity is $-$ *mgh. However, in going from B to A, the work done by gravity is + **mgh. Thus the total work done by a conservative force over a closed path is zero. This leads to another definition of conservative force.

A force is ***conservative*** *if the total work it does on an object is zero when the object moves around any closed path returning to its initial position.*

This definition illuminates an important aspect of a conservative force *viz. work done by a conservative force is recoverable.* Thus in Fig. 12.14, we shall have to do mgh amount of work in taking the body from A to B. However, when body is released at B, we recover mgh of work.

Note. Potential energy is only associated with a conservative force.

Examples. Some common examples of conservative forces are :

(*i*) The gravitational force is a conservative force.

(*ii*) The force exerted by a spring is a conservative force.

(*iii*) The electrostatic force between two charges is a conservative force.

2. Non-conservative Force. *A force is* ***non-conservative*** *if the work done by that force on an object moving between two points depends on the path taken between the points.*

---

* $W = \vec{F}.\vec{S}$. In this situation, angle between $\vec{F}$ and $\vec{S}$ is 180°.

** In this case, angle between $\vec{F}$ and $\vec{S}$ is zero.

The *force of friction is an example of non-conservative force.* Let us illustrate a non-conservative force with an example. Suppose you are to displace a book between two points on a rough horizontal surface, such as a table. If the book is displaced in a straight line between the two points, the work done by friction is simply $*FS$ where :

F = force of friction ; S = distance between the points

However, if the book is moved along *any other* path between the two points (such as a semi-circular path), the work done by friction would *be greater* (in absolute magnitude) than FS. Finally, if the book is moved through any closed path, the work done by friction is never zero**—it is always negative. Thus the work done by a non-conservative force is not recoverable, as it is for a conservative force.

12.15. ELASTIC POTENTIAL ENERGY OF A SPRING

We can store potential energy in a body by changing its shape. For example, when a spring is stretched or compressed, it stores potential energy due to the change in its shape. The greater the distortion in the spring, the greater the potential energy stored in it. This stored potential energy is due to the elastic property of the body and is called elastic potential energy.

Let us see how a spring stores elastic potential energy. When a spring is stretched or compressed, a *restoring force* is developed due to elasticity which tends to bring the spring back to its original length. Therefore, in order to stretch or compress a spring, work has to be done against this restoring force. This work done is stored in the spring in the form of elastic potential energy.

Thus, the energy possessed by a spring due to change in its shape is called elastic potential energy of the spring.

When the spring is in its natural length, its elastic potential energy is considered zero. However, when the spring is stretched or compressed, it acquires elastic potential energy. A spring obeys Hooke's law. According to this law, *if the spring is not stretched or compressed too much, the increase or decrease in the length (x) of the spring is directly proportional to the applied force (F) i.e.,*

$$F \propto x \quad \text{or} \quad F = kx$$

where k is a constant of proportionality and is called *spring constant* or *force constant.* Its value is different for different types of springs. Note that unit of k is N/m ($k = F/x$) and its dimensional formula is $[MT^{-2}]$.

If $x = 1$, then, $F = k \times 1$ or $k = F$

Hence ***spring constant*** *(or* ***force constant****) of a spring is numerically equal to the force required to produce unit displacement (compression or stretch) in the spring.* The value of spring constant depends upon length of spring, radius of spring and nature of material of spring.

12.16. ELASTIC POTENTIAL ENERGY STORED IN A SPRING

Consider a spring having one end attached to rigid support and the other end free as shown in Fig. 12.15. If no stretching force is applied to the free end of the spring, the extension of the spring is zero *i.e.*, $x = 0$. However, when an external stretching force F is applied to the spring, there is an extension x of the spring from its equilibrium position as shown in Fig. 12.15. According to Hook's law,

$$F = k\,x$$

where k is the spring constant. If extension is increased by Δx where Δx is so small that F can be considered constant, then small work ΔW done by the force is

$$\Delta W = F\,\Delta x \quad \text{or} \quad \Delta W = k\,x\,\Delta x$$

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* $W.D. = \vec{F}.\vec{S} = FS \cos 180° = -FS.$

** Since frictional force always opposes motion.

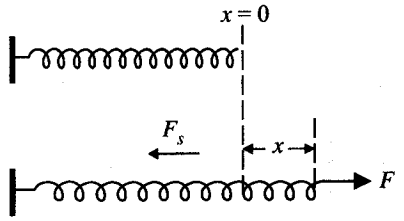


Fig. 12.15

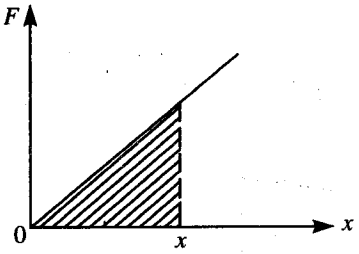


Fig. 12.16

The total work W done in increasing the extension from 0 to x is given by ;

$$W = \int_0^x k\,x\,dx = k\int_0^x x\,dx$$

$$= k\left|\frac{x^2}{2}\right|_0^x = \frac{1}{2}kx^2$$

$$\therefore \qquad W = \frac{1}{2}kx^2$$

Therefore, elastic potential energy U_S (= W) of the spring is

Elastic potential energy of spring, $U_S = \frac{1}{2}kx^2$

Alternate method. Fig. 12.16, shows the graph between the applied stretching force F and extension x of the spring. Since $F \propto x$, the F-x graph is a straight line. The stretching force is zero when $x = 0$ and it is kx when the displacement of the spring is x from the equilibrium position. Therefore, average force F_{av} needed to extend the spring from $x = 0$ to $x = x$ is

$$F_{av} = \frac{0+kx}{2} = \frac{1}{2}kx$$

The displacement of the spring is in the same direction as F_{av} and its magnitude is x. Therefore, work W done in extending the spring is

$$W = F_{av}x\cos 0° = \left(\frac{1}{2}kx\right)x = \frac{1}{2}kx^2$$

This work done is stored in the spring as elastic potential energy U_S of the spring.

Elastic potential energy of spring, $U_S = \frac{1}{2}kx^2$

Note that elastic potential energy of the spring (= $kx^2/2$) is equal to the area (shaded) under force-displacement graph from 0 to x. It is interesting to note that the slope of the graph is k *i.e.*, slope of the graph is equal to the spring constant. Fig. 12.17 shows the graph between elastic potential energy of the spring versus its displacement from the equilibrium position. Note that when the spring is in its natural length (*i.e.*, $x = 0$), the elastic potential energy of the spring is zero. However, as x increaşes, the elastic potential energy of the spring also increases.

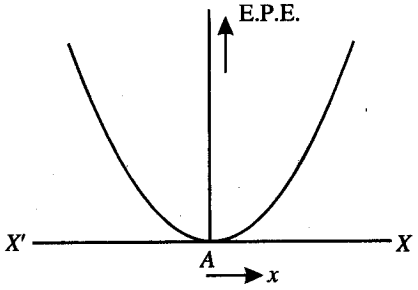


Fig. 12.17

12.17. SPRING FORCE IS A CONSERVATIVE FORCE

We have seen that gravitational force is a conservative force. Another example of conservative force is the force exerted by a spring. We can prove its conservative nature by referring to Fig. 12.18. Your hand must exert a force $F = kx$ to hold the spring stretched a distance x from the equilibrium position. But according to Newton's third law, the spring exerts a force $F_s = -kx$ on your hand. As you stretch the spring an additional amount Δx, the work done by the *spring on your hand* $= F_s \Delta x = -kx\ \Delta x$. However, if you allow the spring to retract this same distance Δx so that displacement is $-\Delta x$, the work done by the spring $= (-kx)(-\Delta x) = kx\ \Delta x$. This is numerically equal to the work done during stretching, but it is opposite in sign. Therefore, the spring does zero net work during the stretching-retraction cycle. *This leads to the conclusion that force exerted by a spring is a conservative force.* There are other conservative forces. The electrostatic force between two electric charges is conservative. So is the nuclear force that holds the nucleous together.

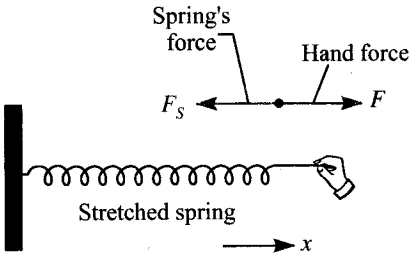


Fig. 12.18

12.18. DIFFERENT FORMS OF ENERGY

Energy has many forms and may be transformed from one form to another. Energy, in all its forms, is measured in the same units as work *i.e.,* joules. Some of the forms of energy are discussed below :

1. **Mechanical energy.** *The mechanical energy (E) of a body is the sum of kinetic energy (K) and potential energy (U) of the body i.e.,*

   Mechanical energy, $E = K + U$

   (*i*) The kinetic energy ($= mv^2/2$) of a body is always positive because both m and v^2 are positive.

   (*ii*) The potential energy may be positive or negative.

   (*iii*) A body may have negative mechanical energy *i.e.,* $K + U$ may be negative. For example, a satellite revolving around the earth has negative mechanical energy. Note that negative mechanical energy indicates bound state. It means the potential energy of the body is negative and its value is more than kinetic energy of the body.

2. **Heat energy.** It is the energy possessed by a body due to the kinetic energy of its molecules. We can produce heat energy by converting mechanical, electric energy etc. into it.

3. **Internal energy.** The internal energy of a body is the sum of kinetic energy and potential energy of its molecules. The kinetic energy of molecules results from their random motions. The potential energy of molecules results from their bonds and interactions with each other.

4. **Electrical energy.** The energy associated with electric current is called electrical energy. In order to move electric charges, work will have to be done. This work done appears as electrical energy.

5. **Chemical energy.** A body such as a chemical compound possesses chemical energy because of chemical bonding of its atoms. A chemical compound has less energy than the parts of which it is made. This difference in energy is called chemical energy.

6. **Nuclear energy.** The energy available from an atomic nucleus is called nuclear energy. We can obtain nuclear energy in two ways *viz.* (*i*) by nuclear fission (*ii*) by nuclear fusion.

In nuclear fission, we split a heavy nucleus into two or more light nuclei. The result of this process is that tremendous amount of energy in released. In fusion, two or more light nuclei are combined to form a heavy nucleus. Again, a huge amount of energy is released.

12.19. TRANSFORMATION OF ENERGY

In all physical processes, energy is transformed from one form to another. For example,

(*i*) A stone held high in the air has potential energy; as it falls, it loses potential energy and at the same time it gains in kinetic energy. In this case, potential energy is being transferred into kinetic energy.

(*ii*) Water at the top of a dam has potential energy which is transformed into kinetic energy as the water falls.

(*iii*) At the base of the dam; kinetic energy of water can be transferred to turbine blades and further transformed into electric energy.

(*iv*) The potential energy stored in the bent bow can be transformed into kinetic energy of an arrow.

In each of these examples, the transfer of energy is accompanied by the performance of work. The water does work on the blades, the bow does work on the arrow and so on. This observation gives us further insight into the relation between work and energy :

Work is done whenever energy is transferred from one object to another.

12.20. LAW OF CONSERVATION OF ENERGY

The importance of the energy concept is that energy is a conserved quantity *i.e.,* in any process energy can be transformed from one form to another but the total energy neither increases nor decreases. This is the law of conservation of energy and is one of the most important principles in physics.

The total energy neither increases nor decreases in any process. Energy can be transformed from one form to another; and transferred from one body to another but the total amount remains the same.

According to law of conservation of energy, the amount of energy in the universe is the same now as it was millions of years ago. How can we make such a universal statement when we cannot even measure the amount of coal and oil in the earth's crust? For over one hundred years, scientists have performed a variety of controlled experiments monitoring the amount of energy involved in various processes. In these experiments, the form of energy changes but the total amount always remains the same. This is a convincing proof of law of conservation of energy.

Illustration. The law of conservation of mechanical energy is best illustrated by a freely falling body. Since the body is under the action of a conservative force alone (*i.e.,* gravitational force), its mechanical energy (sum of K.E. and P.E.) will remain constant. Let a body of mass m fall from a height h above the ground as shown in Fig. 12.19. We shall see that total mechanical energy (K.E. + P.E.) of the body at any point during its downward journey is the same.

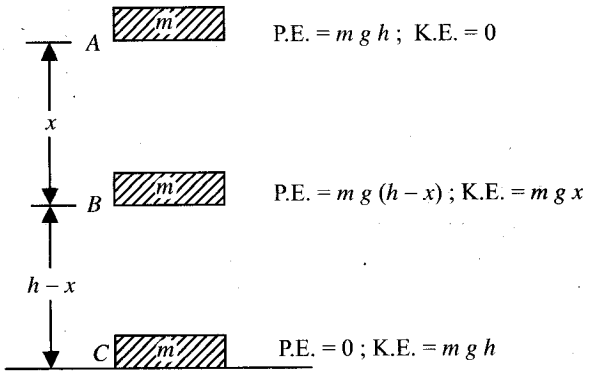


Fig. 12.19

Point *A*. The body starts its downward journey from point A so that its initial velocity (v_0) is zero.

$\therefore$ K.E. = 0 ; P.E. = mgh

Total mechanical energy = K.E. + P.E. = 0 + mgh = mgh

At point *B*. Suppose at point B, the body has velocity v_1.

$\therefore$ $v_1^2 - 0^2 = 2gx$ $\therefore$ $v_1^2 = 2gx$

$$\text{K.E.} = \frac{1}{2}mv_1^2 = \frac{1}{2}m(2gx) = mgx$$

$$\text{P.E.} = mg\ (h - x)$$

$\therefore$ Total mechanical energy = K.E. + P.E. = $mgx + mg\ (h - x) = mgh$

At point *C*. Suppose at point C (*i.e.*, ground), the body has velocity v_2.

$\therefore$ $v_2^2 - 0^2 = 2\ gh$ or $v_2^2 = 2gh$

$\therefore$ $\text{K.E.} = \frac{1}{2}mv_2^2 = \frac{1}{2}m(2gh) = mgh$ and P.E. = 0 $(\because\ h = 0)$

$\therefore$ Total mechanical energy = K.E. + P.E.

$= mgh + 0 = mgh$

Thus the total mechanical energy (K.E. + P.E.) of the body remains the same at all points during the downward journey. As the body falls from height h, its potential energy starts decreasing while the kinetic energy begins to increase; the total mechanical energy (P.E. + K.E.) at any point being the same. When the body reaches the ground, P.E. = 0 and K.E. (= mgh) is equal to the initial potential energy at height h. Fig. 12.20 shows the variation of P.E. and K.E. with height. Note that at any point of journey, the total mechanical energy is mgh as shown by thick horizontal line.

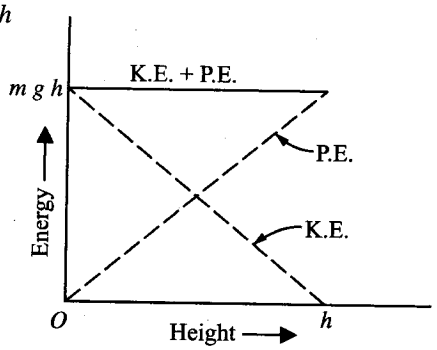


Fig. 12.20

Example 12.17. A body of mass 5 kg initially at rest is subjected to a force of 20 N. What is the kinetic energy acquired by the body at the end of 10 s?

Solution. Mass of body, m = 5 kg ; Initial velocity, v_0 = 0 ; F = 20 N ; t = 10 s

Acceleration of the body, $a = \frac{F}{m} = \frac{20}{5} = 4\ \text{ms}^{-2}$

$$v = v_0 + at = 0 + 4 \times 10 = 40\ \text{ms}^{-1}$$

$\therefore$ Kinetic energy acquired by the body after 10 s

$$= \frac{1}{2}mv^2 = \frac{1}{2} \times 5 \times (40)^2 = \mathbf{4000\ J}$$

Example 12.18. How much work is required to accelerate a 1000 kg car from 20 ms^{-1} to 30 ms^{-1}?

Solution. The work needed is equal to the increase in kinetic energy (work – energy theorem).

Work needed, $W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

Here m = 1000 kg ; v_1 = 20 ms^{-1}; v_2 = 30 ms^{-1}

$\therefore$ $W = \frac{m}{2}\left[v_2^2 - v_1^2\right] = \frac{1000}{2}[(30)^2 - (20)^2] = \mathbf{2.5 \times 10^5 J}$

Example 12.19. How much work is done to move a 1400 kg car from rest to 25 ms^{-1}. If this takes place over a distance of 130 m, what is the average net force?

Solution. According to work-energy theorem, work W done is equal to the increase in kinetic energy.

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Here, $m = 1400$ kg ; $v_1 = 0$; $v_2 = 25$ ms^{-1}

$$\therefore \quad W = \frac{1}{2}mv_2^2 - 0 = \frac{1}{2}\times 1400 \times (25)^2 = \mathbf{4.4 \times 10^5\ J}$$

The average net force F can be found from the relation :

$$W = FS \quad \text{or} \quad F = \frac{W}{S} = \frac{4.4\times 10^5}{130} = \mathbf{3.4 \times 10^3\ N}$$

Example 12.20. An automobile moving horizontally at a speed of 54 km/h reaches the foot of an inclined smooth plane and the engine is switched off. How much distance does the automobile go up the incline before coming to rest? The inclination of the plane to the horizontal is 40°.

Solution. Fig. 12.21 shows the conditions of the problem. When the automobile reaches point O (foot of the incline OB), the engine is switched off. The retarding force $F = mg \sin\theta$ does negative work on the automobile. Consequently, the kinetic energy of the automobile starts decreasing and it becomes zero when car comes to rest.

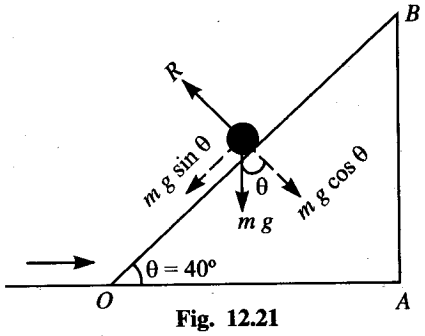


Fig. 12.21

Initial velocity of car, $v_0 = 54$ km/h $= 15$ ms^{-1}

Final velocity of car, $v = 0$

Suppose distance covered by the car before coming to rest is S. Then work done by the opposing force F ($= mg \sin\theta$) is

$$W = -FS = -(mg\sin\theta)\, S$$

According to Work-Energy theorem, net work done on the car is equal to the change in its kinetic energy.

$$\therefore \quad W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}m(0)^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2$$

$$\text{or} \quad -FS = -\frac{1}{2}mv_0^2$$

$$\therefore \quad (mg\sin\theta)S = \frac{1}{2}mv_0^2 \quad \text{or} \quad S = \frac{v_0^2}{2g\sin\theta} = \frac{(15)^2}{2\times 9.8\times \sin 40^\circ} = \mathbf{17.86\ m}$$

Example 12.21. A toy rocket of mass 0.1 kg has a small fuel of mass 0.02 kg which it burns out in 3s. Starting from rest on a horizontal smooth track, it gets a speed of 20 m/s after the fuel is burnt out. What is the approximate thrust of the rocket? What is the energy content per unit mass of the fuel? (Ignore the small mass variation of the rocket during fuel burning).

Solution. Acceleration of rocket, $a = \frac{v - v_0}{t} = \frac{20 - 0}{3} = \frac{20}{3}$ ms^{-2}

$$\text{Thrust of rocket, } F = ma = 0.1\times\frac{20}{3} = \frac{2}{3}\mathbf{N}$$

K.E. gained by rocket $= \frac{1}{2}mv^2 = \frac{1}{2}\times 0.1\times(20)^2 = 20$ J

$\therefore$ Energy content per unit mass of fuel

$$= \frac{\text{Total energy}}{\text{Mass of fuel}} = \frac{20}{0.02} = \mathbf{1000\ J/kg}$$

Example 12.22. A shot travelling at the rate of 100 ms^{-1} is just able to pierce a plank 4 cm thick. What velocity is required to just pierce a plank 9 cm thick?

Solution. The K.E. of the shot is spent in doing work against plank resistance F newtons.

For the first case, $\frac{1}{2}mv_1^2 = F\times S_1$ ...(*i*)

For the second case, $\frac{1}{2}mv_2^2 = F\times S_2$ ...(*ii*)

Dividing eq. (*ii*) by eq. (*i*), we get, $\frac{v_2^2}{v_1^2} = \frac{S_2}{S_1}$ $\therefore$ $v_2 = \sqrt{\frac{S_2}{S_1}}\times v_1 = \sqrt{\frac{9}{4}}\times 100 = \mathbf{150\ ms^{-1}}$

Example 12.23. A small block (See Fig. 12.22) is released from rest and slides down a smooth curved track. Calculate the velocity of the block when it reaches *B*, a vertical distance of 7.5 m below *A*.

Solution. Fig. 12.22 shows the conditions of the problem. We shall solve this problem by making use of energy-conservation principle. Suppose the mass of the block is m. At point A, the potential energy of the block is mgh. The block loses a potential energy mgh in moving from A to B. If the velocity of block on reaching B is v, then kinetic energy gained by block is $mv^2/2$. Since there are no frictional losses, according to energy-conservation principle,

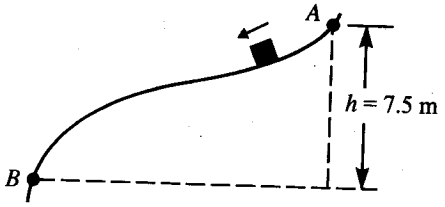


Fig. 12.22

K.E. gained = P.E. lost

or $\frac{1}{2}mv^2 = mgh$

$\therefore$ $v = \sqrt{2gh} = \sqrt{2\times 9.8\times 7.5} = \mathbf{12.12\ ms^{-1}}$

Example 12.24. A boy of mass 40 kg sits in a swing by a rope 6 m long. A person pulls the swing to a side so that the rope makes an angle of 60° with the downward vertical. What is the gain in potential energy of the boy?

Solution. Fig. 12.23 shows the conditions of the problem. Initially, the boy is at position B. When the swing is pulled from B to A, the rope makes an angle 60° with the vertical. The height through which the boy is raised is $h = A'B$.

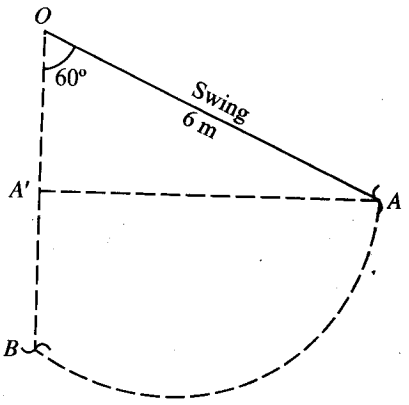


Fig. 12.23

Now $A'B = OB - OA' = OA - OA\cos 60°$

$= OA(1-\cos 60°)$

$= 6\ (1 - \cos 60°) = 3$ m

$\therefore$ Potential energy gained by the body

$$= mgh = 40 \times 9.8 \times 3 = \mathbf{1176\ J}$$

Example 12.25. A ball is dropped from rest at a height of 12 m. If it losses 25% of kinetic energy on striking the ground, what is the height to which it bounces?

Solution. We shall use energy-conservation principle to solve this problem.

Vertical height of the ball, h = 12 m

At a height h (= 12 m), the potential energy of the ball is mgh and its K.E. is zero. As the ball is dropped, its potential energy starts converting into K.E. On reaching the ground, K.E. = mgh and P.E. = 0. Since the ball loses 25% of K.E. on striking the ground,

$\therefore$ Reduced energy = 0.75 mgh

If on striking the ground it bounces to a height h', its potential energy is mgh'. According to energy-conservation principle,

$$mgh' = 0.75\ mgh \quad \text{or} \quad h' = 0.75\ h = 0.75 \times 12 = \mathbf{9\ m}$$

Example 12.26. A 4 kg block (m_4) is on a smooth horizontal table. The block is connected to a second block of mass 1 kg (m_1) by a massless flexible taut chord that passes over a frictionless pulley (See Fig. 12.24). The 1 kg block is 1 m above the floor. The two blocks are released from rest. With what speed does the 1 kg block hit the ground?

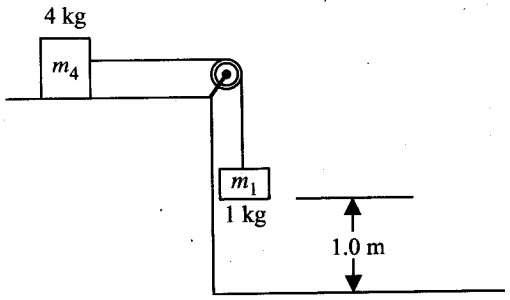


Fig. 12.24

Solution. We shall make use of energy-conservation principle to solve this problem.

Initial K.E. of two blocks, $K_i = 0$ ($\because$ block is at rest)

Initial P.E. of the blocks, $U_1 = {}^*m_1\ gy$

Suppose v_f is the speed of both m_1 and m_4 at the instant when m_1 hits the floor.

Final K.E. of two blocks, $k_f = \frac{1}{2}m_1v_f^2 + \frac{1}{2}m_4v_f^2$

Final P.E. of two blocks, $U_f = 0$

According to energy conservation principle,

$$K_i + U_i = K_f + U_f$$

or
$$0 + m_1gy = \left(\frac{1}{2}m_1v_f^2 + \frac{1}{2}m_4v_f^2\right) + 0$$

$\therefore$
$$v_f = \sqrt{\frac{2m_1gy}{m_1+m_4}} = \sqrt{\frac{2\times1\times9.8\times1}{1+4}} = \mathbf{1.98\ ms^{-1}}$$

Example 12.27. How much work is done to stretch a spring by 15 cm if the spring constant k has the value of 800 N/m?

* We need not include the potential energy of m_4 since it does not change.

Solution. Work done, $W = \frac{1}{2}kx^2$

Here k = 800 N/m ; x = 15 cm = 0.15 m

$\therefore$ $W = \frac{1}{2}(800)\times(0.15)^2 = 9$ J

Example 12.28. **A mass of 2 kg attached to a spring is vibrated horizontally by displacing the mass 40 cm from its equilibrium position and releasing it. Find the maximum velocity of the mass. The spring constant is 24.5 N/m.**

Solution. When mass is displaced through x = 40 cm, its P.E. = $kx^2/2$ and K.E. = 0.

$\therefore$ Total energy = P.E. + K.E. = $\frac{kx^2}{2}+0=\frac{kx^2}{2}$ ...(*i*)

When mass passes through equilibrium position (*i.e.*, x = 0), its P.E. = 0 and K.E. = $mv_m^2/2$ where v_m is the maximum velocity of the mass.

$\therefore$ Total energy = P.E. + K.E. = $0+\frac{1}{2}mv_m^2=\frac{1}{2}mv_m^2$ ...(*ii*)

From eqs. (*i*) and (*ii*), $\frac{1}{2}mv_m^2 = \frac{1}{2}kx^2$

or $v_m = x\sqrt{\frac{k}{m}}=0.4\times\sqrt{\frac{24.5}{2}} = 1.4$ ms^{-1}

PROBLEMS FOR PRACTICE

1. An object A moving horizontally with kinetic energy of 800 J experiences a constant opposing force of 100 N while moving from point X to point Y, where XY = 2 m. What is the kinetic energy of A at Y ? In what further distance will A come to rest if this opposing force continues to act on it ? **[600 J; 6 m]**
2. A force of 5 N acts on a body of mass 2 kg for 10 s. If the body is initially at rest, calculate the kinetic energy acquired. **[1562.5 J]**
3. A 3 kg mass starts at rest at the top of 37° incline which is 5 m long. Its speed as it reaches the bottom is 2 ms^{-1}. Use energy method to find the average frictional force which retarded its motion. **[16.4 N]**
4. How large a force is required to accelerate an electron (m = 9 × 10^{-31} kg) from rest to a speed of 2 × 10^7 ms^{-1} in a distance of 0.5 cm? **[3.64 × 10^{-14} N]**
5. A 500 kg block is at the top of 30° incline. When it is released, it slides 140 cm down along the incline to the bottom. What will be the speed of the block when it reaches the bottom if the frictional force is 0.9 N? How large is the coefficient of friction? **[2.95 ms^{-1}; 0.21]**
6. A 145 g baseball is dropped from a tree 22 m above the ground. With what speed would it hit the ground if air resistance could be ignored? **[21 ms^{-1}]**
7. Calculate the velocity of the bob of a simple pendulum at its mean position if it is able to rise to a vertical height of 10 cm. **[1.4 ms^{-1}]**
8. The bob of a simple pendulum of length 1 m is drawn aside until the string becomes horizontal. After the bob is released, what will be its velocity at the mean position? **[4.43 ms^{-1}]**

9. A car of mass 1.2×10^3 kg moves 300 m up a road which is inclined to the horizontal at an angle θ where $\sin\theta = 1/12$. By how much does the gravitational P.E. of the car increase? **[2.4×10^5 J]**

10. A simple pendulum consisting of a small heavy bob attached to a light string of length 40 cm is released from rest with the string at 60° to the downward vertical. Find the speed of the bob as it passes through the lowest point. **[2 ms^{-1}]**

12.21. MASS-ENERGY RELATION

According to Einstein's theory of relativity, the mass of a body is not constant but increases with the increase in the velocity of the body. In other words, when a body is given kinetic energy, its mass is increased. This shows that mass depends upon energy. Therefore, mass and energy are inter-convertible *i.e.*, they can be changed into each other. According to Einstein, a particle of mass m possesses an equivalent energy E given by ;

$$E = mc^2$$

where c = velocity of light ($= 3 \times 10^8$ ms^{-1})

This relation, although derived strictly from theoretical considerations by Einstein, has been repeatedly verified by experiments and is found to be absolutely correct. The following points may be noted carefully :

(*i*) The mass is another form of energy just as water and ice are two forms of H_2O. Thus, a body of mass m is actually a storage of energy $E = mc^2$ into which it can be converted by suitable means. Once so converted, it will not remain mass at all but only energy.

(*ii*) As a mass of a particle is not constant but depends upon its velocity, therefore, sometimes it is more convenient to talk about mass in terms of equivalent energy. Thus instead of saying that an electron has a rest mass of 9.0×10^{-31} kg, it is more appropriate in nuclear physics to say that an electron (at rest) has an equivalent energy of 0.511 MeV. Similarly, mass of other sub-atomic particles (*e.g.*, protons, neutrons etc.) can be expressed in terms of equivalent energy.

Example 12.29. Find the equivalent energy of (*i*) electron (*ii*) proton and (*iii*) neutron at rest. Given that mass of electron = 9.1×10^{-31} kg; mass of proton = 1.673×10^{-27} kg and mass of neutron = 1.675×10^{-27} kg.

Solution. (*i*) Equivalent energy, $E = m_e c^2 = (9.1\times10^{-31})\times(3\times10^8)^2$ J

$$= \frac{9.1\times10^{-31}\times9\times10^{16}}{1.6\times10^{-19}} \text{ eV}$$

$$= 0.511 \times 10^6 \text{ eV} = \mathbf{0.511 \text{ MeV}}$$

(*ii*) Equivalent energy, $E = m_p c^2 = (1.673\times10^{-27})\times(3\times10^8)^2$ J

$$= \frac{1.673\times10^{-27}\times9\times10^{16}}{1.6\times10^{-19}} \text{ eV}$$

$$= 941.1 \times 10^6 \text{ eV} = \mathbf{941.1 \text{ MeV}}$$

(*iii*) Equivalent energy, $E = m_n c^2 = (1.675\times10^{-27})\times(3\times10^8)^2$ J

$$= \frac{1.675\times10^{-27}\times9\times10^{16}}{1.6\times10^{-19}} \text{ eV}$$

$$= 942.2 \times 10^6 \text{ eV} = \mathbf{942.2 \text{ MeV}}$$

Example 12.30. If 2000 kg of water is heated from 0°C to 100°C, what will be the corresponding increase in the mass of water? Specific heat of water = 1 cal g^{-1} $°C^{-1}$.

Solution. Heat supplied to water, $Q = cm\Delta T$

Here $c = 1$ cal g^{-1} $°C^{-1}$; $\Delta T = 100 - 0 = 100°C$; $m = 2000$ kg $= 2 \times 10^6$ g

$\therefore \quad Q = 1 \times 2 \times 10^6 \times 100 = 2 \times 10^8$ cals $= 2 \times 10^8 \times 4.2$ J $= 8.4 \times 10^8$ J

Increase in mass, $\Delta m = \frac{Q}{c^2} = \frac{8.4 \times 10^8}{(3 \times 10^8)^2} = 0.933 \times 10^{-8}$ kg

Example 12.31. How much mass is converted into energy per day in Tarapur nuclear power plant operated at 10^6 kW?

Solution. Here, $P = 10^6$ kW $= 10^6 \times 10^3$ W $= 10^9$ Js^{-1}; $m = ?$; No. of seconds/day, $t = 24 \times 60 \times 60$ s.

$\therefore$ Energy produced/day, $E = P \times t = 10^9 \times (24 \times 60 \times 60)$J

Now $E = mc^2$; Here c = velocity of light $= 3 \times 10^8 ms^{-1}$

$\therefore \quad m = \frac{E}{c^2} = \frac{10^9 \times (24 \times 60 \times 60)}{(3 \times 10^8)^2} = 0.96 \times 10^{-3}$ kg $= 0.96$ g

12.22. COLLISIONS

Collisions are a common occurrence in everyday life. Some familiar examples are : a tennis racket striking a tennis ball, a cricket bat hitting a cricket ball, the two billiard balls colliding, one car striking another car and a hammer hitting a nail. But a collision can occur even if the objects do not touch. For example, interaction between two positively charged particles is a collision. Here no physical contact occurs between the particles.

*A **collision** is a short-time event and is said to have occurred if two (or more) bodies physically collide against each other or even when the path of motion of one particle is affected by the other.*

Scientists have performed a number of careful experiments to study the effects of collisions. The conclusions can be summed up as under :

(*i*) *The linear momentum is always conserved in a collision i.e.,* linear momentum of the colliding objects after collision is equal to their linear momentum before collision. There is only a sharing of linear momentum between the colliding objects but total linear momentum remains the same.

(*ii*) *The total energy is always conserved in a collision* (law of conservation of energy).

(*iii*) *The kinetic energy in a collision may or may not be conserved.* If kinetic energy (K.E.) is conserved in a collision, it is called **elastic collision.** If kinetic energy is *not* conserved in a collision, it is called **inelastic collision.**

12.23. TYPES OF COLLISIONS

Depending upon whether kinetic energy is conserved in a collision or not, there are broadly two types of collisions *viz.*

1. Elastic collision **2.** Inelastic collision

1. Elastic collision. *A collision in which kinetic energy is conserved is called an **elastic collision.*** Therefore, an elastic collision has the following characteristics :

(*i*) The linear momentum is conserved.

(*ii*) The total energy of the system is conserved.

(*iii*) The kinetic energy is conserved.

Fig. 12.25 shows an elastic collision. The ball is dropped from a height h on a frictionless surface. The ball rebounds to the initial height h. In this case, the K.E. of the ball is conserved. Therefore, the collision is elastic. Let us see how kinetic energy is conserved in an elastic collision. In an elastic collision, some of the initial kinetic energy is converted temporarily during collision to potential energy; after the collision is over, potential energy is entirely converted back to kinetic energy. Clearly, for an elastic collision, the colliding objects must interact by a conservative force.

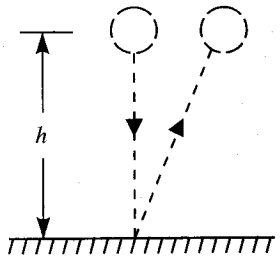


Fig. 12.25

2. Inelastic collision. *A collision in which kinetic energy is not conserved is called an* ***inelastic collision.*** In such a collision, there is some loss of kinetic energy. This energy loss is converted into other forms of energy *e.g.,* heat, sound etc. Therefore, an inelastic collision has the following characteristics :

(*i*) The linear momentum is conserved.

(*ii*) The total energy of the system is conserved.

(*iii*) The kinetic energy is **not** conserved.

Most of the collisions are inelastic. For example, the collision between two cars is inelastic collision. When the two cars collide, a large amount of kinetic energy is lost in doing the work needed to produce the metal distortion and other obvious permanent effects of the collision. Even though kinetic energy is lost during this inelastic collision, linear momentum is conserved.

Perfectly Inelastic Collision. *A collision in which the two objects stick together after the collision is called a* **perfectly inelastic collision**. Since the bodies are stuck together, they must move as a unit and their velocities are in the same direction. The motion after a totally inelastic collision is determined entirely by the conservation of linear momentum principle.

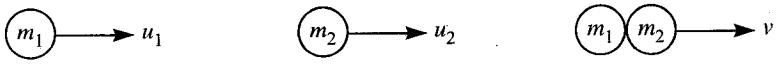


Fig. 12.26

Consider two objects of masses m_1 and m_2 moving in a straight line. Let u_1 and u_2 be their initial velocities. Suppose the objects undergo perfectly inelastic collision. Therefore, after the collision, they stick together to form a single composite body of mass $m_1 + m_2$ and move with the same velocity v (See Fig. 12.26). Conservation of linear momentum states that the initial and final linear momenta of the two-object system must be the same *i.e.,*

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v \quad \text{... Perfectly inelastic collision}$$

Given four of the five quantities (u_1, u_2, m_1, m_2, v), we can solve for the fifth. Often but not always, the unknown is the final velocity v :

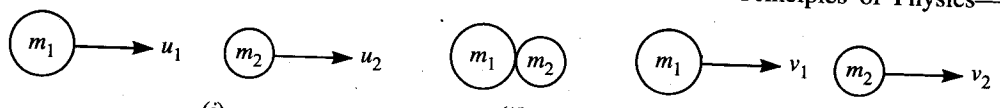
$$\therefore \quad v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

12.24. ELASTIC COLLISION IN ONE DIMENSION

When two bodies moving along the same straight line collide with each other without loss of kinetic energy and continue to move along the same straight line after collision, it is said to be an ***elastic collision in one dimension.***

Consider two masses m_1 and m_2 moving along the same straight line with *velocities u_1 and u_2 respectively as shown in Fig. 12.27 (*i*). The balls will collide only if $u_1 > u_2$.

* Since it is one dimensional motion, the velocity will be assigned +ve and negative signs. If motion is along + x direction, velocity is positive and it is negative if motion is along – x direction.



(*i*) Before collision (*ii*) During collision (*iii*) After collision

Fig. 12.27

After head-on elastic collision [See Fig. 12.27 (*iii*)], they continue to move along a straight line in the same direction with velocities v_1 and v_2 as shown in Fig. 12.27 (*iii*). The two bodies will separate if $v_2 > v_1$.

From law of conservation of linear momentum, we have,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \text{... (i)}$$

Since the collision is assumed to be elastic, kinetic energy is also conserved.

$$\therefore \quad \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad \text{...(ii)}$$

From eq. (*i*), $m_1(u_1 - v_1) = m_2(v_2 - u_2)$ ...(iii)

From eq. (*ii*), $m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$ ...(iv)

Dividing eq. (*iv*) by eq. (*iii*), we have,

$$\frac{u_1^2 - v_1^2}{u_1 - v_1} = \frac{v_2^2 - u_2^2}{v_2 - u_2}$$

or $$u_1 + v_1 = v_2 + u_2$$

or $$u_1 - u_2 = v_2 - v_1 \quad \text{...(v)}$$

This is an interesting result. The quantity on the left of the equation (*v*) [*i.e.*, $u_1 - u_2$] is the relative velocity of approach of the two masses while the quantity on the right ($v_2 - v_1$) is the relative velocity of separation.

Thus for perfectly elastic collision in one dimension, the relative velocity of approach before collision is equal to the relative velocity of recession (or separation) after collision.

If one object is approaching another at a relative speed of 5 ms^{-1}, then after collision, it will be receding at a relative speed of 5 ms^{-1}.

Velocities after collision. The velocities of masses m_1 and m_2 after the collision are v_1 and v_2 respectively. Let us calculate their values.

From eq. (*v*), $v_2 = u_1 - u_2 + v_1$

Putting this value of v_2 in eq. (*i*), we have,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2(u_1 - u_2 + v_1)$$

or $$m_1u_1 + m_2u_2 = m_1v_1 + m_2u_1 - m_2u_2 + m_2v_1$$

or $$v_1(m_1 + m_2) = (m_1 - m_2)u_1 + 2m_2u_2$$

$$\therefore \quad v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2} + \frac{2m_2u_2}{m_1 + m_2} \quad \text{... (vi)}$$

From eq. (*v*), $v_1 = v_2 - u_1 + u_2$

Putting this value of v_1 in eq. (*i*), we have,

$$m_1u_1 + m_2u_2 = m_1(v_2 - u_1 + u_2) + m_2v_2$$

or $$m_1u_1 + m_2u_2 = m_1v_2 - m_1u_1 + m_1u_2 + m_2v_2$$

or $$v_2(m_1+m_2) = 2m_1u_1+(m_2-m_1)u_2$$

$$\therefore \quad v_2 = \frac{2m_1u_1}{m_1+m_2}+\frac{(m_2-m_1)}{m_1+m_2}u_2 \qquad ...(vii)$$

Different Cases.

(*i*) ***When the two bodies are of equal masses*** *i.e.*, $m_1 = m_2 = m$. In that case,

$$v_1 = \frac{(m-m)u_1}{m+m}+\frac{2mu_2}{m+m}=0+u_2$$

$$\therefore \quad v_1 = u_2$$

Also $$v_2 = \frac{2mu_1}{m+m}+\frac{(m-m)u_2}{m+m}=u_1+0$$

or $$v_2 = u_1$$

That is the bodies exchange velocities as a result of the collision; mass m_2 acquires the velocity that mass m_1 had before collision and *vice-versa.*

Therefore, if two bodies of equal masses undergo elastic collision in one dimension, then after collision, the bodies exchange their velocities. This result is made use of for slowing down neutrons in a nulcear reactor.

(*ii*) ***When the target body (i.e., m_2 in this case) is initially at rest.*** This is a very common practical situation, for a moving object to strike one at rest. Putting $u_2 = 0$ in eqs. (*vi*) and (*vii*), we have,

$$v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2}\ ;\ v_2 = \frac{2m_1u_1}{m_1+m_2}$$

Some special cases of this situation are :

(*a*) **When $m_1 = m_2 = m$ *i.e.*, the two bodies are of equal masses.** In this case, the values of v_1 and v_2 are :

$$v_1 = \frac{(m-m)u_1}{m+m}=0\ ;\ v_2 = \frac{2mu_1}{m+m}=u_1$$

That is the striking mass m_1 is stopped dead in its track and the target mass m_2 takes off with a velocity equal to what mass m_1 had before the collision. This effect is well known to billiard players.

Therefore, in a head-on elastic collision (i.e., elastic collision in one dimension) of two identical bodies, one originally at rest, the moving body stops and the other takes on its original speed.

(*b*) **When $m_1 >> m_2$ *i.e.*, a heavy moving body strikes a light body at rest.** Putting $u_2 = 0$ in eqs. (*vi*) and (*vii*) and neglecting m_2 as compared to m_1 (*i.e.*, $m_1 - m_2 \simeq m_1$ and $m_1 + m_2 \simeq m_1$), we have,

$$v_1 = u_1\frac{m_1}{m_1}=u_1\ ;\quad v_2 = u_1\left(\frac{2m_1}{m_1}\right)=2u_1$$

Thus, the velocity of the heavy incoming body is practically unchanged, whereas the light body originally at rest, takes off with twice the velocity of the heavy one. This fact is a common experience. Suppose a ping pong ball initially at rest is struck with a much heavier paddle. The ball leaves the paddle with twice the incoming speed of the paddle (See Fig. 12.28) and the paddle follows through at essentially the same speed it had before it hit the ball.

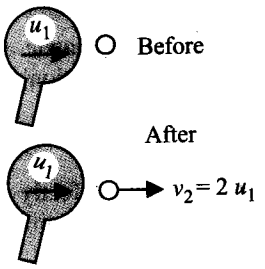
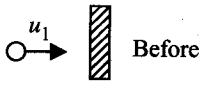


Fig. 12.28

(*c*) **When $m_1 << m_2$, *i.e.*, a light moving object strikes a massive body at rest.** Putting $u_2 = 0$ in eqs. (*vi*) and (*vii*) and neglecting m_1 as compared to m_2 (*i.e.*, $m_1 - m_2 \simeq -m_2$ and $m_1 + m_2 \simeq m_2$), we have,

$$v_1 = -\frac{m_2}{m_2}u_1 = -u_1 \; ; \quad v_2 = \frac{2m_1}{m_2}u_1 \simeq 0$$



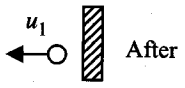


Fig. 12.29

Therefore, the massive object remains essentially at rest and very light incoming object rebounds with essentially its speed but in the opposite direction. For example, a tennis ball colliding with a wall will hardly effect the wall but will bounce back with nearly its incoming speed (See Fig. 12.29).

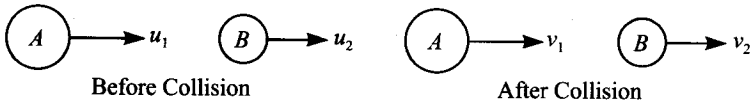
12.25. COEFFICIENT OF RESTITUTION

In actual practice, no collision is perfectly elastic. There is always some loss of kinetic energy in every collision and it depends on the nature of the surfaces of the colliding bodies. The degree of elasticity of a collision is determined by the quantity called *coefficient of restitution (e).*

The coefficient of restitution is defined as the ratio of the velocity of separation to the velocity of approach of the colliding particles. It is denoted by *e*.

$$\text{Coefficient of restitution, } e = \frac{\text{Velocity of separation}}{\text{Velocity of approach}}$$

Consider that two bodies *A* and *B* are moving along a straight line in the same direction with velocities u_1 and u_2 respectively as shown in Fig. 12.30 (*i*). Suppose they collide and continue to move along the same straight line with velocities v_1 and v_2 respectively as shown in Fig. 12.30 (*ii*). Then,



Before Collision (*i*) After Collision (*ii*)

Fig. 12.30

$$\text{Velocity of approach} = u_1 - u_2 \; ; \quad \text{Velocity of separation} = v_2 - v_1$$

$$\therefore \quad \text{Coefficient of restitution, } e = \frac{v_2 - v_1}{u_1 - u_2}$$

The value of *e* determines the type of collision.

(*i*) **For elastic collision.** For elastic collision, the velocity of separation is equal to the velocity of approach *i.e.*,

$$\frac{v_2 - v_1}{u_1 - u_2} = 1 \text{ or } e = 1$$

Hence coefficient of restitution of elastic collision in one dimension is unity.

(*ii*) **For inelastic collision.** As will be shown in Art. 12.26, in an inelastic collision, velocity of separation is always less than the velocity of approach.

$$\therefore \quad \frac{v_2 - v_1}{u_1 - u_2} < 1 \text{ or } e < 1$$

(*iii*) **For perfectly inelastic collision.** In this case, the bodies stick together so that velocity of separation is zero *i.e.*, $v_2 - v_1 = 0$.

$$\therefore \quad \frac{v_2 - v_1}{u_1 - u_2} = \frac{0}{u_1 - u_2} = 0 \text{ or } e = 0$$

Example. 12.32. A 0.1 kg ball moving at a speed of 5 ms^{-1} in + x direction collides head-on with a 0.3 kg ball that is at rest. Assuming that collision is elastic, determine the final velocities of the two balls.

Solution. Fig. 12.31 (*i*) shows the conditions of the problem before the collision.

Here, $m_1 = 0.1$ kg ; $m_2 = 0.3$ kg ; $u_1 = 5\ ms^{-1}$; $u_2 = 0$

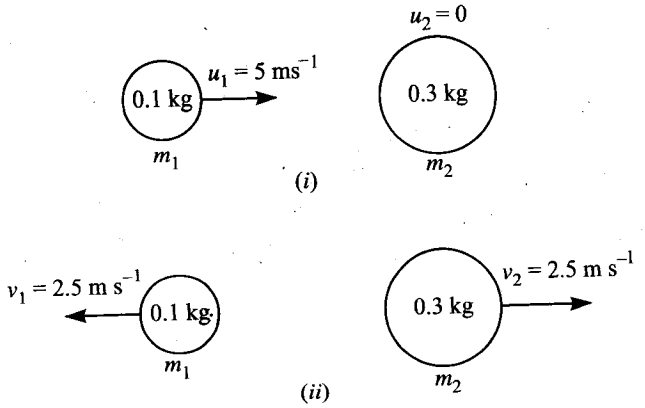


Fig. 12.31

Let the velocities of m_1 and m_2 be v_1 and v_2 respectively after the elastic collision.

$$\therefore \quad v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2} + \frac{2m_2 u_2}{m_1 + m_2}$$

$$= \frac{(0.1 - 0.3)5}{0.1 + 0.3} + \frac{2m_2 \times 0}{m_1 + m_2} = -2.5\ ms^{-2}$$

The negative sign indicates that the velocity is directed to the left.

Also
$$v_2 = \frac{2m_1 u_1}{m_1 + m_2} + \frac{(m_2 - m_1)u_2}{m_1 + m_2}$$

$$= \frac{2 \times 0.1 \times 5}{0.1 + 0.3} + \frac{(m_2 - m_1) \times 0}{m_1 + m_2} = 2.5\ ms^{-2}$$

The positive sign indicates that the velocity is directed to the right. Fig. 12.31(*ii*) shows the movement of the balls after the collision.

Example 12.33. A ball of mass m is moving with initial speed v to the right and undergoes elastic collision with a ball of mass 2 m initially at rest. What are the final velocities of the two balls?

Solution. Fig. 12.32 (*i*) shows the conditions of the problem before collision.

Here $m_1 = m$; $m_2 = 2m$; $u_1 = v$; $u_2 = 0$

Let the final velocities of m and $2m$ be v_1 and v_2 respectively after the collision.

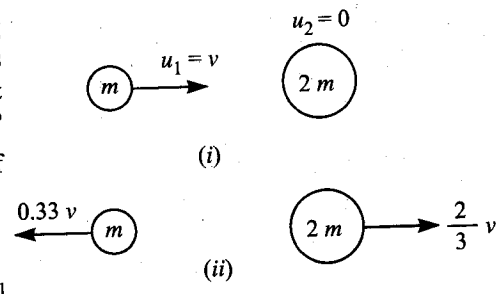


Fig. 12.32

$$\therefore \quad v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2} + \frac{2m_2 u_2}{m_1 + m_2}$$

$$= \frac{(m-2m)v}{m+2m} + \frac{2\times 2m\times 0}{m+2m} = -\frac{1}{3}v + 0$$

$\therefore$ $v_1 = -0.33\ v$

The negative sign indicates that the velocity is directed to the left.

Also $$v_2 = \frac{2m_1u_1}{m_1+m_2} + \frac{(m_2-m_1)u_2}{m_1+m_2}$$

$$= \frac{2mv}{m+2m} + \frac{(2m-m)\times 0}{m+2m} = \frac{2}{3}v + 0$$

$\therefore$ $$v_2 = \frac{2}{3}v$$

The positive sign shows that the velocity is directed to the right. Fig. 12.32 (*ii*) shows the velocities of the balls after the collision.

Example 12.34. Two balls each of mass *m* moving in opposite directions with speed *v* undergo head-on collision. Calculate the velocities of the two balls after the collision.

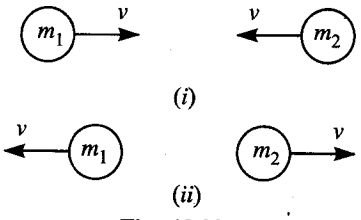


Fig. 12.33

Solution. Fig. 12.33 (*i*) shows the conditions of the problem before collision.

Here $m_1 = m_2 = m$; $u_1 = v$; $u_2 = -v$

Let v_1 and v_2 be the final velocities of m_1 and m_2 respectively after the collision.

$\therefore$ $$v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2} + \frac{2m_2u_2}{m_1+m_2}$$

$$= \frac{(m-m)v}{m+m} + \frac{2m(-v)}{m+m} = 0 - v = -v$$

The negative sign indicates that the velocity is directed to the left.

Also $$v_2 = \frac{2m_1u_1}{m_1+m_2} + \frac{(m_2-m_1)u_2}{m_1+m_2}$$

$$= \frac{2m\times v}{m+m} + \frac{(m-m)\times(-v)}{m+m} = v + 0 = v$$

The positive sign indicates that velocity is directed to right.

Fig. 12.33 (*ii*) shows the velocities of the two balls after the collision. Note that after the collision, the balls bounce back with equal speed *v*.

Example 12.35. In a nuclear experiment, a particle of unknown mass moving at 460 kms^{-1} undergoes a head-on elastic collision with a carbon nucleus (mass = 12 *u*) moving at 220 kms^{-1}. After collision, the carbon nucleus continues in the same direction at 340 kms^{-1}. Calculate the mass and final velocity of the unknown particle.

Solution. The unknown particle is the incoming particle and carbon nucleus is the target particle.

Here $m_1 = ?$; $m_2 = 12u$; $u_1 = 460\ kms^{-1}$; $u_2 = 220\ kms^{-1}$; $v_1 = ?$; $v_2 = 340\ kms^{-1}$

$$v_2 = \frac{2m_1u_1}{m_1+m_2} + \frac{(m_2-m_1)u_2}{m_1+m_2}$$

or $$v_2(m_1+m_2) = u_2(m_2-m_1) + 2m_1u_1$$

or $340(m_1+12) = 220(12-m_1)+2m_1\times460$

$\therefore$ $m_1 = 4u$

The unknown particle is, therefore, α-particle.

$$v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2}+\frac{2m_2u_2}{m_1+m_2}$$

$$= \frac{(4u-12u)460}{4u+12u}+\frac{2\times12u\times220}{4u+12u} = 100 \text{ kms}^{-1}$$

Example 12.36. Suppose the collision in Fig. 12.34 is completely inelastic and the masses and initial velocities are shown. Find the final velocity.

Solution. Total linear momentum before collision

$= m_1 u_1 + m_2 u_2$

$= (5)(2) + 3(-2) = 10 - 6 = 4 \text{ kgms}^{-1}$

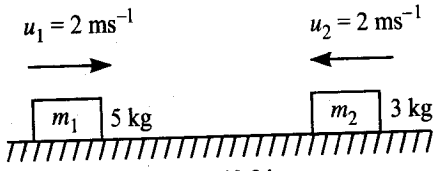


Fig. 12.34

When the bodies undergo perfectly inelastic collision, they stick together to form a single composite body of mass $m_1 + m_2$ and move with the same velocity v.

$\therefore$ Total linear momentum after collision

$= (m_1+m_2)v = (5+3)v = 8v$

$\therefore$ $8v = 4$ or $v = 4/8 = 0.5$ ms^{-1}

Since v is positive, the system moves to the right after the collision.

Example 12.37. A nucleus of radium ($_{88}Ra^{236}$) decays to radon ($_{86}Ra^{232}$) by the emission of an α-particle ($_2He^4$) of energy 4.8 MeV. Calculate the recoil energy of the daughter nucleus ($_{86}Ra^{232}$). Given that mass of $_{86}Ra^{232}$ = 222.0 a.m.u. and mass of α-particle = 4.003 a.m.u.

Solution. The kinetic energy of a particle is given by ;

$$E = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} \quad \therefore \quad p = \sqrt{2mE}$$

Since linear momentum is conserved in this nuclear reaction,

mE = constant

or $m_\alpha E_\alpha = m_D E_D$ ...D stands for daughter

$\therefore$ $E_D = \frac{m_\alpha}{m_D}\times E_\alpha = \frac{4.003}{222}\times4.8 = 0.087$ MeV

Since linear the initial linear momentum of radium nucleus was zero, the final linear momentum must be zero. This is possible only if the two products move in the opposite directions with equal linear momentum. Negative sign is omitted because only magnitude of energy is required.

Example 12.38. A body of mass M at rest is struck by a moving body of mass m. Prove that the fraction of the initial kinetic energy of mass m transferred to the struck body is

$$\frac{4Mm}{(m+M)^2}.$$

Solution. The velocity v_2 of mass M after collision is

$$v_2 = \frac{2m_1u_1}{m_1+m_2}+\frac{(m_2-m_1)u_2}{m_1+m_2}$$

Here $m_1 = m$; $m_2 = M$; $u_1 = u$ (say) ; $u_2 = 0$

$$\therefore \quad v_2 = \frac{2mu}{m+M} + 0 = \frac{2mu}{m+M}$$

K.E. of M after collision, $E_2 = \frac{1}{2}Mv_2^2 = \frac{1}{2}M\left(\frac{2mu}{m+M}\right)^2$

Initial K.E. of m, $E_1 = \frac{1}{2}mu^2$

$\therefore$ Fraction of initial K.E. transferred

$$= \frac{E_2}{E_1} = \frac{2Mm^2u^2}{(m+M)^2 \times \frac{1}{2}mu^2} = \frac{4mM}{(m+M)^2}$$

PROBLEMS FOR PRACTICE

1. A 4 kg block moving to the right at 6 ms^{-1} makes a perfectly elastic collision with a 2 kg block moving to the right at 3 ms^{-1}. Find the final velocities of the blocks.
[4 ms^{-1}; 7 ms^{-1}]
2. Two blocks are sliding along a straight line toward each other on a frictionless surface. One block of mass 10 kg is coming from left at 5 ms^{-1} and the other of mass 6 kg is coming from right at 3 ms^{-1}. Find the velocity of each block if the collision is perfectly elastic.
[– 1 ms^{-1}; + 7 ms^{-1}]
3. A 10 g bullet is fired with a velocity of 300 ms^{-1} into a pendulum bob which has a mass of 990 g. How high does the pendulum bob (plus bullet) swing after the collision?
[0.459 m]
4. A 500 g ball moves along the x-axis with a speed of 20 ms^{-1}. It hits a bat and reverses its direction so its speed along the x-axis is now 30 ms^{-1}. Find its change in linear momentum and its change in kinetic energy.
[25 Ns; 125 J]
5. A 4 kg object moving at 5 ms^{-1} makes a perfectly elastic collision with 1 kg object initially at rest. Find the final velocities of the objects.
[3 ms^{-1}; 8 ms^{-1}]
6. A ball with an initial speed of 4 ms^{-1} has a head-on collision with another ball at rest which is three times as massive. What are the final velocities if the collision is elastic?
[– 2 ms^{-1}; + 2 ms^{-1}]
7. After an elastic collision between two balls of equal masses, one is observed to have a speed of 3 ms^{-1} along the positive x-axis and the other has a speed of 2 ms^{-1} along the negative x-axis. What were the original velocities of the balls?
[– 2 ms^{-1}; 3 ms^{-1}]
8. A ball of mass m makes a head-on elastic collision with a second ball at rest and rebounds with a speed equal to one-third its original speed. What is the mass of the second ball?
[2 m]
9. A block of mass m = 2 kg slides down 30° incline which is 3.6 m high. At the bottom, it strikes a block of mass M = 6 kg which is at rest on a horizontal surface (See Fig. 12.35). If the collision is elastic and friction can be ignored, determine the speed of two blocks after the collision.
[– 4.2 ms^{-1}; 4.2 ms^{-1}]

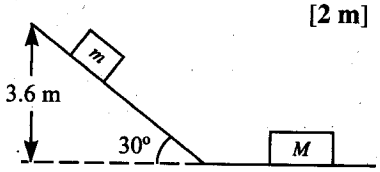


Fig. 12.35

10. A stationary U^{238} nucleus emits an α-particle ($_2He^4$) with a velocity of 1.4×10^7 ms^{-1} and kinetic energy of 4.1 MeV. Calculate the velocity of the residual nucleus.
[– 2.4 × 10^5 ms^{-1}]

12.26. INELASTIC COLLISION IN ONE DIMENSION

In an inelastic collision, linear momentum is conserved but some of the kinetic energy is lost. Consider that two bodies of masses m_1 and m_2 are moving along a straight line in the same direction with velocities u_1 and u_2 respectively as shown in Fig. 12.36 (*i*). Suppose they collide and some of the kinetic energy is lost in the collision. Therefore, the collision is inelastic. Since the collision is head-on, the bodies will move with velocities v_1 and v_2 along the same straight line as shown in Fig. 12.36 (*ii*).

Fig. 12.36

Since linear momentum is conserved in an inelastic collision,

$\therefore$ Linear momentum before collision = Linear momentum after collision

or $$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

If we know the masses (m_1, m_2), initial velocities (u_1, u_2) and one of the final velocities, we can calculate the other final velocity.

Note. In this case, the velocity of separation ($v_2 - v_1$) is always less than the velocity of approach ($u_1 - u_2$). We now prove this fact.

Now $$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

or $$m_2(v_2 - u_2) = m_1(u_1 - v_1) \quad ...(i)$$

Also $$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 < \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$ ...inelastic collision

or $$m_2(v_2^2 - u_2^2) < m_1(u_1^2 - v_1^2) \quad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we have,

$$v_2 + u_2 < u_1 + v_1$$

or $$v_2 - v_1 < u_1 - u_2$$

Thus in an inelastic collision, the velocity of separation is always less than the velocity of approach.

12.27. ELASTIC COLLISION IN TWO DIMENSIONS

When two billiard balls collide on the surface of a billiard table, the general result is that they move off in different directions. Since all velocities lie in the plane of the surface, the collision is two-dimensional. In case of elastic collision in two-dimensions, both linear momentum and kinetic energy are conserved. If the collision is not head-on (as is usually the case), the final linear momenta after the collision will be in different directions than the initial linear momenta and the collision is essentially two dimensional. Since the velocities are not all along a single straight line, the *vector* nature of linear momentum must be recognised. This is most easily accomplished by finding the components of each linear momentum in the *X*- and *Y*-directions. Then linear momentum conservation requires that :

Algebraic sum of *X*-components before collision = Algebraic sum of *X*-components after collision

Algebraic sum of *Y*-components before collision = Algebraic sum of *Y*-components after collision

The energy conservation requires that :

Total K.E. before collision = Total K.E. after collision

Illustration. Consider two perfectly elastic balls of masses m_1 and m_2 moving along the same straight line (say X-axis) with velocities u_1 and u_2 ($u_1 > u_2$). Suppose after the elastic collision, m_1 and m_2 move off in different directions with velocities v_1 and v_2 making angles θ_1 and θ_2 with the incident direction (*i.e*, X-axis) as shown in Fig. 12.37.

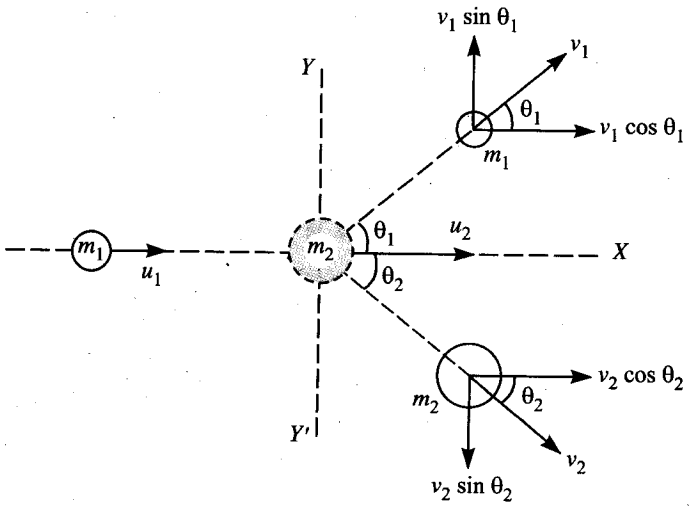


Fig. 12.37

Conservation of linear momentum requires that :

Algebraic sum of X-components before collision = Algebraic sum of X-components after collision

or $$m_1u_1 + m_2u_2 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2 \quad ...(i)$$

Algebraic sum of Y-components before collision = Algebraic sum of Y-components after collision

or $$0 + 0^* = m_1v_1\sin\theta_1 - m_2v_2\sin\theta_2 \quad ...(ii)$$

Since the K.E. is conserved in the collision,

$$\therefore \quad \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad ...(iii)$$

In order to study the final outcome of a two dimensional collision, we require four parameters viz., v_1, v_2, θ_1 and θ_2. Since we have only three independent equations, we cannot determine the values of four parameters. However, if one of these parameters can be specified, either from observation or from some physical consideration, then other three parameters can be uniquely determined.

Example 12.39. Suppose a bomb of mass M is suspended at rest from a cable and detonated exploding into three pieces. As shown in Fig. 12.38, half of its mass ($M/2$) is observed to have a speed v_0 in the $+x$ direction right after the explosion and another piece of mass $M/3$ is seen to have a speed of $2v_0$ at angle of 37° above the horizontal. Determine the velocity of third piece whose mass is $M/6$.

Solution. Fig. 12.38 shows the conditions of the problem. Since the bomb is initially at rest, original linear momentum is zero in both x and y directions. This means that the algebraic sum of x-components as well as algebraic sum of y-components after explosion must be zero.

* There is no initial velocity along Y-axis.

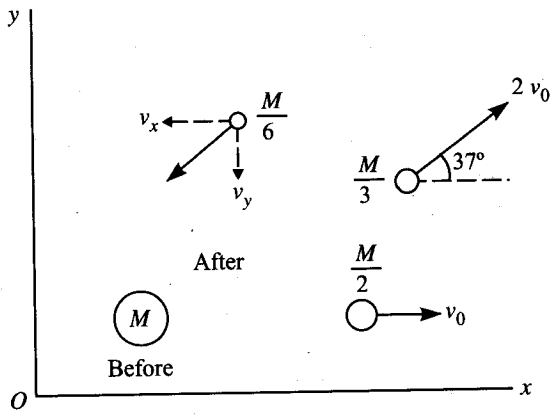


Fig. 12.38

$$\therefore \quad 0 = \frac{M}{2}v_0 + \frac{M}{3}2v_0\cos 37° + \frac{M}{6}v_x \qquad (x\text{-components})$$

or $$0 = \frac{v_0}{2} + \frac{2}{3}v_0\cos 37° + \frac{v_x}{6}$$

$$\therefore \quad v_x = -6.2v_0$$

Similarly, $$0 = \frac{M}{2}(0) + \frac{M}{3}2v_0\sin 37° + \frac{M}{6}v_y \qquad (y\text{-components})$$

or $$0 = 0 + \frac{2v_0\sin 37°}{3} + \frac{v_y}{6}$$

$$\therefore \quad v_y = -2.4\ v_0$$

Magnitude of velocity of $M/6$ is given by ;

$$v = \sqrt{v_x^2 + v_y^2} = v_0\sqrt{(6.2)^2 + (2.4)^2} = \mathbf{6.65}\ v_0$$

The direction of $M/6$ with x-axis is given by ;

$$\tan\theta = \frac{v_y}{v_x} = \frac{-2.4v_0}{-6.2v_0} = \frac{2.4}{6.2}$$

$$\therefore \quad \theta = \tan^{-1}\frac{2.4}{6.2} = \mathbf{21.16°}\ \textit{below x-axis}$$

Example 12.40. In Fig. 12.39, ball 1 has a mass m and is moving at 5 ms^{-1}. It collides with ball 2 of mass $2m$ which is at rest. After collision, ball 1 has a velocity of 2 ms^{-1} at an angle of 50° relative to its original direction as shown. What is the velocity of ball 2 after the collision?

Solution. Fig. 12.39 shows the conditions of the problem. The original linear momentum is zero in y direction and mu_1 in x-direction.

In the x-direction, $mu_1 = mv_1\cos 50° + (2m)v_{2x}$

or $$u_1 = v_1\cos 50° + 2v_{2x}$$

$$\therefore \quad v_{2x} = \frac{u_1 - v_1\cos 50°}{2} = \frac{5 - 2\cos 50°}{2} = +\ 1.86\ ms^{-1}$$

In the y-direction, $0 = mv_1 \sin 50° + (2m)\ v_{2y}$

or $$0 = v_1 \sin 50° + 2\ v_{2y}$$

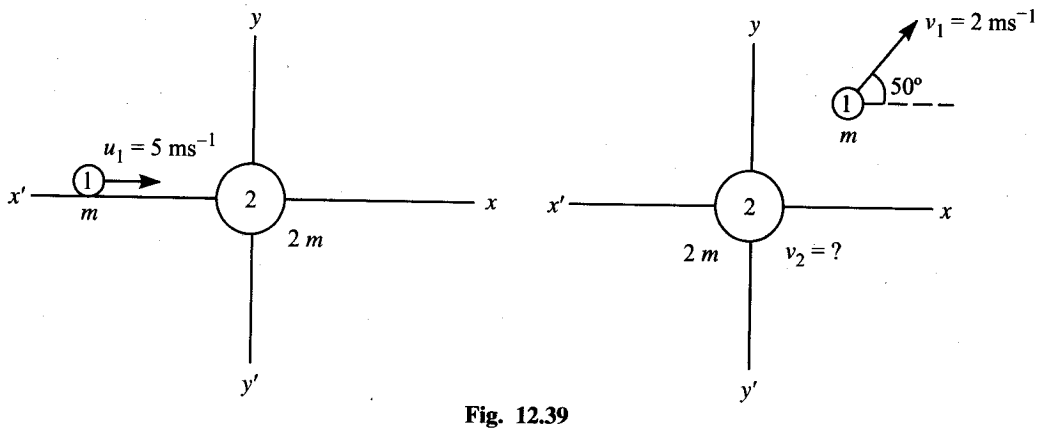


Fig. 12.39

$$\therefore \quad v_{2y} = -\frac{v_1 \sin 50°}{2} = -\frac{2 \sin 50°}{2} = -0.766 \text{ ms}^{-1}$$

Magnitude of velocity of ball 2, $v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{(1.86)^2 + (-0.766)^2} = 2.01$ ms^{-1}

Direction of $\vec{v}_2$ relative to x-axis, $\theta = \tan^{-1}\left(\frac{v_{2y}}{v_{2x}}\right) = \tan^{-1}\left(\frac{-0.766}{1.86}\right) = -22.4°$

PROBLEMS FOR PRACTICE

1. A ball of mass m moving with a speed of 2 m/s collides with a stationary ball of the same mass in an elastic collision. The incident ball is scattered at an angle of 20° from its original direction. Find the final speed of each ball and the direction of the struck ball with the original direction of the striking ball. **[1.88 m/s ; 0.682 m/s ; 70°]**
2. A billiard ball of radius 1.5 cm moving with 1 ms^{-1} hits an identical ball at rest. If the impact parameter of the collision is 1.5 cm, find the speed of each ball after the collision. Assume the collision to be elastic. **[0.5 ms^{-1}; 0.866 ms^{-1}]**

12.28. INELASTIC COLLISION IN TWO DIMENSIONS

Consider two balls moving along the same straight line in the same direction. Suppose after collision they move in different directions and there is a loss of kinetic energy in the collision. Therefore, it is a case of inelastic collision in two dimensions.

Consider two balls of masses m_1 and m_2 moving along the same straight line (say X-axis) with velocities u_1 and u_2 ($u_1 > u_2$). Suppose after the inelastic collision, m_1 and m_2 move off in different directions with velocities v_1 and v_2 making angles θ_1 and θ_2 with the incident direction (*i.e.* X-axis) as shown in Fig. 12.40. Since linear momentum is conserved,

$\therefore$ Algebraic sum of X-components before collision = Algebraic sum of X-components after collision

or $$m_1 u_1 + m_2 u_2 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \quad ...(i)$$

Algebraic sum of Y-components before collision = Algebraic sum of Y-components after collision

or $$0 + 0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2$$

or $$0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2 \quad ...(ii)$$

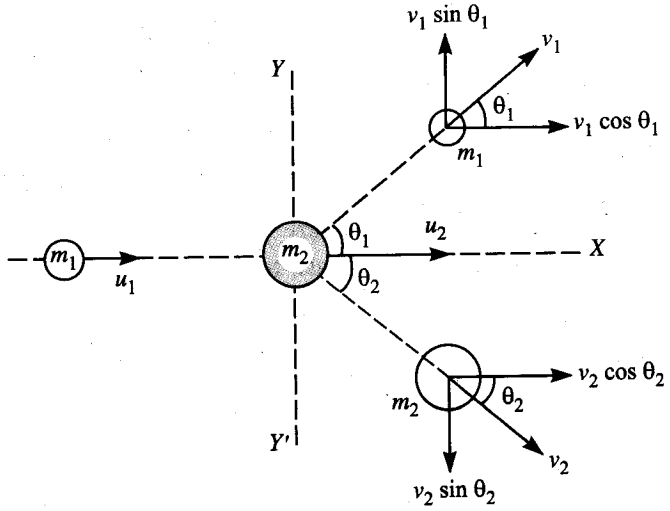


Fig. 12.40

If we know the masses (m_1, m_2), initial velocities (u_1, u_2) and θ_1 and θ_2, we can find v_1 and v_2 from eqs. (*i*) and (*ii*).

12.29. PERFECTLY INELASTIC COLLISION IN TWO DIMENSIONS

It is a two dimensional collision in which the two bodies stick together after the collision. Fig. 12.41 shows perfectly inelastic two dimensional collision. Here a body of mass m_1 is moving along X-axis with velocity u_1. The body of mass m_2 is moving with velocity u_2 at an angle θ_1 with X-axis as shown. When the collision occurs at O, the two bodies stick together and move off with a common velocity v at an angle θ_2 with X-axis. The total linear momentum along X-axis before collision is equal to total linear momentum along X-axis after collision *i.e.*

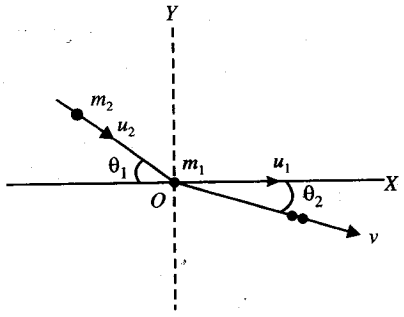


Fig. 12.41

$$m_2 u_2 \cos\theta_1 + m_1 u_1 = (m_1 + m_2) v \cos\theta_2 \quad \ldots (i)$$

Similarly, the total linear momentum along Y-axis before collision is equal to the total linear momentum after collision *i.e.*,

$$m_2 u_2 \sin\theta_1 + 0 = (m_1 + m_2) v \sin\theta_2 \quad \ldots (ii)$$

If we know the masses (m_1, m_2), initial velocities (u_1, u_2) and θ_1, we can find the values of v and θ_2 from eqs. (*i*) and (*ii*).

CONCEPTUAL QUESTIONS

Q.1. A labourer carrying bricks on his head on a level road from one place to another is doing no work on the bricks. Why?

Ans. Work done, $W = \vec{F} . \vec{S} = FS\cos\theta$

The weight (*i.e.*, force) of the bricks is acting vertically downward and the displacement is horizontal *i.e.*, angle θ between $\vec{F}$ and $\vec{S}$ is 90°.

$\therefore$ $W = FS \cos 90° = 0$

Therefore, the labourer is doing no work on the bricks from physics point of view.

Q.2. Is there any work done by the car if it moves with a uniform speed on a smooth level road? Neglect air resistance.

Ans. Work done, $W = FS \cos\theta$.

The weight of the car and displacement are at right angles to each other and there is no friction. Therefore, $W = FS \cos 90° = 0$. Hence no work is done by the car.

Q.3. Is it possible that a body be in accelerated motion under a force acting on the body, yet no work is being done by the force?

Ans. It is possible when the force is perpendicular to motion *i.e.*, $\theta = 90°$. For example, moon revolves around the earth under the force of attraction of the earth. The angle between this force and the direction of motion is 90° everywhere in the moon's orbit. Therefore, earth does no work on the moon. ($W = FS \cos\theta = FS \cos 90° = 0$).

Q.4. If energy is neither created nor destroyed, what happens to the so much energy spent against friction?

Ans. The energy is dissipated in the form of heat. The heat energy so produced is not available for work.

Q.5. When the brakes of a car going 90 km/h are locked, how much farther will it skid than if the brakes are locked at 30 km/h?

Ans. Nine times farther. The car has nine times as much energy when it travels three times as fast :

$$\frac{1}{2}m(3v)^2 = \frac{1}{2}m\,9\,v^2 = 9\left(\frac{1}{2}mv^2\right)$$

The frictional force will ordinary be the same in either case. Therefore, to do nine times the work requires nine times as much sliding distance.

Q.6. A lorry and a car moving with the same K.E. are brought to rest by the application of brakes which provide equal retarding forces. Which of them will come to rest in a shorter distance?

Ans. If K is the K.E. of each vehicle and F is the retarding force, then according to work-energy theorem,

Work done by vehicle = Loss in K.E. or $F \times S = K$ $(\because\ \theta = 0°)$

Since K and retarding force F in the two cases are the same, distance S travelled by them before coming to rest would also be the same.

Q.7. The earth moving around the sun in a circular orbit is acted upon by a force and hence work must be done on the earth. Do you agree with this statement?

Ans. The statement is wrong. The earth revolves around the sun under the force of attraction of the sun. This force (centripetal) is always perpendicular to the motion of the earth. Therefore, $\theta = 90°$ and $W = FS \cos 90° = 0$. Hence, sun does no work on the earth.

Q.8. Two protons are brought toward each other. Will the potential energy of the system decrease or increase?

Ans. The potential energy of the system will increase. It is because in bringing the protons nearer, work will have to be done against the force of repulsion. This work done is stored in proton-proton system in the form of electric potential energy.

Q.9. Can an object have work?

Ans. No, unlike momentum or energy, work is not something an object has. Work is something that an object does to some other object. An object can do work only if it has energy.

Q.10. What are the differences between kinetic energy and momentum of an object?

Ans. Kinetic energy and momentum are both concepts concerning the motion of an object. But they are different.

(*i*) Momentum is a vector quantity whereas kinetic energy is a scalar quantity.

(*ii*) The momentum of one object in collision with another object may add or subtract. But kinetic energy is a scalar quantity like mass and can never be cancelled.

(*iii*) Whereas momentum is proportional to velocity (mv), kinetic energy is proportional to square of velocity ($mv^2/2$). An object that moves with twice the velocity of another object of the same mass has twice the momentum but four times the kinetic energy. It can provide twice the impulse to whatever it encounters but do four times as much work.

(*iv*) Although momenta can be combined in such a way to cancel to zero, there is no way to combine K.E.s. to equal zero.

Q.11. Give two examples of (*i*) positive work (*ii*) negative work and (*iii*) zero work.

Ans. (*i*) **Positive work.** The two examples of positive work are :

(*a*) When a spring is stretched, the work done ($\because \theta = 0°$), by the stretching force is positive.

(*b*) Consider a gas contained in a cylinder fitted with a piston. When the gas expands, the work done by the gas is positive ($\because \theta = 0°$).

(*ii*) **Negative work.** The two examples of negative work done are :

(*a*) When a body is moved over a rough horizontal surface, the work done by the frictional force is negative ($\because \theta = 180°$). However, work done by the applied force is positive.

(*b*) When a positive charge is moved towards another positive charge, the work done by the electrostatic force between them (not the applied force) is negative.

(*iii*) **Zero work.** The two examples of zero work done are :

(*a*) When a body is moved along a circular path by a string, the work done by the tension in the string is zero ($\because \theta = 90°$).

(*b*) When the force applied to a body does not move the body, work done by the force is zero ($\because \vec{S} = 0$).

Q.12. Is total energy conserved in an inelastic collision? Is kinetic energy conserved?

Ans. Yes, total energy is conserved in an inelastic (or elastic) collision. The principle of conservation of energy is a universal law and holds for all phenomena. In an inelastic collision, kinetic energy is not conserved.

Q.13. Give three examples of conservative forces.

Ans. If the work done by a force depends *only* on the end points of the object's position and not on the path taken, the force is said to be conservative. Examples are gravitational force, elastic forces and electrostatic forces.

Q.14. A car is moving with a constant speed in a straight line. Is there any net work done by external forces on the car?

Ans. No, because net work done is equal to the change in kinetic energy. In this case, there is no change in kinetic energy.

Q.15. Is there any law of conservation of any one form of energy?

Ans. There is no law of conservation of any one form of energy. The law of conservation of energy applies to the sum of all forms of energy that may be present in a given case.

Q.16. What are head-on and oblique collisions?

Ans. If the colliding objects move along the straight line joining their centres, the collision is said to be head-on collision. If the colliding objects do not move along the straight line joining their centres, the collision is said to be oblique collision.

VERY SHORT ANSWER QUESTIONS

Q.1. A person holding a 5 kg suitcase is waiting for the bus for 15 minutes. Is he doing any work?

Ans. Work done is zero because displacement is zero.

Q.2. What should be the angle between force and displacement for maximum and minimum work?

Ans. $W = FS \cos\theta$. Therefore, for maximum work, $\theta = 0°$ and for minimum work $\theta = 90°$.

Q.3. A man carrying a bucket of water walks on a horizontal road with uniform velocity. What is the work done on the bucket.

Ans. $W = FS \cos\theta$. The angle θ between $\vec{F}$ and $\vec{S}$ is $90°$ so that work done on the bucket is zero.

Q.4. A man is pushing a rigid wall with a force of 100 N. What is work done by him?

Ans. Work done is zero because $W = FS \cos\theta$ and displacement $S = 0$.

Q.5. What is the work done by earth's gravitational force in keeping the moon in its orbit (assumed circular) ?

Ans. Zero because $W = FS \cos\theta$ and gravitational force (F) is always perpendicular to displacement (S) *i.e.*, $\theta = 90°$.

Q.6. Is work a scalar quantity?

Ans. Yes. The dot product of two vectors ($W = \vec{F} . \vec{S}$) is a scalar quantity.

Q.7. Is the work done by a frictional force negative?

Ans. Yes. Work done by frictional force = $FS \cos\theta = FS \cos 180° = -FS$. Note that angle θ between the frictional force (F) and displacement (S) is $180°$.

Q.8. What is a conservative force? Give one example.

Ans. If the work done by a force depends only on the end points of the objects position and not on the path taken, the force is said to conservative *e.g.*, gravitational force.

Q.9. Does the work done in moving a body depend upon how fast the body is moved?

Ans. No. Work done $W = FS \cos\theta$ and there is no time factor.

Q.10. How much work is done on a body if the increase in kinetic energy is 5 J?

Ans. Work done = Increase in K.E. = 5 J.

Q.11. How do you explain work and energy?

Ans. If a body can do work, we say that the body possesses energy. For example, if a body can do 100 J of work, then energy of the body is 100 J. Infact, energy of a body means stored work.

Q.12. Can the kinetic energy of a body be negative?

Ans. No. It is because K.E. = $mv^2/2$ and m and v^2 can never have negative values. Therefore, K.E. is always positive.

Q.13. Can potential energy of a body be negative?

Ans. Yes when forces involved are attractive. Thus for proton-electron system, the potential energy is negative. But for electron-electron system P.E. is positive.

Q.14. Why is spark produced when two stones are struck against each other?

Ans. The work done in striking the two stones is converted into heat energy which appears as a spark.

Q.15. An object of mass 2 kg is moving with a speed of 10 ms^{-1} on a horizontal road. What is its (*i*) kinetic energy (*ii*) potential energy?

Ans. (*i*) K.E. = $(1/2)\, mv^2 = (1/2) \times 2 \times 10^2 = 100$ J (*ii*) P.E. = $mgh = mg \times 0 = 0$

Q.16. What is the kinetic energy of a body of mass 2 kg and momentum 2 Ns?

Ans. K.E. = $\frac{p^2}{2m} = \frac{(2)^2}{2\times 2} = 1$ J.

Q.17. Can you give an example where force does work on a body without changing its kinetic energy?

Ans. Yes when a body is pulled with a constant velocity on a rough horizontal surface. In this case, work is done against friction but there is no change in K.E.

Q.18. When a spring is compressed, will its potential energy increase or decrease?

Ans. The P.E. will increase but will be negative. Also when a spring is stretched, the P.E. increases but will be positive.

Q.19. A person is lowering a bag containing sugar. Is work done positive or negative?

Ans. Negative. It is because angle θ between the force ($\vec{F}$) and displacement ($\vec{S}$) is 180° ($\because W = FS \cos\theta$).

Q.20. Does kinetic energy depend on direction of motion on the body?

Ans. No. Kinetic energy is a scalar quantity.

Q.21. How much work is done by a centripetal force?

Ans. Zero. It is because centripetal force is always perpendicular to displacement ($W = FS \cos 90° = 0$).

Q.22. In which collision linear momentum/K.E. is conserved?

Ans. Momentum is conserved in all types of collisions but K.E. is conserved only in elastic collisions.

Q.23. Under which condition linear momentum of a system is conserved?

Ans. Linear linear momentum is conserved only in an isolated system.

Q.24. A moving body collides with a stationary body. Is it possible for both to be at rest after collision?

Ans. No, because linear momentum will not be conserved in this case.

Q.25. Is collision possible even without actual contact of colliding particles?

Ans. Yes. For example, interaction between two positively charged particles is a collision. Here no physical contact occurs between the particles.

Q.26. What happens when a very light body in motion collides head-on with a heavy stationary body in an elastic collision?

Ans. The light body is reflected back with the same speed while the heavy body remains stationary.

Q.27. A body moving with a velocity v has elastic head-on collision with another stationary body of the same mass. What will be the velocities of these two masses after collision?

Ans. The first body will come to rest and the second body will move with velocity v. In other words, the bodies interchange their velocities.

Q.28. A spring is cut into two equal halves. What is the spring constant of each portion.

Ans. The spring constant of each half becomes twice the spring constant of the original spring.

Q.29. When is work done by a force zero?

Ans. $W = FS \cos\theta$. Work done is zero when (*i*) $F = 0$ (*ii*) $S = 0$ (*iii*) $\theta = 90°$.

Q.30. What is head-on collision?

Ans. If the colliding objects move along a straight line joining their centres, the collision is said to be head-on collision.

SHORT ANSWER QUESTIONS

Q.1. When is the work done by a force positive, negative and zero?

Ans. Work done, $W = FS \cos\theta$. Here θ is the angle between force (F) and displacement (S). (*i*) If $\theta < 90°$, work done is positive (*ii*) If $\theta > 90°$, work done is negative (*iii*) If $\theta = 90°$, work done is zero.

Q.2. A force $\vec{F} = 5\hat{i} + 6\hat{j} - 4\hat{k}$ acting on a body produces a displacement $\vec{S} = 6\hat{i} + 5\hat{k}$. What is the work done by the force?

Ans. Work done, $W = \vec{F} \cdot \vec{S} = (5\hat{i} + 6\hat{j} - 4\hat{k}) \cdot (6\hat{i} + 0\hat{j} + 5\hat{k})$ = 30 + 0 − 20 = 10 units.

Q.3. A light body and a heavy body have the same momentum (p). Which one has larger kinetic energy?

Ans. $p = mv$; $K = mv^2/2$ $\therefore$ $K = p^2/2m$

Therefore, the light body will have larger kinetic energy.

Q.4. A light body and a heavy body have the same kinetic energy. Which one will have greater momentum (p)?

Ans. $K = p^2/2m$ $\therefore$ $p = \sqrt{2mK}$. Therefore, heavy body will have greater momentum.

Q.5. How will the kinetic energy of a body change if its momentum is doubled?

Ans. K.E. = $p^2/2$m. Therefore, if the momentum of the body is doubled, its kinetic energy will become four times.

Q.6. Give one example when work done by a force is (*i*) positive (*ii*) negative (*iii*) zero.

Ans. Work done by a force is given by: $W = FS \cos\theta$.

(*i*) When a spring is stretched, the work done by the stretching force is positive ($\because$ $\theta = 0°$)

(*ii*) When a body is moved over a rough horizontal surface, the work done by the frictional force is negative ($\because$ $\theta = 180°$). However, work done by the applied force is positive.

(*iii*) When a body is moved with a constant speed in a circular path by a string, the work done by the tension (force) in the string is zero ($\because$ $\theta = 90°$).

Q.7. Distinguish between (*i*) conservative force and (*ii*) non-conservative force with one example each.

Ans. (*i*) If the work done by a force acting on a body depends *only* on the initial and final position of the body and *not* on the path taken, the force is said to be conservative force. Gravitational force is an example. It is because the amount of work done in moving a body against gravity from location A to another higher location B is the same whichever path you may follow in going from A to B.

(*ii*) If the work done by a force acting on a body depends upon the path taken by the body, the force is said to be non-conservative force. Frictional force is an example. The work done against a frictional force depends upon the path taken.

Q.8. Explain the following cases with reasons: (*i*) What will be the change in momentum and K.E. when a bomb explodes in air? (*ii*) What will be the change in P.E. when two protons are brought together? (*iii*) What will be the change in P.E. if a proton and an electron are brought together?

Ans. (*i*) The linear momentum will be conserved because explosion occurs through internal forces. But K.E. will increase due to the energy of explosives.

(*ii*) The protons repel each other and work will have to be done to bring them closer. Therefore, P.E. increases.

(*iii*) Electron and proton attract each other. In this case, electric field does the work to bring them closer. Therefore, P.E. decreases.

Q.9. Can linear momentum of a system be changed without changing its kinetic energy?

Ans. Yes. A familiar example is that of uniform circular motion. Here K.E. remains constant ($\because$ speed is constant) but linear momentum changes due to the change in direction of motion.

Q.10. A compressed spring is dipped in an acid and is allowed to dissolve. Where does the potential energy of the spring go?

Ans. The potential energy of the spring is transferred to the molecules of the acid. As a result, K.E. of molecules increases and temperature of the acid is raised.

Q.11. The momentum of a body is increased by 100%. What is the percentage increase in its kinetic energy?

Ans. K.E., $K = \dfrac{p^2}{2m}$ $\therefore$ $\dfrac{K_2}{K_1} = \left(\dfrac{2p_1}{p_1}\right)^2 = 4$ or $K_2 = 4K_1$

$\therefore$ % age increase in K.E. = $\frac{K_2 - K_1}{K_1} \times 100 = \frac{4K_1 - K_1}{K_1} \times 100 = 300\%$

Q.12. Sand is falling vertically at the rate of 2 kgs^{-1} on a horizontal conveyor belt moving with a velocity of 0.2 ms^{-1}. What is the power required to keep the belt moving?

Ans. Force required, $F = v\frac{dm}{dt}$. Here $v = 0.2$ ms^{-1}; $\frac{dm}{dt} = 2$ kg s^{-1}.

$\therefore$ Power required, $P = Fv = v^2\frac{dm}{dt} = (0.2)^2 \times 2 = 0.08$ W

Q.13. Two masses, one n times heavier than the other, are dropped from the same height.
(*i*) What is the ratio of their linear momentum when they hit the ground?
(*ii*) If the K.E. of the two masses is the same, what is the ratio of their linear momentum when they hit the ground?

Ans. On hitting the ground, each mass has a velocity = $\sqrt{2gh}$

(*i*) $\frac{p_2}{p_1} = \frac{nm\sqrt{2gh}}{m\sqrt{2gh}} = \frac{nm}{m} = n$

(*ii*) $K = p^2/2m \quad \therefore p = \sqrt{2mK} \quad \therefore \frac{p_2}{p_1} = \sqrt{\frac{nm}{m}} = \sqrt{n}$

Q.14. What is the running speed of a pole vaulter who clears a bar of 5 m?

Ans. The pole vaulter's potential energy becomes mgh when he rises to a height h. This energy is obtained from conversion of his kinetic energy which he gains by running.

$\therefore \quad \frac{1}{2}mv^2 = mgh$ or $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 5} = 7\sqrt{2}$ ms^{-1}

Q.15. The power of a water pump is 2 kW. If $g = 10$ ms^{-2}, what is the amount of water it can raise in one minute to a height of 10 m?

Ans. Here $P = 2$ kW $= 2 \times 10^3$ W ; $t = 1$ min $= 60$ s ; $h = 10$ m ; $g = 10$ ms^{-2}

Now, $P = \frac{mgh}{t} \quad \therefore m = \frac{P \times t}{g \times h} = \frac{2 \times 10^3 \times 60}{10 \times 10} = 1200$ kg $= 1200$ litres.

LONG ANSWER QUESTIONS

1. Explain the terms (*i*) work (*ii*) power and (*iii*) energy. **[Refer to Arts. 12.1, 12.7 and 12.8]**
2. Obtain an expression for the work done by a constant force. **[Refer to Art. 12.2]**
3. Obtain mathematically and graphically the work done by a variable force. **[Refer to Art. 12.6]**
4. Explain conservative and non conservative forces with examples. **[Refer to Art. 12.14]**
5. Prove that kinetic energy = $mv^2/2$ and potential energy = mgh where the letters have their usual meanings. **[Refer to Arts. 12.10 and 12.13]**
6. State and explain work-kinetic energy principle. **[Refer to Art. 12.12]**
7. State the principle of conservation of energy. Prove it for the freely falling bodies. **[Refer to Art. 12.20]**
8. Show that power is equal to dot product of force and velocity. **[Refer to Art. 12.7]**
9. What is a collision? Discuss elastic and inelastic collisions of two bodies with examples. **[Refer to Arts. 12.22 and 12.23]**
10. Calculate the velocities of two bodies after elastic collision in one dimension. **[Refer to Art. 12.24]**
11. Prove that in one-dimensional elastic collision between two bodies of equal masses, their velocities are interchanged after collision. **[Refer to Art. 12.24]**
12. Discuss elastic collisions in two dimensions. **[Refer to Art. 12.27]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Work done by a force on an object is

   Work done, $W = \vec{F}.\vec{S} = FS\cos\theta$

   F = force ; S = displacement ; θ = angle between $\vec{F}$ and $\vec{S}$. Even though force and displacement are vectors, work is a scalar.
2. Work can be zero in three ways *viz.* (*i*) force = 0 (*ii*) displacement = 0 (*iii*) $\cos\theta = 0$ *i.e.* force is perpendicular to the direction of motion ($\theta = 90°$).
3. Work can be positive or negative depending upon angle between $\vec{F}$ and $\vec{S}$. If $\theta < 90°$, the work is positive. When $\theta = 90°$, work is zero and if $\theta > 90°$, the work is negative.
4. In case of friction, $\theta = 180°$. This means that work done by a frictional force is $-FS$ where F is the frictional force.
5. If many forces are acting on an object, you can calculate the work done by first finding the resultant force. The total work done is equal to the work done by the resultant force.
6. If the force F doing work is being applied to an object whose velocity is v, the power P delivered by the force is

   $$P = \vec{F}.\vec{v} = Fv\cos\theta$$ where θ is the angle between $\vec{F}$ and $\vec{v}$.
7. Kinetic energy (K.E.) is the energy an object has due to its motion.

   $$\text{K.E.} = \frac{1}{2}mv^2$$

   Since m and v^2 can never have negative values, K.E. is always positive.
8. When a machine gun of power P fires n bullets per second, each with K.E. = E, then $P = nE$.
9. Stopping distance of a vehicle = $\dfrac{\text{K.E.}}{\text{Stopping force}}$
10. Consider two masses m_1 and m_2 moving with velocities v_1 and v_2 respectively. If they are stopped by the same force, their stopping distances are :

    $$\frac{x_1}{x_2} = \frac{K.E._1}{K.E._2} = \frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2}$$

    The stopping times t_1 and t_2 are $\dfrac{t_1}{t_2} = \dfrac{p_1}{p_2} = \dfrac{m_1v_1}{m_2v_2}$
11. For the same stopping force, if the speed of a body increases by a factor n, then stopping distance is increased by a factor n^2.
12. The relation between kinetic energy K and momentum p of a moving object of mass m is

    $$K = p^2/2m$$

    (*i*) If two bodies of unequal mass have same momentum (p), the lighter body will have more kinetic energy.

    (*ii*) If the linear momentum of a body increases by a factor n, its kinetic energy is increased by a factor n^2.

13. The gravitational potential energy is given by; G.P.E. = mgh.
14. If the work done by a force acting on an object depends *only* on the initial and final positions of the object and not on the path taken, the force is said to be conservative. Examples are gravity, elastic forces and electrostatic forces.
15. If the work done by a force depends upon the path taken, the force is called non-conservative. The work done against a frictional force depends upon the path taken. Therefore, frictional force is non-conservative. Other examples are: the resistance of air to the motion of a body and the resistance of a liquid (*e.g.* water) to the motion through it.
16. The work done by/against a conservative force in displacing an object around a closed path is zero. However, the work done by/against a non-conservative force around a closed path is *not* zero.
17. Linear momentum is *always* conserved in all collisions (elastic or inelastic collisions) within isolated systems.
18. In an inelastic collision, some kinetic energy (K.E.) of the system is lost. Usually, the K.E. lost is converted mostly to thermal energy in the system.
19. In a perfectly elastic collision, the K.E. of the system is conserved.
20. **Elastic head–on collisions in one dimension**
    (*i*) If two moving bodies *A* and *B* of equal masses undergo elastic head-on collision, then after collision, the bodies exchange their velocities. If body *B* is at rest initially, the body *A* is stopped dead in its track and body *B* takes off with velocity equal to what body *A* had before collision.
    (*ii*) If a heavy moving body strikes head-on a light body at *rest*, then the velocity of heavy incoming body is practically unchanged whereas the light body takes off with twice the velocity of the heavy body.
    (*iii*) If a light moving body strikes head-on a heavy body at *rest*, the heavy body essentially remains at rest and the light incoming body rebounds with essentially its same speed but in the opposite direction.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A metal ball of mass 2 kg moving with a speed of 36 km/hr has a head-on collision with a stationary ball of mass 3 kg. If after collision, both the balls move together, then loss in kinetic energy due to collision is **[AIIMS 2001]**
   (*a*) 40 J (*b*) 60 J
   (*c*) 100 J (*d*) 140 J
2. A shell in its flight explodes into four unequal parts. Which of the following is conserved? **[CBSE PMT 1997]**
   (*a*) momentum (*b*) kinetic energy
   (*c*) potential energy (*d*) (*b*) and (*c*)
3. The work done in pulling up a block of wood weighing 2 kN for a length of 10 m on a smooth inclined plane inclined at an angle of 15° with the horizontal is **[AFMC-Pune 1999]**
   (*a*) 5.17 kJ (*b*) 4.36 kJ
   (*c*) 8.91 kJ (*d*) 9.82 kJ
4. A body of mass m and velocity 3 km/h collides with a rest body of mass 2 m and sticks to it. Now the combined mass starts to move. What will be the combined velocity? **[CBSE PMT 1996]**
   (*a*) 2 km/h (*b*) 3 km/h
   (*c*) 1 km/h (*d*) 4 km/h
5. A child is sitting on a swing. Its minimum and maximum heights from the ground are 0.75 m and 2m respectively. Its maximum speed will be **[CBSE PMT 2001]**
   (*a*) 10 m/s (*b*) 5 m/s
   (*c*) 8 m/s (*d*) 15 m/s
6. A body of mass 5 kg moving with a velocity of 10 m/s collides with another body of mass

20 kg at rest and comes to rest. The velocity of the second ball due to collision is **[AFMC-Pune 1997]**

(a) 5 m/s (b) 7.5 m/s
(c) 10 m/s (d) 2.5 m/s

7. $(60\hat{i}+15\hat{j}-3\hat{k})$ N force produces velocity $(2\hat{i}-4\hat{j}+5\hat{k})$ m/s in a particle. The value of power at that time will be **[CBSE PMT 2000]**

(a) 95 W (b) 75W
(c) 45 W (d) 100 W

8. If the momentum of a body is increased by 100%, then percentage increase in kinetic energy is **[Pb PMT 1999]**

(a) 150% (b) 200%
(c) 225% (d) 300%

9. A spring 40 mm long is stretched by the application of force. If 10 N force is required to stretch the spring through 1 mm, then work done in stretching the spring through 40 mm is **[AIIMS 1998]**

(a) 8 J (b) 84 J
(c) 48 J (d) 24 J

10. The power of a water pump is 2 kW. If $g = 10$ m/s^2, the amount of water it can raise in one minute to a height of 10 m is **[CPMT 1991]**

(a) 2000 litres (b) 1000 litres
(c) 100 litres (d) 1200 litres

11. A force of 10 N acts on a body of mass 2 kg for 1 m distance. The kinetic energy obtained by the body is **[CBSE PMT 1996]**

(a) 20 J (b) 10 J
(c) 5 J (d) 2.5 J

12. Two bodies of equal masses moving along same straight line with velocities + 3 m/s and – 5 m/s respectively collide elastically. Their velocities after collision will be **[Pb PMT 1997]**

(a) 3 m/s and 4 m/s
(b) –3 m/s and 5 m/s
(c) –4 m/s and 4 m/s
(d) –5 m/s and 3 m/s

13. If a spring extends by x on loading (if T is the tension in the spring and k is spring constant), the energy stored by the spring is **[AIIMS 1997]**

(a) $\frac{T^2}{2x}$ (b) $\frac{T^2}{2k}$
(c) $\frac{2k}{T^2}$ (d) $\frac{2T^2}{k}$

14. Two bodies having kinetic energies in the ratio of 4 : 1 are moving with equal linear momentum. The ratio of their masses is **[CBSE PMT 1999]**

(a) 1 : 2 (b) 1 : 1
(c) 4 : 1 (d) 1 : 4

15. A mass 1 kg is suspended by a thread. It is (*i*) lifted with an acceleration of 4.9 m/s^2 (*ii*) lowered with an acceleration of 4.9 m/s^2. The ratio of tensions is **[CBSE PMT 1998]**

(a) 3 : 1 (b) 1 : 3
(c) 1 : 2 (d) 2 : 1

16. A position dependent force $F = (7 - 2x - 3x^2)$ N acts on a small body of mass 2 kg and displaces it from $x = 0$ to $x = 5$ m. The work done is **[CBSE PMT 1994]**

(a) 35 J (b) – 70 J
(c) – 115 J (d) 270 J

17. A body of mass m is suspended from a string of length l. What is the minimum horizontal velocity that should be given to the body in its lowest position so that it may take one full revolution in vertical plane with the suspension point as centre of the circle? **[AFMC-Pune 1996]**

(a) $v=\sqrt{2gl}$ (b) $v=\sqrt{4gl}$
(c) $v=\sqrt{6gl}$ (d) $v=\sqrt{8gl}$

18. A block of mass 50 kg just slides over a distance of 1m. If the coefficient of friction between their surfaces is 0.2, the work done against friction is **[AIIMS 1996]**

(a) 98 J (b) 72 J
(c) 56 J (d) 34 J

19. The average force necessary to stop a bullet of mass 20 g and speed 250 m/s as it penetrates wood to a distance of 12 cm is **[Pb PMT 1995]**

(a) 2.2×10^3 N (b) 3.2×10^3 N
(c) 4.2×10^3 N (d) 5.2×10^3 N

20. A body, constrained to move in y direction, is subjected to a force given by;

$\vec{F}=(-2\hat{i}+15\hat{j}+6\hat{k})N$

The work done by this force in moving the body through a distance of $10\hat{j}$ m along y axis is **[CBSE PMT 1994]**

(*a*) 190 J (*b*) 160 J

(*c*) 150 J (*d*) 120 J

21. The K.E. acquired by a mass m in travelling a certain distance d, starting from rest, under the action of a constant force is directly proportional to **[CBSE PMT 1994]**

(*a*) m (*b*) $\sqrt{m}$

(*c*) $\frac{1}{\sqrt{m}}$ (*d*) independent of m

22. A ball is dropped from a height of 10 m. It is embedded in sand 1m and stops. **[AFMC-Pune 1996]**

(*a*) only momentum remains conserved

(*b*) only K.E. remains conserved

(*c*) both momentum and K.E. are conserved

(*d*) neither momentum nor K.E. is conserved

23. The kinetic energy of a body of mass 2 kg and momentum 2 N-s is **[AFMC-Pune 1998]**

(*a*) 1 J (*b*) 2 J

(*c*) 3 J (*d*) 4 J

24. A rod elongates by l when a body of mass M is suspended from it. The work done is **[CPMT 1999]**

(*a*) Mgl (*b*) $\frac{1}{2}Mgl$

(*c*) $2\,Mgl$ (*d*) zero

25. A 3 kg ball strikes a heavy rigid wall with a speed of 10 m/s at an angle of 60° with the wall. It gets reflected with the same speed at 60° as shown in Fig. 12.42. If the ball is in contact with the wall for 0.2 s, the average force on the ball by the wall is **[CBSE PMT 2000]**

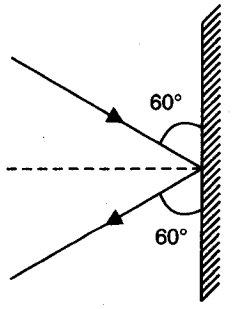


Fig. 12.42

(*a*) 300 N (*b*) 0

(*c*) $150\sqrt{3}$ N (*d*) 150 N

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS				
1. (*b*)	**2.** (*a*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)
6. (*d*)	**7.** (*c*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*b*)	**12.** (*d*)	**13.** (*b*)	**14.** (*d*)	**15.** (*a*)
16. (*c*)	**17.** (*b*)	**18.** (*a*)	**19.** (*d*)	**20.** (*c*)
21. (*d*)	**22.** (*a*)	**23.** (*a*)	**24.** (*b*)	**25.** (*d*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. $m_1 = 2\text{kg}$; $v_1 = 36$ km/hr $= 10$ m/s ; $m_2 = 3$ kg ; $v_2 = 0$

The common velocity v of the balls after collision is

$$v = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = \frac{2\times10 + 3\times0}{2+3} = 4 \text{ m/s}$$

K.E. before collision $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}\times2\times10^2 + \frac{1}{2}\times3\times(0)^2 = 100$ J

K.E. after collision $= \frac{1}{2}(m_1 + m_2)\,v^2 = \frac{1}{2}(2+3)\times(4)^2 = 40$ J

$\therefore$ Loss of K.E. after collision $= 100 - 40 = 60$ J

2. The system is essentially isolated one because the force of explosion is very large as compared to other forces. Therefore, momentum is conserved.

3. Weight of block, mg = 2 kN ; Displacement, S = 10 m
Applied force, $F = mg \sin\theta = 2 \times \sin 15° = 0.5176$ kN
$\therefore$ Work done, $W = F \times S \times \cos 0° = 0.5176 \times 10 =$ **5.17 kJ**

4. If v is the combined velocity, then according to law of conservation of momentum,
$(m_1 + m_2)v = m_1v_1 + m_2v_2$ or $(m+2m)v = m\times 3 + 2m\times 0$ $\therefore$ $v =$ **1 km/h**

5. Maximum height from ground is H = 2 m; Minimum height from ground is h = 0.75 m
Let v be the maximum speed. Note that maximum speed will occur at minimum height.
At the maximum height, the energy will be $E_1 = mgH$
At the minimum height, the energy will be $E_2 = \frac{1}{2}mv^2 + mgh$
According to law of conservation of energy, $E_2 = E_1$.

or $\frac{1}{2}mv^2 + mgh = mgH$ or $\frac{1}{2}v^2 + gh = gH$

$\therefore$ $v = \sqrt{2(H-h)g} = \sqrt{2(2-0.75)\times 10} =$ **5 m/s**

6. Total linear momentum of bodies before collision
$= 5 \times 10 + 20 \times 0 = 50$ kg ms^{-1}
Let v_2 be the velocity of the second body after collision.
Total linear momentum of bodies after collision
$= 5\times 0 + 20\times v_2 = 20\ v_2$ kg ms^{-1}
According to law of conservation of linear momentum,
$20\ v_2 = 50$ $\therefore$ $v_2 = 50/20 =$ **2.5 ms^{-1}**

7. Power, $P = \vec{F}.\vec{v} = (60\hat{i} + 15\hat{j} - 3\hat{k}).(2\hat{i} - 4\hat{j} + 5\hat{k})$
$= 120 - 60 - 15 =$ **45 W**

8. Kinetic energy, $K = \frac{p^2}{2m}$ $\therefore$ $K \propto p^2$ $\therefore$ $\frac{K_2}{K_1} = \left(\frac{p_2}{p_1}\right)^2 = \left(\frac{200}{100}\right)^2 = 4$

$\therefore$ $\frac{K_2 - K_1}{K_1}\times 100 = (4-1)\times 100 =$ **300%**

9. Work done in stretching the spring is given by ;

$W = \frac{1}{2}kx^2$ Here, k = 10N/mm ; x = 40 mm

$\therefore$ $W = \frac{1}{2}\times 10\times (40)^2 = 8000$ N-mm = 8 N-m = **8 J**

10. Power, $P = \frac{mgh}{t}$ $\therefore$ $m = \frac{P\times t}{g\times h}$

Here P = 2 kW = 2 × 10^3 W ; t = 1 min = 60 sec ; g = 10 m/s^2 ; h = 10 m

$\therefore$ $m = \frac{2\times 10^3 \times 60}{10\times 10} = 1200$ kg = **1200 litres**

11. According to work-energy theorem, Work done = Increase in K.E.
$\therefore$ Increase in K.E. = $F \times S = 10 \times 1 =$ **10 J**

12. This is the case of elastic head-on collision. Since the two bodies are of equal masses, they will exchange their velocities after collision. Therefore, their velocities after collision will be **–5 m/s** and **3 m/s**

13. Tension in the spring, $T = F = kx \quad \therefore\ x = T/k$

$$\text{Energy stored in the spring} = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{T}{k}\right)^2 = \frac{T^2}{2k}$$

14. $$\text{Kinetic energy, } K = \frac{p^2}{2m}$$

Since p (= linear momentum) is the same, we have,

$$\frac{K_2}{K_1} = \frac{m_1}{m_2} = \frac{1}{4} \quad \therefore \quad m_1 : m_2 = \mathbf{1:4}$$

15. When mass is lifted, $T_1 = m\,(g + a)$; When mass is lowered, $T_2 = m\,(g - a)$

$$\therefore \quad \frac{T_1}{T_2} = \frac{g+a}{g-a} = \frac{9.8+4.9}{9.8-4.9} = \frac{3}{1} \quad \therefore \quad T_1 : T_2 = \mathbf{3 : 1}$$

16. $$\text{Work done, } W = \int_0^5 F\,dx = \int_0^5 (7-2x-3x^2)\,dx$$

$$= [7x - x^2 - x^3]_0^5 = 35 - 25 - 125 = \mathbf{-\ 115\ J}$$

17. The body should rise through a distance of $2l$ to complete one full revolution.

$\therefore$ Potential energy of body = $2mgl$

If v is the minimum required horizontal velocity, kinetic energy of body = $mv^2/2$.

According to law of conservation of energy, $mv^2/2 = 2\ mgl \quad \therefore\ v = \sqrt{4gl}$

18. Force of friction, $F = \mu_k R = \mu_k mg = 0.2 \times 50 \times 9.8 = 98$ N

$\therefore$ Work done, $W = FS\cos\theta = 98 \times 1 \times \cos 0° = \mathbf{98\ J}$

19. Work done against resistive force = $F \times S = F \times 0.12$ J

$$\text{Loss of kinetic energy} = \frac{1}{2}mv^2 = \frac{1}{2} \times \frac{20}{1000} \times (250)^2 = 625\ \text{J}$$

According to work-energy theorem,

$$F \times 0.12 = 625 \quad \therefore\ F = 625/0.12 = \mathbf{5.2 \times 10^3\ N}$$

20. Work done, $W = \vec{F}.\vec{S} = (-2\hat{i} + 15\hat{j} + 6\hat{k}).(10\hat{j}) = \mathbf{150\ J}$

21. K.E. acquired = Work done = $F \times d$ = constant.

Therefore, K.E. acquired is **independent of mass** m.

22. Since the collision is inelastic, **only momentum is conserved.**

23. $$\text{K.E.} = \frac{p^2}{2m} = \frac{(2)^2}{2 \times 2} = \mathbf{1J}$$

24. $$\text{Work done, } W = \frac{1}{2}kx^2 = \frac{1}{2} \times \frac{F}{x} \times x^2 \qquad (\because\ F = kx)$$

$$= \frac{1}{2}Fx = \frac{1}{2} \times Mg \times l = \frac{1}{2}Mgl$$

25. The components of linear momentum perpendicular to the wall cancel out while the components along the wall add.

$\therefore$ $F \times t$ = Change in linear momentum = $2mv\cos\theta$

$$\therefore \quad F = \frac{2mv\cos\theta}{t} = \frac{2 \times 3 \times 10 \times \cos 60°}{0.2} = \mathbf{150\ N}$$

13. Tension in the spring, $T = F = kx \quad \therefore\ x = T/k$

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16. $$\text{Work done, } W = \int_0^5 F\,dx = \int_0^5 (7 - 2x - 3x^2)\,dx$$

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According to law of conservation of energy, $mv^2/2 = 2\ mgl \quad \therefore\ v = \sqrt{4gl}$

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$$\text{Loss of kinetic energy} = \frac{1}{2}mv^2 = \frac{1}{2} \times \frac{20}{1000} \times (250)^2 = 625\ \text{J}$$

According to work-energy theorem,

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25. The components of linear momentum perpendicular to the wall cancel out while the components along the wall add.

$\therefore$ $F \times t$ = Change in linear momentum = $2mv\cos\theta$

$$\therefore \quad F = \frac{2mv\cos\theta}{t} = \frac{2 \times 3 \times 10 \times \cos 60°}{0.2} = \mathbf{150\ N}$$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A 120 g mass has a velocity $\vec{v} = (2\hat{i} + 5\hat{j})$ m/s at a certain instant. Its K.E. is **[CET Karnataka 1996]**
   (*a*) 3 J (*b*) 4 J
   (*c*) 5 J (*d*) 1.74 J

2. A block of mass 1 kg slides down a rough inclined plane of inclination 60° starting from its top. If coefficient of kinetic friction is 0.5 and length of plane is 1 m, then work done against friction is **[CET Karnataka 2001]**
   (*a*) 19.6 J (*b*) 2.45 J
   (*c*) 9.8 J (*d*) 4.9 J

3. How much work does a pulling force of 40 N do on 20 kg box in pulling it 8 m across the smooth horizontal floor at a constant speed? The pulling force is directed at 60° above the horizontal. **[CET Haryana 1997]**
   (*a*) 784 J (*b*) 277 J
   (*c*) 160 J (*d*) None of above

4. Which of the following is not conserved in inelastic collision? **[CET Chandigarh 1997]**
   (*a*) momentum
   (*b*) K.E.
   (*c*) Both momentum and K.E.
   (*d*) Neither K.E. nor momentum

5. A body of mass 3 kg is under a force which causes a displacement in it given by $S = t^2/3$ (in m). Work done by force in 2 sec is **[BHU 1998]**
   (*a*) 2 J (*b*) 3.8 J
   (*c*) 5.2 J (*d*) 2.6 J

6. A car drives along straight level frictionless road by an engine delivering constant power. Then velocity is directly proportional to **[IIT 1984]**
   (*a*) t (*b*) $1/\sqrt{t}$
   (*c*) $\sqrt{t}$ (*d*) none of these

7. A billiard ball moving with a speed of 5 m/s collides with an identical ball, originally at rest. If the first ball stops dead after collision, then second ball will move forward with a speed of **[SCRA 1998]**
   (*a*) 10 m/s (*b*) 5 m/s
   (*c*) 2.5 m/s (*d*) 1 m/s

8. A body of mass 5 kg is moving with a momentum of 10 kg m/s. A force of 0.2 N acts on it in the direction of motion of the body for 10 sec. The increase in its kinetic energy is **[MP CEE 1999]**
   (*a*) 2.8 J (*b*) 3.2 J
   (*c*) 3.8 J (*d*) 4.4 J

9. An object of mass 2 kg is moving with a velocity of 3 m/s and collides head-on with an object of mass 1 kg moving in the opposite direction with a velocity of 4 m/s. After collision, both objects coalesce so that they move with a common velocity v equal to **[CEET 1998]**
   (*a*) 2/3 m/s (*b*) 1 m/s
   (*c*) 2 m/s (*d*) 3 m/s

10. A light body A and a heavy body B have equal linear momentum. The K.E. of body A **[CET 1998]**
   (*a*) is equal to that of B
   (*b*) is greater than that of B
   (*c*) is less than that of B
   (*d*) is zero

11. A bullet of mass 10 g travelling horizontally with a velocity of 300 m/s strikes a block of wood of mass 290 g which rests on rough horizontal floor. After the impact, the bullet and block move together. If the coefficient of friction between the block and the floor is 1/3, what distance the block will travel before it comes to rest? Take g = 10 m/s^2. **[CET Haryana 1993]**
   (*a*) 7.5 m (*b*) 5 m
   (*c*) 15 m (*d*) 10 m

12. A bullet is fired from a rifle. If the rifle recoils, the K.E. of rifle is **[Meerut 1997]**
   (*a*) greater than that of bullet
   (*b*) less than that of bullet
   (*c*) equal to that of bullet
   (*d*) none of the above

13. A force F applied on a body moves it through a distance S along F. The energy spent is **[CEET 2000]**
   (*a*) $F \times S$ (*b*) F/S
   (*c*) FS^2 (*d*) F/S^2

14. A force $\vec{F} = 5\hat{i} + 6\hat{j} - 4\hat{k}$ acting on a body produces a displacement $\vec{S} = 6\hat{i} + 5\hat{k}$. The work done by the force is **[CET Karnataka 1999]**

(*a*) 18 units (*b*) 15 units
(*c*) 12 units (*d*) 10 units

15. A body of mass 5 *m* initially at rest explodes into three fragments with mass ratio 3 : 1 : 1. Two fragments each of mass *m* are found to move with a speed of 60 m/s in mutually perpendicular directions. The velocity of the third fragment is **[AMU 1999]**

(*a*) $60\sqrt{2}$ m/s (*b*) $20\sqrt{3}$ m/s
(*c*) $10\sqrt{2}$ m/s (*d*) $20\sqrt{2}$ m/s

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*d*)	**2.** (*b*)	**3.** (*c*)	**4.** (*b*)	**5.** (*d*)
6. (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*a*)	**10.** (*b*)
11. (*c*)	**12.** (*b*)	**13.** (*a*)	**14.** (*d*)	**15.** (*d*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. $v^2 = \vec{v}.\vec{v} = (2\hat{i}+5\hat{j}).(2\hat{i}+5\hat{j}) = (2)^2+(5)^2 = 29$ m/s

$\therefore$ K.E. $= \frac{1}{2}mv^2 = \frac{1}{2}\times\frac{120}{1000}\times(29) = 1.74$ J

2. Frictional force, $F_k = \mu_k R = \mu_k mg\cos\theta = 0.5\times1\times9.8\times\cos60° = 2.45$ N

$\therefore$ Work done, $W = F_k \times S = 2.45\times1 = 2.45$ J

3. Work done, $W = \vec{F}.\vec{S} = FS\cos\theta = 40\times8\times\cos60° = 160$ J

4. In an inelastic collision, K.E. is not conserved.

5. $S = \frac{t^2}{3}$; $\frac{dS}{dt} = \frac{2t}{3}$; $\frac{d^2S}{dt^2} = \frac{2}{3}$

$$\text{Work done, } W = \int_0^2 FdS = \int_0^2 m\frac{d^2S}{dt^2}dS = \int_0^2 m\frac{d^2S}{dt^2}\frac{dS}{dt}dt$$

$$= \int_0^2 3\times\frac{2}{3}\times\frac{2t}{3}dt = \frac{4}{3}\left[\frac{t^2}{2}\right]_0^2 = 2.6 \text{ J}$$

6. Power, $P = Fv = m\frac{dv}{dt}v$ or $v\,dv = \frac{P}{m}dt$

Integrating both sides, we have,

$$\frac{v^2}{2} = \frac{P}{m}t + \text{constant} \quad \therefore\ v^2 \propto t \text{ or } v \propto \sqrt{t}$$

7. It is elastic head-on collision between two identical masses. Therefore, the masses will exchange their velocities.

8. Initial K.E. $= \frac{p_1^2}{2m} = \frac{(10)^2}{2\times5} = 10$ J

If p_2 is the final momentum, then, $p_2 - p_1 = F\times t$

or $p_2 - 10 = 0.2 \times 10 \quad \therefore\ p_2 = 12$ kg m/s

(*a*) 18 units (*b*) 15 units
(*c*) 12 units (*d*) 10 units

15. A body of mass 5 *m* initially at rest explodes into three fragments with mass ratio 3 : 1 : 1. Two fragments each of mass *m* are found to move with a speed of 60 m/s in mutually perpendicular directions. The velocity of the third fragment is **[AMU 1999]**

(*a*) $60\sqrt{2}$ m/s (*b*) $20\sqrt{3}$ m/s
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ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*d*)	**2.** (*b*)	**3.** (*c*)	**4.** (*b*)	**5.** (*d*)
6. (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*a*)	**10.** (*b*)
11. (*c*)	**12.** (*b*)	**13.** (*a*)	**14.** (*d*)	**15.** (*d*)

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1. $v^2 = \vec{v}.\vec{v} = (2\hat{i}+5\hat{j}).(2\hat{i}+5\hat{j}) = (2)^2+(5)^2 = 29$ m/s

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$\therefore$ Work done, $W = F_k \times S = 2.45\times1 = 2.45$ J

3. Work done, $W = \vec{F}.\vec{S} = FS\cos\theta = 40\times8\times\cos60° = 160$ J

4. In an inelastic collision, K.E. is not conserved.

5. $$S = \frac{t^2}{3} \; ; \; \frac{dS}{dt} = \frac{2t}{3} \; ; \; \frac{d^2S}{dt^2} = \frac{2}{3}$$

$$\text{Work done, } W = \int_0^2 FdS = \int_0^2 m\frac{d^2S}{dt^2}dS = \int_0^2 m\frac{d^2S}{dt^2}\frac{dS}{dt}dt$$

$$= \int_0^2 3\times\frac{2}{3}\times\frac{2t}{3}dt = \frac{4}{3}\left[\frac{t^2}{2}\right]_0^2 = 2.6 \text{ J}$$

6. Power, $P = Fv = m\frac{dv}{dt}v$ or $v\,dv = \frac{P}{m}dt$

Integrating both sides, we have,

$$\frac{v^2}{2} = \frac{P}{m}t + \text{constant} \quad \therefore \; v^2 \propto t \text{ or } v \propto \sqrt{t}$$

7. It is elastic head-on collision between two identical masses. Therefore, the masses will exchange their velocities.

8. $$\text{Initial K.E.} = \frac{p_1^2}{2m} = \frac{(10)^2}{2\times5} = 10 \text{ J}$$

If p_2 is the final momentum, then, $p_2 - p_1 = F\times t$

or $p_2 - 10 = 0.2 \times 10 \quad \therefore \; p_2 = 12$ kg m/s

$$\text{Final K.E.} = \frac{p_2^2}{2m} = \frac{(12)^2}{2\times 5} = 14.4 \text{ J}$$

$\therefore$ Increase in K.E. = 14.4 – 10 = 4.4 J

9. Total initial momentum = 2 × 3 + 1 × (– 4) = 2 kg m/s

Total final momentum = (2 + 1) v = 3v kg m/s

According to law of conservation of momentum, $3v = 2$ or $v = 2/3$ m/s

10. K.E. = $p^2/2m$

For the same momentum, the lighter body will have more K.E.

11. Total initial momentum = 10 × 300 = 3000 g m/s

Let v m/s be the velocity of the combination after impact.

Total momentum of combination after impact = (10 + 290) v = 300 vg m/s

According to law of conservation of momentum,

$$300\ v = 10 \times 300 \quad \therefore\ v = 10 \text{ m/s}$$

Total K.E. of combination after impact = $\frac{1}{2}\left(\frac{10+290}{1000}\right)\times(10)^2 = 15$ J

If the stopping distance is S metres, then,

$$\text{Work done against friction} = \mu R \times S = \mu\, mg \times S = \frac{1}{3}\times 0.3\times 10\times S$$

According to work-energy theorem, $\frac{1}{3}\times 0.3\times 10\times S = 15 \quad \therefore\ S = 15$ m

12. If m_1 and m_2 are the masses of bullet and rifle and v_1 and v_2 their corresponding velocities, then,

$$m_1 v_1 = m_2\, v_2 \quad \text{or} \quad \frac{v_1}{v_2} = \frac{m_2}{m_1}$$

$$\frac{E_1}{E_2} = \frac{(1/2)\, m_1 v_1^2}{(1/2)\, m_2 v_2^2} = \frac{m_1}{m_2}\times\frac{m_2^2}{m_1^2} = \frac{m_2}{m_1}$$

Since $m_2 > m_1$, therefore, $E_1 > E_2$.

In other words, K.E. of rifle is less than that of bullet.

13. Energy spent = Work done = $F \times S$

14. Work done, $W = \vec{F}.\vec{S} = (5\hat{i} + 6\hat{j} - 4\hat{k}).(6\hat{i} + 0\hat{j} + 5\hat{k})$

= 30 + 0 – 20 = 10 units

15. From principle of conservation of momentum, we have,

$$3m \times v = \sqrt{(m\times 60)^2 + (m\times 60)^2} = m\times 60\sqrt{2}$$

$\therefore$ $$v = \frac{m\times 60\sqrt{2}}{3m} = 20\sqrt{2} \text{ m/s}$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. Calculate the work done in raising a stone of mass 4 kg and specific gravity 2 immersed in water from a depth of 3 m to 1 m below the surface.

Hint. Density of water, d = 1000 kg/m^3; Density of stone, D = 2000 kg/m^3

Force, F = Weight of stone in water

$$= mg\left(1-\frac{d}{D}\right) = 4\times 9.8\left(1-\frac{1}{2}\right) = 19.6 \text{ N}$$

$\therefore$ Work done = $F \times S = 19.6 \times (3 - 1) = \mathbf{39.2\ J}$

2. A ball falls under gravity from a height of 10 m with an initial downward velocity v_0. It collides with ground, loses 50% of energy in collision and then rises back to the same height. Find the initial velocity v_0. [I.I.T.]

Hint. Let v be the velocity of the ball when it just touches the ground. Then,

Increase in K.E. = Decrease in P.E.

or $$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgh \text{ or } v^2 - v_0^2 = 2gh \quad ...(i)$$

Now when the ball hits the ground, 50% of K.E. is lost and the ball rises to the same height h (= 10 m).

$\therefore$ $$\frac{50}{100}\times\frac{1}{2}mv^2 = mgh \text{ or } v^2 = 4gh$$

Putting $v^2 = 4gh$ in eq. (i), we get,

$$4gh - v_0^2 = 2gh \text{ or } v_0 = \sqrt{2gh} = \sqrt{2\times 9.8\times 10} = 14 \text{ ms}^{-1}$$

3. A bullet of mass 0.01 kg and travelling horizontally at a speed of 500 ms^{-1} strikes a block of mass 2 kg which is suspended by a string of length 5 m. The centre of gravity of the block is found to rise a vertical distance of 0.1 m. What is the speed of the bullet after it emerges from the block?[Roorkee Engg]

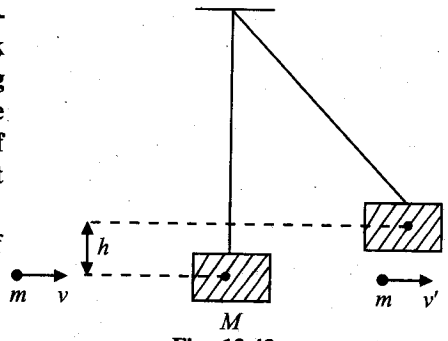

Fig. 12.43

Hint. Mass of bullet, m = 0.01 kg ; Initial speed of bullet, v = 500 ms^{-1}

Mass of block, M = 2 kg

Fig. 12.43 shows the conditions of the problem.

Initially, the block is at rest. When it is struck by the bullet, let its velocity be V. As a result, the block rises to a height h converting K.E. into potential energy.

$\therefore$ $$\frac{1}{2}MV^2 = Mgh$$

or $$V = \sqrt{2gh} = \sqrt{2\times 9.8\times 0.1} = 1.4 \text{ ms}^{-1}$$

Suppose on emerging from the block, the speed of the bullet is v'. Then by law of conservation of linear momentum, we have,

Initial linear momentum of bullet-block = Final linear momentum of bullet-block

or $$mv + M\times 0 = MV + mv'$$

or $$(0.01\times 500) + 0 = (2\times 1.4) + (0.01\times v')$$

$\therefore$ $$v' = \mathbf{220 \text{ ms}^{-1}}$$

4. **A bullet of mass 20 g moving with velocity v is embedded into a block of mass 10 kg. As a result of this collision, the block and the bullet rise to a height of 25 cm from the equilibrium position. Find the original velocity v of the bullet.**

Hint. Mass of bullet, m = 20 g = 20 × 10^{-3} kg ; Mass of block, M = 10 kg

Initially, the block is at rest. Therefore, initial momentum of bullet-block system $= mv + M\times 0 = mv$. Let v' be the velocity of the block when the bullet is embedded in it. Therefore, the final momentum of the bullet-block system = $(m+M)v'$. Then, by the law of conservation of linear momentum, we have,

$$(m+M)v' = mv \quad \text{or} \quad v' = \frac{mv}{m+M} \qquad ...(i)$$

After the collision, the bullet-block system acquires K.E. = $\frac{1}{2}(m+M)v'^2$. The block (along with bullet) rises to a height h (= 25 cm = 25 × 10^{-2} m) and K.E. gained is converted into gravitational P.E. = $(m+M)gh$.

$$\therefore \quad \frac{1}{2}(m+M)v'^2 = (m+M)gh \quad \text{or} \quad v' = \sqrt{2gh}$$

Putting the value of $v' = \sqrt{2gh}$ in eq. (i), we have,

$$\sqrt{2gh} = \frac{mv}{m+M} \text{ or } v = \frac{m+M}{m}\times\sqrt{2gh}$$

$$\therefore \quad v = \frac{20\times10^{-3}+10}{20\times10^{-3}}\times\sqrt{2\times9.8\times25\times10^{-2}} = \mathbf{1109\ ms^{-1}}$$

Note. Note that in this collision, the kinetic energy is not conserved.

$$\text{Initial K.E.} = \frac{1}{2}mv^2$$

$$\text{K.E. after collision} = \frac{1}{2}(m+M)v'^2 = \frac{1}{2}(m+M)\left(\frac{mv}{m+M}\right)^2$$

$$= \frac{1}{2}mv^2\left(\frac{m}{m+M}\right)$$

Note the K.E. after collision is less than the original K.E.

5. **A 1 kg block collides with a horizontal weightless spring of force constant 2 N/m [See Fig. 12.44]. The block compresses the spring 4 m from the rest position. If the coefficient of kinetic friction between the block and the horizontal surface is 0.25, what was the speed of the block at the instant of collision?**

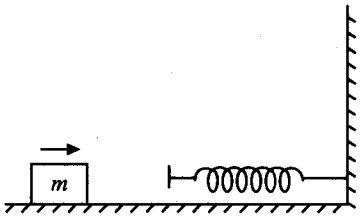

Fig. 12.44

Hint. Let v be the speed of the block at the instant of collision with the spring. Then by the law of conservation of energy, initial K.E. of the block is equal to elastic energy stored in the spring plus energy spent in doing work against friction *i.e.*,

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 + \mu_k mgx$$

Here, m = 1 kg ; x = 4 m ; k = 2 N/m

$$\therefore \quad \frac{1}{2}\times1\times v^2 = \frac{1}{2}\times2\times(4)^2 + 0.25\times1\times9.8\times4$$

or $v^2 = 51.6$ $\therefore$ $v = \sqrt{51.6} = 7.18\ \text{ms}^{-1}$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. Calculate the work done in raising a stone of mass 4 kg and specific gravity 2 immersed in water from a depth of 3 m to 1 m below the surface.

Hint. Density of water, d = 1000 kg/m^3; Density of stone, D = 2000 kg/m^3

Force, F = Weight of stone in water

$$= mg\left(1-\frac{d}{D}\right) = 4\times 9.8\left(1-\frac{1}{2}\right) = 19.6 \text{ N}$$

$\therefore$ Work done = $F \times S$ = 19.6 × (3 − 1) = **39.2 J**

2. A ball falls under gravity from a height of 10 m with an initial downward velocity v_0. It collides with ground, loses 50% of energy in collision and then rises back to the same height. Find the initial velocity v_0. **[I.I.T.]**

Hint. Let v be the velocity of the ball when it just touches the ground. Then,

Increase in K.E. = Decrease in P.E.

or $$\frac{1}{2}mv^2-\frac{1}{2}mv_0^2 = mgh \text{ or } v^2-v_0^2 = 2gh \quad ...(i)$$

Now when the ball hits the ground, 50% of K.E. is lost and the ball rises to the same height h (= 10 m).

$\therefore$ $$\frac{50}{100}\times\frac{1}{2}mv^2 = mgh \text{ or } v^2 = 4gh$$

Putting $v^2 = 4gh$ in eq. (i), we get,

$$4gh-v_0^2 = 2gh \text{ or } v_0 = \sqrt{2gh} = \sqrt{2\times 9.8\times 10} = 14 \text{ ms}^{-1}$$

3. A bullet of mass 0.01 kg and travelling horizontally at a speed of 500 ms^{-1} strikes a block of mass 2 kg which is suspended by a string of length 5 m. The centre of gravity of the block is found to rise a vertical distance of 0.1 m. What is the speed of the bullet after it emerges from the block?[Roorkee Engg]

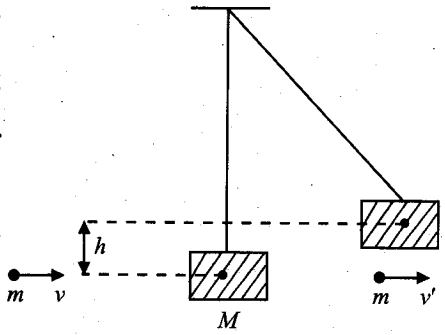

Fig. 12.43

Hint. Mass of bullet, m = 0.01 kg ; Initial speed of bullet, v = 500 ms^{-1}

Mass of block, M = 2 kg

Fig. 12.43 shows the conditions of the problem.

Initially, the block is at rest. When it is struck by the bullet, let its velocity be V. As a result, the block rises to a height h converting K.E. into potential energy.

$\therefore$ $$\frac{1}{2}MV^2 = Mgh$$

or $$V = \sqrt{2gh} = \sqrt{2\times 9.8\times 0.1} = 1.4 \text{ ms}^{-1}$$

Suppose on emerging from the block, the speed of the bullet is v'. Then by law of conservation of linear momentum, we have,

Initial linear momentum of bullet-block = Final linear momentum of bullet-block

or $$mv + M\times 0 = MV + mv'$$

or $$(0.01\times 500) + 0 = (2\times 1.4) + (0.01\times v')$$

$\therefore$ $$v' = \mathbf{220 \text{ ms}^{-1}}$$

6. A bullet of mass m moving with a horizontal velocity v, strikes a stationary block of mass M suspended by a string of length L. The bullet gets embedded in the block. What is the maximum angle made by the string after impact? [Roorkee Engg.]

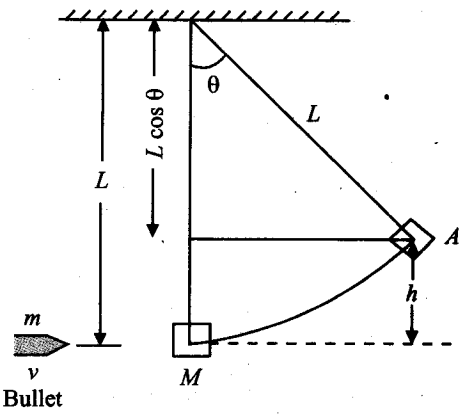

Fig. 12.45

Hint. Fig. 12.45 shows the conditions of the problem. Let the maximum angle made by the string with the vertical be θ. If V is the velocity of the block after the bullet is embedded, then according to the law of conservation of linear momentum, we have,

$$mv = (m + M)V$$

or $$V = \frac{mv}{m+M} \qquad ...(i)$$

By law of conservation of energy, we have, K.E. of combined mass = P.E. of combined mass at the highest point A or $\frac{1}{2}(m+M)V^2 = (m+M)gh$

From Fig. 12.45, $h = L - L\cos\theta = L(1-\cos\theta)$

$\therefore$ $$V^2 = 2gh = 2gL(1-\cos\theta) \qquad ...(ii)$$

Putting the value of V from eq. (i) into eq. (ii), we have,

$$\frac{m^2v^2}{(m+M)^2} = 2gL(1-\cos\theta) \quad \text{or} \quad 1-\cos\theta = \frac{m^2v^2}{2gL(m+M)^2}$$

$\therefore$ $$\theta = \cos^{-1}\left[1 - \frac{m^2v^2}{2gL(m+M)^2}\right]$$

7. A projectile of mass 50 kg is shot vertically upward with an initial velocity of 100 ms^{-1}. After 5 seconds, it explodes into two fragments, one of which having mass 20 kg travels vertically up with a velocity of 150 ms^{-1}.

(i) What is the velocity of the other fragment at that instant?

(ii) Calculate the sum of the momenta of the two fragments 3 s after the explosion. What would have been the momentum of the projectile at this instant if there had been no explosion? [I.I.T.]

Hint. (i) Velocity of the projectile after 5 seconds is

$$v = v_0 - gt = 100 - 9.8\times5 = 51 \text{ ms}^{-1}$$

$\therefore$ Momentum of projectile just before explosion

$$= mv = 50 \times 51 = 2550 \text{ kgms}^{-1}$$

Momentum of the fragments after explosion

$$= m_1v_1 + m_2v_2 = 20\times150 + 30\ v_2 = 3000 + 30v_2$$

By law of conservation of linear momentum, we have,

$$3000 + 30\ v_2 = 2550 \ \therefore\ v_2 = \frac{2550-3000}{30} = -15 \text{ ms}^{-1}$$

Thus, after the collision, the second fragment moves vertically downward.

(*ii*) 3 seconds after the explosion, the upward velocity of fragment m_1 (= 20 kg) is

$$v_1' = v_0' - gt' = 150 - 9.8 \times 3 = 120.6 \text{ ms}^{-1}$$

Downward velocity of fragment m_2 (= 30 kg) after 3s of explosion is

$$v_2' = v_0'' + gt' = 15 + 9.8 \times 3 = 44.4 \text{ ms}^{-1}$$

Sum of momenta = $m_1 v_1' + m_2 v_2' = (20 \times 120.6) + [30 \times (-44.4)]$ = **1080 kgms^{-1}**

If there had been no explosion, the velocity of the projectile after 8 seconds would have been

$$V = v_0 - gt' = 100 - (9.8 \times 8) = 21.6 \text{ ms}^{-1}$$

∴ Momentum of the projectile = Mass × Velocity = 50 × 21.6 = **1080 kgms^{-1}**

8. Two bodies *A* and *B* of masses 100 g and 400 g respectively are moving towards each other with speeds 100 cm/s and 10 cm/s. They suffer a head-on collision and stick together.

(*i*) In which direction will the combined mass move after the collision?

(*ii*) What will be the distance travelled by the combined mass after 10 seconds?

(*iii*) Assuming the two bodies to be originally at the same temperature and specific heat of both the bodies to be 0.1 cal g^{-1} °C^{-1}, calculate the rise in temperature of the combined mass after the collision. **[I.I.T.]**

Hint. Fig. 12.46 shows the conditions of the problem.

Here m_1 = 0.1 kg ; v_1 = 1 m/s ; m_2 = 0.4 kg ; v_2 = 0.1 m/s.

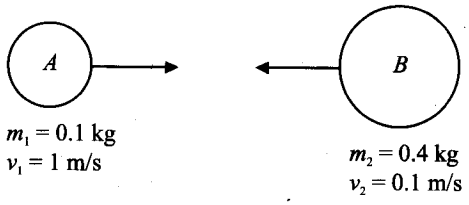

Fig. 12.46

(*i*) Initial momentum of $A = m_1 v_1 = 0.1 \times 1 = 0.1$ kgms^{-1}

Initial momentum of $B = m_2 v_2 = 0.4 \times 0.1 = 0.04$ kgms^{-1}

The initial momentum of *A* is more than the initial momentum of *B*. Therefore, after the collision, the two will move together **in the direction of motion of *A*.**

(*ii*) Total momentum = 0.1 − 0.04 = 0.06 kg ms^{-1}

Velocity of combination, $v = \dfrac{0.06}{m_1 + m_2} = \dfrac{0.06}{0.1 + 0.4} = 0.12 \text{ ms}^{-1}$

Distance travelled in 10 s = $v \times 10 = 0.12 \times 10$ = **1.2 m**

(*iii*) Initial K.E. of the system, $E_1 = \dfrac{1}{2} m_1 v_1^2 + \dfrac{1}{2} m_2 v_2^2$

Final K.E. of the system, $E_2 = \dfrac{1}{2}(m_1 + m_2) v^2$

Loss of K.E., $E = E_1 - E_2$

$$= \left[\frac{1}{2} \times 0.1 \times (1)^2 + \frac{1}{2} \times 0.4 \times (0.1)^2\right] - \left[\frac{1}{2}(0.1 + 0.4)(0.12)^2\right]$$

$$= 0.0484 \text{ J}$$

Let the rise in temperature of the combination be θ°C.

$\therefore$ Heat gained = Loss of K.E.

or $mc\theta \times J = 0.0484 \quad (m = m_1 + m_2)$

$\therefore$ $\theta = \dfrac{0.0484}{mcJ} = \dfrac{0.0484}{500 \times 0.1 \times 4.2} = 2.3 \times 10^{-4}$ °C

9. The graph between resisting force *F* acting on a body and distance covered by the body is shown in Fig. 12.47. The mass of the body is 25 kg and initial speed is 2 m/s. What are the kinetic energies of the body at distances 2 m and 5 m?

Hint. Initial K.E. of the body is

$$K = \frac{1}{2}mv^2 = \frac{1}{2} \times 25 \times (2)^2 = 50 \text{ J}$$

Work done against friction in 2 m distance is

W_1 = Area of $\Delta ab2$

$= \dfrac{1}{2} \times 1 \times 20 = 10$ J

$\therefore$ K.E. of the body at 2 m distance

$= K - W_1 = 50 - 10 = 40$ J

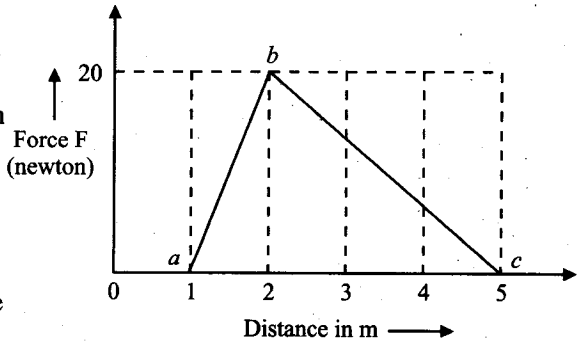

Fig. 12.47

Work done against friction in 5 m distance is

W_2 = Area of $\Delta abc = \dfrac{1}{2} \times 4 \times 20 = 40$ J

$\therefore$ K.E. of the body at a distance of 5 m

$= K - W_2 = 50 - 40 = 10$ J

10. A test tube of mass 10 g closed with a cork of mass 1 g contains some ether. When the test tube is heated, the cork flies out under the pressure of the ether gas. The test tube is suspended by a weightless rigid string of length 5 cm. What is the minimum velocity with which the cork should fly out of the test tube so that the test tube describes a full vertical circle about point *O*? Neglect the mass of the ether.

[I.I.T.]

Hint. Fig. 12.48 shows the conditions of the problem.

Mass of cork, $m = 1 \times 10^{-3}$ kg ; Mass of the test tube, $M = 10 \times 10^{-3}$ kg

Let v be the velocity of the cork and V the velocity of recoil of the test tube. Then,

$$mv = MV$$

$\therefore$ $$V = \frac{mv}{M} = \frac{(1 \times 10^{-3})v}{10 \times 10^{-3}} = \frac{v}{10}$$

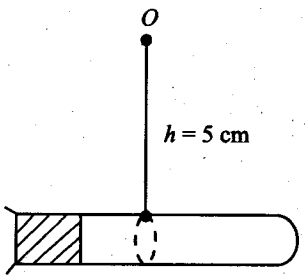

Fig. 12.48

K.E. of test tube = $\dfrac{1}{2}MV^2$

Increase in P.E. of tube = $Mg\ (2h)$

According to principle of conservation of energy, we have,

$$\frac{1}{2}MV^2 = Mg(2h)$$

or $V^2 = 4gh$ or $\frac{v^2}{100} = 4gh$ $\left(\because V = \frac{v}{10}\right)$

or $v^2 = 400\ gh = 400 \times 9.8 \times 0.05 = 196$ $\therefore v = \sqrt{196} = \mathbf{14\ ms^{-1}}$

11. A ball moving with a speed of 9 ms^{-1} strikes an identical stationary ball such that after collision, the direction of each ball makes an angle of 30° with the original line of motion as shown in Fig. 12.49. Find the speeds of the two balls after the collision. Is the kinetic energy conserved in this collision process? [I.I.T.]

Hint. Let m be the mass of each ball. Initially, the balls are along the X-axis.

$\therefore$ Initial momentum of balls along X-axis

$$= m_1u_1 + m_2u_2 = (m\times 9) + (m\times 0) = 9\ m\ \text{kgms}^{-1}$$

Let v_1 and v_2 be the velocities of the balls after collision [See Fig. 12.49].

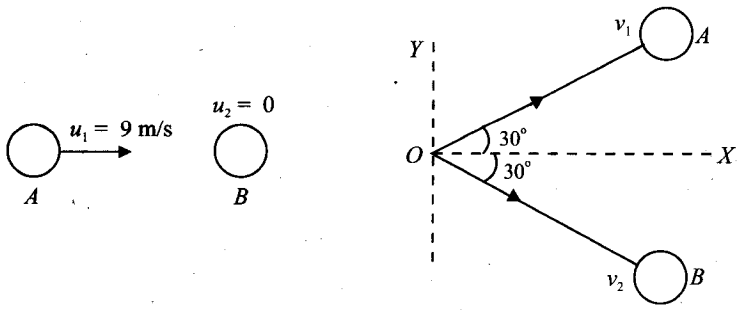

Fig. 12.49

Momentum of the balls along X-axis after collision

$$= m_1v_1\cos 30° + m_2v_2\cos 30°$$

$$= mv_1\times\frac{\sqrt{3}}{2} + mv_2\times\frac{\sqrt{3}}{2} = \frac{m\sqrt{3}}{2}(v_1 + v_2)$$

According to principle of conservation of momentum, the momentum of the balls along X-axis after collision should be equal to their momentum along X-axis before collision.

$\therefore$ $9m = \frac{m\sqrt{3}}{2}(v_1 + v_2)$ or $v_1 + v_2 = 6\sqrt{3}$ ...(*i*)

Initial momentum of the balls along Y-axis = 0.

Momentum of the balls along Y-axis after collision

$$= m_1v_1\sin 30° - m_2v_2\sin 30°$$

$$= mv_1\times\frac{1}{2} - mv_2\times\frac{1}{2} = \frac{m}{2}(v_1 - v_2)$$

Principle of conservation of momentum requires that :

$\frac{m}{2}(v_1 - v_2) = 0$ or $v_1 = v_2$ ...(*ii*)

From eqs. (*i*) and (*ii*), we get, $v_1 = v_2 = \mathbf{3\sqrt{3}\ ms^{-1}}$

Initial K.E. of balls $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}\times m\times(9)^2 + 0 = 40.5m$ joules

Final K.E. of balls $= \frac{1}{2}m(v_1^2 + v_2^2) = \frac{1}{2}m[(3\sqrt{3})^2 + (3\sqrt{3})^2] = 27m$ joules

Note that final K.E. of the balls is not equal to the initial K.E. of the balls. Therefore, **K.E. is not conserved in this collision.**

12. **Two springs have their force constants as k_1 and k_2 ($k_1 > k_2$). On which spring is more work done (*i*) when their lengths are increased by the same amount (*ii*) when they are stretched by the same force?** **[I.I.T]**

Hint. Spring constant, $k = \frac{F}{x} = \frac{mg}{x}$

(*i*) If the extension of the two springs is same (say x), then,

For first spring: $F_1 = k_1 x$; For second spring: $F_2 = k_2 x$

As $k_1 > k_2$, therefore, $F_1 > F_2$.

Therefore, for the same increase in length, work done is more in the case of the **first spring.**

(*ii*) If the stretching force F is the same in case of the two springs, then,

$$F = k_1 x_1 = k_2 x_2$$

Since $k_1 > k_2$; $x_1 < x_2$

Therefore, for the same stretching force, the work done is more in the case of the **second spring.**

13. **A body of mass 0.3 kg is taken up an inclined plane of length 10 m and height 5 m and then allowed to slide down again. The coefficient of friction between the body and the plane is 0.15. Find (*i*) the work done by the gravitational force round the trip (*ii*) work done by the applied force over the upward journey (*iii*) work done by the frictional force over the trip (*iv*) kinetic energy of the body at the end of the trip.**

Hint. Here $m = 0.3$ kg ; $\mu = 0.15$; $\sin\theta = \frac{1}{2}$; $\cos\theta = \frac{\sqrt{3}}{2}$

(*i*) Work done by the gravitational force to move the body up the inclined plane is

$$W_1 = \vec{F}.\vec{S} = FS\cos 180°$$
$$= (mg\sin\theta)S\cos 180°$$
$$= \left(0.3\times 9.8\times\frac{1}{2}\right)\times 10\times\cos 180°$$
$$= -14.7 \text{ J}$$

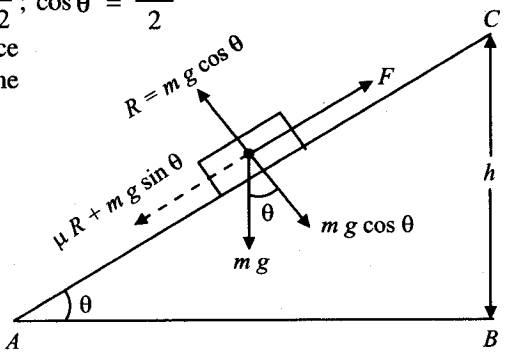

Fig. 12.50

Work done by the gravitational force to move the body down the plane is

$$W_2 = FS\cos 0° = (mg\sin\theta)\times S = 0.3\times 9.8\times\frac{1}{2}\times 10 = 14.7 \text{ J}$$

Work done by gravitational force round the trip

$$= (-14.7) + (14.7) = 0$$

This is expected because gravitational force is a conservative force.

(*ii*) Frictional force, $F_k = \mu_k R = \mu_k mg\cos\theta$

For upward journey (See Fig. 12.50), the frictional force acts along the plane downward. Therefore, the force F required to move the body along the plane upward is

$$F = mg\sin\theta + \vec{F_k} = mg\sin\theta + \mu_k mg\cos\theta$$

$$= mg(\sin\theta + \mu_k \cos\theta) = 0.3\times 9.8\left(\frac{1}{2} + 0.15\times\frac{\sqrt{3}}{2}\right) = 1.85 \text{ N}$$

Work done by the applied force for the upward journey is

$$W_3 = F\times S = 1.85\times 10 = 18.5 \text{ J}$$

(*iii*) The work done by the frictional force is negative. Therefore, work done by the frictional force over the round trip is

$$W_4 = (-\mu_k mg\cos\theta - \mu_k mg\cos\theta)\times S = (-2\mu_k mg\cos\theta)\times S$$

$$= \left(-2\times 0.15\times 0.3\times 9.8\times\frac{\sqrt{3}}{2}\right)\times 10 = -7.6 \text{ J}$$

(*iv*) Kinetic energy of the body at the end of the trip

= Total work done on the body

$$= W_1 + W_2 + W_3 + W_4 = -14.7 + 14.7 + 18.5 - 7.6 = 10.9 \text{ J}$$

Note that answer (*iv*) is the sum of the first three answers.

14. **A bullet having a speed of 153 ms^{-1} crashes through a plank of wood. After passing through the plank, its speed is 130 ms^{-1}. Another bullet, of the same mass and size but travelling at 92 ms^{-1} is fired at the plank. What will be the speed of the second bullet after passing through the plank? Assume that the resistance of the plank is independent of the speed of the bullet.**

Hint. The plank does the same amount of work on the two bullets and, therefore, decreases their kinetic energies equally. If v is the speed of the second bullet after passing through the plank, then,

$$\frac{1}{2}m(153)^2 - \frac{1}{2}m(130)^2 = \frac{1}{2}m(92)^2 - \frac{1}{2}mv^2$$

or $$(153)^2 - (130)^2 = (92)^2 - v^2$$

or $$v^2 = 1955 \quad \therefore \quad v = \sqrt{1955} = 44.2 \text{ ms}^{-1}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 6 : WORK, ENERGY AND POWER

Q.1. The sign of work done by a force on a body is important to understand. State carefully if the following quantities are positive or negative: (*a*) work done by a man in lifting a bucket out of a well by means of a rope tied to the bucket (*b*) work done by gravitational force in the above case (*c*) work done by friction on a body sliding down an inclined plane (*d*) work done by an applied force on a body moving on a rough horizontal plane with uniform velocity (*e*) work done by the resistive force of air on a vibrating pendulum in bringing it to rest.

Ans. Work done $W = FS \cos\theta$ where θ is the smaller angle between force $\vec{F}$ and displacement $\vec{S}$.

(*a*) To lift the bucket, force equal to the weight (= *mg*) of the bucket has to be applied in the vertically upward direction. The displacement is also in the vertically upward direction so that $\theta = 0°$. Therefore, $W = FS \cos 0° = FS$. Clearly, work done is positive.

(*b*) The gravitational force (= *mg*) acts vertically downward while the motion of the bucket is vertically upward so that $\theta = 180°$. Therefore, work done by the gravitational force is $W = FS \cos 180° = -FS$. Obviously, work done is negative.

(*c*) Frictional force always acts in a direction opposite to the direction of motion so that $\theta = 180°$. Therefore, work done by the frictional force is $W = FS \cos 180°. = -FS$. Clearly, work done is negative.

(*d*) Since applied force and the displacement are in the same direction, $\theta = 0°$. Therefore, $W = FS \cos 0° = FS$. Clearly, work done by the applied force is positive.

(*e*) The direction of resistive force is opposite to the direction of motion of the pendulum so that $\theta = 180°$. Hence work done in this case is negative.

Q.2. A body of mass 2 kg initially at rest moves under the action of an applied horizontal force of 7 N on a table with coefficient of kinetic friction = 0.1. Calculate (*a*) work done by the applied force in 10 s (*b*) work done by the frictional force in 10 s (*c*) work done by the net force on the body in 10 s (*d*) change in K.E. of the body in 10 s. Interpret your result.

Ans. Frictional force, $F_k = \mu_k R = \mu_k mg = 0.1 \times 2 \times 9.8 = 1.96$ N

Net force on the body $= F - F_k = 7 - 1.96 = 5.04$ N

$$\text{Acceleration, } a = \frac{\text{Net force}}{\text{Mass}} = \frac{5.04}{2} = 2.52 \text{ ms}^{-2}$$

Distance travelled by the body in 10 seconds is

$$S = v_0 t + \frac{1}{2}at^2 = 0 \times t + \frac{1}{2} \times 2.52 \times (10)^2 = 126 \text{ m}$$

(*a*) Work done by applied force, $W_1 = F \times S = 7 \times 126 = \mathbf{882\ J}$

(*b*) Work done by frictional force, $W_2 = -F_k \times S = -1.96 \times 126 = \mathbf{-246.9\ J}$

(*c*) Work done by net force, $W = W_1 + W_2 = 882 + (-246.9) = \mathbf{635.1\ J}$

(*d*) Velocity of the body after 10 seconds is

$$v = v_0 + at = 0 + 2.52 \times 10 = 25.2 \text{ ms}^{-1}$$

The initial kinetic energy (K.E.) of the body is zero.

$$\therefore \quad \text{Change in K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times (25.2)^2 = \mathbf{635.1\ J}$$

Note that work done by the net force is equal to the change in K.E. of the body. This is the work-energy theorem.

Q.3. Given in Fig. 6.01 are examples of some potential energy functions in one dimension. The total energy *E* of the particle is indicated by a cross on the ordinate axis. In each case, specify the regions, if any, in which the particle cannot be found for the given energy. Also, indicate the minimum total energy the particle must have in each case. Think of simple physical contexts for which these potential energy shapes are relevant.

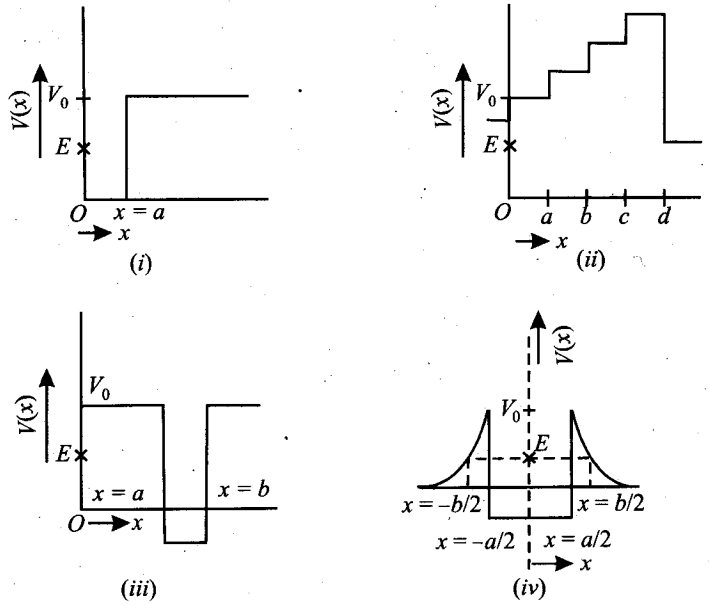

Fig. 6.01

Ans. Total energy of a particle, E = P.E. + K.E. or K.E. = E – P.E. Since K.E. can never be negative, the particle cannot exist in the region where its K.E. would become negative.

(*a*) Refer to Fig. 6.01(*i*). For $x < a$, P.E. is zero so that K.E. is positive in this region. However, for $x > a$, P.E. = $V_0 > E$ so that K.E. is negative in this region. Therefore, the particle cannot exist in the region $x > a$. The particle must have minimum total energy equal to zero.

(*b*) Refer to Fig. 6.01(*ii*). In this case, P.E. = $V_0 > E$ in every region so that K.E. is negative in every region. Therefore, particle cannot exist in any region in Fig. 6.01(*ii*). The particle must have minimum total energy equal to $-V_0$.

(*c*) Refer to Fig. 6.01(*iii*). In this case, for $x < a$ and $x > b$, P.E. = $V_0 > E$ so that in these regions K.E. becomes negative. Therefore, the particle cannot exist in the region $x < a$ and $x > b$. The particle must have minimum total energy equal to V_0.

(*d*) Refer to Fig. 6.01(*iv*). In this case, the particle cannot exist in the region $x = -b/2$ to $x = -a/2$ and $x = a/2$ to $x = b/2$ because P.E. = $V_0 > E$ in this region. The particle must have minimum total energy equal to $-V_0$.

Q.4. The potential energy function for a particle executing linear simple harmonic motion is given by $V(x) = kx^2/2$, where k is the force constant of the oscillator. For $k = 0.5$ Nm^{-1}, the graph of $V(x)$ versus x is shown in Fig. 6.02. Show that a particle of total energy 1J moving under this potential must 'turn back' when it reaches $x = \pm 2$ m.

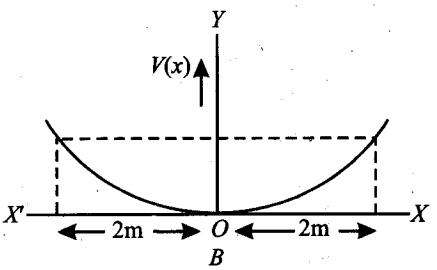

Fig. 6.02

Ans. At any instant, the total energy (E) of the oscillator is equal to sum of its K.E. and P.E. *i.e.*,

$$E = \text{K.E.} + \text{P.E. or } E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

The particle must turn back at the instant its velocity v becomes zero.

$$\therefore \qquad E = 0 + \frac{1}{2}kx^2 \text{ or } 1 = \frac{1}{2} \times 0.5x^2 \quad \therefore \quad x = \pm 2\text{m}$$

Q.5. Answer the following :

(*a*) The casing of a rocket in flight burns up due to friction. At whose expense is the heat energy required for burning obtained? The rocket or the atmosphere?

(*b*) Comets move around the sun in highly elliptical orbits. The gravitational force on the comet due to the sun is not normal to the comet's velocity in general. Yet the work done by the gravitational force over every complete orbit of the comet is zero. Why?

(*c*) An artificial satellite orbiting the earth in very thin atmosphere loses its energy gradually due to dissipation against atmospheric resistance, however small. Why then does its speed increase progressively as it comes closer and closer to the earth?

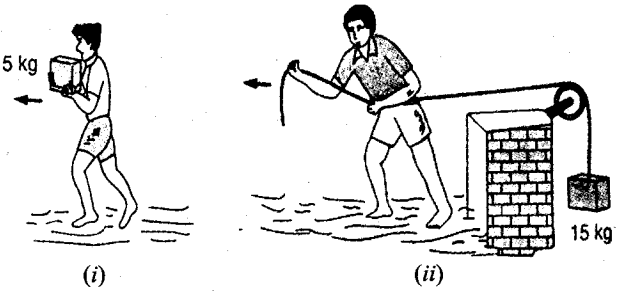

Fig. 6.03

(*d*) In Fig. 6.03(*i*), the man walks 2 m carrying a mass of 15 kg on his hands. In Fig. 6.03(*ii*), he walks the same distance pulling the rope behind him. The rope goes over a pulley and a mass of 15 kg hangs at its other end. In which case is the work done greater?

Ans. (*a*) The heat energy required for burning of the rocket in flight is obtained from the rocket itself and not from the atmosphere. This energy is obtained at the expense of the mass of the rocket and its energy (K.E. and P.E.).

(*b*) The gravitational force on the comet due to the sun is a conservative force. We know that work done by a conservative force over a closed path (regardless of shape of closed path) is always zero. Therefore, work done by the gravitational force over every complete orbit of the comet is zero.

(*c*) According to principle of conservation of energy, the total energy (K.E. + P.E.) of the satellite orbiting the earth remains constant. As the satellite comes closer and closer to earth, its potential energy (P.E.) goes on decreasing. Therefore, the kinetic energy (K.E.) and hence velocity of the satellite goes on increasing. However, total energy of the satellite goes on decreasing a little bit due to the loss of energy against atmospheric friction.

(*d*) Work done $W = FS \cos \theta$. In Fig. 6.03(*i*), the force applied by the man on 15 kg mass is in the vertically upward direction while the displacement is along the horizontal direction. Therefore, $\theta = 90°$ and $W = FS \cos 90° = 0$. In Fig. 6.03(*ii*), the force is applied on 15 kg mass in the vertically upward direction and the mass is also lifted along the vertical direction so that $\theta = 0°$. Therefore, $W = FS \cos \theta = (15 \times 9.8) \times 2 \times \cos 0° = 294$ J. Clearly, work done in the second case is greater.

Q.6. Underline the correct alternative: (*a*) When a conservative force does positive work on a body, the potential energy of the body increases/decreases/remains unaltered. (*b*) Work done by a body against friction always results in a loss of its kinetic/potential energy. (*c*) The rate of change of total momentum of a many particle system is proportional to the external force/ sum of the internal forces on the system. (*d*) In an inelastic collision of two bodies, the quantities which do not change after the collision are the total kinetic energy/total linear momentum/total energy of the system of two bodies.

Ans. (*a*) Positive work means $\theta < 90°$. When a conservative force (*e.g.*, gravitational force) does positive work on a body (*e.g.* freely falling body) potential energy decreases. It is because due to displacement, the body approaches the centre of force (decreasing x).

(*b*) A body does work against friction at the expense of its kinetic energy. Therefore, when a body does work against friction, its kinetic energy decreases.

(*c*) The sum of internal forces in a body is zero so that internal forces cannot change the net momentum of the body. Therefore, the rate of change of total momentum of a system is proportional to the external forces on the system.

(*d*) In an inelastic collision of two bodies, the quantities that do not change after collision are total linear momentum and total energy of the system of two bodies. However, in an inelastic collision, K.E. of the system is not conserved as a part of it may change to other forms.

Q.7. State if each of the following statements is true or false. Give reasons for your answer. (*a*) In an elastic collision of two bodies, the momentum and energy of each body is conserved. (*b*) Total energy of a system is always conserved, no matter what internal and external forces on the body are present. (*c*) Work done in the motion of a body over a closed loop is zero for every force in nature. (*d*) In an inelastic collision, the final kinetic energy is always less than the initial kinetic energy of the system.

Ans. (*a*) False. In an elastic collision, the total momentum and total energy of the system is conserved but not of each body in the system.

(*b*) False. The external force on the system may change the total energy of the system.

(*c*) False. The work done in the motion of a body over a closed loop is zero *only* when the body is moving under the action of conservative forces (*e.g.*, gravitational force, electrostatic force). However, work done over a closed loop is not zero when forces are non-conservative (*e.g.*, frictional force).

(*d*) True. In an inelastic collision, some part of K.E. usually changes to other forms of energy.

Q.8. Answer carefully, with reasons: (*a*) In an elastic collision of two billiard balls, is the total kinetic energy conserved during the short time of collision of the balls (*i.e.*, when they are in contact)? (*b*) Is the total linear momentum conserved during the short time of an elastic collision of two balls? (*c*) What are the answers to (*a*) and (*b*) for an inelastic collision? (*d*) If the potential energy of two billiard balls depends only on the separation distance between their centres, is the collision elastic or inelastic? (Note, we are talking here of potential energy corresponding to the force during collision, not gravitational potential energy).

Ans. (*a*) No. It is because during collision (when balls are in contact), K.E. of the balls gets converted into potential energy. However, K.E. before and after the elastic collision is the same.

(*b*) Yes. The total linear momentum is conserved during the short time of an elastic collision of two balls. There is only a sharing of momentum between the two balls but total momentum remains the same.

(*c*) In an inelastic collision, the total K.E. is not conserved during collision and after the collision is over. However, the total linear momentum is conserved during and after the collision.

(*d*) The collision is elastic. It is because here forces involved are conservative.

Q.9. A body is initially at rest. It undergoes one-dimensional motion with constant acceleration. The power delivered to it at time t is proportional to (*i*) $t^{1/2}$ (*ii*) t (iii) $t^{3/2}$ (*iv*) t^2.

Ans. $v = v_0 + at$. Here $v_0 = 0$ so that $v = at$.

Now $P = F \times v = (ma) \times (at) = ma^2t$

Since m and a are constant, $P \propto t$. Therefore, option (*ii*) is correct.

Q.10. A body is moving unidirectionally under the influence of a source of constant power. Its displacement in time t is proportional to (*i*) $t^{1/2}$ (*ii*) t (iii) $t^{3/2}$ (*iv*) t^2.

Ans. Power = Force × Velocity. It is given that power is constant.

$$\therefore \quad [P] = [MLT^{-2}]\,[LT^{-1}] = [ML^2T^{-3}] = \text{Constant. Now, } M \text{ is constant.}$$

$$\therefore \quad L^2\,T^{-3} = \text{Constant or } \frac{L^2}{T^3} = \text{Constant or } L^2 \propto T^3 \text{ or } L \propto T^{3/2}$$

Therefore, displacement (L) is proportional to $t^{3/2}$.

Q.11. A body constrained to move along the Z-axis of a coordinate system is subjected to a constant force $\vec{F}$ given by; $\vec{F} = (-\hat{i} + 2\hat{j} + 3\hat{k})$ N.

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the X, Y and Z axes of the system respectively. What is the work done by the force in moving the body a distance of 4 m along Z-axis?

Ans. Since the body moves 4 m along Z-axis, $\vec{S} = 4\hat{k}$.

$\therefore$ Work done, $W = \vec{F}.\vec{S} = (-\hat{i}+2\hat{j}+3\hat{k}).(4\hat{k}) = 0 + 0 + 3 \times 4 = 12$ J

Q.12. An electron and a proton are detected in a cosmic ray experiment, the first with kinetic energy 10 keV and the second with 100 keV. Which is faster, the electron or the proton? Obtain the ratio of their speeds. (electron mass = 9.11×10^{-31} kg, proton mass = 1.67×10^{-27} kg, 1 eV = 1.60×10^{-19}J).

Ans. $$E_e = \frac{1}{2}m_e v_e^2 = 10\times1.6\times10^{-16}\text{ J};\quad E_p = \frac{1}{2}m_p v_p^2 = 100\times1.6\times10^{-16}\text{J}$$

$$\therefore \quad \frac{E_e}{E_p} = \frac{m_e v_e^2}{m_p v_p^2} = \frac{1}{10} \quad \therefore \quad \frac{v_e}{v_p} = \sqrt{\frac{1}{10}\frac{m_p}{m_e}} = \sqrt{\frac{1}{10}\times\frac{1.67\times10^{-27}}{9.11\times10^{-31}}} = 13.54$$

Therefore, electron moves faster than the proton.

Q.13. A rain drop of radius 2 mm falls from a height of 500 m above the ground. It falls with decreasing acceleration (due to viscous resistance of the air) until at half its original height, it attains its maximum (terminal) speed and moves with uniform speed thereafter. What is the work done by the gravitational force on the drop in the first and second half of its journey? What is the work done by the resistive force in the entire journey if its speed on reaching the ground is 10 ms^{-1}.

Ans. Gravitational force on the rain drop = mg. Since mass m of the drop and g remain constant, the drop experiences a constant gravitational force throughout the journey. Therefore, work done by the gravitational force on the drop in each half of the journey is the same.

$$\text{Mass of drop, } m = \frac{4}{3}\pi r^3\rho$$

Here, ρ = density of water = 10^3 kgm^{-3}; r = 2 mm = 2×10^{-3}m

$$\therefore \quad m = \frac{4}{3}\pi\times(2\times10^{-3})^3\times10^3 = 33.5\times10^{-6}\text{ kg}$$

Work done by gravitational force during each half is

$$W = FS\cos\theta$$

Here, $$F = mg = 33.5\times10^{-6}\times9.8\text{ N};\quad S = \frac{500}{2} = 250\text{ m};\quad \theta = 0°$$

$$\therefore \quad W = (33.5\times10^{-6}\times9.8)\times250\times\cos0° = \mathbf{0.082\text{ J}}$$

Original energy of drop, $E_1 = mgh = 33.5 \times 10^{-6} \times 9.8 \times 500 = 0.164$ J

Final energy of drop, $E_2 = \frac{1}{2}mv^2 = \frac{1}{2}\times33.5\times10^{-6}\times(10)^2 = 1.675 \times 10^{-3}$ J

$\therefore$ W.D. by resistive force = $E_1 - E_2 = 0.164 - 1.675 \times 10^{-3} = \mathbf{0.1623}$ J

Q.14. A molecule in a gas container hits a horizontal wall with speed 200 ms^{-1} and angle 30° with the normal, and rebounds with the same speed. Is momentum conserved in the collision? Is the collision elastic or inelastic?

Ans. Fig. 6.04 shows the conditions of the problem. Here m is the mass of the gas molecule and M the mass of the wall of the container. We know that momentum is conserved in all types of collision (elastic or inelastic). Therefore, momentum is conserved in this collision. We now check whether K.E. is conserved or not. As the mass of the wall is very large as compared to that of the gas molecule, the gas molecule will recoil with the same velocity (*i.e.*, 200 ms^{-1}) but the wall remains at rest.

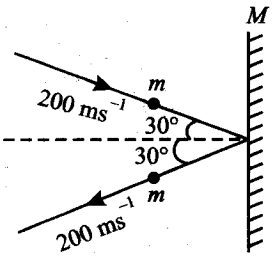

Fig. 6.04

$\therefore$ Total K.E. after collision= $\frac{1}{2}m(200)^2 + \frac{1}{2}M(0)^2 = 2 \times 10^4\ m$ J

Total K.E. before collision = $\frac{1}{2}m(200)^2 = 2 \times 10^4\ m$ J

Thus the K.E. of the molecule after collision is the same as before collision. Hence, collision is elastic.

Q.15. A pump on the ground floor of a building can pump up water to fill a tank of volume 30 m^3 in 15 minutes. If the tank is 40 m above the ground and the efficiency of the pump is 30%, how much electric power is consumed by the pump?

Ans. Mass of water pumped, m = volume × density = 30 × 1000 = 3 × 10^4 kg

$$\therefore \text{ Output power, } P_0 = \frac{\text{Work done}}{\text{Time}} = \frac{mgh}{t} = \frac{3\times10^4\times9.8\times40}{15\times60} = 13070 \text{ W}$$

$$\therefore \text{ Input power } = \frac{P_0}{\eta} = \frac{13070}{0.3} = 43.6\times10^3 \text{ W} = 43.6 \text{ kW}$$

Q.16. Two identical ball bearings in contact with each other and resting on a frictionless table are hit head-on by another ball bearing of the same mass moving initially with a speed v. If the collision is elastic, which of the following (See Fig. 6.05) is a possible result after collision?

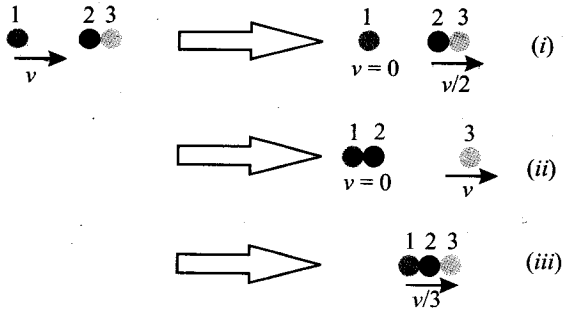

Fig. 6.05

Ans. Let m be the mass of each ball bearing.

Total K.E. of the system before collision = $\frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

Let us see in which case K.E. is conserved after collision.

For case (i) : $E_1 = \frac{1}{2}(2m)(v/2)^2 = \frac{1}{4}mv^2$

For case (ii) : $E_2 = \frac{1}{2}mv^2$

For case (iii) : $E_3 = \frac{1}{2}(3m)(v/3)^3 = \frac{1}{6}mv^2$

We see that K.E. is conserved only in case (*ii*). Hence case (*ii*) is the only possible result.

Q.17. The bob A of a pendulum released from 30° to the vertical hits another bob B of the same mass at rest on a table as shown in Fig. 6.06. How high does the bob A rise after the collision? Neglect the size of the bobs and assume the collision to be elastic.

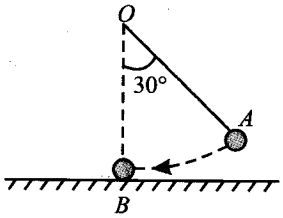

Fig. 6.06

Ans. The bob A of the pendulum will not rise. The reason is simple. We know that when two bodies of the same mass undergo an elastic collision, their velocities are interchanged. Therefore, after collision, the bob A will come to rest and the bob B would move with the velocity of bob A acquired at the instant of collision.

Q.18. The bob of a pendulum is released from a horizontal position A as shown in Fig. 6.07. If the length of the pendulum is 1.5 m, what is the speed with which the bob arrives at the lowermost point B? Given that it dissipates 5% of its initial energy against air resistance.

Ans. Fig. 6.07 shows the conditions of the problem. We shall solve this example using energy conservation principle.

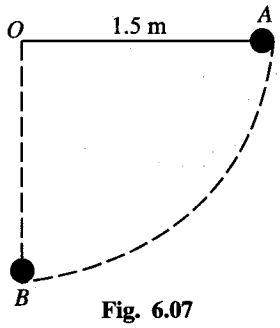

Fig. 6.07

Vertical height of bob, h = 1.5 m

Suppose m is the mass of the bob. When the bob is at position A, the energy is mgh and it is wholly potential energy. As the bob is released from position A, the potential energy of bob starts converting into kinetic energy. At the lowest position B, the energy of the bob is wholly kinetic. Since 5% energy is lost against friction,

$\therefore$ K.E. of bob at B = 0.95 (mgh) ... (i)

If v is the velocity acquired by the bob at B, then,

$$\text{K.E. of bob at } B = \frac{1}{2}mv^2 \quad \ldots (ii)$$

From eqs. (i) and (ii), $\frac{1}{2}mv^2 = 0.95(mgh)$ $\therefore v = \sqrt{2\times 0.95\, gh} = \sqrt{2\times 0.95\times 9.8\times 1.5} = 5.3 \text{ ms}^{-1}$

Q.19. A trolley of mass 300 kg carrying a sand bag of 25 kg is moving uniformly with a speed of 27 km/h on a frictionless track. After a while, sand starts leaking out of a hole on the floor of the trolley at the rate of 0.05 kg s^{-1}. What is the speed of the trolley after the entire sand bag is empty ?

Ans. Since the trolley carrying the sand bag is moving with a uniform speed, external force on the system is zero ($F = ma$ and $a = 0$). If the sand leakes out of a hole on the floor of the trolley, it does not lead to application of any external force on the trolley. Therefore, the speed of the trolley will not change even after the sand bag is empty.

Q.20. A body of mass 0.5 kg travels in a straight line with velocity $v = ax^{3/2}$ where $a = 5\ m^{-1/2}\ s^{-1}$. What is the work done by the net force during its displacement from x = 0 to x = 2 m?

Ans. Here, m = 0.5 kg ; $v = ax^{3/2}$; $a = 5\ ms^{-1/2}\ s^{-1}$

Initial velocity at x = 0 m is $v_1 = a \times 0 = 0$

Final velocity at x = 2 m is $v_2 = a\ 2^{3/2} = 5 \times 2^{3/2} = 10\sqrt{2}$ ms^{-1}

Work done, W = Change in K.E. = $\frac{1}{2}m(v_2^2 - v_1^2) = \frac{1}{2}\times 0.5\left[(10\sqrt{2})^2 - 0^2\right]$ = 50 J

Q.21. The blades of a windmill sweep out a circle of area A. (a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through it in time t? (b) What is the kinetic energy of the air? (c) Assume that the windmill converts 25% of the wind's energy into electrical energy, and that A = 30 m^2, v = 36 km/h and the density of air is 1.2 kg m^{-3}. What is the electrical power produced?

Ans. (a) Volume of air passing through the circular area per second, $V = Av$ and mass of air passing through the circular area per second = $Av\rho$ where ρ is the density of air.

$\therefore$ Mass of air passing in time $t = Av\rho t$

(b) K.E. of air = $\frac{1}{2}mv^2 = \frac{1}{2}(Av\rho t)\,v^2 = \frac{1}{2}Av^3\rho t$

(c) Electrical energy produced = 25% of K.E. of air = $0.25\times\frac{1}{2}Av^3\rho t = \frac{1}{8}Av^3\rho t$

$\therefore$ Electric power produced, $P = \frac{\text{Energy}}{\text{Time}} = \frac{1}{8}Av^3\rho$

Here A = 30 m^2; v = 36 km/h = 10 ms^{-1}; ρ = 1.2 kg/m^3

$\therefore$ $P = \frac{1}{8}\times 30\times(10)^3\times 1.2$ = 4500 W = **4.5 kW**

Q.22. A person trying to lose weight (dieter) lifts a 10 kg mass, one thousand times, to a height of 0.5 m each time. Assume that the potential energy lost each time she lowers the mass is dissipated. (*a*) How much work does she do against the gravitational force? (*b*) Fat supplies 3.8×10^7 J of energy per kilogram which is converted to mechanical energy with a 20% efficiency rate. How much fat will the dieter use up?

Ans. (*a*) Work done W against the gravitational force is

$$W = n(mgh) = 1000\ (10 \times 9.8 \times 0.5) = \mathbf{49000\ J}$$

(*b*) Mechanical energy supplied by 1 kg of fat

$$= 20\% \text{ of } 3.8 \times 10^7 \text{J} = 0.76 \times 10^7 \text{J/kg}$$

$$\therefore \quad \text{Fat used up by dieter} = \frac{49000}{0.76 \times 10^7} = 6.45 \times 10^{-3} \text{ kg}$$

Q.23. A family uses 8 kW of power. (*a*) Direct solar energy is incident on the horizontal surface at an average rate of 200 W per square metre. If 20% of this energy can be converted to useful electrical energy, how large an area is needed to supply 8 kW? (*b*) Compare this area to that of the roof of a typical house.

Ans. (*a*) Let the required surface area be A square metres.

$\therefore$ Solar energy incident per second = 200 A watts

Electric power produced = 20% of 200 A = 40 A watts

$$\therefore \quad 40 \text{ A} = 8000 \text{ or } \text{A} = 8000/40 = \mathbf{200\ m^2}$$

(*b*) This area is comparable to the area of roof of a small house built in a 10 m × 25 m plot.

Q.24. A bullet of mass 0.012 kg and horizontal speed 70 ms^{-1} strikes a block of wood of mass 0.4 kg and instantly comes to rest with respect to the block. The block is suspended from the ceiling by means of thin wires. Calculate the height to which the block rises. Also, estimate the amount of heat produced in the block.

Ans. $m_1 = 0.012$ kg ; $m_2 = 0.4$ kg ; $u_1 = 70\ \text{ms}^{-1}$; $u_2 = 0$.

When the bullet strikes the block of wood, the bullet comes to rest *w.r.t.* the block and the combination behaves as a single unit of mass $(m_1 + m_2)$. Let v be the velocity acquired by the combination. Then, according to principle of conservation of linear momentum, we have,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v \quad \text{or} \quad m_1u_1 + m_2 \times 0 = (m_1 + m_2)v$$

$$\therefore \quad v = \frac{m_1u_1}{m_1 + m_2} = \frac{0.012 \times 70}{0.012 + 0.4} = 2.04\ \text{ms}^{-1}$$

Let the block rise to a height h. Then, according to principle of conservation of energy, we have,

K.E. of combination = P.E. of combination

$$\text{or} \quad \frac{1}{2}(m_1 + m_2)v^2 = (m_1 + m_2)gh$$

$$\therefore \quad h = \frac{v^2}{2g} = \frac{(2.04)^2}{2 \times 9.8} = \mathbf{0.212\ m}$$

The loss of K.E. of the system will appear as heat in the block.

$\therefore$ Heat produced = Initial KE. of bullet – Final K.E. of combination

$$= \frac{1}{2}m_1u_1^2 - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2} \times 0.012 \times (70)^2 - \frac{1}{2} \times (0.012 + 0.4) \times (2.04)^2 = \mathbf{28.54\ J}$$

Q.25. Two inclined frictional tracks, one gradual and the other steep meet at A from where two stones are allowed to slide down from rest, one on each track (See Fig. 6.08). Will the stones reach the bottom at the same time? Will they reach there with the same speed? Explain. Given $\theta_1 = 30°$, $\theta_2 = 60°$ and $h = 10$ m, what are the speeds and times taken by the two stones?

Ans. As shown in Fig. 6.08, the two inclined planes (AB and AC) have the same height h. The two stones of masses m_1 and m_2 are allowed to slide down from rest from point A along the tracks AB and AC respectively. As the stones reach the bottom of the inclined planes, their entire P.E. is converted into K.E.

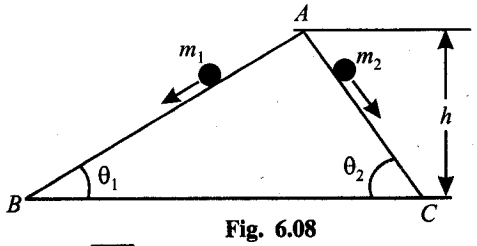

Fig. 6.08

For stone of mass m_1, $m_1gh = \frac{1}{2}m_1v_1^2 \quad \therefore \quad v_1 = \sqrt{2gh}$

For stone of mass m_2, $m_2gh = \frac{1}{2}m_2v_2^2 \quad \therefore \quad v_2 = \sqrt{2gh}$

As $v_1 = v_2 = \sqrt{2gh}$, the two stones will reach the bottom with the same speed.

For the stone m_1 sliding down the plane AB, acceleration $a_1 = g\sin\theta_1$ and for the stone m_2 sliding down the plane AC, acceleration $a_2 = g\sin\theta_2$.

As $\theta_2 > \theta_1$, therefore, $a_2 > a_1$.

Now $\qquad v = v_0 + at \quad \text{or} \quad v = 0 + at \quad \therefore t = v/a$

Since v is the same for the two stones, $t \propto 1/a$.

Now $\qquad t_1 \propto \frac{1}{a_1}$ and $t_2 \propto \frac{1}{a_2} \quad \therefore \frac{t_1}{t_2} = \frac{a_2}{a_1}$

Since $a_2 > a_1$, $t_2 < t_1$ *i.e.*, the stone (m_2) sliding down the plane AC will reach the bottom earlier. Therefore, the two stones will reach the bottom at different times.

Speed of each stone $= \sqrt{2gh} = \sqrt{2\times9.8\times10} = 14\ \text{ms}^{-1}$

Time taken by m_1, $t_1 = \frac{v}{g\sin\theta_1} = \frac{14}{9.8\sin30°} = 2.86\ \text{s}$

Time taken by m_2, $t_2 = \frac{v}{g\sin\theta_2} = \frac{14}{9.8\sin60°} = 1.65\ \text{s}$

Q.26. A 1 kg block situated on a rough incline is connected to a spring of spring constant 100 Nm^{-1} as shown in Fig. 6.09. The block is released from rest with the spring in the unstretched position. The block moves 10 cm down the incline before coming to rest. Find the coefficient of friction between the block and the incline. Assume that the spring has a negligible mass and the pulley is frictionless.

Ans. Let μ be the coefficient of friction between the block and the incline. To start with, the spring is in the unstretched position. As the block moves down the incline from rest, the spring is stretched. The work done in moving the block down the plane is stored in the spring in the form of potential energy (P.E.). Let us first find the work done on the block in moving it down the plane through distance x = 10 cm = 0.1 m. The component $mg\sin\theta$ of weight of block ($= mg$) acts down the plane and force of friction f ($= \mu R = \mu\, mg\cos\theta$) acts up the plane.

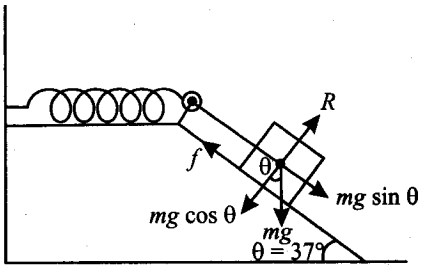

Fig. 6.09

$\therefore$ Net force on the block, $F = mg\sin\theta - f = mg\sin\theta - \mu\, mg\cos\theta = mg(\sin\theta - \mu\cos\theta)$

$\therefore$ Work done on the block, $W = Fx = mgx(\sin\theta - \mu\cos\theta)$

The work done W on the block is equal to the potential energy ($= kx^2/2$) stored in the spring, *i.e.*, $W = kx^2/2$.

$\therefore \qquad mgx(\sin\theta - \mu\cos\theta) = \frac{1}{2}kx^2$

or $1 \times 9.8 \times 0.1(\sin 37° - \mu \cos 37°) = \frac{1}{2} \times 100 \times (0.1)^2$

On solving, we get, $\mu = 0.112$

Q.27. A bolt of mass 0.3 kg falls from the ceiling of an elevator moving down with uniform speed of 7 ms^{-1}. It hits the floor of the elevator (length of the elevator = 3 m) and does not rebound. What is the heat produced by the impact? Would your answer be different if the elevator were stationary?

Ans. Here $m = 0.3$ kg ; $v_0 = 7$ ms^{-1}; $h = 3$ m.

As the elevator is moving down with uniform speed, the acceleration is zero. Therefore, the bolt strikes the floor of the elevator with the same speed (*i.e.*, 7 ms^{-1}).

P.E. of bolt at ceiling = $mgh = 0.3 \times 9.8 \times 3 = 8.82$ J

On impact, only the P.E. of the bolt is converted into heat energy. Since the bolt does not rebound, entire P.E. of the bolt is converted into heat.

$\therefore$ Heat produced = P.E. lost by bolt = **8.82 J**

The answer will not be different if the elevator were stationary. It is because in this case also the acceleration is zero.

Q.28. A trolley of mass 200 kg moves with a uniform speed of 36 km/h on a frictionless track. A child of mass 20 kg runs on the trolley from one end to the other (10 m away) with a speed of 4 ms^{-1} relative to the trolley in a direction opposite to trolley's motion and jumps out of the trolley. What is the final speed of the trolley? How much the trolley has moved from the time the child begins to run?

Ans. Mass of trolley, $M = 200$ kg; Speed of trolley, $v = 36$ km/h = 10 ms^{-1}; Mass of child, $m = 20$ kg.

Momentum of the system before the child starts running is

$$p_1 = (M+m)v = (200+20)10 = 2200 \text{ kgms}^{-1}$$

Suppose v' is the final speed of the trolley when the child starts running at a speed of 4 ms^{-1} relative to the trolley in a direction opposite to the motion of the trolley. Clearly, the speed of child *w.r.t.* earth is $(v' - 4)$.

$\therefore$ Momentum of the system when child is running

$$p_2 = Mv' + m(v'-4) = 200v' + 20(v'-4) = 220v' - 80$$

Since there is no external force, according to principle of conservation of momentum,

$$p_2 = p_1 \text{ or } 220v' - 80 = 2200 \quad \therefore \; v' = 2280/220 = \mathbf{10.36} \text{ ms}^{-1}$$

The time t taken by the child to run from one end to the other is $t = 10/4 = 2.5$ s

$\therefore$ Distance moved by the trolley in 2.5 s = $10.36 \times 2.5 \simeq 25.9$ m

Q.29. Which of the following potential energy curves in Fig. 6.010 cannot possibly describe the elastic collision of two billiard balls? Here r is the distance between centres of the balls.

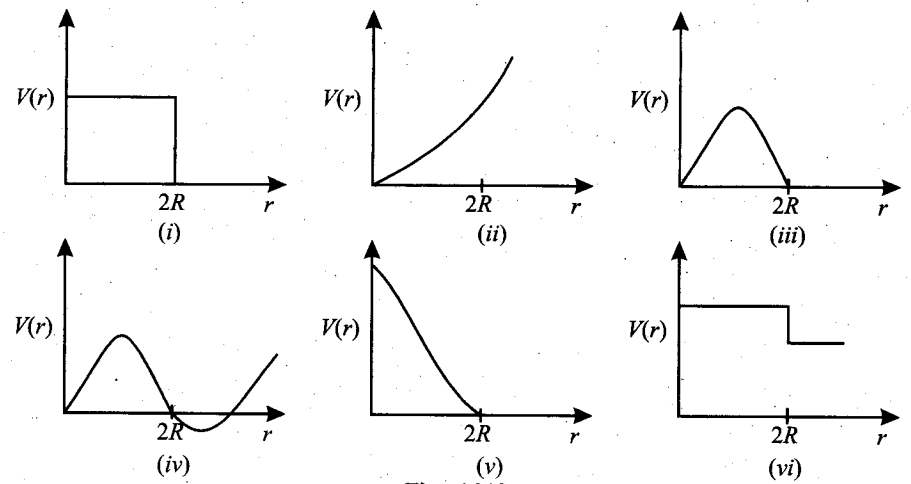

Fig. 6.010

Ans. The potential energy $[V(r)]$ of a system of two masses varies inversely as the distance (r) between their centres *i.e.*, $V(r) \propto 1/r$. When the two balls touch each other, then P.E. becomes zero *i.e.*, at $r = R + R = 2R$, $V(r) = 0$. As can be seen from Fig. 6.010, only curve (v) meets these two conditions. Therefore, all potential energy curves except curve (v) cannot describe the elastic collision of two billiard balls.

Q.30. Consider the decay of a free neutron at rest: $n \rightarrow p + e^{-1}$. Show that the two-body decay of this type must necessarily give an electron of fixed energy and, therefore, cannot account for the observed continuous energy distribution in the β-decay of a neutron or a nucleus (See Fig. 6.011).

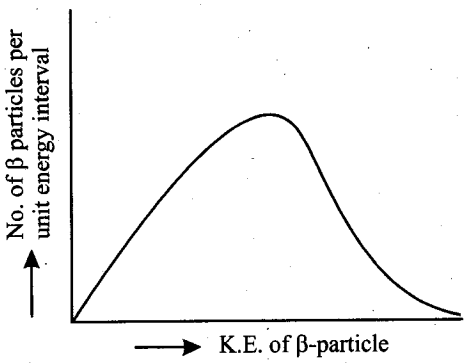

Fig. 6.011

Ans. In the given decay process, $n \rightarrow p + e^-$

Energy of emitted electron = $\Delta m\ c^2$

Here Δm = Mass defect = Mass of neutron – Mass of proton and electron

Therefore, according to the given decay, the electron $(\bar{e})$ coming out of the nucleus should always possess a fixed amount of kinetic energy. However, the graph in Fig. 6.011 shows that the emitted electron can have any value of energy between zero and maximum value. Therefore, the given decay process cannot account for the observed continuous energy spectrum in the β-decay of neutron or nucleus.

Unit VI

MOTION OF SYSTEM OF PARTICLES AND RIGID BODY

- Centre of Mass and Rotational Motion
- Moment of Inertia
- N.C.E.R.T. Textbook Exercises

13

Centre of Mass and Rotational Motion

INTRODUCTION

In the previous chapters, we treated an object as a point object or a particle. However, actual objects are not point objects. They have a finite shape and are made up of a very large number of particles. In order to study the motion of a body (or object), we have to apply Newton's second law of motion. But it is not possible to apply Newton's second law of motion to every individual particle of the body. In order to overcome this difficulty, we have introduced the important concept of *centre of mass* (C.M.) of a body or system of particles. We can show that the motion of the centre of mass describes the motion of the body as a whole. Thus, the concept of centre of mass enables us to treat a real object as a point object. In this chapter, we shall deal with centre of mass and rotational motion.

13.1. EXTENDED BODIES

A body whose mass is distributed in space is called an extended body.

A real or extended body consists of a large number of particles. The forces exerted by the various particles on one another are called *internal forces.* The internal forces between any two particles make up an action-reaction pair and by Newton's third law of motion, are equal in magnitude but opposite in direction. As a result, the sum of all the internal forces in a body is zero.

In order to move or stop a body, we have to apply force on the body from outside. The force exerted on the body from outside is called the *external force.* The motion of the body is governed by external forces only.

13.2. CENTRE OF MASS

The centre of mass of a body (or system of particles) is defined as a point at which the entire mass of the body is supposed to be concentrated.

Consider a body of mass m. It can be shown that if net external force $\vec{F}$ is applied at the centre of mass (C.M.) of the body, then the C.M. moves just as a *particle* of mass m would move if it were acted on by a force $\vec{F}$. It means the centre of mass (C.M.) of a body is that point at which all the *mass* of the body can be considered to be concentrated while applying Newton's second law. In other words, the motion of the centre of mass represents the motion of the body as a whole. Thus we can replace the whole body by a single particle located at the centre of mass. It is for this reason that we have been free and easy about treating extended bodies (trains, cars etc.) as if they were particles.

For a given object, the centre of mass is the average position of all the particles of mass that constitute the object. For example, the centre of mass of a symmetrical object with uniform composition (*e.g.,* sphere, cylinder, rod etc.) is at its geometrical centre. In contrast, an irregularly shaped object such as a baseball bat has more of its mass toward one end. Therefore, the centre of mass of a baseball bat is toward the thicker end.

13.3. CONCEPT OF CENTRE OF MASS

The position of the C.M. of a body can be determined rather easily. For simplicity, we consider the motion along X-axis ; vector notation need not be used. Suppose a system consists of a large number of particles having masses m_1, m_2, m_3 ... A few of these particles are shown in Fig. 13.1. For the various particles of the body, we have $F_1 = m_1 a_1$, $F_2 = m_2 a_2$, $F_3 = m_3 a_3$ and so on. Here F_1 is the net external force on mass m_1 and produces acceleration a_1 and so on.

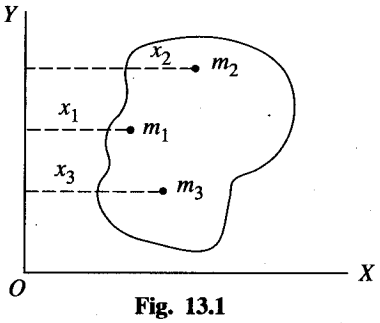

Fig. 13.1

$$\therefore \quad F_1 + F_2 + F_3 + ... = m_1 a_1 + m_2 a_2 + m_3 a_3 + ...$$

Each acceleration is the second derivative of the corresponding displacement.

$$\therefore \quad F_1 + F_2 + F_3 + .. = m_1 \frac{d^2 x_1}{dt^2} + m_2 \frac{d^2 x_2}{dt^2} + m_3 \frac{d^2 x_3}{dt^2} + ...$$

or $$\Sigma F = \frac{d^2}{dt^2}(m_1 x_1 + m_2 x_2 + m_3 x_3 + ...)$$

or $$\frac{\Sigma F}{\Sigma m} = \frac{d^2}{dt^2}\left(\frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ...}{m_1 + m_2 + m_3 + ...}\right) \quad ... (i)$$

From this we see that x-coordinate of centre of mass is defined as :

$$x_{C.M.} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ...}{m_1 + m_2 + m_3 + ...}$$

Therefore eq. (*i*) becomes :

$$\frac{\Sigma F}{\Sigma m} = \frac{d^2}{dt^2}(x_{C.M.})$$

or $$\frac{\Sigma F}{\Sigma m} = a_{C.M.}$$

or $$a_{C.M.} = \frac{\Sigma F}{\Sigma m} = \frac{\text{Net force on the system}}{\text{Total mass of the system}}$$

Similar formulas can be written for the coordinates $y_{C.M.}$ and $z_{C.M.}$

Thus we have proved that the acceleration of C.M. is equal to the net force acting on the system divided by the total mass of the system.

This result tells us that the system of particles (or a body) moves as if all of its mass were concentrated at the C.M. and all external forces acted at that point. Thus we can replace a body of mass m by a single particle of mass m located at the centre of mass. In other words, we can treat the linear motion of any body as the motion of a particle.

13.4. CENTRE OF MASS OF TWO PARTICLE SYSTEM

Any extended body consists of a large number of tiny particles. For simplicity, we first find the position of the centre of mass for a system made up of only two particles. Consider a system

of two particles of masses m_1 and m_2 and having position vectors $\vec{r}_1$ and $\vec{r}_2$ *w.r.t.* the origin O of the coordinate system as shown in Fig. 13.2. The position vector $\vec{R}_{C.M.}$ of the centre of mass of this system is given by ;

$$\vec{R}_{C.M.} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \ldots (i)$$

or $$(m_1 + m_2)\, \vec{R}_{C.M.} = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

Thus the product of the total mass of the system and the position vector of the centre of mass is equal to the sum of the products of individual masses and their respective position vectors.

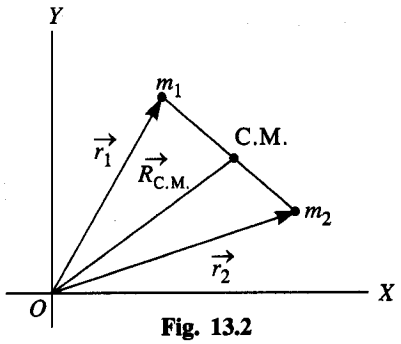

Fig. 13.2

Let us now derive eq. (*i*). Let $\vec{f}_1$ and $\vec{f}_2$ be the external forces acting on m_1 and m_2 respectively. There will be internal forces * $\vec{F}_{12}$ and $\vec{F}_{21}$ which the particles exert on each other. By Newton's third law,

$$\vec{F}_{12} = -\vec{F}_{21} \quad \text{or} \quad \vec{F}_{12} + \vec{F}_{21} = 0$$

Note that the sum of internal forces is zero. This is true even if the system consists of any number of particles.

Now total force on particle of mass m_1 is $\vec{F}_{12} + \vec{f}_1$. If $\vec{v}_1$ is the instantaneous velocity of the particle of mass m_1, then according to Newton's second law,

$$\frac{d}{dt}\left(m_1 \vec{v}_1\right) = \vec{F}_{12} + \vec{f}_1 \quad \ldots (ii)$$

Similarly, for particle m_2, we have,

$$\frac{d}{dt}\left(m_2 \vec{v}_2\right) = \vec{F}_{21} + \vec{f}_2 \quad \ldots (iii)$$

Adding eqs. (*ii*) and (*iii*), we have,

$$\frac{d}{dt}\left(m_1 \vec{v}_1\right) + \frac{d}{dt}\left(m_2 \vec{v}_2\right) = \left(\vec{F}_{12} + \vec{f}_1\right) + \left(\vec{F}_{21} + \vec{f}_2\right)$$

or $$\frac{d}{dt}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2\right) = \vec{f}_1 + \vec{f}_2 \qquad \left(\because \vec{F}_{12} + \vec{F}_{21} = 0\right)$$

Now $\vec{f}_1 + \vec{f}_2 =$ total net external force $\vec{F}_{ext}$ on the system.

$$\therefore \quad \vec{F}_{ext} = \frac{d}{dt}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2\right)$$

The velocity of a particle is defined as the time rate of change of its position vector.

$$\therefore \quad \vec{v}_1 = \frac{d}{dt}\left(\vec{r}_1\right) \text{ and } \vec{v}_2 = \frac{d}{dt}\left(\vec{r}_2\right)$$

$$\therefore \quad \vec{F}_{ext} = \frac{d}{dt}\left[m_1 \frac{d}{dt}\left(\vec{r}_1\right) + m_2 \frac{d}{dt}\left(\vec{r}_2\right)\right]$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\left(m_1 \vec{r}_1 + m_2 \vec{r}_2\right)\right]$$

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* $\vec{F}_{12}$ denotes the force exerted on particle 1 (m_1) by particle 2 (m_2) and $\vec{F}_{21}$ on particle 2 by particle 1.

$$\therefore \quad \vec{F}_{ext} = \frac{d^2}{dt^2}\left(m_1 \vec{r}_1 + m_2 \vec{r}_2\right) \qquad ...(iv)$$

Multiplying and dividing the R.H.S. of eq. (*iv*) by $(m_1 + m_2)$, we have,

$$\vec{F}_{ext} = (m_1 + m_2)\frac{d^2}{dt^2}\left(\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}\right)$$

or $$\vec{F}_{ext} = (m_1 + m_2)\frac{d^2 \vec{R}_{C.M.}}{dt^2} \qquad ...(v)$$

where $$\vec{R}_{C.M.} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad ...(vi)$$

Eq. (*v*) has the same form as Newton's second law for a single particle of mass $m_1 + m_2$ located at $\vec{R}_{C.M.}$ and under the influence of net external force $\vec{F}_{ext.}$. Thus eq. (*vi*) gives the position of centre of mass (C.M.) of the two particle system.

Discussion. The following points may be noted :

(*i*) The centre of mass moves like a particle of mass $M(=m_1 + m_2)$ under the influence of the net external force on the system. The internal forces cancel out in pairs.

(*ii*) Eq. (*vi*) expresses the centre of mass in the vector form. We can also express it in component form as

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \; ; \; y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

Here x and y are the coordinates of the centre of mass.

(*iii*) The centre of mass of two particle system always lies between the two particles and on the line joining them. This can be readily proved. If the centre of mass of the two particles were at the origin, then $\vec{R}_{C.M.} = 0$.

$$\therefore \quad \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0$$

or $$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

or $$\vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$$

Thus if r_1 is negative, then r_2 is positive. It means if m_1 lies to the left of the origin (C.M.), then m_2 lies to the right of the origin. Therefore, the centre of mass of a system of two particles always lies on the straight line joining these particles. Moreover, if $m_1 > m_2$ then $\vec{r}_1 < \vec{r}_2$. Thus the centre of mass of two particle system lies closer to the heavier particle.

(*iv*) If the masses of the two particles are equal, then,

$$\vec{R}_{C.M.} = \frac{m\vec{r}_1 + m\vec{r}_2}{m+m} \text{ or } \vec{R}_{C.M.} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

$$x = \frac{mx_1 + mx_2}{m+m} \text{ or } x = \frac{x_1 + x_2}{2}$$

$$y = \frac{my_1 + my_2}{m+m} \text{ or } y = \frac{y_1 + y_2}{2}$$

13.5. CENTRE OF MASS—GENERALISATION TO N PARTICLES

Consider a *system of N particles of masses $m_1, m_2, m_3....m_N$ and let their position vectors be $\vec{r}_1, \vec{r}_2, \vec{r}_3 ... \vec{r}_N$ respectively. If $\vec{R}_{\text{C.M.}}$ is the position vector of the centre of mass of the system of N particles, then,

$$(m_1 + m_2 + m_3 + ... + m_N)\vec{R}_{\text{C.M.}} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_N\vec{r}_N$$

or $$\vec{R}_{\text{C.M.}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_N\vec{r}_N}{m_1 + m_2 + m_3 + ... + m_N}$$

Suppose N particles of the system of masses $m_1, m_2, m_3 ... m_N$ are acted upon by external forces $\vec{f}_1, \vec{f}_2, \vec{f}_3 ... \vec{f}_N$ respectively. Every particle is also subjected to internal forces due to interactions with other particles. According to Newton's third law, for each internal force acting on one particle, there is an equal but opposite force acting on another particle. For example, if a particle m_1 exerts a force on particle m_2, then particle m_2 exerts an equal and opposite force on m_1. When we sum over all the particles in the system, the internal forces cancel out and we are left only with external forces.

$\therefore$ Net external force $\vec{F}_{ext}$ on the system of N particles is

$$\vec{F}_{ext} = \vec{f}_1 + \vec{f}_2 + \vec{f}_3 + ... + \vec{f}_N$$

$$= \frac{d}{dt}\left[m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_N\vec{v}_N\right]$$

$$= \frac{d}{dt}\left[m_1\frac{d}{dt}\left(\vec{r}_1\right) + m_2\frac{d}{dt}\left(\vec{r}_2\right) + m_3\frac{d}{dt}\left(\vec{r}_3\right) + ... + m_N\frac{d}{dt}\left(\vec{r}_N\right)\right]$$

$\therefore$ $$\vec{F}_{ext} = \frac{d^2}{dt^2}\left[m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_N\vec{r}_N\right] \quad ...(i)$$

Multiplying and dividing the R.H.S. of eq. (i) by total mass $M\left(=m_1 + m_2 + m_3 + ... + m_N\right)$, we have,

$$\vec{F}_{ext} = M\frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_N\vec{r}_N}{m_1 + m_2 + m_3 + ... + m_N}\right)$$

or $$\vec{F}_{ext} = M\frac{d^2\vec{R}_{\text{C.M.}}}{dt^2} \quad ... (ii)$$

where $$\vec{R}_{\text{C.M.}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_N\vec{r}_N}{m_1 + m_2 + m_3 + ... + m_N}$$

$$= \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{M} \quad ...(iii)$$

Eq. (ii) has the same form as Newton's second law for a single particle of mass $M(=m_1 + m_2 + m_3 + ... + m_N)$ located at $\vec{R}_{\text{CM}}$ under the influence of net external force $\vec{F}_{ext}$. Note that eq. (iii) gives the position of centre of mass (C.M.) of the system of N particles.

* This system could be extended body as being made up of N particles.

Conclusions. The following points are worth noting :

(*i*) The position of centre of mass depends upon the shape and size of the body and the distribution of its mass.

(*ii*) The centre of mass of the body may lie within or outside the body.

(*iii*) The net external force applied at the centre of mass produces only linear motion but no rotation.

(*iv*) The motion of the centre of mass represents the motion of the body as a whole. Therefore, we can replace the whole body by a single particle located at the centre of mass.

Note. For the general case of a system of N particles in space, we can express the co-ordinates of centre of mass in terms of the co-ordinates of the location of points of the N particles. Consider that N particles having masses $m_1, m_2, m_3 \ldots, m_N$ are distributed in space at the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) ..., (x_N, y_N, z_N) respectively. Then the co-ordinates (x, y, z) of the centre of mass of this system of N particles in space are given by

$$x_{\text{C.M.}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots m_Nx_N}{m_1 + m_2 + m_3 + \ldots\ m_N} = \frac{\sum_{i=1}^{i=N} m_i\, x_i}{\sum_{i=1}^{i=N} m_i}$$

$$y_{\text{C.M.}} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \ldots m_N\, y_N}{m_1 + m_2 + m_3 + \ldots\ m_N} = \frac{\sum_{i=1}^{i=N} m_i\, y_i}{\sum_{i=1}^{i=N} m_i}$$

$$z_{\text{C.M.}} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \ldots m_N\, z_N}{m_1 + m_2 + m_3 + \ldots\ m_N} = \frac{\sum_{i=1}^{i=N} m_i\, z_i}{\sum_{i=1}^{i=N} m_i}$$

Example 13.1. Two bodies of masses 0.8 kg and 1.2 kg are located at (1, –2) and (–3, 4) respectively. Find the coordinates of the centre of mass.

Solution. Here $m_1 = 0.8$ kg ; $m_2 = 1.2$ kg ; $x_1 = 1$; $y_1 = -2$; $x_2 = -3$; $y_2 = 4$.

Coordinates of centre of mass (C.M.) are :

$$x_{\text{C.M.}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{0.8\times1 + 1.2\times(-3)}{0.8 + 1.2} = \mathbf{-1.4}$$

$$y_{\text{C.M.}} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{0.8\times(-2) + 1.2\times4}{0.8 + 1.2} = \mathbf{1.6}$$

Therefore, coordinates of C.M. are (–1.4, 1.6).

Example 13.2. There is a bar 1.5 m long. (*i*) Suppose there are equal 60 kg masses at either end. (*ii*) Someone puts unequal masses of 50 kg and 80 kg on the ends. Where is the centre of mass in each case? Assume that the size of each of the masses is negligible compared with the distance between them.

Solution. (*i*) Fig. 13.3 (*i*) shows the conditions of the problem. If we choose the origin to lie at one of the masses, then $x_1 = 0$ and $x_2 = 1.5$ m.

$$\therefore \quad x_{C.M.} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{60\times0 + 60\times1.5}{60+60} = \mathbf{0.75\ m}$$

Clearly, centre of mass lies at the centre of the system.

Fig. 13.3

(*ii*) Fig. 13.3 (*ii*) shows the conditions of the problem. We choose the origin to lie at the 50 kg mass. Therefore, $x_1 = 0$ and $x_2 = 1.5$ m.

$$\therefore \quad x_{C.M.} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{50\times0 + 80\times1.5}{50+80} = \mathbf{0.92\ m}$$

The answer makes sense; the centre of mass is closer to the more massive constituent of the system. Our choice of origin at one of the masses was convenient because it simplifies calculations. But that choice was purely for convenience; it cannot influence the actual physical location of the centre of mass.

Example 13.3. Three particles each of mass 2.5 kg are located at the corners of a right angled triangle whose sides are 2 m and 1.5 m long as shown in Fig. 13.4. Locate the centre of mass.

Solution. We choose our coordinate system as shown (to simplify calculations) with m_1 at the origin, m_2 on the x-axis. Then m_1 has coordinates $x_1 = 0$; $y_1 = 0$; m_2 has coordinates $x_2 = 2$ m, $y_2 = 0$ and m_3 has coordinates $x_3 = 2$ m; $y_3 = 1.5$ m. We are given that $m_1 = m_2 = m_3 = 2.5$ kg.

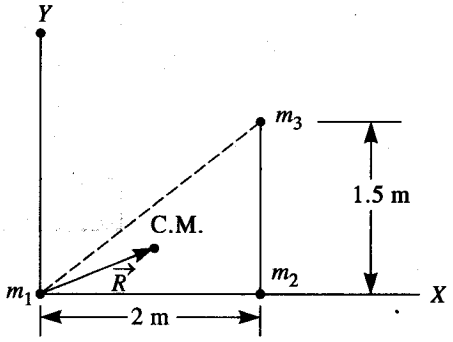

Fig. 13.4

$$x_{C.M.} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{2.5\times0 + 2.5\times2 + 2.5\times2}{2.5+2.5+2.5} = \mathbf{1.33\ m}$$

$$y_{C.M.} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{2.5\times0 + 2.5\times0 + 2.5\times1.5}{2.5+2.5+2.5} = \mathbf{0.5\ m}$$

Thus the x and y coordinates of C.M. are 1.33 m, 0.5 m. The C.M. and position vector $\vec{R}$ are shown in Fig. 13.4.

Example 13.4. Four equal masses m lie at the corners of a rectangle of width a and length b. Where is the centre of mass ?

Solution. A convenient coordinate system has two sides of the rectangle along x-axis and y-axis with one corner at the origin as shown in Fig. 13.5.

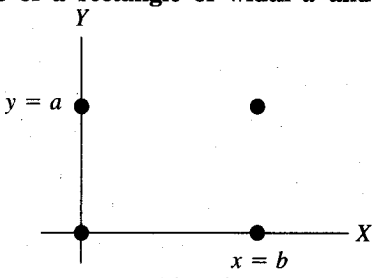

Fig. 13.5

$$x_{C.M.} = \frac{\Sigma m_i x_i}{M} = \frac{m(0+b+b+0)}{4m} = \frac{b}{2}$$

$$y_{C.M.} = \frac{\Sigma m_i y_i}{M} = \frac{m(0+0+a+a)}{4m} = \frac{a}{2}$$

In both cases, we summed going counter-clockwise around the rectangle, starting at the origin. The answer is hardly surprising; the centre of mass lies at the centre of the rectangle.

Example 13.5. Two bodies of masses 100 g and 300 g have position vectors $2\hat{i}+5\hat{j}+13\hat{k}$ and $-6\hat{i}+4\hat{j}-2\hat{k}$ respectively. Find (*i*) position vector of centre of mass (*ii*) distance of centre of mass from the origin.

Solution. Here m_1 = 100 g ; m_2 = 300 g ; $\vec{r}_1 = 2\hat{i}+5\hat{j}+13\hat{k}$; $\vec{r}_2 = -6\hat{i}+4\hat{j}-2\hat{k}$

(*i*) The position vector of the centre of mass is

$$\vec{R}_{\text{C.M.}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{100(2\hat{i}+5\hat{j}+13\hat{k}) + 300(-6\hat{i}+4\hat{j}-2\hat{k})}{100+300}$$

$$= -4\hat{i} + \frac{17}{4}\hat{j} + \frac{7}{4}\hat{k} \text{ units}$$

(*ii*) The distance of centre of mass from the origin is

$$|\vec{R}| = \sqrt{(-4)^2 + (17/4)^2 + (7/4)^2} = 6.09 \text{ units}$$

Example 13.6. Two balls of masses 3 *m* and *m* are separated by a distance *l*. Find the position of centre of mass.

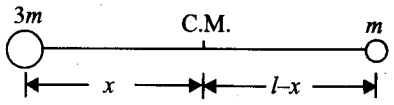

Fig. 13.6

Solution. Let C.M. of the system be at a distance x from the ball of mass $3m$ (See Fig. 13.6). Then distance of the ball of mass m from C.M. is $l - x$. According to the definition of centre of mass,

$$3m \times x = m(l - x)$$

$$\therefore \quad x = l/4$$

PROBLEMS FOR PRACTICE

1. The distance between the centres of carbon and oxygen atoms in carbon monoxide gas molecule is 1.13 Å. Locate the centre of mass of the molecule relative to carbon atom.
[0.602 Å]

2. Two bodies of masses 2 g and 10 g have position vectors $3\hat{i}+2\hat{j}-\hat{k}$ and $\hat{i}-\hat{j}+3\hat{k}$ respectively. Find the position vector and distance of C.M. from the origin.
$\left[\frac{1}{6}(18\hat{i}-3\hat{j}+14\hat{k});\ 2.73 \text{ units}\right]$

3. A 2 kg body and a 3 kg body are moving along the x-axis. At a particular instant, the 2 kg body is 1 m from the origin and has a velocity of 3 m/s and the 3 kg body is 2 m from the origin and has a velocity of –1 m/s. Find the position of the centre of mass.
[x = 1.6 m]

13.6. MOTION OF CENTRE OF MASS OF A SYSTEM

Consider a system of N particles of total mass $M(=m_1+m_2+m_3+.....+m_N)$. If $\vec{R}_{CM}$ is the position vector of the centre of mass, then,

$$M\vec{R}_{\text{CM}} = \Sigma m_i \vec{r}_i \qquad ...(i)$$

Here m_i is the mass of the ith particle and $\vec{r}_i$ is the position vector of the ith particle *w.r.t.* the origin O of the coordinate system.

Differentiating eq. (*i*) *w.r.t.* time, we have,

$$M\frac{d\vec{R}_{\text{C.M.}}}{dt}=\Sigma m_i\frac{d\vec{r}_i}{dt}$$

or $$M\vec{V}_{\text{C.M.}}=\Sigma m_i\vec{v}_i \quad \ldots (ii)$$

Here $\vec{v}_i = d\vec{r}_i/dt$ is the velocity of the ith particle and $\vec{V}_{C.M.}$ is the velocity of the centre of mass.

Differentiating equation (*ii*) *w.r.t.* time, we have,

$$M\frac{d\vec{V}_{\text{C.M.}}}{dt}=\Sigma m_i\frac{d\vec{v}_i}{dt}$$

or $$M\vec{a}_{\text{C.M.}}=\Sigma m_i\vec{a}_i$$

Here $\vec{a}_i = d\vec{v}_i/dt$ is the acceleration of ith particle and $\vec{a}_{C.M.}$ is the acceleration of the centre of mass. By Newton's second law, $m_i\vec{a}_i=\vec{f}_i$ where $\vec{f}_i$ is the net force on the ith particle. Since internal forces on the particles cancel out,

$\therefore$ $M\vec{a}_{\text{C.M.}}=$ Net external force on the system of particles

or $$M\vec{a}_{\text{C.M.}}=\vec{f}_1+\vec{f}_2+\vec{f}_3+\ldots+\vec{f}_N=\Sigma\vec{f}_i$$

or $$M\vec{a}_{\text{C.M.}}=\Sigma\vec{f}_i=\vec{F}_{ext}=\text{ Net external force on the system.}$$

Thus the centre of mass moves like a particle of mass $M = \Sigma m_i$ under the influence of the net external force F_{ext} on the system.

Note that motion of C.M. represents the motion of the body (or system of particles) as a whole. Thus we can replace the whole body by a single particle of mass M located at the C.M. of the body.

13.7. LINEAR MOMENTUM OF THE SYSTEM

Consider a system of N particles of total mass $M = m_1 + m_2 + m_3 + \ldots + m_N$. Assume that the particles have linear momentum $\vec{p}_1=m_1\vec{v}_1$, $\vec{p}_2=m_2\vec{v}_2\ldots\vec{p}_N=m_N\vec{v}_N$ where $\vec{v}_1$, $\vec{v}_2$ $\ldots\vec{v}_N$ are the velocities of the individual particles. Then the total linear momentum $\vec{P}$ of the system is

$$\vec{P}=m_1\vec{v}_1+m_2\vec{v}_2+\ldots\ldots+m_N\vec{v}_N$$

Now, $$M\vec{V}_{\text{C.M.}}=\Sigma m_i\vec{v}_i=\Sigma\vec{p}_i \quad \text{[See Art. 13.6]}$$

$\therefore$ $$\vec{P}=M\vec{V}_{C.M.} \quad \ldots(i)$$

Thus the total linear momentum of a system of particles is equal to the product of the total mass M and the velocity of the centre of mass of the system.

If we differentiate eq. (*i*) *w.r.t.* time, we get,

$$\frac{d\vec{P}}{dt}=M\frac{d\vec{V}_{\text{C.M.}}}{dt}=M\vec{a}_{\text{CM}}=\vec{F}_{ext}$$

$\therefore$ $$\frac{d\vec{P}}{dt}=\vec{F}_{ext}=M\vec{a}_{\text{C.M.}} \quad \ldots (ii)$$

Here $\vec{F}_{ext}$ is the net external force on the system. Eq. (*ii*) is *Newton's second law for a system of particles.*

13.8. LINEAR MOMENTUM CONSERVATION AND CENTRE OF MASS

We know that $\vec{F}_{ext} = M\vec{a}_{C.M.} = M\dfrac{d\vec{V}_{C.M.}}{dt}$

If net external force $\vec{F}_{ext} = 0$, then,

$$M\frac{d\vec{V}_{C.M.}}{dt} = 0$$

or $$\frac{d}{dt}\left(M\vec{V}_{C.M.}\right) = 0$$

or $$M\vec{V}_{C.M.} = \vec{P} = \Sigma m_i \vec{v}_i = \text{Constant}$$

Thus when no net external force is acting, the total linear momentum of the system of particles and the velocity of the centre of mass are constant. This important result is known as the *law of conservation of linear momentum.*

If the net external force on a system of particles is zero, the velocity of the centre of mass of the system is constant and the total momentum of the system is constant (conserved).

Let us interpret this important result. It means that if net external force acting on a system of particles is zero, its centre of mass will continue to move with constant velocity no matter what individual particles composing the system are doing. In other words, if the linear momentum of a system is constant, its centre of mass will move with a constant speed in a straight line.

Example 13.7. Two particles of mass 2 kg and 1 kg are moving along the same straight line with speeds 2 m/s and 5 m/s respectively. What is the velocity of centre of mass of the system if both the particles are moving (*i*) in the same direction (*ii*) in opposite directions?

Solution. Here m_1 = 2 kg ; m_2 = 1 kg

(*i*) For this case, v_1 = 2 m/s ; v_2 = 5 m/s ; $v_{C.M.}$ = ?

Now $$\vec{v}_{C.M.} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{2\times 2 + 1\times 5}{2+1} = 3 \text{ ms}^{-1}$$

(*ii*) For this case, v_1 = 2 ms^{-1} ; v_2 = –5 m/s ; $v_{C.M.}$ = ?

Again $$\vec{v}_{C.M.} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{2\times 2 + 1\times(-5)}{2+1} = -\frac{1}{3}\text{ms}^{-1}$$

The negative sign shows that velocity of centre of mass is in the direction of motion of 1 kg part.

13.9. EXAMPLES OF MOTION OF CENTRE OF MASS

An extended body consists of a large number of particles. Generally, the individual particles interact with each other through internal forces, thereby changing their individual velocities and momenta as time goes on. However, these interactions will not influence the motion of the centre of mass (C.M.). *In other words, the motion of the centre of mass of a body cannot be affected by the internal forces.* In the absence of external forces, the C.M. of a body at rest remains at rest and if the C.M. is in motion, it will maintain that motion. Moreover, if net external force does act, the C.M. will move according to Newton's second law of motion. We now illustrate the motion of centre of mass through some examples.

(*i*) If a baseball bat is tossed in the air, the motion of most parts of the bat is quite complex. The complicated nature of the path is not unexpected because a particle at the end of the bat is subjected to not only the force of gravity but also to the forces exerted on it by

the rest of the bat. But there is one point — the *centre of mass* — that follows a simple parabolic path (See Fig. 13.7). This is the same parabolic path that a projected particle follows when acted on only by the force of gravity (*i.e.* projectile motion).

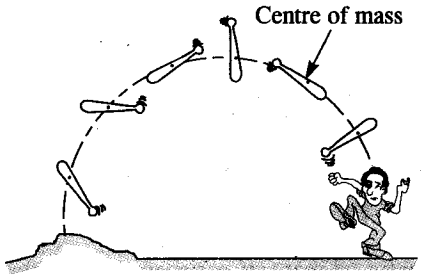

Fig. 13.7

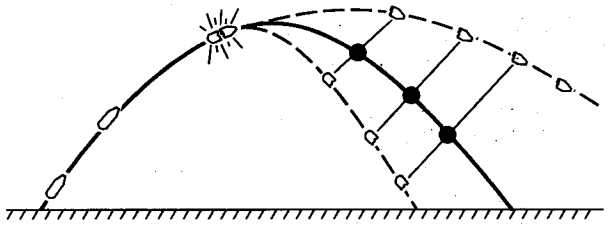

Fig. 13.8

(*ii*) Suppose a shell following parabolic path explodes in flight, splitting into two fragments of equal mass (See Fig. 13.8). The fragments follow new parabolic paths but the centre of mass continues in the same path as the shell's path before explosion. The internal forces during the explosion were certainly very large. However, they cancel in pairs because of action and reaction and therefore cannot affect the motion of the centre of mass.

(*iii*) A radioactive nucleus of uranium-235 decays spontaneously to thorium-231 by emitting an α-particle ($_2He^4$).

$$^{235}U \longrightarrow {}^{231}Th + \alpha$$

The uranium nucleus is initially at rest. Therefore, to conserve linear momentum, α-particle and Th^{231} fly off in opposite directions. Since the process of radioactive decay is due to internal forces, the centre of mass of the system remains at rest.

13.10. RIGID BODY

A **perfectly rigid body** *is defined as that body which does not undergo any change in shape and volume when external forces are applied on it.* The distance between any two particles of such a body will remain the same no matter how large force is applied on the body.

No body is perfectly rigid in the real world; every solid object is deformed to some degree by the force applied to it. Even a solid lying at rest on a surface is deformed by its own weight downward and upward normal force. *For all practical purposes, solid bodies are considered as perfectly rigid bodies.*

Note. The difference between a system of N particles and a rigid body may be noted. In a system of N particles, the particles can change their positions relative to one another. However, in a rigid body, the particles remain at their respective positions. Note that "a system of particles" includes a rigid body as a special case.

13.11. CENTRE OF MASS OF RIGID BODIES

The centre of mass of a rigid body is a point whose position is fixed w.r.t. the body. It does not change with time because the positions of particles in a rigid body remain fixed. The position of the centre of mass depends upon the shape and distribution of mass in the body. *It may lie within the body or outside it.* For a uniform solid sphere, the centre of mass lies at its geometrical centre. In this case, the centre of mass lies within the body. For a circular ring, the centre of mass lies at its geometrical centre. Note that at the centre of mass, there is no mass at all. In this case, the centre of mass lies outside the body.

(*i*) For symmetrically shaped bodies of uniform composition (*e.g.*, spheres, cylinders, rectangular solids), the centre of mass (C.M.) is located at the geometrical centre of the body.

(*ii*) For irregularly shaped bodies, the centre of mass can be found experimentally. This is done by making use of the fact that the centre of mass of any freely suspended body will lie on a vertical line directly below (or at) any point from which it is suspended. If the object is two dimensional, it needs only be hung from two different points and the respective vertical lines drawn. Then the centre of mass will be at the intersection of the two lines.

13.12. CENTRE OF MASS OF SOLID OBJECTS

We have seen that the position vector $\vec{R}_{C.M.}$ of the centre of mass of a system of particles is given by ;

$$\vec{R}_{C.M.} = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i} \qquad ...(i)$$

where $\vec{r}_i$ is the position and m_i the mass of the ith particle of the system and the sum is over all the particles in the system. Since Σm_i is the total mass M of the system, we have,

$$\vec{R}_{C.M.} = \frac{\Sigma m_i \vec{r}_i}{M}$$

In component form this become :

$$x_{C.M.} = \frac{\Sigma m_i x_i}{M} \; ; \; y_{C.M.} = \frac{\Sigma m_i y_i}{M} \; ; \; z_{C.M.} = \frac{\Sigma m_i z_i}{M} \qquad ...(ii)$$

where $x_{C.M.}$, $y_{C.M.}$ and $z_{C.M.}$ are the co-ordinates locating the centre of mass and x_i, y_i and z_i are the co-ordinates that locate the ith particle.

Often the system of particles will not consist of discrete particles for which a sum can be done. It may be that the system is a continuous object *e.g.*, a book. For a rigid body, it is often convenient to think of it as made up of a continuous distribution of matter. In other words, we consider it as made up of n particles where n is allowed to approach infinity. Then the summations in equation (*ii*) become integrals :

$$x_{C.M.} = \frac{1}{M}\int x\, dm \; ; \; y_{C.M.} = \frac{1}{M}\int y\, dm \; ; \; z_{C.M.} = \frac{1}{M}\int z\, dm$$

We can express the vector position of the centre of mass of a rigid body in the form :

$$\vec{R}_{C.M.} = \frac{1}{M}\int \vec{r}\, dm$$

13.13. CENTRE OF MASS OF UNIFORM ROD

Consider a uniform rod of mass M and length L placed along the x-axis as shown in Fig. 13.9. By symmetry, we can see that $y_{C.M.} = z_{C.M.} = 0$. Therefore, we are interested only in $x_{C.M.}$. If the mass per unit length is λ (the linear mass density), then $\lambda = M/L$ for a uniform rod.

Consider a very small element dx of the rod at a distance x from the origin O. Then mass of this element is $dm = \lambda dx$.

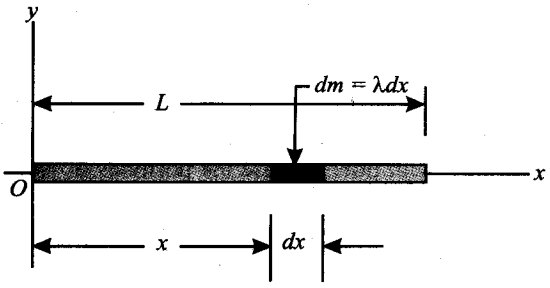

Fig. 13.9

$$\therefore \quad x_{C.M.} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx$$

$$= \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_0^L = \frac{\lambda L^2}{2M} = \frac{(M/L)}{2M}L^2 = \frac{L}{2}$$

$$\therefore \quad x_{C.M.} = L/2$$

This is a reasonable result because our first guess would be that the centre of mass of a uniform rod should be at the centre of the rod.

Non-uniform rod. Suppose the rod is *non-uniform* and the mass per unit length varies linearly with x according to the expression $\lambda = \beta x$ where β is a constant.

$$\therefore \quad x_{C.M.} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{\beta}{M}\int_0^L x^2\,dx = \frac{\beta L^3}{3M}$$

Now $$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \beta x\,dx = \frac{\beta L^2}{2}$$

Putting the value of M ($= \beta L^2/2$) in the expression for $x_{C.M.}$, we have,

$$x_{C.M.} = \frac{\beta L^3}{3(\beta L^2/2)} = \frac{2}{3}L$$

Since the mass of the rod increases as x, we are not surprised to find that the centre of mass of the rod is to the right of its centre.

13.14. CENTRE OF MASS OF SEMICIRCULAR HOOP

Consider a thin strip of mass m of a material bent into the shape of a semicircle of radius R as shown in Fig. 13.10. From the symmetry of the object, we conclude that the centre of mass must lie on the *y-axis (*i.e.*, **$x_{C.M.} = 0$).

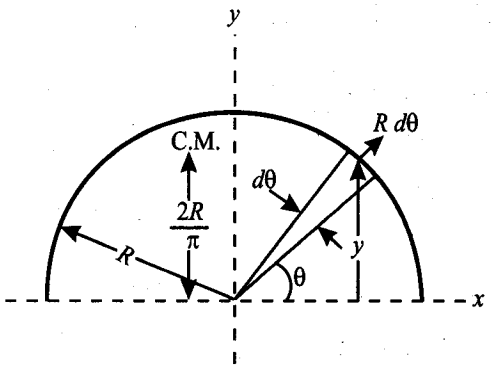

Fig. 13.10

Consider a mass element of length $R\,d\theta$ at height $y = R\sin\theta$. The mass of this element is $dm = \lambda R\,d\theta$ where $\lambda = M/\pi R$ = Mass per unit length.

$$\therefore y_{C.M.} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^\pi (R\sin\theta)\lambda R\,d\theta$$

$$= \frac{R^2\lambda}{M}\int_0^\pi \sin\theta\,d\theta = \frac{R^2\lambda}{M}[-\cos\theta]_0^\pi = \frac{2R^2\lambda}{M}$$

Putting $\lambda = M/\pi R$ in the above expression, we have,

$$y_{C.M.} = \frac{2R^2(M/\pi R)}{M} = \frac{2R}{\pi} = 0.637\,R$$

Thus the centre of mass is roughly two-third of radius along the y-axis. Note that the centre of mass is not within the body in this case.

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* All the mass is at positive y ; thus $\int y\,dm$ cannot be zero.

** $x_{C.M.} = 0$ because for every mass element at $+x$, there is an equal one at $-x$.

13.15. CENTRE OF MASS OF UNIFORM CONE

Consider a uniform cone of height h and radius R as shown in Fig. 13.11. We choose our co-ordinate system such that the origin is at the tip of the cone and the z-axis is along the line of symmetry. Then $x_{C.M.} = y_{C.M.} = 0$ because the centre of mass must lie on the line of symmetry. To find $z_{C.M.}$, we divide the cone into an infinite number of cylinders of thickness dz, one of which is shown in Fig. 13.11. The mass of each infinitesimal cylinder is $dm = \rho dv = \rho\pi r^2 dz$ where ρ* is the density (mass per unit volume) and $dV = \pi r^2 dz$.

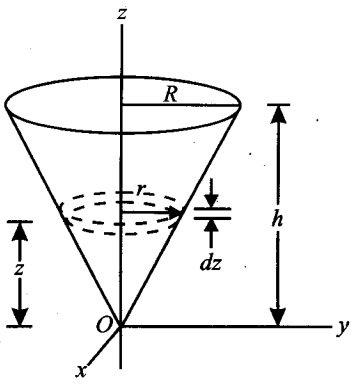


Fig. 13.11

$$\therefore \quad z_{C.M.} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int_0^h z\,\rho\,\pi r^2\,dz$$

Now r (= the radius of each infinitesimal cylinder) is related to z by the ratio $r/z = R/h$ so that $r = Rz/h$.

$$\therefore \quad z_{C.M.} = \frac{1}{M}\int_0^h \frac{\rho\pi R^2}{h^2} z^3\,dz = \frac{\rho\pi R^2}{Mh^2}\left[\frac{z^4}{4}\right]_0^h = \frac{\rho\pi R^2 h^2}{4M}$$

Total mass of cone, $M = \rho \times$ Total volume of the cone

$$= \rho \times \frac{\pi R^2 h}{3} = \frac{\rho\pi R^2 h}{3}$$

Putting the value of M (= $\rho\pi R^2 h/3$) in the expression for $z_{C.M.}$, we have,

$$z_{C.M.} = \frac{3}{4}h$$

Therefore, the centre of mass is $(3/4)h$ from the tip of the cone or $(1/4)h$ from its base.

13.16. CENTRE OF MASS OF SYMMETRICALLY SHAPED BODIES (UNIFORM COMPOSITION)

For symmetrically shaped bodies of uniform composition, such as *spheres, cylinders, rings, discs* and *rectangular solids*, the centre of mass is located at the geometric centre of the body. To see why this is true, consider a uniform circular cylinder such as a solid wheel. We expect the C.M. to be at the centre of the circle. To show that it is, we first choose a co-ordinate system whose origin is at the centre of the circle with z-axis perpendicular to the cylinder (See Fig. 13.12). When we take the sum $\Sigma m_i x_i$, their is as much mass at any $+x_i$ as there is at $-x_i$; so all terms cancel out in pairs and $x_{C.M.} = 0$; the same is true for $y_{C.M.}$. In the vertical (z) direction, the C.M. must lie halfway between the circular faces, since if we choose our origin of co-ordinates at that point there is as much mass at any $+z_i$ as at $-z_i$, so $z_{C.M.} = 0$. For other uniform symmetrically shaped bodies, we can make similar arguments to show that the C.M. must lie on a line of symmetry. If a symmetric body is not uniform, then these arguments do not hold.

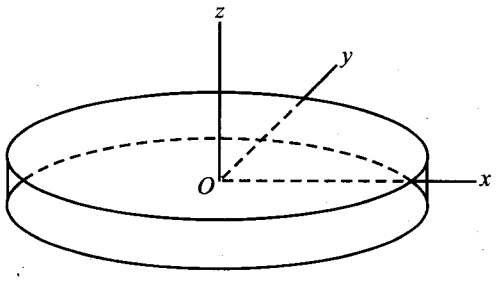
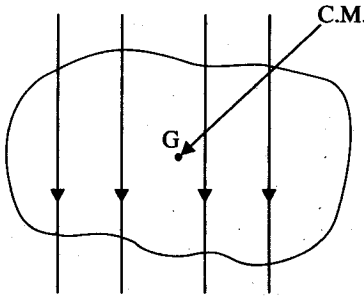


Fig. 13.12

* is constant because the cone is uniform.

13.17. CENTRE OF MASS AND CENTRE OF GRAVITY

The centre of mass C.M. of a body is a point where the entire mass of the body is supposed to be concentrated. *The centre of gravity G of a body is the point where the whole weight of the body is supposed to act on the body.* The centre of gravity coincides with the centre of mass in uniform gravitational field such as those found close to the surface of the earth [See Fig. 13.13]. However, this is not true in non-uniform gravitational field.



Uniform gravitational field

Fig. 13.13

Note. The sum of the moments about the centre of mass of a body is zero. Therefore, if a body is suspended from any point and is in equilibrium, the centre of mass must lie on a vertical line through the point of suspension.

13.18. THE GENERAL MOTION OF A RIGID BODY

A body is said to have general motion if it has translational motion as well as rotational motion. Suppose you push with one finger against a book that is lying on the table. You will notice that the resulting motion depends not only on the direction and magnitude of the force you apply but also on the point of application of the force. If the line of action of force passes through the C.M. (centre of mass) of the book, the book moves forward without rotation *i.e.*, it has only translational motion. If push is applied near the edge of the book, the book will have translational as well as rotational motion.

Consider a rigid body capable of making translational and rotational motion in space. To describe the general motion of the body, we should know its *position* and *orientation* at every instant of time. To do so, choose a convenient point P in the body. The point P may or may not be the centre of mass (See Fig. 13.14).

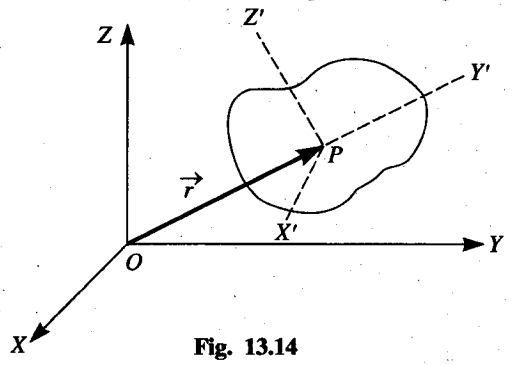


Fig. 13.14

(*i*) In order to describe the position of the body at any time, we select an inertial frame of reference with origin at O (coordinate frame $OXYZ$). Let the position vector of particle P *w.r.t.* origin O be $\vec{r}$. Any change in the position of vector $\vec{r}$ will tell us about the translational motion of the body. In other words, the coordinates of P (x, y, z) at any time in the coordinate frame $OXYZ$ give the position of the body at that instant.

(*ii*) In order to study the orientation of the body, we fix the coordinate frame in the body itself with P as the origin (coordinate frame $PX'Y'Z'$). When the body rotates in space, the orientation of coordinate frame $PX'Y'Z'$ changes *w.r.t.* the inertial frame $OXYZ$.

Consequently, at any instant of time, the orientation of $PX'Y'Z'$ frame *w.r.t.* $OXYZ$ frame will give the orientation of the body at that instant.

13.19. ROTATIONAL MOTION

In everyday life, we often encounter rigid bodies that rotate *e.g.*, the rotation of a wheel about its axis. Before we discuss rotation, it is helpful to distinguish clearly between translational and rotational motion of a rigid body. A rigid body executes translational motion when each particle of the body has the same displacement in the same time interval. For example, the motion of the body of a car travelling along a straight road is translational. A rigid object executes rotational

motion when each particle of the object (except those on the axis of rotation) moves in a circle. For example, the motion of a door being opened or closed is rotational.

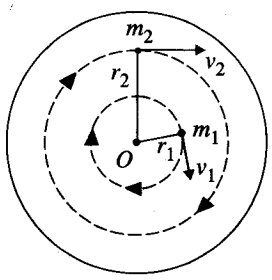


Fig. 13.15

Consider a wheel being rotated at constant angular speed about its axis. The wheel is composed of a very large number of particles; two particles of masses m_1 and m_2 are shown in Fig. 13.15. Note that the particles are moving in circles of different radii but the centres of these circles lie on a line called the *axis of rotation.* In this case, axis of rotation is passing through the centre O of the wheel.

Discussion. The following points may be noted :

(*i*) The different particles of the wheel have the same angular velocity ω but different linear velocities. The farther the particle is situated from the axis of rotation, the greater is its linear velocity $(v = \omega r)$.

(*ii*) In many practical situations, rotational motion is somehow related to linear motion. For example, the rotation of a bicycle's wheels results in its translation. The faster the rotation, the faster the translation. Obviously, there must be some relation between these two motions.

(*iii*) Translational motion may be one, two or three dimensional motion but the rotational motion of a rigid body can be either two or three dimensional. It is because to describe the rotation of a body, atleast two dimensions are required.

(*iv*) The rotational motion in two dimensions is important for two reasons. First, it can be easily represented on the paper or blackboard. Secondly, many rigid bodies execute rotation in two dimensions.

13.20. EQUATIONS FOR UNIFORMLY ACCELERATED ROTATIONAL MOTION

In chapter 7, we derived the important equations that relate acceleration, velocity and displacement for the situation of uniform linear acceleration. Those equations were derived from the definitions of linear velocity and acceleration assuming constant acceleration. The definitions of angular velocity and angular acceleration are the same as for their linear counterparts except that S has been replaced by angular displacement θ, v by ω and a by α. Therefore, angular equations for constant angular acceleration will be analogous to their linear counterparts with S replaced by θ, v by ω and a by α and can be derived in exactly the same way. We summarise them here, opposite their linear counterparts.

	Angular	**Linear**	
(*i*)	$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	(a, α constant)
(*ii*)	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$S = v_0 t + \frac{1}{2} a t^2$	(a, α constant)
(*iii*)	$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2aS$	(a, α constant)

Note that we have chosen the initial position $\theta_0 = 0$ (also $x_0 = 0$) for simplicity; ω_0 represents the angular velocity at t = 0 whereas θ and ω represent the angular position and velocity at time t.

Note. If the angular velocity is constant, $\alpha = 0 =$ constant. Therefore, above equations are also valid for constant angular velocity. For constant angular velocity ω_0, these equations (putting α = 0) become:

(*i*) $\omega = \omega_0$ (*ii*) $\theta = \omega_0 t$ (*iii*) $\bar{\omega} = \omega$

13.21. DERIVATION OF EQUATIONS OF ROTATIONAL MOTION

Consider a rigid body rotating about a given axis with *uniform angular acceleration.*

Let ω_0 = initial angular velocity *i.e.* at $t = 0$

ω = final angular velocity after time t

α = uniform angular acceleration

θ = angular displacement after time t

(*i*) **To derive $\omega = \omega_0 + \alpha t$**

We know that $\alpha = \dfrac{d\omega}{dt}$

or $d\omega = \alpha\, dt$ ...(*i*)

At $t = 0$, $\omega = \omega_0$ and at $t = t$, $\omega = \omega$. Therefore, integrating eq. (*i*) with proper limits, we have,

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt$$

or $[\omega]_{\omega_0}^{\omega} = \alpha[t]_0^t$

or $\omega - \omega_0 = \alpha t$

$\therefore$ $\omega = \omega_0 + \alpha t$

(*ii*) **To derive $\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$**

We know that $\omega = \dfrac{d\theta}{dt}$

or $d\theta = \omega dt$ ...(*ii*)

At $t = 0$, $\theta = 0$ and at $t = t$, $\theta = \theta$. Therefore, integrating eq. (*ii*) with proper limits, we have,

$$\int_0^{\theta} d\theta = \int_0^t \omega dt = \int_0^t (\omega_0 + \alpha t) dt \qquad [\because \omega = \omega_0 + \alpha t]$$

or $\int_0^{\theta} d\theta = \int_0^t \omega_0 dt + \int_0^t \alpha t\, dt$

or $[\theta]_0^{\theta} = \omega_0 [t]_0^t + \alpha \left[\dfrac{t^2}{2}\right]_0^t$

or $\theta - 0 = \omega_0 t + \dfrac{1}{2}\alpha t^2$

$\therefore$ $\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$

(*iii*) **To derive $\omega^2 - \omega_0^2 = 2\alpha\theta$**

We know that $\omega = \dfrac{d\theta}{dt}$ and $\alpha = \dfrac{d\omega}{dt} = \dfrac{d\omega}{d\theta}\left(\dfrac{d\theta}{dt}\right)$

$\therefore$ $\alpha = \dfrac{d\omega}{d\theta}\omega$

or $\omega d\omega = \alpha d\theta$ ...(*iii*)

When $\theta = 0$, $\omega = \omega_0$ and when $\theta = \theta$, $\omega = \omega$. Therefore, integrating eq. (*iii*) with proper limits; we have,

$$\int_{\omega_0}^{\omega} \omega \, d\omega = \int_0^{\theta} \alpha \, d\theta$$

or $$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha[\theta]_0^{\theta}$$

or $$\frac{\omega^2}{2} - \frac{\omega_0^2}{2} = \alpha\theta$$

$\therefore$ $$\omega^2 - \omega_0^2 = 2\alpha\theta$$

Example 13.8. A wheel is turning at a speed of 2 revolutions per second and is then allowed to come to rest. If it does so in 30 s, how far did it turn in the process? Assume uniform deceleration.

Solution. $\omega_0 = 2$ rev/s ; $\omega = 0$; $t = 30$ s

Average angular velocity, $\bar{\omega} = \frac{\omega_0 + \omega}{2} = \frac{2+0}{2} = 1$ rev/s

Now $\theta = \bar{\omega} t$

Since 1 rev $= 2\pi$ rad $= 360°$

$\therefore$ $\theta = 2\pi \times 30 = 60\pi$ rad or $10{,}800°$

Example 13.9. A vehicle with 50 cm diameter wheels is moving at a speed of 20 ms^{-1}. (*i*) How fast its wheels are turning? (*ii*) How many revolutions the wheels will turn if the car uniformly comes to rest in 30 s?

Solution. (*i*) As shown in Fig. 13.16, the distance s rolled is equal to $r\theta$. Therefore, the speed of the wheels along the ground is equal to its tangential speed; so tangential speed is 20 ms^{-1}. Now tangential speed v_T is related to angular speed as :

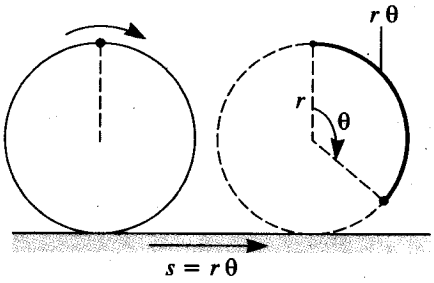


Fig. 13.16

$$v_T = \omega r$$

$\therefore$ $$\omega = \frac{v_T}{r} = \frac{20}{0.25} = 80 \text{ rad/s}$$

To find the speed in revolutions/second, we recall that 1 rad $= \frac{1}{2\pi}$ revolution.

$\therefore$ $$\omega = 80 \times \left(\frac{1}{2\pi}\right) = \frac{40}{\pi} \text{ rev/s}$$

(*ii*) $\omega_0 = 80$ rad/s $= \frac{40}{\pi}$ rev/s ; $\omega = 0$; $t = 30$ s

Average angular velocity, $\bar{\omega} = \frac{\omega_0 + \omega}{2} = \frac{1}{2}\left(\frac{40}{\pi} + 0\right) = \frac{20}{\pi}$ rev/s

Now $\theta = \bar{\omega} t = \frac{20}{\pi} \times 30 = \frac{600}{\pi}$ revolutions

Example 13.10. Starting from rest, a fan takes 5 seconds to attain the maximum speed of 400 r.p.m. Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.

Solution. Maximum angular velocity, ω_m = 400 r.p.m. = $400 \times \frac{2\pi}{60} = \frac{40\pi}{3}$ rad/s

Constant angular acceleration, $\alpha = \frac{\omega_m - 0}{\text{Time}} = \frac{(40\pi/3) - 0}{5} = \frac{8\pi}{3}$ rad/s^2

Now $\omega = \omega_0 + \alpha t$

Here $\omega = \frac{\omega_m}{2} = \frac{20\pi}{3}$ rad/s ; $\alpha = \frac{8\pi}{3}$ rad/s^2 ; $\omega_0 = 0$

$\therefore$ $\frac{20\pi}{3} = 0 + \frac{8\pi}{3} t$ or $t = 2.5$ s

Example 13.11. The wheel of a motor accelerated uniformly from rest rotates through 2.5 radians during the first second. Find the angle rotated in the next second.

Solution. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

Here $\theta = 2.5$ rad. ; $\omega_0 = 0$; $t = 1$ s ; $\alpha = ?$

$\therefore$ $2.5 = 0 \times t + \frac{1}{2}\alpha \times (1)^2$ $\therefore$ $\alpha = 5$ rad/s^2

Angle turned during the first two seconds

$$= \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 5 \times (2)^2 = 10 \text{ rad}$$

$\therefore$ Angle turned during the 2nd second

$$= 10 \text{ rad} - 2.5 \text{ rad} = \mathbf{7.5 \text{ rad}}$$

13.22. TORQUE (OR MOMENT OF FORCE)

Torque is the turning effect of a force and is defined as under :

The torque produced by a force about the axis of rotation is equal to the product of magnitude of force and the perpendicular distance of the line of action of the force from the axis of rotation. It is denoted by Greek letter τ (Tau).

Torque, τ = Force × Perpendicular distance of force from axis of rotation

The perpendicular distance of the line of action of the force from the axis of rotation is called **lever arm** (or **moment arm**) of the force.

$\therefore$ Torque, τ = Force × Lever arm

Thus in Fig. 13.17, d is the lever arm of the force. It is clear that greater the magnitudes of force and lever arm, the greater is the torque due to a force. It is a common experience that we can open or close a door easily by applying force near the edge of the door. By doing so, we increase the lever arm of the force and hence the torque due to the force. Similarly, to unscrew a nut fitted tightly to a bolt, we need a wrench with a long arm.

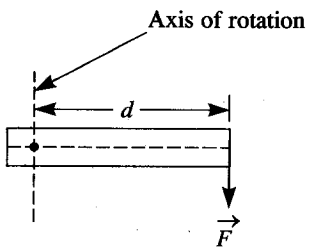


Fig. 13.17

By convention, torque tending to produce anticlockwise rotation is assigned positive value and that tending to produce clockwise rotation is assigned negative value. Note that torque is a **vector quantity** *i.e.,* it has magnitude as well as direction.

Units. Torque, τ = Force × Lever arm

The SI unit of force is 1N and that of distance is 1 m. Therefore, SI unit of torque is 1 Nm and dimensional formula is $[ML^2 T^{-2}]$. Note that unit for torque is the same as that for energy (or work). But two quantities are very different. An obvious difference is that energy is

a scalar whereas torque is a vector. The special name joule (1 Nm = 1 joule) is used *only* for energy (and for work), never for torque.

13.23. TORQUE ON A PARTICLE IN A PLANE (ROTATIONAL MOTION OF PARTICLE)

Consider a particle of mass m moving in the xy plane in a circle of radius r about point O as shown in Fig. 13.18. Suppose at any time, the particle is located at point P. The position vector of the particle is $\vec{r}(=\overrightarrow{OP})$ and its coordinates are (x, y) or (r, θ) in cartesian coordinates or polar coordinates respectively; θ is the angle between OP and x-axis [See Fig. 13.18]. Suppose a force $\vec{F}$ is applied to the particle at point P. Then,

$$\vec{F} = \hat{i}F_x + \hat{j}F_y$$

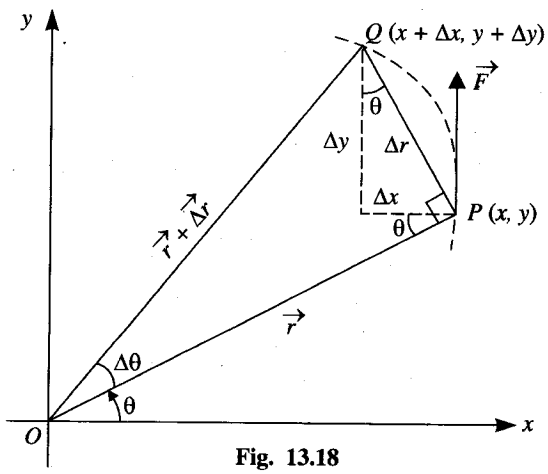


Fig. 13.18

Under the action of force $\vec{F}$, the particle moves to point Q in a small time Δt such that $\angle POQ = \Delta\theta$ and $\overrightarrow{OQ} = \vec{r} + \overrightarrow{\Delta r}$. Now the position vector of the particle is $\vec{r} + \overrightarrow{\Delta r}$ and its coordinates are $(x+\Delta x,\ y+\Delta y)$ or $(r+\Delta r,\ \theta+\Delta\theta)$. The displacement $\overrightarrow{\Delta r}$ of the particle is given by ;

$$\overrightarrow{\Delta r} = \hat{i}\Delta x + \hat{j}\Delta y$$

The small work done by the applied force is

$$\Delta W = \vec{F}.\overrightarrow{\Delta r} = (\hat{i}F_x + \hat{j}F_y).(\hat{i}\Delta x + \hat{j}\Delta y)$$

or $$\Delta W = F_x\Delta x + F_y\Delta y \qquad ...(i)$$

As the particle moves from P to Q, distances x and y change by Δx and Δy respectively. Now $PQ = r\Delta\theta$ and Δx = projection of $r\Delta\theta$ in the $-x$ direction.

$\therefore$ $$\Delta x = -(r\Delta\theta)\sin\theta$$

$$= -r\Delta\theta\frac{y}{r} = -y\Delta\theta \qquad \left(\because \sin\theta = \frac{y}{r}\right)$$

or $$\Delta x = -y\Delta\theta$$

Similarly, Δy is the projection of $r\Delta\theta$ in the $+y$ direction.

$\therefore$ $$\Delta y = r\Delta\theta\cos\theta = r\Delta\theta\left(\frac{x}{r}\right) = x\Delta\theta$$

or $$\Delta y = x\Delta\theta$$

Putting the values of Δx and Δy in eq. (i), we have,

$$\Delta W = F_x(-y\Delta\theta) + F_y(x\Delta\theta)$$

or $$\Delta W = [xF_y - yF_x]\Delta\theta$$

or $$\Delta W = \tau\Delta\theta \qquad ...(ii)$$

where Torque $\tau = x\,F_y - yF_x$ ...(iii)

Eq. (*iii*) gives the torque about point O due to force $\vec{F}$.

Thus in rotational motion, the work done by torque is equal to the product of torque and angular displacement. Note that in linear motion, work done by the force is equal to the product of force and linear displacement.

13.24. EXPRESSION FOR TORQUE IN POLAR COORDINATES

Suppose the position vector of the particle at point P (x, y) is $\vec{r}$ and it makes an angle θ with x-axis. Let the angle between the force vector $\vec{F}$ and position vector $\vec{r}$ be ϕ. If $\vec{F}$ makes an angle α with the positive direction of x-axis, then [See Fig. 13.19 (*i*)],

$$\phi = \alpha - \theta$$

Now $F_x = F\cos\alpha$ and $F_y = F\sin\alpha$

Also $x = r\cos\theta$ and $y = r\sin\theta$

We have seen that : $\tau = xF_y - yF_x$

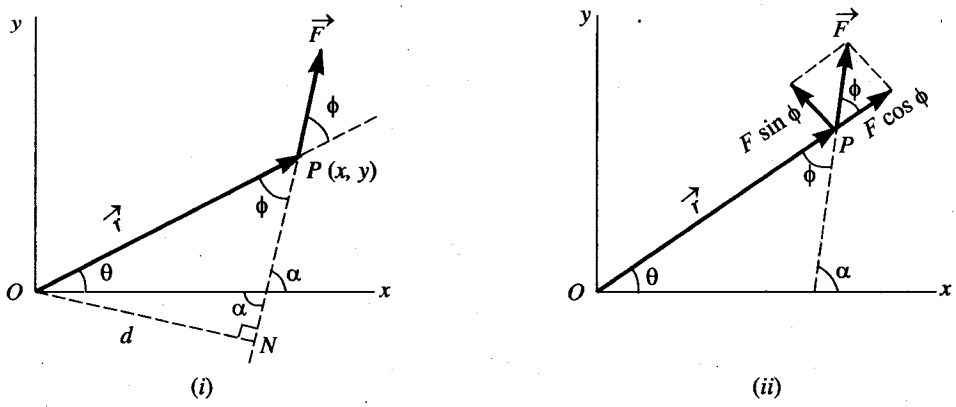


Fig. 13.19

Putting the various values in this equation, we have,

$$\begin{aligned}\tau &= (r\cos\theta)(F\sin\alpha) - (r\sin\theta)(F\cos\alpha)\\ &= rF(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\\ &= rF\sin(\alpha-\theta)\\ &= rF\sin\phi \qquad (\because \phi = \alpha-\theta)\end{aligned}$$

$$\therefore \quad \tau = rF\sin\phi \qquad \ldots(i)$$

Eq. (*i*) gives the expression for torque in polar co-ordinates. Note that torque due to a force $\vec{F}$ depends upon the magnitude of force and displacement $(\vec{r})$ of the force from the axis of rotation (point O).

Interpretation of torque equation. The torque equation can be interpreted in the following two ways :

(*i*) $\tau = F(r\sin\phi)$

If d (= ON) is the perpendicular distance from point O (axis of rotation) to the line of action of the force, then,

$$d = r\sin\phi = \text{Lever arm of force}$$

$$\therefore \quad \tau = Fd = \text{Force} \times \text{Lever arm}$$

Thus torque produced by a force is equal to the product of magnitude of force and the lever arm of the force.

(*ii*) $\tau = rF\sin\phi = r\,(F\sin\phi)$

As shown in Fig. 13.19 (*ii*), the force $\vec{F}$ is split into components parallel and perpendicular to $\vec{r}$. The parallel component $F\cos\phi$ goes through point O (axis of rotation) and has zero lever arm. Therefore, torque due to $F\cos\phi$ about O is zero. But the perpendicular component $F\sin\phi$ has a lever arm r so torque due to it is $\tau = r\,(F\sin\phi) = r\,F\sin\phi$.

Thus torque produced by a force is equal to the product of the magnitude of the perpendicular component of force (i.e., component $\perp \vec{r}$) and the magnitude of the displacement from the axis of rotation to the point of application of the force.

Note that both approaches give the same value of torque.

Special Cases

$$\tau = rF\sin\phi$$

(*i*) When $\phi = 0°$ (*i.e.*, $\vec{r}$ and $\vec{F}$ are in the same direction), then,

$$\tau = rF\sin 0° = 0$$

(*ii*) When $\phi = 90°$ (*i.e.*, angle between $\vec{r}$ and $\vec{F}$ is 90°), then,

$$\tau = rF\sin 90° = rF \quad \textit{...maximum value}$$

Thus for given $\vec{r}$ and $\vec{F}$, torque will be maximum when angle between $\vec{r}$ and $\vec{F}$ is 90°.

The opening or closing of a door hinged at one end is a beautiful example of the application of torque. The most effective way to open the door is to apply force farthest from the hinges (*i.e.*, maximum r) and push at right angles ($\phi = 90°$) to the door.

13.25. GENERAL DEFINITION OF TORQUE

It has already been mentioned that torque is a vector quantity *i.e.*, it has magnitude and direction. We shall now give general definition of torque which takes care of its magnitude as well as direction.

$$\text{Torque, } \tau = r\,F\sin\phi = r\,(F\sin\phi)$$

Note that the two quantities involved in the definition of torque are the vectors $\vec{r}$ and $\vec{F}$. We see that magnitude of torque produced is equal to the product of the magnitude of one vector (r) and the magnitude of the perpendicular component of the other vector ($F\sin\phi$). This, by definition, is the vector or cross product of $\vec{r}$ and $\vec{F}$.

$$\therefore \quad \text{Torque, } \vec{\tau} = \vec{r}\times\vec{F}$$

Therefore, torque produced by a force is equal to the cross product of the position vector $(\vec{r})$ of the force and the force $(\vec{F})$.

Since torque is the vector (or cross) product of two vectors ($\vec{r}$ and $\vec{F}$), it is a *vector quantity*. The magnitude of torque is $rF\sin\phi$ where ϕ is the smaller angle between $\vec{r}$ and $\vec{F}$. The direction of torque is always perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ (by definition of vector product) and its sense is given by right-hand rule.

Right-hand rule. *Curl the fingers of the right hand in the sense that would rotate the first vector $\vec{r}$ into the second vector $\vec{F}$ through the smaller angle. Then extended thumb gives the direction of torque.*

The right hand rule is illustrated in Fig. 13.20. The axis of rotation is perpendicular to the plane of the disc and is passing through the centre of the disc. The two vectors $\vec{r}$ and $\vec{F}$ are in the plane of the disc. By right-hand rule, the direction of torque is upward along the axis of rotation. Note that torque vector acts along the axis of rotation (not along the force). This definition may seem a little strange since all the action is in the plane perpendicular to the torque. But there is no unique vector direction within the plane of rotation; in fact only the direction perpendicular to that plane specifies the orientation of that plane.

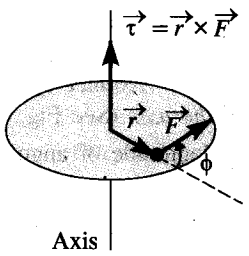


Fig. 13.20

Note. If a force $\vec{F}$ rotates the particle in *XY* plane in anticlockwise direction, then by right hand rule, the direction of torque will be along the positive direction of *Z*-axis.

13.26. RECTANGULAR COMPONENTS OF TORQUE

When a force $\vec{F}$ rotates a particle in a plane, say *XY* plane (two dimensions), the torque acts along *Z*-axis *i.e.,* the torque has only *Z*-component while the *X*-component and *Y*-component of torque are zero. However, when the applied force $\vec{F}$ rotates the body in three dimensions, we have three rectangular components of torque *viz* τ_x, τ_y and τ_z. The *x*, *y* and *z*-components of torque can be obtained easily as under :

$$\vec{\tau} = \vec{r} \times \vec{F}$$

or $$\hat{i}\,\tau_x + \hat{j}\,\tau_y + \hat{k}\,\tau_z = (\hat{i}\,x + \hat{j}\,y + \hat{k}\,z) \times (\hat{i}\,F_x + \hat{j}\,F_y + \hat{k}\,F_z)$$

or $$\hat{i}\,\tau_x + \hat{j}\,\tau_y + \hat{k}\,\tau_z = \hat{i}(yF_z - zF_y) + \hat{j}(zF_x - xF_z) + \hat{k}(xF_y - yF_x)$$

Equating the rectangular components on two sides, we have,

$$\tau_x = yF_z - zF_y \; ; \; \tau_y = zF_x - xF_z \; ; \; \tau_z = xF_y - yF_x$$

13.27. POWER ASSOCIATED WITH TORQUE

We have already seen that work done by a constant torque τ (See Art. 13.23) for angular displacement $\Delta\theta$ is

$$\Delta W = \tau \, \Delta\theta$$

If this work is done in a small time interval Δt, then dividing both sides of Δt, we have,

$$\frac{\Delta W}{\Delta t} = \frac{\tau \, \Delta\theta}{\Delta t}$$

Now $\Delta W/\Delta t$ is the average power P associated with the torque and $\Delta\theta/\Delta t$ is the angular velocity ω of the rotating body.

$\therefore$ Power, $P = \tau \, \omega$

Hence power developed by an agent exerting a torque is equal to the product of torque (τ) and angular velocity (ω) of the rotating body. This is analogy of $P = Fv$ for linear motion.

13.28. ANGULAR MOMENTUM

A mass moving in a straight line has linear momentum. When a mass rotates about some axis, there is momentum associated with rotational motion, called *angular momentum.* Just as net external force is required to change the linear momentum of a particle, an external net torque is required to change the angular momentum of a particle.

The angular momentum of a particle about an axis of rotation is the product of its linear momentum and the perpendicular distance of the line of action of linear momentum from the axis (See Fig. 13.21). It is denoted by L.

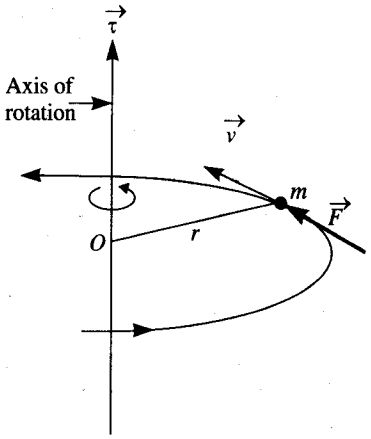


Fig. 13.21

Magnitude of angular momentum, $L = r\ m\ v$

where $m\ v = p$ = linear momentum of the particle.

r = perpendicular distance of the line of action of linear momentum from the axis of rotation.

The perpendicular distance of the line of action of linear momentum from the axis of rotation is called *lever arm for linear momentum.*

$\therefore$ Angular momentum, $L = p \times$ Lever arm for linear momentum

Note that the definition of angular momentum is analogous to the definition of torque (or moment of force). Indeed, angular momentum is sometimes called moment of linear momentum. Note that angular momentum is a **vector quantity** *i.e.*, it has magnitude as well as direction.

Units. Angular momentum, $L = r\ m\ v$. Therefore, the SI unit of angular momentum is kg m^2 s^{-1} and its dimensional formula is $[ML^2T^{-1}]$.

Note. Since the definition of L $(= r\ m\ v)$ contains r, the value of L depends upon the point O (axis of rotation) about which it is calculated. Any time an angular momentum is discussed, the point about which it is calculated must be clearly mentioned.

13.29. EXPRESSION FOR ANGULAR MOMENTUM

We know that torque due to a force F rotating the particle in xy-plane is given by ;

$$\tau = xF_y - yF_x \qquad ...(i)$$

According to Newton's second law of motion,

$$\vec{F} = \frac{d}{dt}(\vec{p}) = \frac{d}{dt}(m\vec{v})$$

where $\vec{p}$ is the linear momentum of the particle.

In component form, we have,

$$F_x = \frac{d}{dt}(p_x) = \frac{d}{dt}(mv_x) = m\frac{dv_x}{dt}$$

$$F_y = \frac{d}{dt}(p_y) = \frac{d}{dt}(mv_y) = m\frac{dv_y}{dt}$$

Here $p_x = mv_x$ = x-component of the linear momentum of the particle.

$p_y = m\ v_y$ = y-component of the linear momentum of the particle.

Putting the values of F_x and F_y in eq. (i), we have,

$$\tau = xm\frac{dv_y}{dt} - ym\frac{dv_x}{dt} = m\left[x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right] \qquad ...(ii)$$

Now $$\frac{d}{dt}(xv_y - yv_x) = \left[x\frac{dv_y}{dt} + v_y\frac{dx}{dt}\right] - \left[y\frac{dv_x}{dt} + v_x\frac{dy}{dt}\right]$$

$$= \left[x\frac{dv_y}{dt} + v_y v_x\right] - \left[y\frac{dv_x}{dt} + v_x v_y\right]$$

or $$\frac{d}{dt}(xv_y - yv_x) = x\frac{dv_y}{dt} - y\frac{dv_x}{dt} \quad ...(iii)$$

From eqs. *(ii)* and *(iii)*, we have,

$$\tau = m\frac{d}{dt}(xv_y - yv_x) = \frac{d}{dt}(xmv_y - ymv_x)$$

or $$\tau = \frac{d}{dt}(xp_y - yp_x)$$

The quantity $(xp_y - yp_x)$ is the moment of the linear momentum of the particle about the given axis of rotation and is, therefore, the angular momentum of the particle about the given axis.

$\therefore$ Angular momentum, $L = xp_y - yp_x$

$\therefore$ $$\tau = \frac{dL}{dt} \quad ...(iv)$$

Thus the time rate of change of angular momentum of a particle is equal to the net torque applied to the particle.

Eq. *(iv)* gives the relation between torque (τ) and angular momentum (L). You may recall that a similar relation exists between force and linear momentum *viz* $F = dp/dt$. Thus the net torque causes change in angular momentum just as net force causes change in linear momentum.

13.30. ANGULAR MOMENTUM IN POLAR CO-ORDINATES

Suppose $P(x, y)$ is the position of a particle of mass m and linear momentum $\vec{p}$ rotating in the XY plane about the Z-axis. Let $\vec{r}$ be the position vector of the particle making an angle θ with X-axis. Let the angle between linear momentum vector $\vec{p}$ and position vector $\vec{r}$ be ϕ. If $\vec{p}$ makes an angle α with the positive direction of X-axis, then [See Fig. 13.22],

$$\phi = \alpha - \theta$$

Now $p_x = p\cos\alpha$ and $p_y = p\sin\alpha$

Also $x = r\cos\theta$ and $y = r\sin\theta$

We have seen that $L = xp_y - yp_x$

Putting the various values in this equation, we have,

$$L = (r\cos\theta)(p\sin\alpha) - (r\sin\theta)(p\cos\alpha)$$

$$= rp(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$$

$$= rp\sin(\alpha - \theta)$$

$\therefore$ $$L = rp\sin\phi \qquad (\because \alpha - \theta = \phi) \quad ...(i)$$

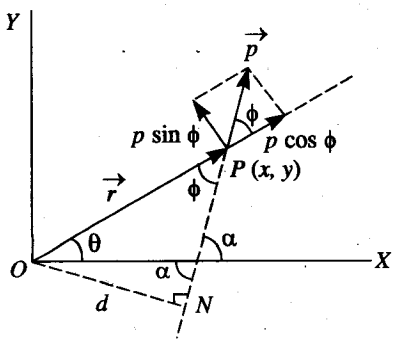


Fig. 13.22

Eq. *(i)* is the expression for angular momentum of a particle in polar co-ordinates. Note that angular momentum depends on the magnitude of linear momentum (p) of the particle and magnitude of displacement (r) of the particle from the axis of rotation (point O).

Interpretation of Angular Momentum Equation. The angular momentum equation can be interpreted in the following two ways :

(*i*) $L = p\ (r \sin\phi)$

If d (= ON) is the perpendicular distance from point O (axis of rotation) to the line of action of the linear momentum vector, then,

$d = r \sin\phi$ = *Lever arm for momentum

$\therefore$ $L = pd$ = Linear momentum × Lever arm for momentum

Thus angular momentum is the product of linear momentum and the lever arm for momentum.

(*ii*) $L = rp \sin\phi = r\ (p \sin\phi)$

As shown in Fig. 13.22, the linear momentum vector $\vec{p}$ is split into components parallel and perpendicular to $\vec{r}$. The parallel component $p \cos\phi$ goes through point O (axis of rotation) and has zero lever arm for momentum. Therefore, angular momentum due to $p \cos\phi$ is zero. But the perpendicular component $p \sin\phi$ has a lever arm for momentum equal to r so angular momentum due to this component is $L = r\ (p \sin\phi) = rp \sin\phi$.

Hence the angular momentum produced by a torque is equal to the product of the magnitude of the perpendicular component of linear momentum (i.e., component $\perp \vec{r}$*) and the magnitude of the displacement of the particle from the axis of rotation.*

Note that both approaches give the same value of angular momentum.

13.31. GENERAL DEFINITION OF ANGULAR MOMENTUM

It has already been mentioned that angular momentum is a vector quantity *i.e.,* it has magnitude and direction. We shall now give general definition of angular momentum which takes care of its magnitude as well as direction.

Angular momentum, $L = r\,p \sin\phi = r\ (p \sin\phi)$

Note that the two quantities involved in the definition of angular momentum are the vectors $\vec{r}$ and $\vec{p}$. We see that the magnitude of the angular momentum is equal to the product of the magnitude of one vector (r) and the magnitude of the perpendicular component of the other vector ($p \sin\phi$). This, by definition, is the vector or cross product of $\vec{r}$ and $\vec{p}$.

$\therefore$ Angular momentum, $\vec{L} = \vec{r} \times \vec{p}$

Therefore, the angular momentum produced by a torque is equal to the cross product of position vector $\vec{r}$ *of the linear momentum and the linear momentum* ($\vec{p}$).

Since angular momentum is the vector product of two vectors ($\vec{r}$ and $\vec{p}$), it is a *vector quantity.* The magnitude of angular momentum is $rp \sin\phi$ where ϕ is the smaller angle between $\vec{r}$ and $\vec{p}$. The direction of angular momentum is always perpendicular to the plane containing $\vec{r}$ and $\vec{p}$ (by definition of vector product) and its sense is given by **right-hand rule.

Right-hand rule. *Curl the fingers of your right hand in the sense that would rotate the first vector* $\vec{r}$ *into the second vector* $\vec{p}$ *through the smaller angle. Then extended thumb gives the direction of the angular momentum.*

The right-hand rule is illustrated in Fig. 13.23. Here the particle is in circular motion about the axis through O in the XY plane. By right-hand rule, the angular momentum points in the

* Note that lever arm for momentum is different from the lever arm for force. It is because momentum and force may not be parallel.

** Same used to find the direction of torque.

+Z direction. *Note that angular velocity* $(\vec{\omega})$ *is also a vector quantity.* Both $\vec{L}$ and $\vec{\omega}$ are in the same direction. We have seen that linear momentum of bodies is very useful in dealing with linear motion of bodies. Similarly, angular momentum is a useful concept in dealing with rotational motion. Like linear momentum, angular momentum is also a conserved quantity.

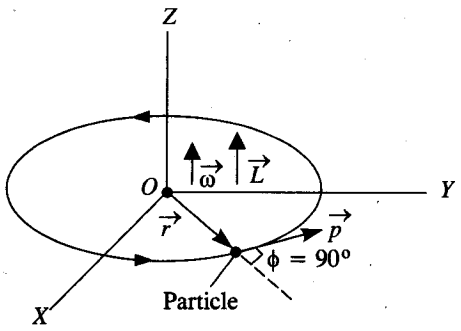


Fig. 13.23

Note. If the body rotates in three dimensions, the three rectangular components of angular momentum can be obtained as under :

$$\vec{L} = \vec{r} \times \vec{p}$$

or $$\hat{i}\,L_x + \hat{j}\,L_y + \hat{k}\,L_z = (\hat{i}\,x + \hat{j}\,y + \hat{k}\,z) \times (\hat{i}\,p_x + \hat{j}\,p_y + \hat{k}\,p_z)$$

or $$\hat{i}\,L_x + \hat{j}\,L_y + \hat{k}\,L_z = \hat{i}\,(y\,p_z - z\,p_y) + \hat{j}(z\,p_x - x\,p_z) + \hat{k}(x\,p_y - y\,p_x)$$

Equating the rectangular components on two sides, we have,

$$L_x = y\,p_z - z\,p_y\,;\; L_y = z\,p_x - x\,p_z\,;\; L_z = x\,p_y - y\,p_x$$

If the particle moves in a plane, say *XY* plane (two dimensions), the angular momentum acts along *Z*-axis *i.e.,* its angular momentum has only *Z*-component while *X*-component and *Y*-component of angular momentum are zero.

13.32. TORQUE AND ANGULAR MOMENTUM FOR A SYSTEM OF PARTICLES

Consider a system of *N* particles of masses m_1, m_2, m_3 ... m_N having respective position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots \vec{r}_N$. Suppose external forces $\vec{F}_1, \vec{F}_2, \vec{F}_3 \ldots \vec{F}_N$ act on the respective particles in addition to the internal forces. The sum of the torques due to internal forces is *zero. Therefore, we are to consider the torques on the system of particles due to external forces only. Thus referring to Fig. 13.24, particle 1 has mass m_1 and velocity $\vec{v}_1$ and the resultant force on it is $\vec{F}_1$. The position vector of particle 1 *w.r.t.* origin *O* is $\vec{r}_1$. The resultant torque on the particle 1 is $\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1$ and its angular momentum about *O* is $\vec{L}_1 = \vec{r}_1 \times m_1 \vec{v}_1$. Similarly for particles 2, 3 and so on. We have the same origin for all position vectors, angular momenta and torques. Therefore, for each particle in the system, we have,

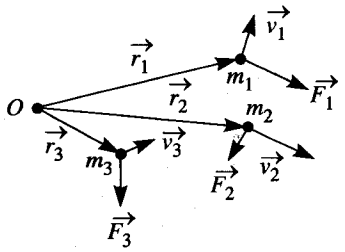


Fig. 13.24

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* Consider two particles 1 and 2 in the system. They interact through internal forces $\vec{F}_{12}$ and $\vec{F}_{21}$. According to Newton's third law, $\vec{F}_{12} = -\vec{F}_{21}$, *i.e.,* the particles interact through central forces. Therefore, both the forces have the same moment arm. Since these forces are equal in magnitude and opposite in direction, the resultant torque is zero. This is true of any pair of particles.
Suppose the internal forces were not zero. Then there could be spontaneous angular momentum change even in the absence of external torques. But this has never been observed constituting the best indirect proof that internal forces are central forces.

$$\vec{\tau}_1 = \frac{d\vec{L}_1}{dt}\ ;\ \vec{\tau}_2 = \frac{d\vec{L}_2}{dt};.....$$

Adding all these equations yields

$$\vec{\tau}_1+\vec{\tau}_2+\vec{\tau}_3+...+\vec{\tau}_N = \frac{d\vec{L}_1}{dt}+\frac{d\vec{L}_2}{dt}+\frac{d\vec{L}_3}{dt}+...+\frac{d\vec{L}_N}{dt}$$

or $$\vec{\tau}_{total} = \frac{d\vec{L}_1}{dt}+\frac{d\vec{L}_2}{dt}+\frac{d\vec{L}_3}{dt}+...+\frac{d\vec{L}_N}{dt}$$

or $$\vec{\tau}_{total} = \frac{d}{dt}(\vec{L}_1+\vec{L}_2+\vec{L}_3+.....+\vec{L}_N)$$

Thus the net external torque acting on a system of particles is equal to the time rate of change of the total angular momentum of the system.

For a rigid body. In *N*-particle system, the particles can change their positions relative to one another. However, in a rigid body, the particles remain at their respective positions. Note that "a system of particles" includes a rigid body as a special case. Therefore, the above treatment is also applicable to a rigid body. For a rigid body,

$$\vec{\tau}_{total} = \frac{d\vec{L}}{dt}$$

13.33. GEOMETRICAL MEANING OF ANGULAR MOMENTUM

Consider a particle of mass m rotating in the XY plane about Z-axis. Suppose at any instant, the particle is at point A and its position vector is $\vec{r}$. Let after small time interval Δt the particle be at point B where its position vector is $\vec{r}+\Delta\vec{r}$ (See Fig. 13.25). Then displacement of the particle during the time interval Δt is

$$\vec{AB} = (\vec{r}+\Delta\vec{r})-\vec{r} = \Delta\vec{r}$$

If $\vec{v}$ is the velocity of the particle at point A, then the small displacement covered in time Δt is

$$\Delta\vec{r} = \vec{v}\Delta t$$

The area $\Delta\vec{A}$ swept out by the position vector in time interval Δt is shown shaded. Note that this area is half of the area of parallelogram $OABC$.

Area of parallelogram $OABC = |\vec{r}\times\Delta\vec{r}|$

Area swept by the position vector of the particle in time Δt is

$$\Delta\vec{A} = \text{Area of triangle } OAB = \frac{1}{2}|\vec{r}\times\Delta\vec{r}|$$

$$\therefore\quad \Delta\vec{A} = \frac{1}{2}|\vec{r}\times\Delta\vec{r}| = \frac{1}{2}\vec{r}\times\Delta\vec{r}$$

If $\vec{p}$ is the linear momentum of the particle, then,

$$\vec{v} = \frac{\vec{p}}{m}$$

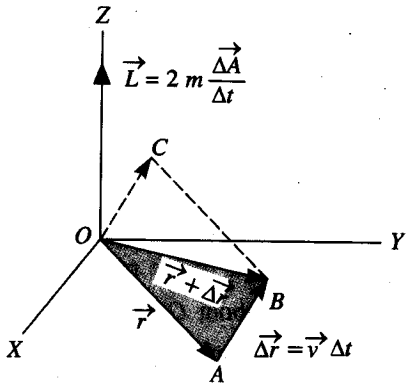


Fig. 13.25

$$\therefore \quad \Delta\vec{A} = \frac{1}{2}\vec{r}\times\frac{\vec{p}}{m}\Delta t$$

or $$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2m}\vec{r}\times\vec{p}$$

But $\vec{r}\times\vec{p} = \vec{L}$, the angular momentum of the particle about Z-axis.

$$\therefore \quad \frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{L}}{2m}$$

or $$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t}$$

Now $\Delta\vec{A}/\Delta t$ is the area swept by the position vector of the particle per unit time and is called *areal velocity* of the position vector of the particle.

$\therefore$ Angular momentum = 2 × Mass × Areal velocity

We now have the geometrical interpretation of the angular momentum as under :

The angular momentum of a particle about a given axis is equal to twice the product of the mass of the particle and the areal velocity of the position vector of the particle.

13.34. ANGULAR MOMENTUM AND KEPLER'S SECOND LAW

We can derive Kepler's second law for planetary motion from angular momentum consideration.

Kepler's second law states that *each planet moves so that a line from the sun to the planet sweeps out equal areas in equal intervals of time.*

For the motion of earth and other planets around the sun, the centripetal force is provided by the gravitational attraction of the sun. The gravitational force is a central force *i.e.*, it acts along the line joining the centres of the sun and the planet. Thus the torque on the planets in their orbits is zero *i.e.*,

$$\vec{\tau} = \vec{r}\times\vec{F} = 0 \qquad (\because \text{angle between } \vec{r} \text{ and } \vec{F} \text{ is zero})$$

Now $$\vec{\tau} = \frac{d\vec{L}}{dt} \quad \text{or} \quad 0 = \frac{d\vec{L}}{dt}$$

$\therefore$ $\vec{L}$ = a constant vector

Therefore, the angular momentum of any planet around the sun is constant.

Consider a planet of mass m moving in an elliptical orbit around the sun. The angular momentum of the planet is given by ;

$$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t}$$

As angular momentum $\vec{L}$ of the planet around the sun is constant,

$$\therefore \quad \frac{\Delta\vec{A}}{\Delta t} = \text{Constant}$$

This means that the time rate at which a planet sweeps out an area is constant (equal areas in equal times). Thus in Fig. 13.26, if the planet takes the same time to go from A to B and from C to D, then the shaded areas are the same. Hence the line joining the sun and a planet sweeps out equal areas in equal intervals of time (Kepler's second law for planetary motion). Note that this means that the planet must be moving faster when it is closer to the sun.

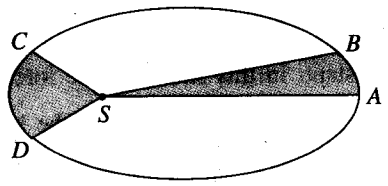


Fig. 13.26

Note. We can make additional observation about the planet. The constant angular momentum of the planet not only means that the magnitude of angular momentum is constant but also that its direction in space is constant. This constancy of direction in space leads to the conclusion that the plane of a planet's orbit is fixed in space.

13.35. LAW OF CONSERVATION OF ANGULAR MOMENTUM

If net torque on a system is zero, then change in its angular momentum with time is also zero *i.e.*, if

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta t} = 0, \text{ then, } \Delta \vec{L} = 0$$

or $\vec{L}$ = Constant vector

This is known as the law of conservation of angular momentum and may be stated as under:

If the net torque acting on a system is zero, then the total angular momentum of the system remains constant.

It applies to a particle or a system of particles or a rigid body. If no net torque acts on a system of N particles, then,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ... + \vec{L}_N = \text{Constant vector}$$

The two important situations in which torque $\vec{\tau}(= \vec{r} \times \vec{F})$ on a particle (or body) is zero are :

(*i*) **When no external force acts on the particle.** In that case, the angular momentum of the particle is constant. With $\vec{F} = 0, \vec{\tau} = \vec{r} \times \vec{F} = 0$ and $\vec{L} = \vec{r} \times m\vec{v} =$ constant. Note that in this case, linear as well as angular momentum remains the same. Consequently, the particle moves with a uniform speed in a straight line.

(*ii*) **The force on the particle happens to be parallel or antiparallel to position vector $\vec{r}$.** Then $\vec{\tau} = \vec{r} \times \vec{F} = 0$ and the particle's angular momentum, $\vec{L} = \vec{r} \times m\vec{v}$ is constant. In other words, if a force on the particle is radial, the angular momentum of the particle remains constant.

13.36. SIMPLE PENDULUM

An object suspended so that it can swing to and fro about a horizontal axis is called a ***pendulum.***

Fig. 13.27 shows a simple pendulum. It consists of a string of length l suspended from a fixed point O and carrying a bob of mass m at its free end. When the bob is slightly displaced from the equilibrium position and then released, it executes rotational motion about an axis through the suspension point O. Therefore, it is profitable to describe the motion in terms of torque. Suppose at any time t, the bob is at point P making an angle θ with the vertical. Let ω be the angular velocity and α be the angular acceleration at that instant. In this situation, the forces acting on the bob are :

(*i*) The weight mg of the bob acting vertically downward.

(*ii*) The tension T in the string acting along PO.

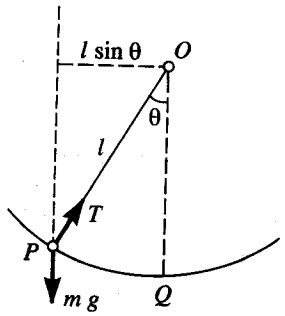


Fig. 13.27

Let τ be the torque due to the two forces acting on the bob at time t. Since the tension T in the string is acting along the line between the suspension point and the mass of bob, it exerts no torque. The only force which exerts a torque on the bob is that due to the weight $m\ g$ of the bob.

$\therefore$ $\tau = mg \times$ lever arm of mg

or $\tau = -mg \times l \sin\theta$ ...(*i*)

The negative sign is used since this is a restoring torque *i.e.*, a torque which forces the bob toward $\theta = 0°$.

The angular momentum of the bob about the axis of rotation at the considered instant (*i.e.*, at point P) is

$$L = r\,m\,v = l\,m(\omega l) = ml^2\omega \qquad (\because v = \omega r = \omega l)$$

Now, $$\tau = \frac{dL}{dt} = \frac{d}{dt}(ml^2\omega) = ml^2\frac{d\omega}{dt}$$ ...(*ii*)

From eqs. (*i*) and (*ii*), we have,

$$-m\,g\,l\sin\theta = ml^2\frac{d\omega}{dt}$$

or $$\frac{d\omega}{dt} = -\frac{g}{l}\sin\theta$$

Now $\frac{d\omega}{dt} = \alpha$, the angular acceleration of the bob at time t.

$\therefore$ $$\alpha = -\frac{g}{l}\sin\theta$$ ...(*iii*)

Eq. (*iii*) governs the oscillations of a simple pendulum in the vertical plane.

Discussion. The following points may be noted :

(*i*) When $\theta = 0°$, $\alpha = -(g/l)\sin 0° = 0$. Therefore, the bob has zero angular acceleration at the equilibrium position.

(*ii*) The negative sign in eq. (*iii*) shows that the tangential acceleration is directed toward the mean position.

(*iii*) If θ is small (as usually is the case), $\sin\theta = \theta$. Therefore, eq. (*iii*) becomes :

$$\alpha = -\frac{g}{l}\theta \quad \text{or} \quad \alpha \propto -\theta$$

Thus angular acceleration is directly proportional to the negative of angular displacement. Such a motion is called simple harmonic motion (S.H.M.). It can be shown that time period for a simple pendulum is given by ;

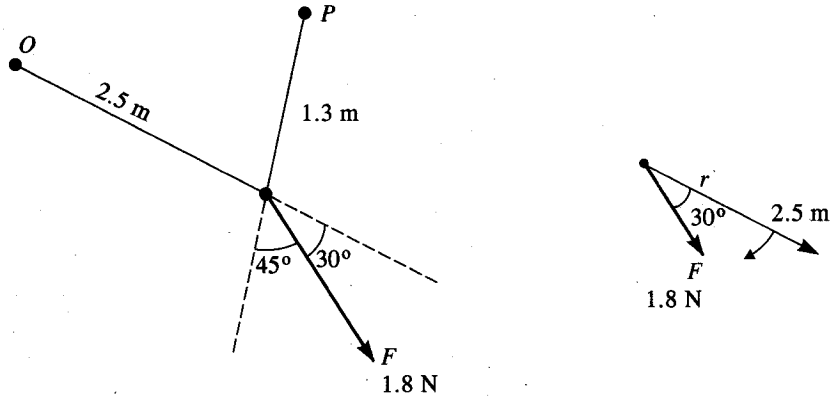
$$T = 2\pi\sqrt{\frac{l}{g}}$$

Note that time period of a simple pendulum is independent of its mass. It depends only on the length of the pendulum and acceleration due to gravity.

Example 13.12. Fig. 13.28 (*i*) shows 1.8 N force applied 2.5 m from a point *O* and 1.3 m from a point *P*. What are the torques about *O* and *P* due to $\vec{F}$?

Solution. The magnitude of torque about point O is

$$\tau_0 = rF\sin\phi = (2.5)(1.8)\sin 30° = 2.3 \text{ Nm}$$



(*i*) (*ii*)

Fig. 13.28

The magnitude of torque about point P is

$$\tau_P = rF\sin\phi = (1.3)(1.8)\sin 45° = \mathbf{1.7\ Nm}$$

To determine the direction of torque about O, we place two vectors $\vec{r}$ and $\vec{F}$ tail to tail [See Fig. 13.28 (*ii*)] and apply the right-hand rule. It is clear that torque vector is into the page. Similarly, the torque vector about P is out of the page.

Example 13.13. A force of 36 N is applied to a particle located at 0.15 m from the axis of rotation. What is the magnitude of torque about this axis, if the angle between the direction of the applied force and the radius vector is (*i*) 60° (*ii*) 90° ?

Solution. Torque $\tau = rF\sin\phi$. Here $F = 36$ N ; $r = 0.15$ m

(*i*) When $\phi = 60°$; $\tau = rF\sin\phi = 0.15 \times 36 \times \sin 60° = \mathbf{4.68\ Nm}$

(*ii*) When $\phi = 90°$; $\tau = rF\sin\phi = 0.15 \times 36 \times \sin 90° = \mathbf{5.4\ Nm}$

Example 13.14. A wheel of grindstone has applied at its axle 2 cm in radius a constant tangential force of 600 N. Find the torque acting on it.

Solution. Torque, $\tau = rF\sin\phi$

Here $r = 2$ cm $= 0.02$ m ; $F = 600$ N ; $\phi = 90°$

$\therefore$ $\tau = 0.02 \times 600 \times \sin 90° = \mathbf{12\ Nm}$

Example 13.15. An electron of mass 9×10^{-31} kg revolves in a circle of 0.53 Å around the nucleus of hydrogen atom with a velocity of 2.2×10^6 ms^{-1}. Find the angular momentum of the electron about the centre of the circle.

Solution. Angular momentum of electron is given by ;

$$L = r\,m\,v$$

Here $r = 0.53$ Å $= 0.53 \times 10^{-10}$m ; $m = 9 \times 10^{-31}$ kg ; $v = 2.2 \times 10^6$ ms^{-1}

$\therefore$ $L = (0.53 \times 10^{-10}) \times (9 \times 10^{-31}) \times (2.2 \times 10^6)$

$= \mathbf{1.0494 \times 10^{-34}\ Js}$

Example 13.16. In Fig 13.29, find the angular momentum of the particle relative to the coordinate. origin.

Solution. Referring to Fig. 13.29, the particle is in the xy plane moving in a straight line parallel to the x-axis. The magnitude of the particle's angular momentum relative to the origin O is

$$L = r\,p\sin\phi = r\,m\,v\sin\phi$$
$$= m\,v\,(r\sin\phi) = mvb$$

By right-hand rule, the direction of $\vec{L}$ for the particle is in the $-Z$ direction. Note that a particle with a straight line path has non-zero angular momentum about any point that is not on its path. If particle's speed is constant, its angular momentum is constant. But if the particle accelerates along a straight line, its angular momentum changes with time.

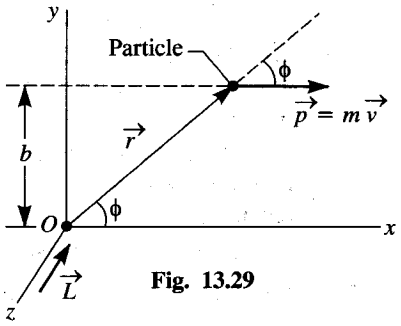


Fig. 13.29

Example 13.17. A torque of 20 Nm is applied on a wheel initially at rest. Calculate the angular momentum of the wheel after 3 sec.

Solution. Torque, $\tau = \dfrac{dL}{dt} = \dfrac{L - L_0}{t}$

Here, initial angular momentum, $L_0 = 0$; $\tau = 20$ Nm ; $\tau = 3$ s

$$\therefore \qquad 20 = \frac{L-0}{3} \text{ or } L = 60 \text{ kg m}^2\text{s}^{-1}$$

Example 13.18. A car of mass 900 kg is moving around a circular path of radius 300 m with a steady speed of 72 km/h. Calculate its angular momentum.

Solution. Here $m = 900$ kg ; $r = 300$ m ; $v = 72$ km/h $= 20$ ms^{-1}

Angular momentum, $L = m\,v\,r = 900 \times 20 \times 300 = 5.4 \times 10^6$ kg m^2 s^{-1}

CONCEPTUAL QUESTIONS

Q.1. If no external torque acts on a body, will its angular velocity remain conserved?

Ans. We know that :

$$\tau = \frac{dL}{dt}$$

If $\tau = 0$, then $dL/dt = 0$ or L = constant. Thus it is the angular momentum ($=I/\omega$) that is conserved and not the angular velocity (ω) of the body.

Q.2. A particle is executing uniform circular motion. Is the particle in rotational equilibrium?

Ans. It is true. In uniform circular motion, the angular velocity of the particle is constant in magnitude and direction. Hence angular acceleration of the particle is zero. We know that a particle is said to be in rotational equilibrium if its angular acceleration is zero.

Q.3. Is any work done by the centripetal force?

Ans. No work is done by the centripetal force.

Work done $= FS\cos\theta = FS\cos 90° = 0$.

Q.4. What is the power needed to maintain uniform circular motion?

Ans. No power is needed to maintain uniform circular motion.

Power required $= \vec{F}.\vec{v} = F\,v\cos\theta = F\,v\cos 90° = 0$

Q.5. What is the torque due to the centripetal force about the axis of rotation?

Ans. The torque due to the centripetal force about the axis of rotation is zero.

Torque, $\tau = F \times d = F \times 0 = 0$

Q.6. Why centripetal force cannot do work?

Ans. The centripetal force cannot do work on the object since it is always perpendicular to the velocity.

Q.7. Is the kinetic energy of a body moving in a vertical circle same at every point?

Ans. The velocity of a body moving in a vertical circle is different at different points. Therefore, kinetic energy of the body will be different at different points on the circle.

Q.8. A stone of mass m tied to the end of a string is whirled around in a horizontal circle (neglect force due to gravity). The length of string is reduced gradually keeping the angular momentum of the stone about the centre of the circle constant. Then the tension in the string is given by $T = Ar^n$, where A is a constant and r is instantaneous radius of the circle. Show that $n = -3$.

Ans. For circular motion, tension in the string is $T = \frac{mv^2}{r}$ ...(*i*)

Angular momentum, $L = m\,v\,r$ = constant = K $\therefore$ $v = K/mr$

Putting the value of v (= K/mr) in eq. (*i*), we have,

$$T = \frac{m}{r}\left(\frac{K}{mr}\right)^2 = \frac{K^2}{m}r^{-3}$$

As $T = Ar^n$ $\therefore$ $n = -3$

Q.9. A particle moves in a circular path with decreasing speed. What happens to its angular momentum?

Ans. The angular momentum of a particle of mass m moving along a circular path of radius r with linear velocity $\vec{v}$ is given by ;

$$\vec{L} = \vec{r} \times m\vec{v}$$

If the speed (v) of the particle decreases along the circular path, the magnitude of angular momentum (= $m\,v\,r$) decreases. However, the direction of angular momentum remains unchanged.

VERY SHORT ANSWER QUESTIONS

Q.1. What is the centre of mass of a body?

Ans. The centre of mass of a body (or system of particles) is the point where the entire mass of the body (or system of particles) is supposed to be concentrated.

Q.2. Where does the centre of mass of a two particle system lie?

Ans. The centre of mass of a two particle system lies between and on the line joining the two particles.

Q.3. Where does the centre of mass of a two particle system lie when one particle is more massive to the other?

Ans. The centre of mass is closer to the heavier particle.

Q.4. Does the centre of mass of a rigid body always lie within the body?

Ans. No, it is not necessary. The centre of mass of a body may lie within or outside the body.

Q.5. Upon what factors the centre of mass of a body depend?

Ans. The centre of mass of a body depends upon the shape and size of the body and the distribution of its mass.

Q.6. Can you give one example where the centre of mass lies inside the body and outside the body?

Ans. The centre of mass of a solid sphere lies at its centre which is within the body. However, the centre of mass of a hollow sphere is also at its centre but is outside the body.

Q.7. What is an isolated system?

Ans. The system is said to be *isolated* if nothing (energy, matter, light) crosses the boundary of the system to or from the environment and if net external force on the system is zero.

Q.8. Where does the centre of mass of a rectangle lie?

Ans. It lies at the point of intersection of the diagonals of the rectangle.

Q.9. Can the centre of mass of a body lie where there is no mass?

Ans. Yes, it is possible. For example, in case of a uniform circular ring, the centre of mass lies at the centre of ring where there is no mass.

Q.10. Give examples where the centre of mass coincides with the geometrical centre of the body.

Ans. For symmetrically shaped bodies of uniform composition (*e.g.*, spheres, cylinders, rectangular solids), the centre of mass is located at the geometrical centre of the body.

Q.11. What is meant by torque in rotational motion?

Ans. The turning effect of a force about an axis of rotation is called torque about that axis. It is measured by the product of the magnitude of force and the perpendicular distance of the line of action of the force from the axis of rotation.

Q.12. What is the lever arm or moment arm of a force?

Ans. It is the perpendicular distance of the line of action of the force from the axis of rotation.

Q.13. What is the SI unit of torque?

Ans. Torque, τ = Force × Lever arm. The SI unit of force is 1N and that of distance is 1m. Therefore, SI unit of torque is 1 Nm.

Q.14. Give one example of torque.

Ans. The opening or closing of a door hinged at one end is an example of application of torque.

Q.15. Is torque a vector quantity?

Ans. Yes. It is because it is the cross product of force $(\vec{F})$ and lever arm of force $(\vec{r})$ *i.e.*, torque, $\vec{\tau} = \vec{F} \times \vec{r}$.

Q.16. What is angular momentum?

Ans. When a body rotates about an axis, there is momentum associated with rotational motion. This is called angular momentum. Recall that a body moving in a straight line has linear momentum.

Q.17. How is angular momentum measured?

Ans. The angular momentum of a body about a given axis is equal to the product of linear momentum $(m\vec{v})$ and the perpendicular distance $(\vec{r})$ of the line of action of linear momentum vector from the axis of rotation.

Linear momentum, $\vec{L} = \vec{r} \times m\vec{v} = r\,mv$ (in magnitude)

Q.18. What is the SI unit of angular momentum?

Ans. Angular momentum, $L = r\,m\,v$. Therefore, the SI unit of angular momentum is m(kg) (ms^{-1}) or kg m^2s^{-1}.

Q.19. What is the dimensional formula of angular momentum?

Ans. $L = rmv$. Therefore, $[L] = [LM\ LT^{-1}] = [ML^2T^{-1}]$.

Q.20. Is angular momentum a vector quantity?

Ans. Yes. It is because angular momentum is the cross product of position vector $(\vec{r})$ of the linear momentum and the linear momentum $(\vec{p})$ *i.e.*, $\vec{L} = \vec{r} \times \vec{p}$.

SHORT ANSWER QUESTIONS

Q.1. What is the significance of centre of mass of a body?

Ans. The centre of mass of a body is the point where the whole mass of the body can be considered to be concentrated. Therefore, we can replace the whole body by a single particle located at the centre of mass. The motion of the centre of mass represents the motion of the body as a whole.

Q.2. What is the difference between centre of mass and centre of gravity?

Ans. The centre of mass (C.M.) of a body is a point where the whole mass of the body is supposed to be concentrated. On the other hand, centre of gravity (G) of a body is the point where the whole weight of the body is supposed to act on the body.

Q.3. Do centre of mass and centre of gravity of a body coincide?

Ans. They coincide only in uniform gravitational field as found close to the surface of earth. However, this is not true in non-uniform gravitational field.

Q.4. Show that the centre of mass of two particles divides the line joining the particles in the inverse ratio of their masses.

Ans. The centre of mass is the point about which the two masses are in equilibrium. Thus,

$$F_1 r_1 = F_2 r_2 \text{ or } m_1 g\, r_1 = m_2 g\, r_2 \therefore \frac{m_1}{m_2} = \frac{r_2}{r_1}$$

Q.5. How does angular momentum differ from linear momentum?

Ans. In this regard, the following two points may be noted :

(*i*) A body moving in a straight line has linear momentum. However, when a body rotates about an axis of rotation, the momentum associated with rotational motion is called angular momentum.

(*ii*) A net external force is required to change the linear momentum of a body while a net external torque is required to change the angular momentum of the body.

Q.6. State the law of conservation of angular momentum.

Ans. Refer to Art. 13.35.

Q.7. To tighten the bolt of a truck wheel, we prefer to use a wrench with a long arm. Why?

Ans. Torque, $\vec{\tau} = \vec{r} \times \vec{F}$ = Lever arm × Force. When the lever arm $(\vec{r})$ is long, the torque (*i.e.*, turning effect of force) is increased. Thus, a smaller force can produce a large turning effect.

Q.8. How will you determine the direction of torque?

Ans. Refer to Art. 13.25.

Q.9. You use a thumb and finger to open a bottle cork but turn the door by a single finger. Why?

Ans. In either case, we need a couple (two equal and opposite forces having different lines of action) to open the bottle cork and to turn the door. In case of cork, the thumb and finger provide the couple. In case of door, the force of finger and reaction at the hinge provide the necessary couple.

Q.10. A rope walker holds a long uniform stick in his hands. If he bends to right, in which direction should he turn the stick to keep himself in the balanced position on the rope?

Ans. When the walker leans towards to right, he produces a clockwise torque. To keep himself in balance, he should turn the stick in such a way to produce anticlockwise torque. To do so, he should turn the left end of the stick downward and right end upward.

LONG ANSWER QUESTIONS

1. Define centre of mass of (*i*) a body (*ii*) two-particle system (*iii*) *N*-particle system. **[Refer to Arts. 13.2, 13.4 and 13.5]**
2. Derive an expression for the position vector of the centre of mass of a two-particle system. **[Refer to Art. 13.4]**
3. Derive an expression for the position vector of the centre of mass of a system of *N* particles. **[Refer to Art. 13.5]**
4. A system consists of *N* particles of total mass *M*. Show that the centre of mass moves like a particle of total mass *M* under the influence of the net external force. **[Refer to Art. 13.5]**
5. Show that the total linear momentum of a system of particles is equal to the product of total mass of the system and the velocity of centre of mass of the system. **[Refer to Art. 13.7]**
6. Show that if the net external force on a system of particles is zero, the velocity of the centre of mass of the system is constant. **[Refer to Art. 13.8]**
7. Derive an expression for the work done by a force on a particle in a plane. How does it lead to the definition of torque due to a force? **[Refer to Art. 13.23]**
8. Show that torque produced by a force is equal to the cross product of the position vector of the force and the force. **[Refer to Art 13.25]**
9. Derive the relation between torque and angular momentum of a particle about an axis. **[Refer to Art. 13.29]**
10. Give the geometrical meaning of angular momentum. **[Refer to Art. 13.33]**
11. Deduce Kepler's second law of planetary motion from angular momentum considerations. **[Refer to Art. 13.34]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The centre of mass of a system of particles moves as if it were a particle with the same mass of the whole system subject to the forces external to the system.
2. The instantaneous angular velocity of a particle rotating in a circle or of a rigid body rotating about a fixed axis is given by ;

$$\omega = \frac{d\theta}{dt}$$

3. The angular acceleration of a rigid body about a fixed axis is given by :

$$\alpha = \frac{d\omega}{dt}$$

When a rigid body rotates about a fixed axis, every part of the body has the same angular velocity and angular acceleration. However, different parts of the body have different linear velocities and linear accelerations.

4. When a rigid body rotates about a fixed axis, the angular velocity and angular acceleration are related to the linear velocity and linear acceleration as under :

$$v = r\omega \; ; \qquad a_t = r\alpha$$

5. In rotational motion, angular velocity, angular acceleration, angular momentum and torque are all vectors directed along the axis of rotation.
6. In non-uniform circular motion, both centripetal acceleration (a_c) and tangential acceleration (a_t) are present. This is illustrated in Fig. 13.30.

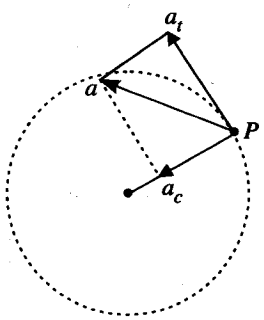


Fig. 13.30

Total acceleration, $\vec{a} = \vec{a}_t + \vec{a}_c$

The magnitude of total acceleration of the point P on rotating body is given by ;

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

(*i*) For uniform circular motion $\vec{a}_t = 0$ and there is only centripetal acceleration.

$$\therefore \qquad \vec{a} = 0 + \vec{a}_c = \vec{a}_c$$

(*ii*) For uniform translatory motion both $\vec{a}_t$ and $\vec{a}_c$ are zero. Therefore, $\vec{a} = 0$.

(*iii*) For accelerated translatory motion, $\vec{a}_c = 0$ and $\vec{a}_t \neq 0$.

7. The torque $\vec{\tau}$ due to a force $\vec{F}$ is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

8. The angular momentum $\vec{L}$ of a particle of linear momentum $\vec{p} = m\vec{v}$ is given by ;

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$$

where $\vec{r}$ = position vector of the particle relative to the origin.

9. If ϕ is the angle between $\vec{r}$ and $\vec{p}$, the magnitude of $\vec{L}$ is given by ;

$$L = mvr\sin\phi$$

10. The net external torque acting on a particle or rigid body is equal to time rate of change of angular momentum.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

11. The work done by external forces in rotating a rigid body about a fixed axis is

Work done, $W = \tau\theta$

12. The rate at which work is being done by external forces in rotating a rigid body about a fixed axis or power delivered is

Power delivered, $P = \tau\omega$

13. Angular momentum = 2 × Mass × Areal velocity

or $$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t}$$

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Torque produces **[CBSE PMT 1997]**

(*a*) linear motion
(*b*) rotational motion
(*c*) both linear and rotational motion
(*d*) none of the above

2. Find the torque of a force $\vec{F} = -3\hat{i} + \hat{j} + 5\hat{k}$ acting at a point $\vec{r} = 7\hat{i} + 3\hat{j} + \hat{k}$ **[CBSE PMT 1997]**

(*a*) $14\hat{i} - 38\hat{j} + 16\hat{k}$ (*b*) $4\hat{i} + 4\hat{j} + 6\hat{k}$
(*c*) $-21\hat{i} + 4\hat{j} + 4\hat{k}$ (*d*) $-14\hat{i} + 34\hat{j} - 16\hat{k}$

3. Which of the following has unit joule-sec? **[CBSE PMT 1994]**

(*a*) moment of inertia (*b*) torque
(*c*) angular momentum (*d*) power

4. A bomb travelling in a parabolic path explodes in mid air. The centre of mass of fragments will **[CBSE 2002]**

(*a*) move vertically upwards and then downwards
(*b*) move vertically downwards
(*c*) move irregularly
(*d*) move in parabolic path, the unexploded bomb would have travelled

5. Consider a system of two identical particles. One of the particles is at rest and the other has an acceleration $\vec{a}$. The centre of mass has an acceleration **[Manipal 2001]**

(*a*) $\vec{a}$ (*b*) $\frac{1}{2}\vec{a}$
(*c*) $2\vec{a}$ (*d*) zero

6. A couple produces a **[Pre-Medical Dental 1997]**

(*a*) pure linear motion
(*b*) pure rotational motion
(*c*) both linear and rotational motion
(*d*) no motion

7. A wheel is rotating at 900 r.p.m. about its axis. When power is cut off, it comes to rest in 1 minute. The angular retardation in rad/s^2 is **[MP PMT 1999]**

(*a*) $\pi/2$ (*b*) $\pi/4$
(*c*) $\pi/6$ (*d*) $\pi/8$

8. An earth satellite is moving around the earth in a circular orbit. In such case, what is conserved? **[DPMT 2001]**

(*a*) velocity
(*b*) linear momentum
(*c*) angular momentum
(*d*) none of the above

9. A particle moves along a circle of radius $(20/\pi)$ m with constant tangential acceleration. If the velocity of the particle is 80 m/s at the end of the second revolution after motion has begun, the tangential acceleration is **[CBSE 2003]**

(*a*) 160 π m/s^2 (*b*) 40 π m/s^2
(*c*) 40 m/s^2 (*d*) 640 π m/s^2

10. In an orbital motion, the angular momentum vector is **[AIIMS 2004]**

(*a*) in the orbital plane
(*b*) parallel to linear momentum
(*c*) along the radius vector
(*d*) perpendicular to the orbital plane

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*d*)	**5.** (*b*)
6. (*b*)	**7.** (*a*)	**8.** (*c*)	**9.** (*c*)	**10.** (*d*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Torque produces only **rotational motion.**

2. Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

or $$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix} = \hat{i}(15-1) - \hat{j}(3+35) + \hat{k}(7+9) = 14\hat{i} - 38\hat{j} + 16\hat{k}$$

3. **Angular momentum** has the unit joule-sec.

4. When the bomb explodes in air, the centre of mass of all fragments would move in a **parabolic path along which unexploded bomb would have moved.**

5. The two particles are identical. Therefore, the acceleration of C.M. is the average acceleration of the two particles *i.e.*, $(0 + \vec{a})/2 = \vec{a}/2$.

6. The couple produces turning effect so that **pure rotational motion** takes place.

7. Angular retardation of the wheel is given by;

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{2\pi(n_2 - n_1)}{t} = \frac{2\pi(0 - 900/60)}{60} = \pi/2 \text{ rad/s}^2$$

8. Since no external torque is acting on the system, the **angular momentum is conserved.**

9. $r = \frac{20}{\pi}$ m ; $\omega = \frac{v}{r} = \frac{80}{20/\pi} = 4\pi$ rad/s

Now $\omega^2 = \omega_0^2 + 2\alpha\theta$ or $(4\pi)^2 = (0)^2 + 2\alpha \times 4\pi$ $\therefore \alpha = 2\pi$ rad/s^2

$\therefore$ Tangential acceleration $a_T = r\alpha = (20/\pi) \times 2\pi = 40$ m/s^2

10. $\vec{L} = \vec{r} \times \vec{p}$. Therefore, $\vec{L}$ is **perpendicular to the plane containing** $\vec{r} \times \vec{p}$ *i.e.*, **orbital plane.**

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A flywheel gains a speed of 540 r.p.m. in 6 sec. Its angular acceleration is
[CET Karnataka 2001]
(*a*) 3π rad/sec^2 (*b*) 9π rad/sec^2
(*c*) 18π rad/sec^2 (*d*) 54π rad/sec^2

2. A shell fired from a gun at an angle θ to the horizontal explodes in mid-air. Then centre of mass of the shell fragments will move
[CET Haryana 2001]
(*a*) along tangent to the parabolic path of the intact shell at the point of explosion.
(*b*) along the same parabolic path along which the 'intact' shell was moving
(*c*) vertically downward
(*d*) horizontally

3. A stone of mass *m* tied to a string of length *l* is rotating along a circular path with constant speed *v*. The torque on the stone is
[PET Rajasthan 1997]
(*a*) *mvl* (*b*) zero
(*c*) mv^2l (*d*) *mv*/*l*

4. In rotary motion, the linear velocities of all the particles of the body are
(*a*) same (*b*) different
(*c*) zero (*d*) cannot say

5. A mass m is moving with a constant velocity along a line parallel to the x-axis away from the origin. Its angular momentum *w.r.t.* the origin is **[IIT 1997]**
   (*a*) zero (*b*) constant
   (*c*) goes on increasing
   (*d*) goes on decreasing
6. Two particles having mass M and m are moving in a circular path having radius R and r. If their time periods are same, then the ratio of angular velocities will be **[DCE 2004]**
   (*a*) r/R (*b*) R/r
   (*c*) 1 (*d*) $\sqrt{R/r}$
7. When a torque acting upon a system is zero, which of the following will be constant? **[J and K C.E.T. 2002]**
   (*a*) force (*b*) linear momentum
   (*c*) linear impulse (*d*) none of these
8. Angular momentum of a body is defined as the product of **[Kerala P.E.T. 2002]**
   (*a*) mass and angular velocity
   (*b*) centripetal force and radius
   (*c*) linear velocity and angular velocity
   (*d*) moment of inertia and angular velocity
9. A body A of mass M while falling vertically downwards under gravity breaks into two parts: a body B of mass $M/3$ and a body C of mass $2M/3$. The centre of mass of bodies B and C taken together shifts compared to that of body A towards **[AIEEE 2005]**
   (*a*) depends on height of breaking
   (*b*) does not shift
   (*c*) body C
   (*d*) body B
10. Two identical particles move towards each other with velocity $2v$ and v respectively. The velocity of centre of mass is **[AIEEE 2002]**
   (*a*) v (*b*) $v/3$
   (*c*) $v/2$ (*d*) zero

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS				
1. (*a*)	**2.** (*b*)	**3.** (*b*)	**4.** (*b*)	**5.** (*b*)
6. (*c*)	**7.** (*d*)	**8.** (*d*)	**9.** (*b*)	**10.** (*c*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Angular speed, $\Delta\omega$ = 540 r.p.m. = 18π rad/sec
   Time taken, $\Delta t = 6$ sec
   $\therefore$ Angular acceleration, $\alpha = \frac{\Delta\omega}{\Delta t} = \frac{18\pi}{6} = 3\pi$ **rad/sec^2**
2. Since no significant external forces act, the centre of mass will continue to maintain its original motion.
3. Torque, $\tau = rF \sin\phi = rF \sin 0° = 0$.
4. In rotary motion, the angular velocity is the same for all particles of the body. However, the linear velocity v (= $r\,\omega$) will depend upon the radius. Since different particles are moving in a circle of different radii, the linear velocities will be different.
5. Angular momentum = $mv \times r$. Since mv and r are constant, the angular momentum is constant.
6. Angular velocity, $\omega = 2\pi/T$. Since the time period of the two particles is the same, their angular velocities will also be the same. Therefore, ratio of angular velocities is 1.
7. Torque, $\tau = \frac{dL}{dt} = 0$ or L = constant. In other words, angular momentum is constant.
8. Angular momentum = Moment of inertia × Angular velocity.
9. Since no external force is acting on the system after breaking, the centre of mass of two parts B and C will follow the original path *i.e.,* no shift compared to body A.
10. $m_1 = m_2 = m$; $v_1 = 2v$; $v_2 = -v$.
    Let V be the velocity of the centre of mass. According to principle of conservation of linear momentum,
    $$(m + m)V = m \times 2v + m(-v) \quad \therefore \quad V = v/2$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **A car accelerates uniformly from rest to a speed of 15 ms^{-1} in a time of 20 s. Find the angular acceleration of one of its wheel and the number of revolutions turned by the wheel in the process. The radius of the car wheel is 1/3 metre.**

Hint. Linear acceleration, $a = \frac{v - v_0}{t} = \frac{15 - 0}{20} = 0.75\ ms^{-2}$

Average linear velocity, $\bar{v} = \frac{v_0 + v}{2} = \frac{0 + 15}{2} = 7.5\ ms^{-1}$

$\therefore$ Total distance travelled, $S = \bar{v}t = 7.5 \times 20 = 150$ m

Angular acceleration, $\alpha = \frac{a}{r} = \frac{0.75}{1/3} = 2.25\ rad/s^2$

Total angle turned, $\theta = \frac{S}{r} = \frac{150}{1/3} = 450$ rad

Total number of rev. turned $= \frac{450}{2\pi} = 72$ rev

2. **A wheel starting from rest is rotating with a constant angular acceleration of 3 rad/s^2. An observer notes that it traces an angle of 120 radians in 4 sec interval. For how long the wheel had been rotated when the observer started his observation?**

Hint. Let ω be the angular velocity of the wheel when the observer started his observation.

Now $\theta = \omega t + \frac{1}{2}\alpha t^2$

or $120 = \omega \times 4 + \frac{1}{2} \times 3 \times (4)^2 \quad \therefore \quad \omega = 24$ rad/s

Now $\omega = \omega_0 + \alpha t \quad$ or $\quad t = \frac{\omega - \omega_0}{\alpha} = \frac{24 - 0}{3} = 8$ s

3. **A particle moves in a circle of radius 20 cm. Its linear speed is given by $v = 2t$ where t is in seconds and v in ms^{-1}. Find the radial and tangential acceleration at $t = 3$ s.**

Hint. Linear speed at $t = 3$ s is $v = 2t = 2 \times 3 = 6\ ms^{-1}$

Radial acceleration at $t = 3$ s is $a_r = \frac{v^2}{r} = \frac{(6)^2}{0.2} = 180\ ms^{-2}$

Tangential acceleration is $a_t = \frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}(2t) = 2\ ms^{-2}$

4. **A wheel of radius 1.5 m is rotating at a constant angular acceleration of 10 rad/s^2; its initial (at $t = 0$) angular speed being (60/π) rev/min. Find its angular speed, angular displacement and linear velocity and acceleration at a point on the rim at time t = 2s.**

Hint. $\omega = \omega_0 + \alpha t$

Here $\omega_0 = \frac{60}{\pi}$ rev/min $= \frac{60}{\pi} \times 2\pi$ rad / min $= \frac{60}{\pi} \times 2\pi \times \frac{1}{60}$ rad/s $= 2$ rad/s

Here $\alpha = 10\ rad/s^2$; $t = 2$s

$\therefore$ $\omega = 2 + 10 \times 2 = 22$ rad/s

Now $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 2 \times 2 + \frac{1}{2} \times 10 \times (2)^2 = 24$ rad

$v_t = r\omega = 1.5 \times 22 = 33\ ms^{-1}$

$a_t = r\alpha = 1.5 \times 10 = 15\ ms^{-2}$

5. **Three particles of masses 3 kg, 4 kg and 5 kg are located at the corners of an equilateral triangle of sides 1 m. Locate the centre of mass.**

Hint. We choose our coordinate system as shown (to simplify calculations) with m_1 at the origin and m_2 on the X-axis [See Fig. 13.31].

Coordinates of m_1 are $x_1 = 0$, $y_1 = 0$

Coordinates of m_2 are $x_2 = 1$ m, $y_2 = 0$

Coordinates of m_3 are $x_3 = 0.5$ m, $y_3 = \frac{\sqrt{3}}{2}$ m

Let the coordinates of the centre of mass be x and y.

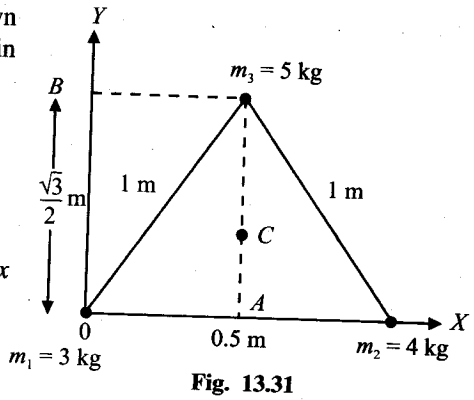


Fig. 13.31

$$\therefore \quad x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{3\times0 + 4\times1 + 5\times0.5}{3+4+5} = \frac{6.5}{12} = \mathbf{0.54\ m}$$

and $$y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{3\times0 + 4\times0 + 5\times\sqrt{3}/2}{3+4+5} = \frac{5\times\sqrt{3}}{12} = \mathbf{0.36\ m}$$

6. **Find the centre of mass of metallic letter E of uniform density whose dimensions are given in Fig. 13.32 (*i*). Take the origin at the bottom of the left corner. The width of the letter is 4 cm everywhere.**

Hint. We can divide the letter E into three rectangles and one square as shown by the shaded regions in Fig. 13.32 (*ii*). The centres of mass of these four parts are C_1, C_2, C_3 and C_4 as shown.

Coordinates of C_1 are $x_1 = 6$ cm ; $y_1 = 2$ cm

Coordinates of C_2 are $x_2 = 2$ cm ; $y_2 = 10$ cm

Coordinates of C_3 are $x_3 = 6$ cm ; $y_3 = 18$ cm

Coordinates of C_4 are $x_4 = 6$ cm ; $y_4 = 10$ cm

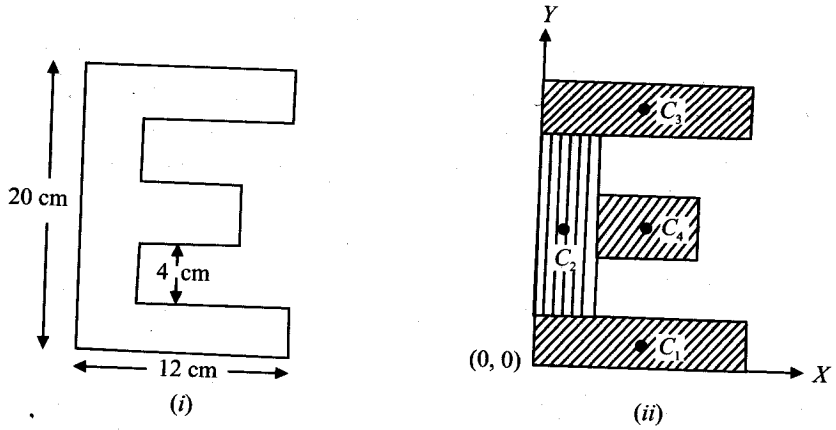


Fig. 13.32

Area of each rectangle = $12 \times 4 = 48$ cm^2

Area of square = $4 \times 4 = 16$ cm^2

Suppose m is the mass per unit area. Let the coordinates of the centre of mass of the letter E be x and y.

$$\therefore \quad x = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{48m \times 6 + 48m \times 2 + 48m \times 6 + 16m \times 6}{48m + 48m + 48m + 16m} = \frac{768m}{160m} = 4.8 \text{ cm}$$

$$y = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{48m \times 2 + 48m \times 10 + 48m \times 18 + 16m \times 10}{48m + 48m + 48m + 16m} = \frac{1600m}{160m} = 10 \text{ cm}$$

7. A 2 kg body and a 3 kg body are moving along the X-axis. At a particular instant, the 2 kg body is 1 m from the origin and has a velocity of 3 ms^{-1} and the 3 kg body is 2 m from the origin and has a velocity of –1 ms^{-1}. Find the position and velocity of centre of mass. Also find the total momentum.

Hint. The x-coordinate of centre of mass is

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{2 \times 1 + 3 \times 2}{2 + 3} = \frac{8}{5} = 1.6 \text{ m}$$

The velocity of the centre of mass is

$$V = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = \frac{2 \times 3 + 3 \times (-1)}{2 + 3} = \frac{3}{5} = 0.6 \text{ ms}^{-1}$$

Total momentum = $(m_1 + m_2)\ V = (2 + 3) \times 0.6 = 3 \text{ kg ms}^{-1}$

8. A mass of 4 kg is located at $x_1 = 0$ m ; $y_1 = 0$ m and a mass of 6 kg is located at $x_2 = 5$ m, $y_2 = 4$ m. Where should one place a third mass of 8 kg so that the coordinates of the centre of mass of the system is at $x = 2$ m, $y = 2$ m ?

Hint. $m_1 = 4$ kg ; $m_2 = 6$ kg ; $m_3 = 8$ kg

$$\therefore \quad x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{4 \times 0 + 6 \times 5 + 8 \times x_3}{4 + 6 + 8} = \frac{30 + 8x_3}{18}$$

or $$x_3 = \frac{18x - 30}{8} = \frac{18 \times 2 - 30}{8} = \frac{6}{8} = 0.75 \text{ m}$$

Now $$y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{4 \times 0 + 6 \times 4 + 8 \times y_3}{4 + 6 + 8} = \frac{24 + 8y_3}{18}$$

$$\therefore \quad y_3 = \frac{18y - 24}{8} = \frac{18 \times 2 - 24}{8} = \frac{12}{8} = 1.5 \text{ m}$$

14

Moment of Inertia

INTRODUCTION

The tendency of a body to resist change in its linear velocity is called *linear inertia* or simply *inertia.* In other words, objects do not change their state of linear motion unless acted upon by some net external force. Similarly, the tendency of a body to resist change in its angular velocity is called *rotational inertia* or *moment of inertia.* In other words, objects do not change their rotational motion unless acted upon by some net external torque. Thus, when you switch off a fan, its blades continue to rotate for some time. It is due to the rotational inertia (or moment of inertia) of the fan blades. The fan blades slowly come to rest on account of frictional force of air and the axial bearings. In this chapter, we shall discuss the important role of moment of inertia of bodies in rotational motion.

14.1. CONCEPT OF MOMENT OF INERTIA

The **moment of inertia** *(or* **rotational inertia***) of a body about an axis is a measure of the difficulty in starting or stopping or changing rotation of the body about that axis.* It is denoted by *I.*

The greater the difficulty in starting or stopping or changing rotation of a body about an axis, the greater is the moment of inertia of the body about that axis and *vice-versa.* A body rotates under the action of a net external torque. The rotation of the body takes place about a line, called *axis of rotation.* For example, when you open a door, the vertical line passing through the hinges is the axis of rotation. Just as linear inertia of a body is its resistance to change in linear motion, similarly the moment of inertia (or rotational inertia) of a body about an axis is its resistance to change in rotational motion about that axis. The linear inertia of a body depends upon its mass; the greater the mass of the body, the greater is its linear inertia. However, the moment of inertia of a body about an axis not only depends upon the mass of the body but also upon the distribution of mass of the body *w.r.t.* axis of rotation. As we shall see, moment of inertia plays the same role in rotational motion as mass plays in linear motion. The greater the moment of inertia (I) of a body about an axis of rotation, the greater is the torque required to rotate or stop or change rotation of the body about that axis and *vice-versa.*

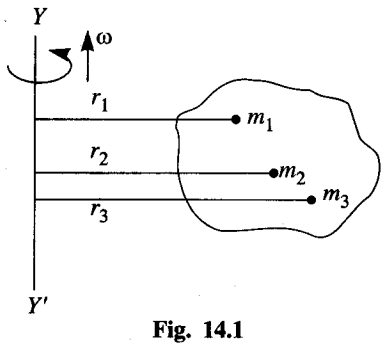


Fig. 14.1

14.2. MOMENT OF INERTIA OF A RIGID BODY

Consider a rigid body rotating about the axis YY' with an angular speed ω as shown in Fig. 14.1. Suppose the body is made up of a large number (n) of small particles of masses

m_1, m_2, m_3, ... m_n situated at perpendicular distances r_1, r_2, r_3, ... r_n respectively from the axis of rotation YY'. As the body rotates, each particle of the body follows a circular path around the axis. Although each particle of the body has the same angular speed ω, the linear velocity (v) of each particle depends upon particle's distance from the axis of rotation. Thus particle of mass m_1 follows a circular path of radius r_1. The linear velocity of this particle is $v_1 (= \omega r_1)$.

$\therefore$ Rotational kinetic energy of the particle of mass m_1

$$= \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (\omega r_1)^2 = \frac{1}{2} m_1 r_1^2 \omega^2$$

The total kinetic energy K_r of the rotating body is the sum of the kinetic energies of all the particles of which the body is composed.

$$\therefore \quad K_r = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots$$

$$= \frac{1}{2} \left(\sum_{i=1}^{i=n} m_i r_i^2 \right) \omega^2$$

where m_i is the mass of the ith particle and r_i is its perpendicular distance from the axis of rotation.

The quantity $\sum_{i=1}^{i=n} m_i r_i^2$ is the sum of the products of the mass of each particle and the square of its perpendicular distance from the axis of rotation (YY'). It is called **moment of inertia** or **rotational inertia** of the body about the axis of rotation and is represented by the symbol I.

$$\therefore \quad \text{Moment of inertia, } I = \sum_{i=1}^{i=n} m_i r_i^2$$

$$\therefore \text{ Total K.E. of rotating body, } K_r = \frac{1}{2} I \omega^2$$

Note that K.E. of rotating body is analogous to the translational K.E. $mv^2/2$. Angular velocity ω replaces linear velocity v and moment of inertia I replaces mass m.

Thus the **moment of inertia** *of a rigid body about a given axis of rotation is the sum of the products of the masses of its particles and the squares of their respective perpendicular distances from the axis of rotation.*

Discussion. The following points may be noted :

(*i*) *The moment of inertia* I $(= \Sigma m_i r_i^2)$ *of a body about an axis of rotation is directly proportional to the total mass of the body.* The more massive the body, the more difficult it will be to start it rotating or stop it from rotating.

(*ii*) *For a given mass, the moment of inertia of a body depends upon the distribution of the mass from the axis of rotation.* The larger the distance of the mass from the axis of rotation, the larger will be its rotational inertia.

(*iii*) *The moment of inertia of a body is not a fixed quantity but depends on the location of the axis about which the moment of inertia is calculated.* A particular body does have a unique mass; it does *not* have a unique moment of inertia. Its moment of inertia depends not only on its total mass and the distribution of that mass in space (*i.e.*, shape of the body) but also on the orientation and position of the axis of rotation. Therefore, one must specify the axis of rotation (its location and orientation) relative to which I is measured.

Units of moment of inertia. Moment of inertia, $I = mr^2$. Therefore, the SI unit of moment of inertia is kg m^2. The dimensional formula of M.I. is $[ML^2T^0]$.

14.3. RADIUS OF GYRATION

The **radius of gyration** *of an object is defined as the distance from the given axis of rotation at which if whole mass of the object were supposed to be *concentrated, the moment of inertia would be the same as with the actual distribution of mass.*

The radius of gyration is denoted by the symbol K. The moment of inertia of a body of mass M and radius of gyration K is given by :

$$I = M K^2$$

For example, the moment of inertia of a thin rod of mass M and length L about an axis through its centre and perpendicular to its length (See Art. 14.19) is

$$I = \frac{ML^2}{12}$$

or $$MK^2 = \frac{ML^2}{12}$$

$\therefore$ $$K = \frac{L}{\sqrt{12}}$$

This means that the rod has the same moment of inertia as a particle of mass M rotating at a disance $L/\sqrt{12}$ from the considered axis would have. The radius of gyration is a measure of the distribution of mass of a body relative to a given axis of rotation. A large radius of gyration means that, on the average, the mass is relatively far from the given axis of rotation. The SI unit of radius of gyration is metre.

Another definiton of K. Suppose a body consists of n particles each of mass m. Then total mass of the body is $M = mn$.

Now $$I = \Sigma m r^2 = m\left[r_1^2 + r_2^2 + \ldots\ldots\ldots\ldots + r_n^2\right]$$

or $$MK^2 = mn\left[\frac{r_1^2 + r_2^2 + \ldots\ldots\ldots\ldots + r_n^2}{n}\right]$$

But $mn = M$ = mass of the body

$\therefore$ $$MK^2 = M\left[\frac{r_1^2 + r_2^2 + \ldots\ldots\ldots\ldots + r_n^2}{n}\right]$$

or $$K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots\ldots\ldots\ldots + r_n^2}{n}}$$

= root mean square distance of the particles from the axis of rotation.

Thus the radius of gyration of a body about a given axis may be defined as the root mean square distance of the various particles of the body from the axis of rotation.

Example 14.1. Four particles each of mass m are connected by massless rods to form a rectangle of sides $2a$ and $2b$ as shown in Fig. 14.2. The system rotates about an axis in the plane of the figure through the centre. Find the moment of inertia about this axis.

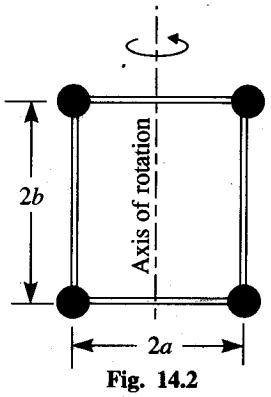


Fig. 14.2

* Not to be confused with the centre of mass.

Solution. It is clear from Fig. 14.2 that distance of each particle from the axis of rotation is a. Therefore, the moment of inertia of each particle about this axis is $m\,a^2$. Hence the total moment of inertia of the body about the axis is

$$I = m\,a^2 + m\,a^2 + m\,a^2 + m\,a^2$$

or $$I = 4\,m\,a^2 \text{ kg m}^2$$

Example 14.2. Two particles of masses 5 kg and 7 kg are mounted 4 m apart on a light rod (whose mass can be ignored) as shown in Fig. 14.3 (*i*). Calculate the moment of inertia of the system (*i*) when rotated about an axis passing half-way between the masses and (*ii*) when the system rotates about an axis located 0.5 m to the left of the 5 kg mass.

Solution. (*i*) Fig. 14.3 (*i*) shows the conditions of the problem. Both particles are at the same distance from the axis of rotation. Therefore, moment of inertia of the system about the shown axis is

$$I = m_1r^2 + m_2r^2 = 5\times2^2 + 7\times2^2 = 48 \text{ kg m}^2$$

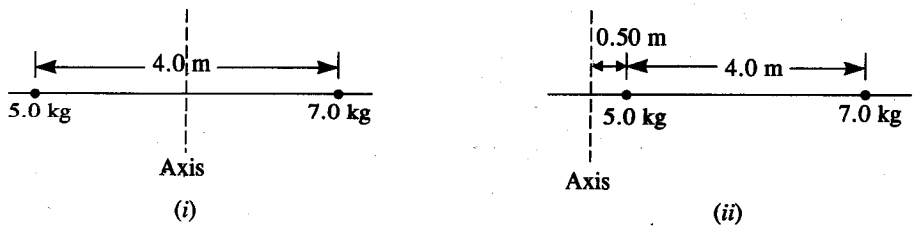


Fig. 14.3

(*ii*) Fig. 14.3 (*ii*) shows the conditions of the problem. Now 5 kg mass is 0.5 m from the axis and 7 kg mass is 4.5 m from the axis. Therefore, moment of inertia of the system about the shown axis is

$$I = m_1r_1^2 + m_2r_2^2 = 5\times(0.5)^2 + 7\times(4.5)^2 = 143 \text{ kg m}^2$$

This example illustrates two important points. First, the moment of inertia of a given system is different for different axes of rotation. Secondly, part (*ii*) illustrates the fact that mass close to the axis of rotation contributes little to the moment of inertia; in this example, 5 kg mass contributes less than 1% to the total.

Example 14.3. Four particles of masses 4 kg, 2 kg, 3 kg and 5 kg are fixed at the four corners *A*, *B*, *C* and *D* of a square of each side 1 m. Calculate the moment of inertia of the system about (*i*) an axis passing through the point of intersection of the diagonals and perpendicular to the plane of the square (*ii*) the side *AB* (*iii*) the diagonal *BD*.

Solution. Fig. 14.4 shows the conditions of the problem.

$$AB = BC = CD = DA = 1 \text{ m}$$

$$\therefore \quad OA = OB = OC = OD = \frac{1}{\sqrt{2}} \text{m}$$

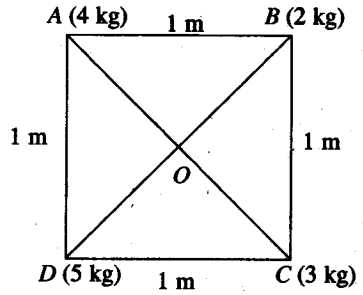


Fig. 14.4

(*i*) M.I. of the system about the axis passing through O and perpendicular to the plane of the square is

$$I_O = 4\times(OA)^2 + 2\times(OB)^2 + 3\times(OC)^2 + 5(OD)^2$$

$$= 4\times\frac{1}{2} + 2\times\frac{1}{2} + 3\times\frac{1}{2} + 5\times\frac{1}{2} = 7 \text{ kg m}^2$$

(*ii*) M.I. of the system about the side *AB*

$$= 3\times(BC)^2 + 5\times(AD)^2 = 3\times1 + 5\times1 = 8 \text{ kg m}^2$$

(*iii*) M.I. of the system about the diagonal *BD*

$$= 4\times(OA)^2+3\times(OC)^2 = 4\times\frac{1}{2}+3\times\frac{1}{2} = \mathbf{3.5\ kg\ m^2}$$

Example 14.4. Three point masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a as shown in Fig. 14.5. Find the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1.

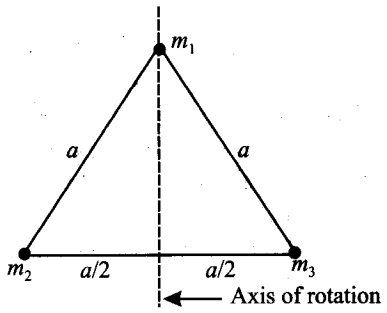


Fig. 14.5

Solution. Fig. 14.5 shows the conditions of the problem. The perpendicular distance of m_1 from the axis of rotation $r_1 = 0$. But perpendicular distances of m_2 and m_3 from the axis of rotation are $r_2 = a/2$ and $r_3 = a/2$. Therefore, moment of inertia I of the three particle system is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$= m_1\times 0 + m_2\times(a/2)^2 + m_3\times(a/2)^2 = \frac{a^2}{4}(m_2 + m_3)$$

PROBLEMS FOR PRACTICE

1. Three mass points m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1? $\left[a^2(m_2+m_3)/4\right]$
2. Four equal masses (M) are located in a plane at the four corners of a square of side a. What is the moment of inertia of the system about an axis passing through one of the masses perpendicular to the plane? $[4Ma^2]$
3. A 2.4 kg ball on the end of a light rod is rotated in a horizontal circle of radius 1.2 m. Calculate the moment of inertia of the ball. **[3.5 kg m^2]**
4. A dumbell has two 1 kg point masses at the ends of a massless rod 1 m long. Find the moment of inertia of the rod about an axis (*i*) perpendicular to the rod and passing through its centre (*ii*) perpendicular to the rod and passing through one end. What is the radius of gyration for part (*ii*) ? **[(*i*) 0.5 kg m^2 (*ii*) 1 kg m^2 ; 0.71 m]**
5. A thin light wooden metre stick is loaded with two masses as follows : 5 kg at 30 cm mark and 7 kg at 70 cm mark. What is the moment of inertia about an axis through the zero end of the metre stick? **[3.88 kg m^2]**

14.4. ROTATIONAL KINETIC ENERGY

The kinetic energy possessed by a body on account of its rotation about a given axis is called rotational kinetic energy of the body. It is denoted by K_r.

Consider a rigid body rotating about the axis YY' with an angular speed ω as shown in Fig. 14.6. Suppose the body is made up of a large number of particles of masses m_1, m_2, m_3,..... situated at perpendicular distances r_1, r_2, r_3,..... respectively from the axis of rotation. As the body rotates, each particle within the body follows a circular path around the axis. Although each particle within the body has the same angular speed ω, the linear velocity of each particle depends upon particle's position *w.r.t.*

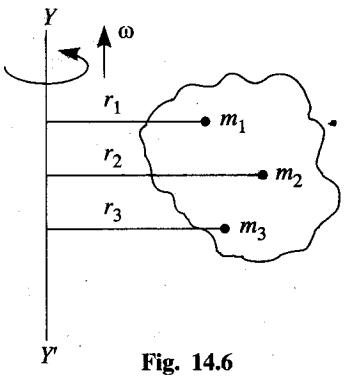


Fig. 14.6

axis of rotation. Let us find the kinetic energy (K_r) of this rotating body. The particle of mass m_1 follows a circular path of radius r_1. The magnitude of the linear velocity of this particle on the circle is v_1.

$\therefore$ Rotational kinetic energy of the particle of mass m_1

$$= \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1(r_1\omega)^2 = \frac{1}{2}m_1r_1^2\omega^2$$

Similarly, the rotational kinetic energy of particles of masses $m_2, m_3, \ldots\ldots$ are $\frac{1}{2}m_2r_2^2\omega^2$, $\frac{1}{2}m_3r_3^2\omega^2 \ldots\ldots$ respectively. The rotational kinetic energy K_r of the body is equal to the sum of the rotational kinetic energies of all the particles.

$\therefore$ Rotational K.E. of the body, $K_r = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \ldots\ldots$

$$= \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots)\omega^2$$

$$= \frac{1}{2}(\Sigma m_ir_i^2)\omega^2$$

But $\Sigma m_ir_i^2$ is the moment of inertia I of the body about the given axis of rotation.

$\therefore$ Rotational K.E. of the body, $K_r = \frac{1}{2}I\omega^2$

Thus the rotational K.E. of a body is equal to half the product of the moment of inertia of the body and the square of the angular velocity of the body about the given axis of rotation.

Now $$K_r = \frac{1}{2}I\omega^2$$

If $\omega = 1$, the $I = 2K_r$.

Thus the moment of inertia of a rigid body about a given axis of rotation is numerically equal to twice the rotational kinetic energy of the body when rotating with unit angular velocity about that axis.

Physical significance of moment of inertia. Let us now turn to physical significance of moment of inertia (I). We have seen that kinetic energy K_r of a rotating body is

$$K_r = \frac{1}{2}I\omega^2$$

We also know that kinetic energy K of a body for linear motion is

$$K = \frac{1}{2}mv^2$$

Note that both expressions for kinetic energy are one-half the product of property of body and square of velocity term. We know that linear velocity v is an analogy of angular velocity ω in rotational motion. Therefore, mass (m) of the body is an analogue of moment of inertia (I) of the body in rotational motion. *Hence we arrive at a very important conclusion that moment of inertia (I) plays the same role in rotational motion as the mass (m) plays in linear motion.*

14.5. TORQUE AND MOMENT OF INERTIA

Suppose a body rotates about the YY' axis under the action of a constant torque τ as shown in Fig. 14.7. Let the constant angular acceleration produced in the body be α. The angular acceleration of all the particles of the body will be the same (α) but the linear acceleration (a) of each particle will depend upon particle's position *w.r.t.* the axis of rotation. The

particle of mass m_1 follows a circular path of radius r_1. Therefore, the magnitude a_1 of the linear acceleration of this particle is

$$a_1 = r_1 \alpha$$

If F_1 is the net external force acting on this particle, then,

$$F_1 = m_1 a_1 = m_1(r_1\alpha) = m_1 r_1 \alpha$$

The magnitude of torque due to this force on this particle

$$= F_1 r_1 = (m_1 r_1 \alpha) r_1 = m_1 r_1^2 \alpha$$

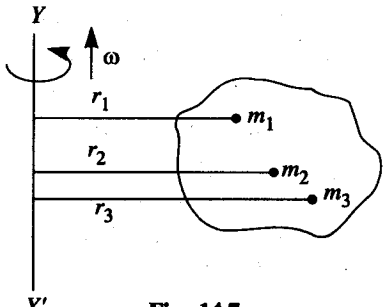


Fig. 14.7

Similarly, the magnitudes of torque on the particles of masses m_2, m_3..... are $m_2 r_2^2\alpha$, $m_3 r_3^2\alpha$..... respectively. By right-hand rule, the torques on all the particles act in the same direction. Therefore, the magnitude of the total torque τ on the body is just the sum of individual torques on the particles *i.e.*,

Torque on the body, $\tau = m_1 r_1^2\alpha + m_2 r_2^2\alpha + m_3 r_3^2\alpha + ...$

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...)\,\alpha$$

$$= (\Sigma m_i r_i^2)\,\alpha$$

But $\Sigma m_i r_i^2$ is the moment of inertia I of the body about the axis of rotation.

$$\therefore \qquad \tau = I\alpha \qquad ...(i)$$

Equation (*i*) is **basic relation for rotational motion**. It is analogous to Newton's second law of linear motion *viz.* $F = ma$. The torque τ is analogous to force F; the moment of inertia I is analogous to mass m and the angular acceleration α is analogous to the linear acceleration a.

Now $\tau = I\alpha$

If $\alpha = 1$, then $\tau = I$.

Hence the moment of inertia of a body about a given axis is equal to the torque required to produce unit angular acceleration in the body about that axis.

Note that $\vec{\tau}$ and $\vec{\alpha}$ are vector quantities. Since $\vec{\tau}$ and $\vec{\alpha}$ act in the same direction, we can write eq. (*i*) in the vector form as :

$$\vec{\tau} = I\vec{\alpha}$$

14.6. ANGULAR MOMENTUM AND MOMENT OF INERTIA

Consider a rigid body rotating about the axis *YY'* with an angular speed ω as shown in Fig. 14.8. Suppose the body is made up of a large number of particles of masses m_1, m_2, m_3..... situated at perpendicular distances r_1, r_2, r_3 ..... respectively from the axis of rotation. As the body rotates, each particle within the body follows a circular path around the axis. Although each particle within the body has the same angular speed ω, the linear velocity of each particle depends upon particle's position *w.r.t.* axis of rotation.

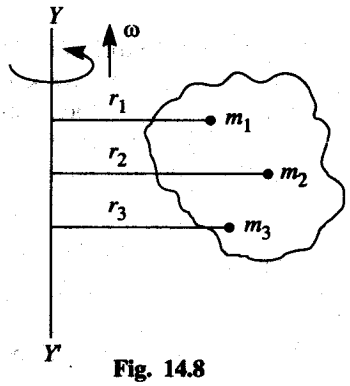


Fig. 14.8

The particle of mass m_1 follows a circular path of radius r_1 about the axis of rotation. Therefore, the magnitude v_1 of the linear velocity of this particle is

$$v_1 = r_1 \omega$$

The magnitude of the angular momentum of this particle about the axis of rotation

$$= \text{linear momentum} \times \text{radius } r_1$$

$$= m_1 v_1 \times r_1 = m_1(\omega r_1) \times r_1 = m_1 r_1^2 \omega$$

Similarly, the magnitudes of angular momenta of particles of masses m_2, m_3,..... are $m_2 r_2^2 \omega$, $m_3 r_3^2 \omega$,..... respectively. By right hand rule, the angular momenta of all the particles in the rigid body point in the same direction, parallel to the angular velocity vector. Therefore, the magnitude of total angular momentum L of the body about the axis of rotation is just the sum of individual momenta of the particles *i.e.*,

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega + ...$$

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...)\omega$$

$$= (\Sigma m_i r_i^2)\omega$$

But $\Sigma m_i r_i^2$ is the moment of inertia I of the body about the axis of rotation.

$$\therefore \qquad L = I\,\omega \qquad ...(i)$$

Thus the magnitude of angular momentum of a body about a given axis is equal to the product of moment of inertia (I) of the body and its angular speed (ω) *about that axis.*

If $\omega = 1$, then, $I = L$.

Hence the moment of inertia of a rigid body about a given axis of rotation is numerically equal to the angular momentum of the body when rotating with unit angular velocity about that axis.

Units of angular momentum. $L = I\omega$. The SI unit of I is kg m^2 and that of ω is s^{-1}. Therefore, SI unit of angular momentum is kg m^2 s^{-1}. Its dimensional formula is $[M\ L^2\ T^{-1}]$. Note that angular momentum is a vector quantity *i.e.,* it has magnitude as well as direction.

Discussion. The following points may be noted :

(*i*) $\vec{L}$ and $\vec{\omega}$ are vector quantities. Since $\vec{L}$ and $\vec{\omega}$ are in the same direction, we can write eq. (*i*) in vector form as :

$$\vec{L} = I\vec{\omega}$$

(*ii*) Eq. (*i*) reveals an interesting aspect. Moment of inertia of a body is its tendency to resist change in its angular velocity just as linear inertia of a body is its tendency to resist change in its linear velocity.

(*iii*) We know that magnitude of linear momentum is $p = mv$. However, the magnitude of angular momentum is $L = I\,\omega$. Thus replacing m by I and v by ω, we get the angular momentum $L = I\omega$. *Thus if you know the equation for linear motion, you can easily guess the rotational analogy.*

14.7. RELATION BETWEEN TORQUE AND ANGULAR MOMENTUM

The magnitude of angular momentum of a rigid body about an axis is

$$L = I\,\omega$$

Differentiating both sides *w.r.t. t,* we get,

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha \quad \left(\because \alpha = \frac{d\omega}{dt}\right) \qquad ...(i)$$

But $I\alpha$ = Torque τ so that from eq. (*i*), we have,

$$\tau = \frac{dL}{dt}$$

Thus the torque acting on a body is equal to the time rate of change of angular momentum of the body.

14.8. WORK-ENERGY THEOREM IN ROTATIONAL MOTION

In linear motion, the work-energy theorem states that the net work done on a body by the external force is equal to the change in body's linear kinetic energy. Similarly, in rotational motion, the work done by a force in rotating the body about an axis is equal to the change in body's rotational kinetic energy. Therefore, work-energy theorem in rotational motion may be stated as under :

The net work done by the external force in rotating a rigid body about an axis is equal to the change in body's rotational kinetic energy i.e.,

Work done on body = Change in body's rotational K.E.

To prove $\tau = I\alpha$. We now use work-energy theorem in rotational motion to prove the relation $\tau = I\alpha$. Consider a body rotating with an angular speed ω_0. Suppose a torque τ is suddenly applied to the body. The torque acts while the body turns through an angle θ and is then removed. Suppose the angular speed at that time is ω_f. Work done by the torque is $\tau\theta$. According to work-energy theorem,

Work done on body = Change in body's rotational K.E.

or $$\tau\theta = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_0^2$$

or $$\tau\theta = \frac{1}{2}I(\omega_f^2 - \omega_0^2)$$

Now $\omega_f^2 - \omega_0^2 = 2\alpha\theta$, where α is the angular acceleration produced in rad s^{-2}.

$\therefore$ $$\tau\theta = \frac{1}{2}I(2\,\alpha\,\theta)$$

or $$\tau = I\alpha$$

14.9. POWER IN ROTATIONAL MOTION

Consider a rigid body rotating about an axis due to a constant applied torque τ. If the body rotates through a small angle $\Delta\theta$ is small time Δt, then, small work ΔW done on the body is

$$\Delta W = \tau\,\Delta\theta$$

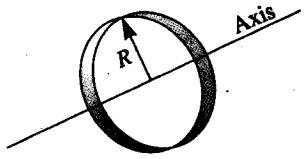
Now $$\text{Power, } P = \frac{\Delta W}{\Delta t} = \frac{\tau\Delta\theta}{\Delta t} = \tau \times \frac{\Delta\theta}{\Delta t} = \tau\,\omega \qquad \left(\because \omega = \frac{\Delta\theta}{\Delta t}\right)$$

$\therefore$ $$P = \tau\,\omega$$

Thus power of a rotating body is equal to the product of torque and angular velocity.

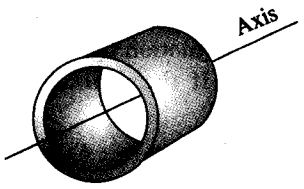
14.10. M.I. OF SOME COMMONLY USED BODIES

We will derive M.I. for some bodies in Arts. 14.17 to 14.23. The M.I. for other shapes about various axes can be found in a similar way. Below is listed M.I. for some common shapes. Note that more than one M.I. is listed for some shapes since M.I. depends upon the axis of rotation.



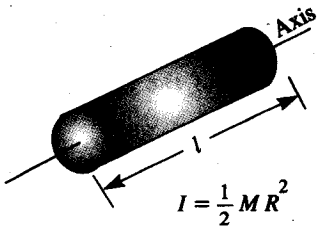
$I = MR^2$

(*a*) Hoop about central axis



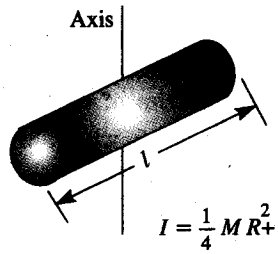
$I = \frac{1}{2} M (R_1^2 + R_2^2)$

(*b*) Annular cylinder (or ring) about central axis



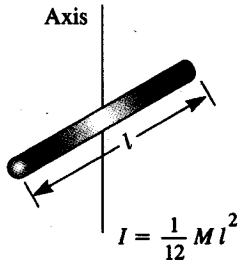
$I = \frac{1}{2} MR^2$

(*c*) Solid cylinder (or disc) about central axis



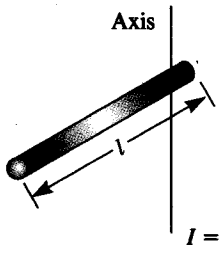
$I = \frac{1}{4} MR^2 + \frac{1}{12} Ml^2$

(*d*) Solid cylinder (or disc) about central diameter



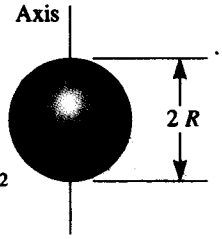
$I = \frac{1}{12} Ml^2$

(*e*) Thin rod about axis through centre perpendicular to length



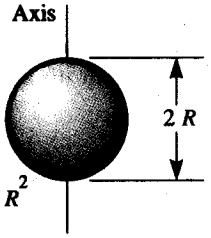
$I = \frac{1}{3} Ml^2$

(*f*) Thin rod about axis through one end perpendicular to length



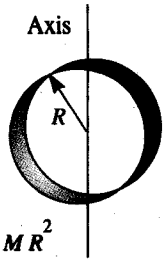
$I = \frac{2}{5} MR^2$

(*g*) Solid sphere about any diameter



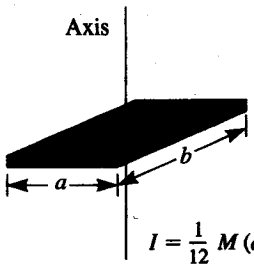
$I = \frac{2}{3} MR^2$

(*h*) Thin spherical shell about any diameter



$I = \frac{1}{2} MR^2$

(*i*) Hoop about any diameter



$I = \frac{1}{12} M (a^2 + b^2)$

(*j*) Slab about perpendicular axis through centre

Fig. 14.9

Example 14.5. A solid flywheel has a moment of inertia of 0.1 kg m^2 about its axis of rotation. It is set in motion by applying a tangential force of 19.6 N with a rope wound round its circumference. If the radius of the wheel is 10 cm, calculate the angular acceleration produced.

Solution. I = 0.1 kg m^2; F = 19.6 N ; R = 10 cm = 0.1 m

Applied torque, $\tau = F \times R = 19.6 \times 0.1 = 1.96$ Nm

Now, $\tau = I\alpha$

$\therefore$ Angular acceleration, $\alpha = \dfrac{\tau}{I} = \dfrac{1.96}{0.1} = 19.6$ rad s^{-2}

Example 14.6. A flywheel of mass 25 kg has a radius of 0.2 m. It is making 240 r.p.m. What is the torque necessary to bring it to rest in 20 s? If the torque is due to a force applied tangentially on the rim of the flywheel, what is the magnitude of the force ?

Solution. Mass of flywheel, M = 25 kg ; Radius of flywheel, R = 0.2 m

Moment of inertia of the flywheel about the axis through its centre and perpendicular to its plane is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(25) \times (0.2)^2 = 0.5 \text{ kg m}^2$$

Initial angular velocity, $\omega_0 = 2\pi f_0 = 2\pi \times \dfrac{240}{60} = 8\pi$ rad s^{-1}

Final angular velocity, $\omega = 0$

Now $\omega = \omega_0 + \alpha t$

$\therefore$ $\alpha = \dfrac{\omega - \omega_0}{t} = \dfrac{0 - 8\pi}{20} = -0.4\pi$ rad s^{-2}

Torque on the flywheel, $\tau = I\alpha = (0.5) \times (-0.4\pi) = -0.2\,\pi$ Nm

$\tau = F\,R$

$\therefore$ Tangential force, $F = \dfrac{\tau}{R} = \dfrac{0.2\pi}{0.2} = \pi$ N

Example 14.7. What constant torque should be applied to a disc of mass 16 kg and diameter 0.5 m so that it acquires an angular velocity of 4π rad s^{-1} in 8 s ? The disc is initially at rest and rotates about an axis through the centre of the disc and in a plane perpendicular to the disc.

Solution. M= 16 kg; $R = \dfrac{0.5}{2} = 0.25$ m ; $\omega = 4\pi$ rad s^{-1}; $t = 8$ s ; $\omega_0 = 0$

M.I. of the disc about an axis through its centre and perpendicular to its plane is

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times (16) \times (0.25)^2 = 0.5 \text{ kg m}^2$$

Angular acceleration, $\alpha = \dfrac{\omega - \omega_0}{t} = \dfrac{4\pi - 0}{8} = \dfrac{\pi}{2}$ rad s^{-1}

Now, $\tau = I\alpha = (0.5) \times \left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{4}$ Nm

Example 14.8. A circular metal disc of mass 4 kg and diameter 0.4 m makes 10 revolutions per second about an axis passing through its centre and perpendicular to its plane. (*i*) What is the angular momentum about the same axis? (*ii*) Calculate the magnitude of the torque which will increase the angular momentum by 20% in 10 seconds.

Solution. M = 4 kg ; R = 0.2 m; $\omega = 2\pi f = 2\pi \times 10 = 20\pi$ rad s^{-1}

(*i*) M.I. of the disc about the given axis is

$$= \frac{1}{2}MR^2 = \frac{1}{2}\times 4\times (0.2)^2 = 0.08 \text{ kg m}^2$$

$\therefore$ Magnitude of angular momentum of the disc about the given axis is

$$L = I\omega = 0.08\times 20\pi = 5 \text{ kg m}^2 \text{ s}^{-1}$$

(*ii*) $$\tau = \frac{dL}{dt}$$

$dL = 20\ \%\ of\ 5 = 5\times\frac{20}{100} = 1 \text{ kg m}^2 \text{ s}^{-1}$; $dt = 10s$

$\therefore$ $$\tau = \frac{1}{10} = 0.1 \text{ Nm}$$

Example 14.9. **A flywheel of mass 500 kg and diameter 1m makes 500 revolutions per minute. Assuming the mass to be concentrated along the rim, find (*i*) angular velocity (*ii*) the moment of inertia about the axis of rotation and (*iii*) energy of flywheel.**

Solution. Mass of flywheel, M = 500 kg

$$\text{Frequency, } n = \frac{500}{60} = \frac{25}{3} \text{ r.p.s.}$$

(*i*) Angular velocity, $\omega = 2\pi n = 2\pi\times\frac{25}{3} = 52.38$ rad/s

(*ii*) It is given that the mass of the flywheel is concentrated at the rim (edge).

$\therefore$ Radius of gyration, K = Distance of rim from axis of rotation

$$= \frac{1}{2}\times\text{diameter of wheel} = \frac{1}{2}\times 1 = 0.5 \text{ m}$$

Moment of inertia I about the axis of rotation is

$$I = MK^2 = 500\times(0.5)^2 = 125 \text{ kgm}^2$$

(*iii*) Rotational K.E. of flywheel is

$$K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\times 125(52.38)^2 = 17.15\times 10^4 \text{J}$$

Example 14.10. **Energy of 484 J is spent in increasing the speed of a flywheel from 60 r.p.m. to 360 r.p.m. Find the M.I. of the wheel.**

Solution. ω_0 = 60 r.p.m. = 1 r.p.s.= 2π rad s^{-1} ; ω = 360 r.p.m.= 6 r.p.s.= 12π rad s^{-1}

According to work-energy principle,

Energy spent = Increase in rotational K.E.

or $$484 = \frac{I}{2}\left[\omega^2 - \omega_0^2\right] = \frac{I}{2}\left[(12\pi)^2 - (2\pi)^2\right] = \frac{I}{2}\times 140\pi^2$$

$\therefore$ $$I = \frac{2\times 484}{140\pi^2} = 0.7 \text{ kg m}^2$$

Example 14.11. **The moment of inertia of a wheel is 1000 kg m^2. At a given instant, its angular velocity is 10 rad/s. After the wheel rotates through an angle of 100 radians, the wheel's angular velocity is 100 rad/s. Calculate (*i*) the torque applied on the wheel (*ii*) the increase in rotational kinetic energy of the wheel.**

Solution. $\omega_0 = 10$ rad/s ; $\omega = 100$ rad/s ; $\theta = 100$ rad ; $I = 1000$ kg m^2

(*i*) Angular acceleration, $\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{(100)^2 - (10)^2}{2\times100} = 49.5$ rad/s^2

$\therefore$ Torque applied, $\tau = I\ \alpha = 1000 \times 49.5 = 4.95 \times 10^4$ Nm

(*ii*) Increase in rotational K.E. $= \frac{I}{2}(\omega^2 - \omega_0^2) = \frac{1000}{2}\ [(100)^2 - (10)^2] = 4.95 \times 10^6$ J

PROBLEMS FOR PRACTICE

1. A disc of mass 2 kg and radius 20 cm is free to rotate about an axis through its centre and perpendicular to the disc. If a force of 50 N is applied tangentially, calculate the angular acceleration. **[250 rad s^{-2}]**
2. The angular velocity of a wheel increases from 1200 r.p.m. to 4500 r.p.m. in 10 s. Calculate its angular acceleration and the number of revolutions made during this time. **[11π rad s^{-2}; 475]**
3. Energy of 484 J is spent in increasing the speed of a flywheel from 60 r.p.m. to 360 r.p.m. Find the moment of inertia of the wheel. **[0.7 kg m^2]**
4. If a flywheel of mass 20 kg and diameter 1 m is rotating at 300 revolutions per minute, what is its kinetic energy? **[2465 J]**
5. A uniform circular disc of mass 200 g and radius 4 cm is rotated about one diameter at an angular speed of 10 rad s^{-1}. Find the kinetic energy of the disc and its angular momentum about the axis of rotation. **[4 × 10^{-3} J ; 8 × 10^{-4} J-s]**

14.11. LAW OF CONSERVATION OF ANGULAR MOMENTUM

Angular momentum obeys a conservation law much like the one obeyed by linear momentum. The law of conservation of angular momentum can be stated as follows :

If no net external torque acts on a body or system, its angular momentum remains constant in both magnitude and direction i.e.,

$$I\omega = \text{Constant if } \Sigma\ \tau_{ext} = 0$$

It is very important to keep in mind that it is Iω — the product of moment of inertia and the angular velocity — that remains constant and not the angular velocity ω. In many interesting situations, the moment of inertia of the body or system changes through the action of some internal forces *e.g.,* an ice skater who starts a spin with outstretched arms and then brings the arms closer to the body. In this case, since the mass is being redistributed closer to the axis of rotation, the *moment of inertia is decreasing* even though the mass remains the same. Since this change occurs without any external torque, $I\omega$ must remain constant, requiring the angular velocity ω to increase. Similarly, if moment of inertia were to increase, the angular velocity would have to decrease proportionately so that $I\omega$ is constant. Thus when there is zero net external torque on a body and the body is rotating about an axis, we can write :

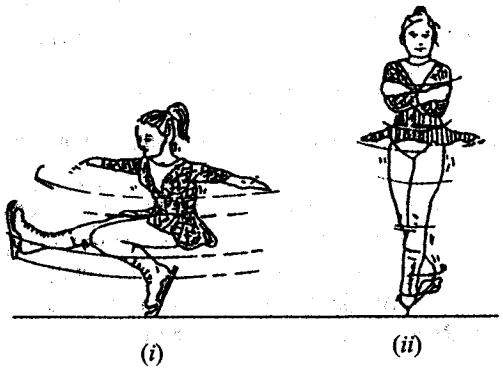
$$I_1\omega_1 = I_2\omega_2 = \text{Constant}$$

where I_1 and ω_1 are the moment of inertia and the angular velocity about that axis at some initial time and I_2 and ω_2 are their values at some other time. The parts of the body may alter their positions relative to one another so that I changes. Then ω will change in such proportion that $I\omega$ remains constant.

14.12. EXAMPLES OF CONSERVATION OF ANGULAR MOMENTUM

Many interesting everyday phenomena can be understood on the basis of conservation of angular momentum. A few of them are discussed below by way of illustration :

(*i*) **A skater doing a spin on the ice rotates at a relatively low speed when she stretches her arms and a leg outward** [See Fig. 14.10 (*i*)]; but when she folds her arms and brings the stretched leg closer to the other [See Fig. 14.10 (*ii*)], she suddenly spins faster. The reason is simple. When she brings her arms and legs closer to the axis of rotation, her moment of inertia about the axis is reduced. Since the angular momentum ($I\omega$) remains *constant, if I decreases, then angular velocity ω must increase. Reverse happens should she increase her moment of inertia about the axis of rotation by outstretching her arms and legs.



(*i*) (*ii*)

Fig. 14.10

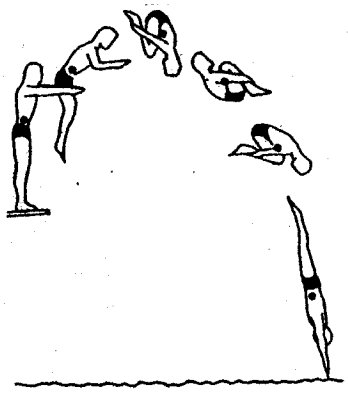
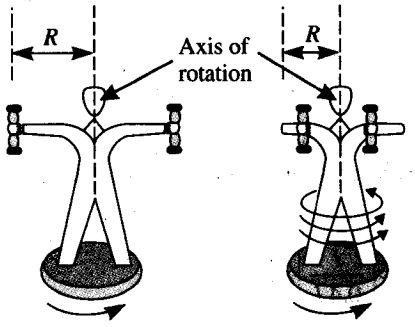


Fig. 14.11

(*ii*) **A diver rotates faster when arms and legs are tucked in than when outstretched.** The push as he leaves the board gives him an initial angular momentum about his centre of mass. During the journey, his angular momentum ($I\omega$) remains the same. When he reduces his moment of inertia about C.M. by curling himself, he rotates quickly [See Fig. 14.11]. Before entering water, he stretches out. This increases his moment of inertia which reduces his angular speed to a low value. The change in moment of inertia from the straight position to the tuck position can be a factor of as much as 3.5.

(*iii*) Fig. 14.12 (*i*) **shows a man standing on a rotating table with his arms outstretched and holding heavy dumbell in each hand.** As he pulls in his arms close to the axis of rotation [See Fig. 14.12 (*ii*)], his moment of inertia about the axis of rotation is reduced.



(*i*) (*ii*)

Fig. 14.12

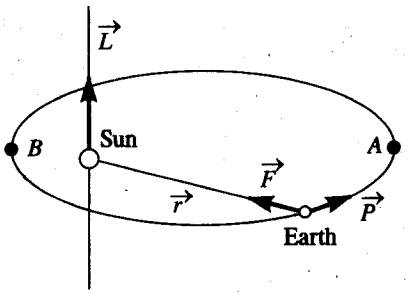


Fig. 14.13

* We ignore small torque due to friction.

Since the angular momentum ($I\omega$) remains constant, the angular velocity ω must increase. If he reduces his moment of inertia by a factor of 2, he will then rotate with twice the previous angular velocity.

(*iv*) **Another example of the law of conservation of angular momentum is found in the motions of the planets and moons.** Consider the earth's orbit around the sun as shown in Fig. 14.13. Its path is elliptical with the sun at one focus of the ellipse. Let us take the axis of rotation as shown, line through the sun and perpendicular to the plane of the orbit. Because the gravitational force acting on the earth is radial, it exerts zero torque about this axis. As a result, the angular momentum $\vec{L}(= I\vec{\omega})$ of the earth around the sun remains constant (in magnitude as well as direction). This tells us two important facts about the motion of planets :

(*a*) Since the direction of $\vec{L}$ remains unchanged, the *orbit of earth lies in a plane.* To see this, note that $\vec{L}$ is always perpendicular to the plane defined by $\vec{r}$ and $\vec{p}$. Since direction of $\vec{L}$ does not change, $\vec{r}$ and $\vec{p}$ must remain in a single plane. So the orbit is confined to a plane.

(*b*) Since the magnitude of angular momentum of earth about the sun is constant ($L = I\ \omega$ = constant), the earth travels faster at the points closer to the sun. When the planet is near the sun, its moment of inertia about the axis of rotation decreases. Since the angular momentum of the earth about the sun remains constant, the angular velocity of earth must increase. Reverse happens should the earth travel away from the sun.

(*v*) **The angular speed (ω) of inner layers of the whirl-wind (tornado) is alarmingly high.** When a whirl-wind is formed, the air rotates in the form of concentric layers. The moment of inertia (I) of inner layers is very small. Since angular momentum ($= I\omega$) remains constant, the inner layers rotate with alarming high angular speed (ω).

(*vi*) **All helicopters are provided with two propellers.** If there were one propeller, then according to law of conservation of angular momentum, the helicopter would rotate itself in the opposite direction.

Note. Note that in all the above cases, angular momentum L ($=I\omega$) is conserved. However, kinetic energy ($K_r=I\omega^2/2$) is not conserved.

Example 14.12. A ballet dancer spins about a vertical axis at 1 r.p.s. with arms outstretched. With her arms folded, her moment of inertia about the vertical axis decreases by 60%. Calculate the new rate of revolution.

Soluton. Initial angular velocity, ω_1 = 1 r.p.s.

Initial M.I. of dancer, $I_1 = I$ (say)

Final M.I. of dancer, $I_2 = 0.4\ I$

Let ω_2 r.p.s. be the new angular velocity. Since angular momentum remains constant,

$$\therefore \quad I_2\omega_2 = I_1\omega_1$$

or
$$\omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \left(\frac{I}{0.4I}\right)\times 1 = 2.5 \text{ r.p.s.}$$

Example 14.13. The two discs are rotating about their axes, normal to the discs and passing through the centres of the discs. The larger disc has mass 2 kg, radius 0.2 m and initial angular velocity 50 rad s^{-1}. The smaller disc has mass 4 kg, radius 0.1 m and initial angular velocity 200 rad s^{-1}. The two discs are brought in contact face to face, with their axes

of rotation coincident. What is the angular speed of the two-disc system. Is kinetic energy conserved during this process?

Solution. Before contact.

$$\text{M.I. of larger disc, } I_1 = \frac{1}{2}M_1R_1^2 = \frac{1}{2}(2)\times(0.2)^2 = 0.04\,\text{kg m}^2$$

$$\text{M.I. of smaller disc, } I_2 = \frac{1}{2}M_2R_2^2 = \frac{1}{2}(4)\times(0.1)^2 = 0.02\,\text{kg m}^2$$

Total angular momentum of the two discs

$$= I_1\omega_1 + I_2\omega_2 = 0.04 \times 50 + 0.02 \times 200 = 6\ \text{kg m}^2\ \text{s}^{-1}$$

Total initial rotational K.E. of the two discs is

$$K_i = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$$

$$= \frac{1}{2}(0.04)\times(50)^2 + \frac{1}{2}(0.02)\times(200)^2 = 450\ \text{J}$$

After contact. Since the discs are brought into contact face to face with their axes of rotation coincident, the moment of inertia I of the two-disc system is $I_1 + I_2$ *i.e.*,

$$I = I_1 + I_2 = 0.04 + 0.02 = 0.06\ \text{kg m}^2$$

If ω is the angular velocity of the two-disc system, then angular momentum

$$= (I_1 + I_2)\omega = I\omega = 0.06 \times \omega\ \text{kg m}^2\ \text{s}^{-1}$$

Since angular momentum is conserved in the system,

$$\therefore \quad I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$

or $$6 = 0.06\ \omega \quad \therefore\ \omega = \frac{6}{0.06} = \mathbf{100\ rad\ s^{-1}}$$

Final rotational kinetic energy of the two-disc system is

$$K_f = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.06)\times(100)^2 = 300\,\text{J}$$

Note that initial rotational K.E. of the two discs was 450 J and on contact, the two-disc system has rotational K.E. of 300 J. Thus one-third of K.E. is lost during this "angular collision" which is the rotational analog of an inelastic collision. The loss of energy is due to the friction between the discs. Note that angular momentum is conserved in the process because frictional torque is an internal torque.

Example 14.14. A disc of moment of inertia 5×10^{-4} kg m^2 is rotating freely about axis O through its centre at 40 r.p.m. (See Fig. 14.14). Calculate the new r.p.m. if some wax of mass 0.02 kg is dropped gently on to the disc 0.08 m from the axis of rotation.

Solution. Magnitude of initial angular momentum of the disc about the axis of rotation is

$$L_1 = I_1\,\omega_1 = 5 \times 10^{-4}\ \omega_1$$

where ω_1 is the angular velocity corresponding to 40 r.p.m.

When the wax of mass m (= 0.02 kg) is dropped gently on to the disc at a distance r (= 0.08 m) from the centre O, the disc slows down. Suppose the angular velocity is now ω_2. The magnitude of total angular momentum about O of the disc plus wax is

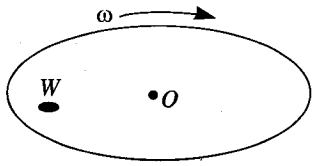


Fig. 14.14

$$L_2 = I_1\omega_2 + mr^2\omega_2$$

$$= 5 \times 10^{-4}\omega_2 + 0.02 \times (0.08)^2\omega_2 = 6.28 \times 10^{-4}\omega_2$$

Since the angular momentum of the system is conserved in the process,

$$\therefore \quad L_2 = L_1$$

or $$6.28 \times 10^{-4}\ \omega_2 = 5 \times 10^{-4}\ \omega_1$$

$$\therefore \quad \frac{\omega_2}{\omega_1} = \frac{5}{6.28} = \frac{500}{628} = \frac{n_2}{40}$$

where n_2 is the r.p.m of the disc with wax because angular velocity is proportional to the r.p.m.

$$\therefore \quad n_2 = \frac{500}{628} \times 40 \simeq 32 \text{ r.p.m.}$$

Example 14.15. **A star of mass twice the solar mass and radius 10^6 km rotates about its axis with an angular speed of 10^{-6} s^{-1}. What is the angular speed of the star when it collapses (due to inward gravitational forces) to a radius of 10^4 km ? Solar mass = 1.99×10^{30} kg.**

Solution. During collapse, the total angular momentum of an isolated star is conserved.

$$\therefore \quad I_1 \omega_1 = I_2 \omega_2$$

or $$\frac{*2}{5} M R_1^2 \omega_1 = \frac{2}{5} M R_2^2 \omega_2 \quad \text{or} \quad R_1^2 \omega_1 = R_2^2 \omega_2$$

$$\therefore \quad \omega_2 = \frac{R_1^2}{R_2^2} \omega_1$$

Here, $R_1 = 10^6$ km ; $R_2 = 10^4$ km ; $\omega_1 = 10^{-6}$ s^{-1}

$$\therefore \quad \omega_2 = \frac{(10^6)^2}{(10^4)^2} \times 10^{-6} = 10^{-2} \text{ s}^{-1}$$

Example 14.16. **If the earth were to suddenly contract to half its present radius (without any external torque acting on it), by how much would the day be decreased ?**

Solution. During contraction, the angular momentum of the isolated earth is conserved.

$$\therefore \quad I_1 \omega_1 = I_2 \omega_2$$

or $$\frac{2}{5} M R_1^2 \omega_1 = \frac{2}{5} M R_2^2 \omega_2 \quad \text{or} \quad R_1^2 \omega_1 = R_2^2 \omega_2$$

Here $$R_2 = \frac{R_1}{2}; \ \omega_1 = \frac{2\pi}{24}; \ \omega_2 = \frac{2\pi}{T}$$

$$\therefore \quad R_1^2 \times \frac{2\pi}{24} = \left(\frac{R_1}{2}\right)^2 \times \frac{2\pi}{T} \quad \text{or} \quad T = 6 \text{ hours}$$

This means the day will be of 6 hours duration. Hence the day will decrease by 24 – 6 = 18 hours.

PROBLEMS FOR PRACTICE

1. A person stands, hands at the side, on a platform that is rotating at a rate of 1.2 r.p.s. If the person now raises her arms, the speed of rotation decreases to 0.8 r.p.s. By what factor has the moment of inertia of the person changed ? **[1.5]**
2. A mass of 2 kg is rotating on a circular path of radius 0.8m with angular velocity of 14 rad s^{-1}. If the radius of the path becomes 1m, what will be the value of angular velocity? **[28.16 rad s^{-1}]**

* The M.I. of a uniform solid sphere about an axis along a diameter is $\frac{2}{5} MR^2$ where R is the radius of the sphere.

3. A 4.5 m-diameter merry-go-round is rotating freely with an angular velocity of 0.7 rad s^{-1}; its total moment of inertia is 1750 kg m^2. Four people standing on the ground, each of 65 kg mass, suddenly step onto the edge of the merry-go-round. What will be the angular velocity of the merry-go-round now ? **[0.04 rad s^{-1}]**
4. A small block is rotating in a horizontal circle at the end of a thread which passes down through a hole at the centre of the table top. If the system is rotating at 2.5 r.p.s. in a circle of radius 30 cm, what will be the speed of rotation when the thread is pulled inwards to decrease the radius to 10 cm? Neglect friction. **[22.5 rev/s]**

14.13. THEOREMS ON MOMENT OF INERTIA

We can often simplify the calculations of the moments of inertia for various bodies by using two general theorems relating the moment of inertia about one axis of the body to that about another axis. The following are the two important theorems on moment of inertia :

(*i*) Theorem of parallel axes (*ii*) Theorem of perpendicular axes.

14.14. THEOREM OF PARALLEL AXES

If I_{CM} is the moment of inertia of a body about an axis $P'Q'$ passing through the centre of mass of the body, then moment of inertia I of the body about any axis PQ parallel to the centre of mass axis $P'Q'$ is given by ;

$$I = I_{CM} + Mh^2 \qquad ...(i)$$

where M = Total mass of the body.

h = Perpendicular distance between the parallel axes.

Eq. (*i*) is known as *theorem of parallel axes* and may be stated as under :

The moment of inertia of a rigid body about any given axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.

This is a very simple and useful theorem. For objects of regular shapes (*e.g.*, rod, sphere, disc, cylinder etc.), it is quite easy to find the moment of inertia of the object (I_{CM}) about an axis passing through the centre of mass of the object. By using theorem of parallel axes, we can calculate the moments of inertia of these objects about *any other axis* which is parallel to the centre of mass axis.

Proof. Consider a body of mass M as shown in Fig. 14.15. Let C be the centre of mass of the body. $P'Q'$ is an axis passing through the centre of mass of the body while axis PQ is parallel to $P'Q'$.

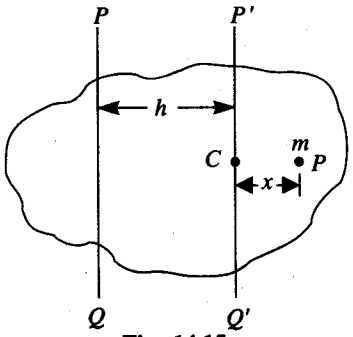


Fig. 14.15

Let I_{CM} = M.I. of the body about axis $P'Q'$

I = M.I. of the body about axis PQ

h = perpendicular distance between the parallel axes PQ and $P'Q'$

Consider a particle of mass m at point P at a perpendicular distance x from the axis $P'Q'$.

Moment of inertia of the particle about the axis $PQ = m\,(h + x)^2$

Moment of inertia of the whole body about axis PQ is

$$I = \sum m(h+x)^2 = \sum m(h^2 + x^2 + 2hx)$$

or $$I = \sum mx^2 + \sum mh^2 + \sum 2mxh$$

Now $$\sum mh^2 = h^2 \sum m = Mh^2$$ where M is the total mass of the body

and $\sum mx^2 = I_{CM}$ = M.I. of the body about axis $P'Q'$

Also $\sum 2mxh = 2h\sum mx = 0$

since $\sum mx$ is the sum of the moments of the various particles about the axis through the centre of mass and is equal to zero.

$\therefore$ $I = I_{CM} + Mh^2$

Thus theorem of parallel axes is proved.

14.15. THEOREM OF PERPENDICULAR AXES

The theorem of parallel axes can be applied to any body. But theorem of perpendicular axes can be applied only to plane figures (*i.e.*, two dimensional bodies *e.g.*, a lamina) or bodies of uniform thickness. whose thickness can be neglected as compared to the other dimensions. This theorem is stated below :

The moment of inertia of a plane body about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the plane body about any two mutually perpendicular axes in the plane of the body which intersect the first axis i.e.,

$I_Z = I_X + I_Y$ ... plane body in XY plane

This theorem is most useful when considering a plane body (*i.e.*, two dimensional body) which is of regular form (symmetrical) about two of the three axes. If the moment of inertia about these axes is known, then that about the third axis can be calculated.

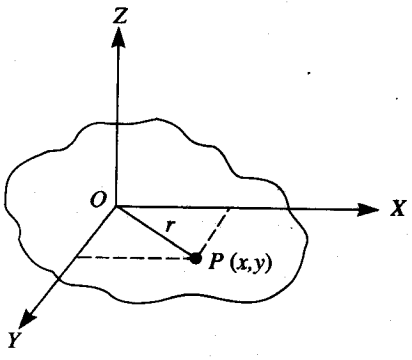


Fig. 14.16

Proof. Consider a plane lamina in the XY plane as shown in Fig. 14.16. Here OX and OY are two mutually perpendicular axes lying in the plane of the lamina. The axis OZ passes through O and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles each of mass m. Consider one such particle at point P (x, y) situated at a distance r from the point of intersection (point O) of the axes.

$\therefore$ $r^2 = x^2 + y^2$

The moment of inertia of the particle at P about OZ axis = $m\,r^2$

The moment of inertia of the whole lamina about OZ axis is

$$I_Z = \sum mr^2 = \sum m(x^2 + y^2)$$
$$= \sum mx^2 + \sum my^2$$

But $\sum mx^2$ = M.I. of the whole lamina about Y-axis

and $\sum my^2$ = M.I. of the whole lamina about X-axis

$\therefore$ $I_Z = I_Y + I_X$

or $I_Z = I_X + I_Y$

Thus theorem of perpendicular axes is proved.

14.16. CALCULATION OF MOMENT OF INERTIA OF RIGID BODIES

When an object consists of a number of discrete mass points, its moment of inertia about a given axis is equal to the sum of the moments of inertia of individual mass points about that axis *i.e.*,

$$I = \sum m_i r_i^2$$

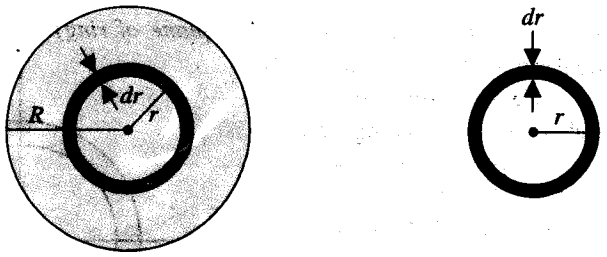
14.18. M.I. OF UNIFORM CIRCULAR DISC

(About an axis passing through its centre and perpendicular to plane of disc)

Consider a *uniform* disc of mass M and radius R as shown in Fig. 14.18(*i*). The disc is in the XY plane so that we are to find moment of inertia of the disc about Z-axis (*i.e.*, I_Z). Not all parts of the disc are at the same distance from the axis. Since we already know the rotational inertia of a ring, we can divide the disc into mass elements that are themselves rings as shown in Fig. 14.18 (*ii*). Consider a ring of radius r, width dr and mass dm [See Fig. 14.18 (*ii*)]. Its moment of inertia is $r^2\,dm$ so that the total moment of inertia of the disc is

$$I_Z = \int_{r=0}^{r=R} r^2 dm$$

where the limits are chosen to include all ring-shaped mass elements in the entire disc.



(*i*) (*ii*)

Fig. 14.18

To evaluate the integral, we must relate the variables m and r. The mass element dm is a tiny fraction of the total mass M. Since the disc is uniform, the area of the considered ring ($2\pi r dr$) is the same fraction of the total area of the disc (πR^2) as dm of M *i.e.*,

$$\frac{dm}{M} = \frac{2\pi r dr}{\pi R^2} \text{ or } dm = \frac{2Mr}{R^2} dr$$

$$\therefore \quad I_Z = \int_{r=0}^{r=R} r^2 \frac{2Mr}{R^2} dr = \frac{2M}{R^2} \int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2} \left| \frac{r^4}{4} \right|_0^R = \frac{1}{2} MR^2$$

$$\therefore \quad I_Z = \frac{1}{2} MR^2$$

Note that for the same mass M and radius R, the moment of inertia I_Z of a circular ring is more than the moment of inertia I_Z of the disc. It is because the mass of ring is at a greater distance from the axis whereas the mass of disc is distributed from the axis out to the edge.

M.I. about any diameter of disc. We can easily find the M.I. of the disc about its diameter. According to theorem of perpendicular axes,

$$I_Z = I_X + I_Y$$

Here $I_Z = \frac{MR^2}{2}$; $I_X = I_Y$ = M.I. of disc about any diameter

$$\therefore \quad \frac{MR^2}{2} = 2I_X \quad \text{or} \quad I_X = \frac{MR^2}{4}$$

Here m_i is the mass of the ith particle and r_i is its perpendicular distance from the axis of rotation.

When an object consists of a continuous distribution of mass, then we consider a large number of very small mass elements Δm_i throughout the object and sum the individual moments of inertia $\Delta m_i r_i^2$ over the entire object. In the limit $\Delta m_i \to 0$, the sum becomes an integral and, we have,

$$I = \int r^2 dm$$

where the limits of integration are chosen to include the entire object. Here dm represents the mass of any infinitesimal particle of the body and r is the perpendicular distance of this particle from the axis of rotation. We now find the moment of inertia of some bodies of simple geometrical shape.

14.17. M.I. OF A THIN CIRCULAR RING ABOUT ITS AXIS

(Line passing through centre and perpendicular to plane of ring)

Consider a thin circular ring of mass M and radius R as shown in Fig. 14.17. We are to find the moment of inertia of this ring about an axis passing through the centre of the ring and perpendicular to the plane of the ring. The ring is in the XY plane so that we are to find moment of inertia of the ring about Z-axis (*i.e.*, I_Z) through O. We divide the ring into mass elements dm, one of which is shown in Fig. 14.17. All mass elements in the ring are at the same distance R from the axis of rotation. Therefore, M.I. of the ring about the axis of rotation (*i.e.*, Z-axis) is $I_Z = \int r^2 dm = \int R^2 dm = R^2 \int dm$ where the integration is over the ring. But the sum of the mass elements over the ring is just the total mass M of the ring.

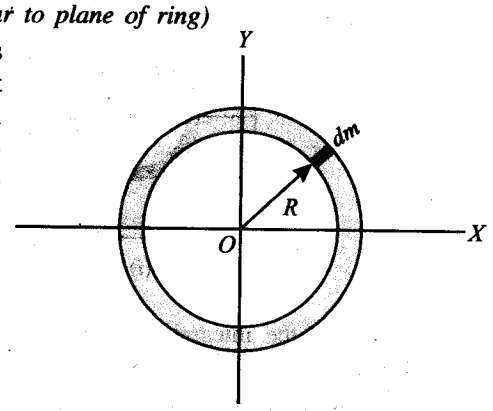


Fig. 14.17

$$\therefore \quad I_Z = MR^2$$

Thus M.I. of the ring is the same as if all the mass were concentrated in one place at a distance R from the axis of rotation ; the angular distribution of the mass about the axis does not matter.

(*i*) M.I. about any diameter of ring. When the ring is in the XY plane, $I_Z = MR^2$ as calculated above. According to theorem of perpendicular axes, we have,

$$I_Z = I_X + I_Y$$

From symmetry, $I_X = I_Y$ = M.I. of ring about any diameter

$$\therefore \quad MR^2 = 2I_X \quad \text{or} \quad I_X = \frac{MR^2}{2}$$

Thus M.I. of the ring about any diameter is $MR^2/2$.

(*ii*) M.I. about tangent perpendicular to plane of ring. $I_Z = I_{C.M.} = MR^2$. According to theorem of parallel axes, M.I. I of the ring about tangent perpendicular to the plane of the ring is

$$I = I_{C.M.} + M(R^2) = MR^2 + MR^2 = 2MR^2$$

(*iii*) M.I. about tangent in the plane of the ring. $I_Z = MR^2 \quad \therefore \quad I_X = I_Y = \frac{I_Z}{2} = \frac{MR^2}{2}$.

According to theorem of parallel axes, M.I. I of the ring about a tangent in the plane of the ring is

$$I = I_X + M(R^2) = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

14.19. M.I. OF A THIN UNIFORM ROD

(About an axis through its centre and perpendicular to its length)

Consider a thin uniform rod AB of length l and mass M, free to rotate about an axis CD through its centre O and perpendicular to its length. It is desired to find the moment of inertia of this rod about the axis CD.

Let the rod coincide with the X-axis, with origin at the centre of the rod. Consider a mass element of mass dm and length dx as shown in Fig. 14.19. Suppose x is the distance of mass element from the axis of rotation (CD). Therefore, M.I. of the rod about the axis CD is

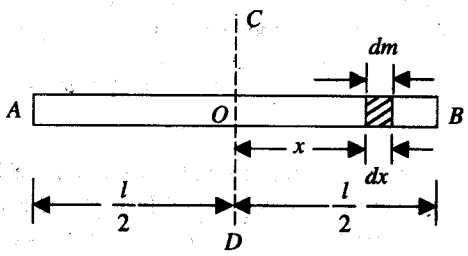


Fig. 14.19

$$I = \int_{x=-l/2}^{x=l/2} x^2 dm$$

Here we have chosen the limits to include the entire rod. In order to carry out the integration, we must relate x and dm so that we have a single variable under the integral sign. The length dx of the mass element is some tiny fraction of the total length l. Since the rod is uniform, the mass dm is the same fraction of the total mass M *i.e.*,

$$\frac{dm}{M} = \frac{dx}{l} \text{ or } dm = \frac{M}{l}dx$$

$$\therefore \quad I = \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx = \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}$$

or $$I = \frac{Ml^2}{12}$$ ...M.I. of rod about bisector

Radius of gyration, $K = \sqrt{\frac{I}{M}} = \sqrt{\frac{Ml^2}{12 \times M}} = \frac{l}{\sqrt{12}}$

M.I. of rod about axis through its end and perpendicular to length. According to theorem of parallel axes, the M.I. of the rod I about an axis through its end and perpendicular to its length is

$$I = I_{CM} + Mh^2$$

Here, $$I_{CM} = \frac{Ml^2}{12}; \quad h = \frac{l}{2}$$

$$\therefore \quad I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

or $$I = \frac{Ml^2}{3}$$

Radius of gyration, $K = \sqrt{\frac{I}{M}} = \sqrt{\frac{Ml^2}{3M}} = \frac{l}{\sqrt{3}}$

14.20. M.I. OF THIN HOLLOW CYLINDER ABOUT ITS AXIS

We now find the moment of inertia of a thin hollow cylinder about an axis through its centre along its length. Consider a thin hollow cylinder of mass M and radius R as shown in Fig. 14.20. Every element dm of the cylinder is at the same perpendicular distance R from the axis of the cylinder. Therefore, M.I. of the cylinder about its axis is

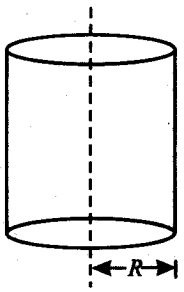


Fig. 14.20

$$I = \int r^2\, dm = \int R^2\, dm = R^2 \int dm = MR^2$$

$$\therefore \quad I = MR^2$$

Note that this formula of moment of inertia is similar to that of a circular ring about an axis passing through the centre of the ring and perpendicular to the plane of the ring. This is expected because a hollow cylinder consists of identical circular rings stacked one over the other.

14.21. M.I. OF UNIFORM SOLID CYLINDER ABOUT ITS AXIS

We shall find the moment of inertia of a uniform solid cylinder about an axis through its centre, along its length *i.e.* the Z-axis in Fig. 14.21.

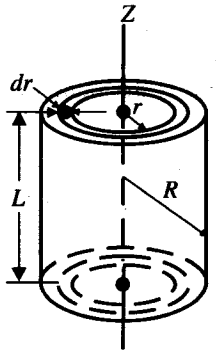


Fig. 14.21

Let R = radius of the cylinder

M = mass of the cylinder

L = length of the cylinder

ρ = uniform density of cylinder material

Consider a cylindrical shell of radius r, thickness dr and length L as shown in Fig. 14.21. This element is chosen because all the mass within a such an element is essentially the same distance (*i.e.* r) from the axis of rotation (*i.e.* Z-axis).

Volume of the considered cylindrical shell is

$$dV = (\text{Cross-sectional area}) \times \text{Length}$$

$$= (2\pi r \times dr) \times L = (2\pi r dr)L$$

Mass of the considered cylindrical shell is

$$dm = \rho\, dV = \rho\, 2\pi\, rLdr$$

$$\therefore \text{ Moment of inertia, } I_Z = \int r^2 dm = 2\pi\rho L \int_0^R r^3 dr = \frac{\pi\rho L R^4}{2}$$

Now $$\rho = \frac{M}{\text{Total volume of cylinder}} = \frac{M}{\pi R^2 L}$$

$$\therefore \quad I_Z = \pi \times \frac{M}{\pi R^2 L} \times \frac{LR^4}{2} = \frac{1}{2} MR^2$$

Note that moment of inertia does not depend upon the length of the cylinder. This expression is also valid for a disc-shaped object such as a phonograph record (neglecting the hole) and for a long thin object such as a log with uniform radius. This is because a solid cylinder is made of uniform circular discs placed one over the other.

14.22. M.I. OF A UNIFORM HOLLOW SPHERE ABOUT A DIAMETER

Consider a hollow sphere with centre O as shown in Fig. 14.22. Since the sphere is hollow, the mass is spread over the surface of the sphere.

Let M = mass of the sphere

R = radius of the sphere

$\therefore$ Surface area of the sphere = $4\pi R^2$

Mass per unit area of sphere $= \dfrac{M}{4\pi R^2}$

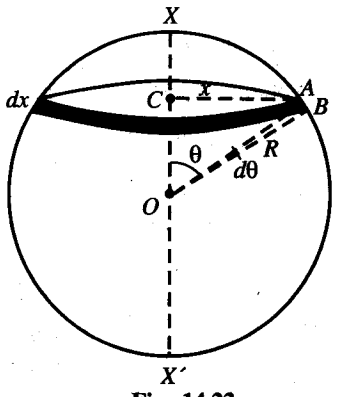


Fig. 14.22

We have to calculate the moment of inertia of this hollow sphere about the diameter XX'. Consider a small element of the hollow sphere in the form of a ring of radius x with centre C and width dx. Let $\angle AOC = \theta$ and $\angle AOB = d\theta$. The radius of the considered ring is $x = R\sin\theta$ and its circumference $= 2\pi R\sin\theta$.

Area of elementary ring = Circumference × Width = $(2\pi R\sin\theta)\times Rd\theta$

Mass of elementary ring = Mass per unit area × Area

$$= \frac{M}{4\pi R^2}\times(2\pi R\sin\theta)\times Rd\theta = \frac{M}{2}\sin\theta d\theta$$

The moment of inertia of the elementary ring about XX' axis is

$$dI = \left(\frac{M}{2}\sin\theta d\theta\right)(R\sin\theta)^2 = \frac{M}{2}R^2\sin^3\theta d\theta$$

As θ increases from 0 to π, the elemental rings cover the whole spherical surface. Therefore, the moment of inertia I of the hollow sphere about XX' axis is

$$I = \int_0^{\pi}\frac{M}{2}R^2\sin^3\theta d\theta = \frac{MR^2}{2}\int_0^{\pi}(1-\cos^2\theta)\sin\theta d\theta$$

$$= \frac{MR^2}{2}\int_0^{\pi}-(1-\cos^2\theta)d(\cos\theta)$$

$$= -\frac{MR^2}{2}\left|\cos\theta-\frac{\cos^3\theta}{3}\right|_0^{\pi} = \frac{2}{3}MR^2$$

$\therefore$ $$I = \frac{2}{3}MR^2$$

Using theorem of parallel axes, the moment of inertia I' of hollow sphere about a tangent to the sphere is

$$I' = I + MR^2 = \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2$$

14.23. M.I. OF A UNIFORM SOLID SPHERE ABOUT A DIAMETER

We now find the moment of inertia of a uniform solid sphere about the diameter *i.e.*, dotted line in Fig. 14.23.

Let M = mass of the sphere

R = radius of the sphere

ρ = density of the sphere

Consider a small disc of thickness dy at a distance y from the centre O of the sphere.

Radius of disc, $r = \sqrt{R^2 - y^2}$

Volume of disc, $dV = \pi r^2 dy$

Mass of disc, $dm = \rho\pi r^2 dy = \rho\pi(R^2 - y^2)dy$

$(\because r^2 = R^2 - y^2)$

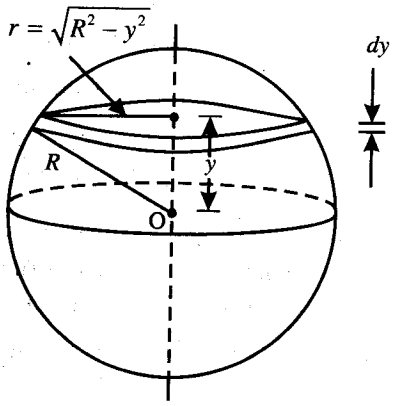


Fig. 14.23

$\therefore$ Moment of inertia of the disc is

$$dI = \frac{1}{2}dmr^2 = \frac{\rho\pi}{2}(R^2 - y^2)^2 dy$$

$$= \frac{\rho\pi}{2}(R^4 - 2R^2y^2 + y^4)dy$$

$\therefore$ Moment of inertia of the sphere is given by ;

$$I = \int_{-R}^{R} dI = \frac{\rho\pi}{2}\int_{-R}^{R}(R^4 - 2R^2y^2 + y^4)dy$$

$$= \frac{\rho\pi}{2}\left[R^4 y - \frac{2}{3}R^2y^3 + \frac{y^5}{5}\right]_{-R}^{R} = \frac{8}{15}\rho\pi R^5 \quad ...(i)$$

Volume of sphere, $V = \frac{4}{3}\pi R^3$

$\therefore$ Density, $\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$

Putting the value of ρ in eq. (*i*), we get,

$$I = \frac{8}{15}\times\frac{3M}{4\pi R^3}\times\pi\times R^5 = \frac{2}{5}MR^2$$

This is the expression for the moment of inertia of a uniform solid sphere about any of its diameter.

Using theorem of parallel axes, the moment of inertia I' of solid sphere about a tangent to the sphere is

$$I' = I + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

Example 14.17. **Calculate moment of inertia of a circular disc of diameter 40 cm, thickness 7 cm and uniform density 9 g/cm^3 about a transverse axis through the centre of the disc.**

Solution. Here D = 40 cm ; x = 7 cm ; ρ = 9 g/cm^3 ; R = 40/2 = 20 cm

Mass of disc, $M = \left(\frac{\pi}{4}D^2\right)\times x\times\rho$

$$= \left[\frac{\pi}{4}\times(40)^2\right]\times 7\times 9 = 7.92\times 10^4 \text{ g}$$

$\therefore$ Moment of inertia I of the disc about a transverse axis through the centre of the disc is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}\times(7.92\times10^4)\times(20)^2 = \mathbf{1.58\times 10^7\ g\ cm^2}$$

Example 14.18. **A thin uniform rod of mass 1 kg and length 1 m is rotating about an axis through its centre and perpendicular to its length. Calculate: (*i*) moment of inertia (*ii*) radius of gyration.**

Soluion. (*i*) M.I. of rod, $I = \frac{Ml^2}{12} = \frac{1\times(1)^2}{12} = 0.0833$ kg m^2

(*ii*) $I = M K^2$

$\therefore$ Radius of gyration, $K = \sqrt{\frac{I}{M}} = \sqrt{\frac{0.0833}{1}} = 0.288$ m

Example 14.19. Calculate the moment of inertia of a uniform disc of mass 200 g and radius 5 cm about an axis passing through its edge and perpendicular to the plane of the disc.

Solution. M = 200 g = 0.2 kg ; R = 5 cm = 0.05 m

The M.I. of a uniform disc about an axis passing through its centre and perpendicular to its plane is

$$I_{CM} = \frac{1}{2}MR^2$$

According to theorem of parallel axes, the M.I. of the disc about an axis passing through its edge and perpendicular to the plane of the disc is

$$I = I_{CM} + Mh^2$$

Here, $I_{CM} = \frac{MR^2}{2}$; $h = R$

$\therefore$ $I = \frac{MR^2}{2} + MR^2 = \frac{3}{2}M R^2 = \frac{3}{2}\times(0.2)\times(0.05)^2 = 7.5 \times 10^{-4}$ kg m^2

Example 14.20. Radius of gyration of a body about an axis at a distance 6 cm from its centre of mass is 10 cm. Find its radius of gyration about a parallel axis through its centre of mass.

Solution. M.I. of the body about the given axis is

$$I = M K^2 \qquad \text{(Here } K = 10 \text{ cm)}$$

M.I. of the body through a parallel axis through centre of mass is

$$I_{CM} = M k^2$$

According to theorem of parallel axes,

$$I = I_{CM} + Mh^2$$

Here, $I = M K^2$; $I_{CM} = Mk^2$; h = 6 cm

$\therefore$ $MK^2 = Mk^2 + Mh^2$

or $K^2 = k^2 + h^2$

$\therefore$ $k = \sqrt{K^2 - h^2} = \sqrt{(10)^2 - (6)^2} = \sqrt{64} = 8$ cm

Example 14.21. A symmetric lamina of mass *M* consists of a square shape with semicircular section over each of the edge of the square as shown in Fig. 14.24. The side of the square is 2*a*. The moment of inertia of the lamina about an axis through the centre of mass and perpendicular to the plane is 1.6 *M a*^{2}. Find the moment of inertia of the lamina about the tangent *AB* in the plane of the lamina.

Solution. Fig. 14.24 shows the conditions of the problem. The moment of inertia of the lamina about an axis through C.M. and perpendicular to its plane is

$$I_{CM} = 1.6Ma^2$$

Let I_X and I_Y be the moments of inertia about the axes XX' and YY' respectively. Then according to the theorem of perpendicular axes,

$$I_X + I_Y = I_{CM}$$

Since the lamina is symmetrical about the axes XX' and YY',

$$\therefore \quad I_X = I_Y$$

or $$I_X = \frac{I_{CM}}{2} = \frac{1.6Ma^2}{2} = 0.8Ma^2$$

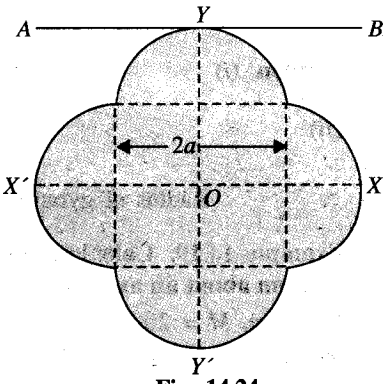


Fig. 14.24

According to theorem of parallel axes, we have,

$$I_{AB} = I_X + M(OY)^2 = 0.8Ma^2 + M(2a)^2 = 4.8\ Ma^2$$

Example 14.22. The diameter of a flywheel increases by 1%. What will be the percentage increase in moment of inertia about axis of symmetry ?

Solution. M.I. of flywheel about the axis of symmetry is

$$I = M R^2$$

Taking log, we have, $\log_e I = \log_e M + 2 \log_e R$

Differentiating, $$\frac{dI}{I} = 0 + \frac{2dR}{R}$$

$\therefore$ Percentage increase in M.I. of flywheel

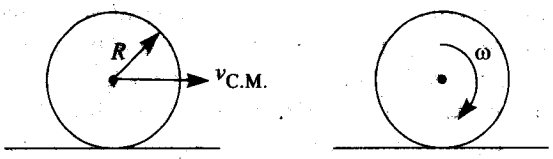
$$= \frac{dI}{I} \times 100 = 2\left(\frac{dR}{R} \times 100\right) = 2 \times 1\% = 2\%$$

PROBLEMS FOR PRACTICE

1. A solid cylinder of mass 5 kg and radius 30 cm is free to rotate about its central axis. Find the moment of inertia of the cylinder. **[0.225 kg m^2]**
2. Calculate the moment of inertia about a transverse axis through the centre of a disc whose radius is 20 cm. Its density is 9 g /cm^3 and its thickness is 7 cm. **[1.584 × 10^7 g cm^2]**
3. A uniform circular disc of mass 100 g and diameter 10 cm is rotating about an axis passing through its centre and perpendicular to the plane of the disc. Find the moment of inertia of the disc. **[1.25 × 10^{-4} kg m^2]**

14.24. COMBINED ROTATION AND TRANSLATION

Consider a wheel rolling on a flat surface without slipping. Each particle of the wheel is undergoing two types of motion at the same time. The centre of the wheel, which is wheel's centre of mass, is moving horizontally with speed v_{CM} [See Fig. 14.25 (*i*)]. At the same time, the wheel is rotating about its centre of mass with angular speed ω [See Fig. 14.25 (*ii*)]. Thus rolling is the superposition of two motions :



(*i*) Fig. 14.25 (*ii*)

(*i*) *Translation of object's centre of mass* at velocity v_{CM} along a straight line.

(*ii*) *Rotation about the centre of mass* at angular velocity ω.

Therefore, the wheel possesses both translational and rotational kinetic energy. It is possible to show that the total kinetic energy of the wheel can be expressed quite simply if we restrict ourselves to the rotation about one particular axis *i.e.,* through the wheel's centre of mass. This is the usual axis for a rolling object. Then we can state :

The total kinetic energy of an object undergoing both translational and rotational motion about its centre of mass axis is equal to the sum of translation K.E. of the centre of mass and the rotational K.E. about the centre of mass i.e.,

$$\text{K.E.}_{total} = K_t + K_r$$

or $$\text{K.E.}_{total} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I\omega^2$$

where M = mass of the object

v_{CM} = velocity of object's centre of mass

I = moment of inertia of the object about the centre of mass axis.

Example 14.23. A billiard ball of mass M rolls along a surface at speed V. The ball's radius in not known. What is ball's total kinetic energy?

Solution. Since the ball rolls without slipping, $v_{CM} = V$. The moment of inertia of a uniform solid sphere about a diametrical axis is

$$I = \frac{2}{5} MR^2 \qquad \text{... } R \text{ is the radius of sphere}$$

The angular velocity of the ball is $\omega = V/R$.

$$\therefore \quad \text{K.E.}_{total} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} MV^2 + \frac{1}{2}\left(\frac{2}{5} MR^2\right)\left(\frac{V}{R}\right)^2$$

$$\therefore \quad \text{K.E.}_{total} = \frac{7}{10} MV^2$$

Note that all uniform spheres with same M and V have the same total kinetic energy, independent of radius R.

Now $$\frac{\text{Translation K.E.}}{\text{Total K.E.}} = \frac{\frac{1}{2} MV^2}{\frac{7}{10} MV^2} = 0.71$$

Thus 71% of the total K.E. is translational K.E. and 29 % is rotational K.E.

14.25. ROLLING CYLINDER (WITHOUT SLIPPING) ON AN INCLINED PLANE

It is an example of simultaneous rotational and linear motion of the centre of mass of a body. Consider a solid cylinder of radius R and mass M rolling down a plane inclined at an angle θ to the horizontal as shown in Fig. 14.26. The various forces acting on the cylinder are :

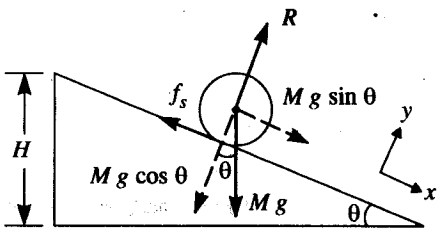


Fig. 14.26

(*i*) Weight Mg of the cylinder acting vertically downward.

(*ii*) The frictional force $*f_s$ between the cylinder and the surface of the inclined plane acting in a direction opposite to motion.

* The frictional force is static; the cylinder rolls without slipping and there is no relative motion between the cylinder and the inclined plane at the point of contact.

(*ii*) The linear acceleration is just 2/3 as large as it would be if the cylinder could slide without friction down the slope ($a = g \sin\theta$).

Minimum μ_s required for rolling

From eq. (*iv*), $f_s = \dfrac{Ia}{R^2}$

Now $I = \dfrac{1}{2}MR^2$ and $a = \dfrac{2}{3} g \sin\theta$

$\therefore$ $f_s = \dfrac{1}{2}MR^2 \times \dfrac{2}{3} g \sin\theta \times \dfrac{1}{R^2}$

or $f_s = \dfrac{1}{3}Mg \sin\theta$

If there were no friction between the cylinder and the incline, the cylinder would have slipped instead of rolled. Friction must be present to make the cylinder roll. To prevent slipping, the coefficient of static friction must be at least as great as

$$\mu_s = \frac{f_s}{R} = \frac{\frac{1}{3}Mg \sin\theta}{Mg \cos\theta} = \frac{1}{3} \tan\theta$$

or $$\mu_s = \frac{1}{3} \tan\theta \qquad ...(vi)$$

If the plane is tilted only slightly (*i.e.*, θ is small), only a small value of μ_s is needed to prevent slipping. But as the angle θ increases, the required value of μ_s increases.

Note. Note that rolling motion is possible only if a frictional force is present between the object and the incline to produce a net torque about the centre of mass. If there were no friction between the cylinder and the plane, the cylinder would have slid instead of rolled. We did not takes it into account in the energy equation because it is static friction ($\because$ the contact point is at rest relative to the surface at any instant) and does no work.

Example 14.24. A solid uniform sphere of mass M_s and radius R_s and a solid uniform cylinder of mass M_c and radius R_c start from rest at the same point on an inclined plane. The two roll down without slipping. Will they arrive at the bottom of the incline at the same instant?

Solution. For an object rolling down an inclined plane, the linear acceleration is given by;

$$a = \frac{Mg \sin\theta}{M + \frac{I}{R^2}}$$

For a uniform solid sphere of mass M_s and radius R_s, $I_s = \dfrac{2}{5} M_s R_s^2$.

$\therefore$ $$a_s = \frac{M_s g \sin\theta}{M_s + \frac{2}{5}\frac{M_s R_s^2}{R_s^2}} = \frac{5}{7} g \sin\theta$$

For a uniform solid cylinder of mass M_c and radius R_c, $I_c = \dfrac{1}{2} M_c R_c^2$.

$\therefore$ $$a_c = \frac{M_c g \sin\theta}{M_c + \frac{1}{2}\frac{M_c R_c^2}{R_c^2}} = \frac{2}{3} g \sin\theta$$

Since $a_s > a_c$, the solid sphere will arrive first.

Example 14.25. A solid cylinder of radius 4 cm and mass 250 g rolls down an inclined plane (1 in 10). Find the linear acceleration and total kinetic energy after 5 s.

Solution. R = 4 cm = 0.04 m ; M = 250 g = 0.25 kg ; $\sin\theta = 1/10$; t = 5 s

The linear acceleration of a body rolling down an inclined plane is given by ;

$$a = \frac{Mg\sin\theta}{M + \frac{I}{R^2}}$$

But the M.I.of solid cylinder about its symmetry axis is

$$I = \frac{1}{2}MR^2$$

$$\therefore \quad a = \frac{Mg\sin\theta}{M + \frac{1}{2}\frac{MR^2}{R^2}} = \frac{2}{3}g\sin\theta = \frac{2}{3}\times 9.8\times\frac{1}{10} = 0.653 \text{ ms}^{-2}$$

Now $v = u + at = 0 + 0.653\times 5 = 3.265 \text{ ms}^{-1}$

As the cylinder is rolling, total K.E. of the cylinder is the sum of linear K.E. and rotational K.E.

$$\therefore \quad \text{Total K.E.} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 \qquad \left(\because \omega = \frac{v}{R}\right)$$

$$= \frac{3}{4}Mv^2 = \frac{3}{4}\times 0.25\times(3.625)^2 = 2\text{J}$$

Example 14.26. A cylinder of mass 5 kg and radius 30 cm and free to rotate about its axis, receives an angular impulse of 3 kg m^2 s^{-1} initially followed by a similar impulse after every 4s. What is the angular speed of the cylinder 30s after the initial impulse? The cylinder is at rest initially.

Solution. M = 5 kg ; R= 30 cm = 0.3 m ; L = 3 kg m^2 s^{-1} ; t = 4 s

Moment of inertia of the solid cylinder about its axis of symmetry is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}\times(5)\times(0.3)^2 = 0.225 \text{ kg m}^2$$

The initial angular impulse is 3 kg m^2 s^{-1} *i.e.*, at t = 0, the angular momentum of the cylinder = 3 kg m^{2}s^{-1}. The cylinder receives angular impulse after every 4 s. Therefore, after 4s, the angular momentum of the cylinder = 3 + 3 = 6 kg m^2 s^{-1}. After 8 s, the angular momentum of the cylinder = 6 + 3 = 9 kg m^2 s^{-1}. During 28 seconds, the cylinder will receive 7 impulses (no impulse is received between 28 s and 30 s).

$\therefore$ Total angular momentum L of the cylinder after 30 s is

$$L = 3 + 3\times 7 = 24 \text{ kg m}^2 \text{ s}^{-1}$$

If ω is the angular speed of the cylinder after 30 s, then,

$$L = I\,\omega$$

or $$\omega = \frac{L}{I} = \frac{24}{0.225} = 106.67 \text{ rad s}^{-1}$$

Example 14.27. A solid cylinder rolls down an inclined plane. Its mass is 2 kg and radius 0.1 m. If the height of the inclined plane is 4 m, what is the rotational kinetic energy when it reaches the foot of the plane? Assume that the surfaces are smooth. Given moment of inertia of solid cylinder about its axis = $MR^2/2$.

Solution. At the top of the inclined plane, the solid cylinder has P.E. = Mgh. As the cylinder rolls down the inclined plane, its P.E. starts converting into K.E. of translation and K.E. of rotation. Therefore, at the bottom of the inclined plane, according to principle of conservation of energy,

$$Mgh = \text{K.E. of translation} + \text{K.E. of rotation}$$

or $$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

or $$Mgh = \frac{1}{2}M(\omega R)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 \qquad (\because v = \omega R)$$

or $$Mgh = \frac{3}{4}MR^2\omega^2 \quad \therefore \quad \omega^2 = \frac{4gh}{3R^2}$$

$\therefore$ $$\text{K.E. of rotation, } K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\left(\frac{1}{2}MR^2\right)\times\frac{4gh}{3R^2}$$

or $$K_r = \frac{Mgh}{3} = \frac{2\times9.8\times4}{3} = 26.13 \text{ J}$$

PROBLEMS FOR PRACTICE

1. A solid sphere and a solid cylinder having the same mass and radius roll down the same inclined plane. What is the ratio of their linear acceleration ? **[15 : 14]**
2. A hoop and a disc, both 0.5 m radius and 2 kg mass, are released from the top of an inclined plane 3 m high and 8 m long. What is the speed of each when it reaches the bottom? Assume that both roll without slipping. **[Hoop 5.4 ms^{-1} ; disc 6.3 ms^{-1}]**
3. A hoop released from the top of an inclined plane has a speed of 15 ms^{-1} at the bottom. How high is the upper end of the inclined plane? Assume that hoop rolled without slipping. **[23 m]**
4. A disc of moment of inertia of 0.1 kg m^2 about its centre and radius 0.2 m is released from rest on a plane inclined at 30° to the horizontal. If the mass of the disc is 5 kg, calculate the angular velocity after it has rolled 2 m down the plane. **[18.3 rad s^{-1}]**
5. A cylinder of mass 10 kg is rolling perfectly on a plane inclined at 30° to the horizontal. Find the force of friction between the cylinder and the surface of the inclined plane. **[16.3 N]**

14.26. LAWS OF ROTATIONAL MOTION

Like Newton's three laws for linear motion, we can state the three laws of rotational motion.

1. First law. *Every body continues in its state of rest or of uniform rotational motion about an axis until it is compelled by some external torque to change that state.*

2. Second law. *The rate of change of angular momentum about an axis is directly proportional to the impressed external torque and the change in angular momentum takes place in the direction of the applied torque. i.e.,*

$$\tau \propto \frac{I\omega_2 - I\omega_1}{t}$$

or $$\tau \propto I\frac{(\omega_2 - \omega_1)}{t}$$

or $\tau \propto I\alpha \qquad \left(\because \alpha = \frac{(\omega_2 - \omega_1)}{t}\right)$

or $\tau = kI\alpha$

The SI unit of torque is defined in such a way that the value of k is unity.

$\therefore \qquad \tau = I\alpha$

3. Third law. *To every external torque applied, there is an equal and opposite restoring torque.*

4.27. LINEAR AND ANGULAR QUANTITIES

The table below summarises the analogy between linear and angular quantities.

S.No.	Linear quantity or Equation	Angular quantity or equation	Relation between linear and angular quantities
1.	Position x	Angular position θ	
2.	Linear velocity v	Angular velocity ω	$v = \omega r$
3.	Linear acceleration a	Angular acceleration α	$a = r\alpha$
4.	Mass m	Moment of inertia I	$I = \int r^2 dm$
5.	Force $\vec{F}$	Torque $\vec{\tau}$	$\vec{\tau} = \vec{r} \times \vec{F}$
6.	Newton's second law for linear motion $F = ma$	Newton's second Law for rotational motion $\tau = I\alpha$	
7.	Work $= \vec{F} \cdot \vec{S}$	Work $= \vec{\tau} \cdot \vec{d\theta}$	
8.	Power $= \vec{F} \cdot \vec{v}$	Power $= \vec{\tau} \cdot \vec{\omega}$	
9.	K.E. $= \frac{1}{2}mv^2 = \frac{p^2}{2m}$	K.E. $= \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$	
10.	Linear momentum $\vec{p} = m\vec{v}$	Angular momentum $\vec{L} = I\vec{\omega}$	$\vec{L} = \vec{r} \times \vec{p}$
11.	Newton's second law in terms of linear momentum $\vec{F} = \frac{\vec{dp}}{dt}$	Newton's second law in terms of angular momentum $\vec{\tau} = \frac{\vec{dL}}{dt}$	
12.	Conservation of linear momentum on collision if no external force acts.	Conservation of angular momentum on collision if no external torque acts.	
13.	**Equations of Motion** $S = \left(\frac{v_0 + v}{2}\right)t$ $v = v_0 + at$ $S = v_0 t + \frac{1}{2}at^2$ $v^2 = v_0^2 + 2aS$	**Equations of Motion** $\theta = \left(\frac{\omega_0 + \omega}{2}\right)t$ $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$	

14.28. EQUILIBRIUM OF RIGID BODIES

A rigid body will be in equilibrium if the following two conditions are met :

(*i*) **The vector sum of the forces acting on the body must be zero.**

(*ii*) **The net torque acting on the body must be zero.**

The first condition ensures that the linear (or translational) acceleration is zero *i.e.,* either the body is at rest or it is moving with uniform linear velocity. Under this condition, the body is said to be in *translational equilibrium.* The second condition ensures that the angular acceleration is zero *i.e.* either the body is at rest or it is moving with uniform angular velocity. Under this condition, the body is said to be in *rotational equilibrium.* A body satisfying both conditions is said to be in equilibrium. If such a body is stationary, it is said to be in *static equilibrium* and if it is moving with uniform linear or angular velocity, it is said to be in *dynamic equilibrium.*

1. Translational equilibrium. For an object to be in translational equilibrium, the vector sum of the forces acting on it must be zero *i.e.*

Net external force, $\vec{F} = 0$

Stating that the vector sum of forces acting on an object is zero is equivalent to :

$$\Sigma \vec{F}_x = 0 \; ; \; \Sigma \vec{F}_y = 0 \; ; \; \Sigma \vec{F}_z = 0$$

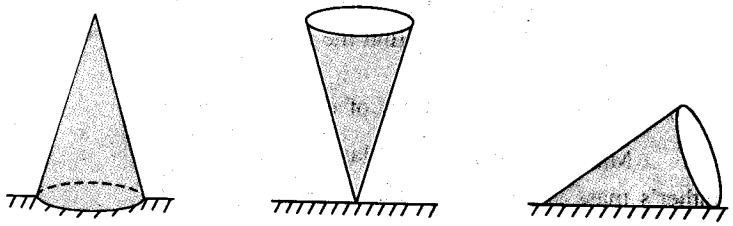
It means that all the forces along the x-axis add to zero. Same is true for forces along y and z axes. According to Newton's second law of motion, $\vec{F} = m\vec{a}$. For translational motion, $\vec{F} = 0$.

$$\therefore \qquad 0 = m\vec{a}$$

or Acceleration, $\vec{a} = 0$ *i.e.,* $\vec{v} = 0$ or constant

Therefore, when a body is in translational equilibrium, it is either at rest ($\vec{v} = 0$) or it is moving with constant velocity ($\vec{v}$ = constant). There are three types of translational equilibrium *viz.*

(*i*) **Stable equilibrium.** A body is in stable equilibrium if it returns to its equilibrium position after it has been displaced slightly [See Fig. 14.27(*i*)].



Stable (*i*) Unstable (*ii*) Neutral (*iii*)

Fig. 14.27

(*ii*) **Unstable equilibrium.** A body is in unstable equilibrium if it does not return to its equilibrium position and does not remain in the displaced position after it has been displaced slightly [See Fig. 14.27(*ii*)].

(*iii*) **Neutral equilibrium.** A body is in neutral equilibrium if it stays in the displaced position after it has been displaced slightly [See Fig. 14.27(*iii*)].

2. Rotational equilibrium. *A rigid body is said to be in rotational equilibrium if the net external torque acting on the body is zero i.e.,*

Net external torque, $\vec{\tau} = 0$ ...For rotational equilibrium

When the body is in rotational equilibrium (*i.e.*, $\vec{\tau} = 0$), either the body does not rotate or rotates with a constant angular velocity ($\vec{\omega}$). We can also define rotational equilibrium in another alternative way :

A body is said to be in rotational equilibrium if the algebraic sum of moments of all the forces acting on the body about a fixed point/axis is zero.

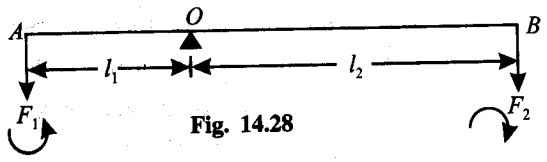


Fig. 14.28

Consider a rod *AB* of negligible mass pivoted at point *O* as shown in Fig. 14.28. The rod is under the action of two parallel forces F_1 and F_2. The rod will be in rotational equilibrium if the algebraic sum of moments of F_1 and F_2 about *O* is zero *i.e.*,

$$F_1 \times l_1 - F_2 \times l_2 = 0$$

Here $F_1 \times l_1 = +\tau_1$ (anticlockwise) ; $F_2 \times l_2 = -\tau_2$ (clockwise)

$$\therefore \quad \tau_1 + (-\tau_2) = 0 \quad \text{or} \quad \tau_1 - \tau_2 = \text{Net torque} = 0$$

Note. If a body is in rotational static equilibrium, it will not rotate about any axis. We are, therefore, free to choose any possible rotation axis for the purpose of computing torques.

14.29. BINARY SYSTEM

The term *binary system* means two objects that have observable effect on each other's motion due to their mutual gravitational force. For example, earth-moon system is a binary system. The *barycentre* is the centre of mass of the binary (two-body) system. If you can imagine the bodies at either end of an imaginary rod, the barycentre of the system is the point at which the system would balance like a *seesaw*. If we know the distances of the two bodies (binary system) from the barycentre, we can find the comparative masses of the two bodies. To illustrate this concept, suppose that a father tried to seesaw with his young daughter and they found that they could balance only if the daughter sat three times as far from the pivot (*fulcrum*) as the father [See Fig. 14.29]. We can easily find the relative masses of the two. Referring to Fig. 14.29,

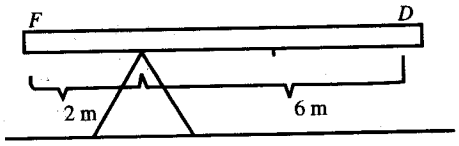


Fig. 14.29

$$\text{Mass of father } (F) \times 2 = \text{Mass of daughter } (D) \times 6$$

$$\therefore \quad \text{Mass of father} = 3 \times \text{Mass of daughter}$$

Therefore, father's mass is three times that of the daughter.

(*i*) **Earth-Moon System.** We often describe the motion of the earth and moon separately. We say that the earth orbits the sun and the moon orbits the earth. In reality, the earth and moon are better thought of as a binary (two-body) system orbiting the sun. Fig. 14.30 shows the earth-moon system. The barycentre of the system is approximately 4800 km from the centre of the earth and 384,000 km from the centre of the moon.

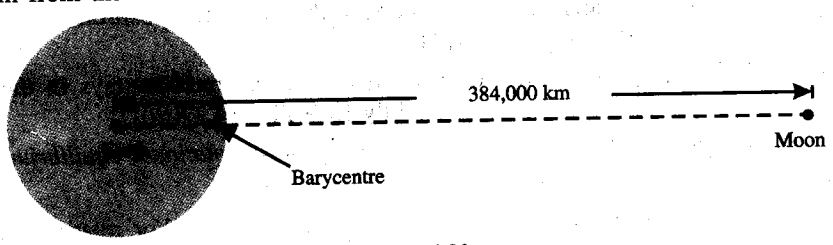


Fig. 14.30

We can find the comparative masses of the earth and moon. Imagine the earth and moon placed at opposite ends of a rod which is 388,800 km (= 384,000 + 4800) long. The point at which this system would balance is the barycentre. If M_E and M_M are the masses of the earth and moon respectively, then, $M_E \times 4{,}800 = M_M \times 3{,}84{,}000$

or $$\frac{M_M}{M_E} = \frac{4800}{384{,}000} = \frac{1}{80}$$

We conclude that the mass of the moon is only 1/80 the mass of the earth. The barycentre is the point around which both the earth and moon revolve while continuing to revolve around the sun.

(*ii*) **Diatomic Molecules.** A diatomic molecule has two atoms separated by some distance. We can represent such a molecule by the model shown in Fig. 14.31. Its two atoms act as though they are joined by a spring with spring constant k. The spring represents the bond between the two atoms. Since a diatomic molecule extends appreciably in space, it has an appreciable moment of inertia. It may therefore have kinetic energy of rotation as well as of translation. Let us now consider a diatomic gas. When two of its molecules collide, they will, in general tend to rotate as well as to deflect each other. In some collisions, energy will be transferred from the translations of the molecules to their rotations ; in others from the rotations to the translations. We may assume, then, that the internal energy of a diatomic gas is shared between the rotations and translations of its molecules.

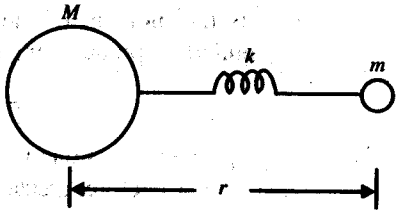


Fig. 14.31

In a diatomic molecule (*i.e.*, hydrogen, oxygen, nitrogen etc.), the two atoms are held together by interatomic forces. They orbit about a common centre of mass with a common time period of revolution. It is due to this reason that a diatomic molecule is regarded as a binary system.

(*iii*) **Binary star system.** Two stars that are bound to each other by gravity and orbit about a common centre of mass are called binary stars. Fig. 14.32 shows the binary star system. Here two stars of masses M_1 and M_2 revolve along circular paths of radii r_1 and r_2 respectively. The centre of mass of the binary system is C. From the definition of centre of mass,

$$M_1 r_1 = M_2 r_2$$

From Kepler's third law, it can be shown that :

$$M_1 + M_2 = \frac{4\pi^2}{G}\frac{r^3}{T^2} \qquad ...(i)$$

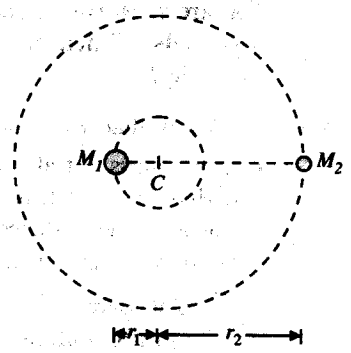


Fig. 14.32

where $r = r_1 + r_2$ = distance between two stars

T = common time period of revolution

We can use eq. (*i*) to find the sum of the masses of the two stars of the binary system. If mass of one star is known, the mass of the other star can be found out. It is important to note that nearly half the stars around us form binary star system.

CONCEPTUAL QUESTIONS

Q.1. Why are spokes fitted in the cycle wheel?

Ans. The spokes of cycle wheel increase its moment of inertia. The greater the moment of inertia, the greater is the opposition to any change in uniform rotational motion. As a result, the cycle runs smoother and steadier. If the cycle wheels had no spokes, the cycle would be driven in jerks and hence unsafe.

Q.2. How a swimmer jumping from a height is able to increase the number of loops made in the air?

Ans. The swimmer can increase the number of loops in air by pulling his arms and legs inward *i.e.*, by decreasing the moment of inertia. By doing so, the angular velocity ω increases because angular momentum ($= I\omega$) remains constant.

Q.3. Why is it more difficult to revolve a stone by tieing it to a longer string than by tieing it to a shorter string?

Ans. If I is the moment of inertia of the stone about the axis of rotation, then torque τ required to produce angular acceleration α is

$$\tau = I\alpha \quad \text{where} \quad I = m r^2$$

If r is greater (*i.e.*, stone is farther from the axis of rotation), the moment of inertia will be greater. Therefore, greater torque is required to produce the same angular acceleration.

Q.4. A man standing on the turn table raises his hands suddenly. What would happen?

Ans. The angular velocity ω of the turn table will decrease. It is because when the man raises his hands, the moment of inertia of the system increases. Since the angular momentum ($= I\omega$) of the system remains the same, ω will decrease *i.e.*, the table slows down.

Q.5. You are given two circular discs of equal masses and thicknesses but made of different materials. Which one will have larger moment of inertia about its central axis and why?

Ans. We know that $I = \sum m r^2$. Thus the moment of inertia of a body not only depends upon its mass but also depends on the distribution of mass from the axis of rotation. The disc made of less dense material will have greater surface area *i.e.*, its particles are at a greater distance from the axis of rotation. Therefore, the disc with less dense material will have greater moment of inertia.

Q.6. A circular platform is rotating with a uniform angular velocity. What will be the change in the motion of the platform if a person sits near the edge and then starts moving towards the centre of the platform?

Ans. The system tends to keep its angular momentum L ($=I\omega$) constant. When the person sits near the edge of the platform, the moment of inertia of the platform increases. Since L ($= I\omega$) remains constant, the angular velocity ω decreases *i.e.*, the platform slows down. When the person starts moving towards the centre of the platform, the moment of inertia decreases. Consequently, the angular velocity ω of the platform increases.

Q.7. Why is ladder more apt to slip when you are high up on it than when you just begin to climb?

Ans. When a man is high up on the ladder, the moment of force tending to rotate the ladder about the base increases because of greater lever arm. However, when the man begins to climb, the lever arm is small and the moment of force is insufficient to cause slipping.

Q.8. If two circular discs of different radii and of the same weight and thickness are made from metals of different densities, which disc will have the larger moment of inertia about the central axis?

Ans. $$I_1 = \frac{1}{2} mR_1^2 \; ; \; I_2 = \frac{1}{2} mR_2^2 \quad \therefore \quad \frac{I_1}{I_2} = \frac{R_1^2}{R_2^2}$$

Now $$m = \pi R_1^2 t \rho_1 = \pi R_2^2 t \rho_2 \quad \therefore \quad \frac{R_1^2}{R_2^2} = \frac{\rho_2}{\rho_1}$$

or $$\frac{I_1}{I_2} = \frac{\rho_2}{\rho_1} \quad \text{or} \quad I \propto \frac{1}{\rho}$$

Thus the disc which has greater density will have smaller rotational inertia.

Q.9. The moments of inertia of two rotating bodies A and B about the same axis are I_A and I_B ($I_A > I_B$) and their angular momentum are equal. Which has greater rotational kinetic energy?

Ans. Rotational $K.E.$; $K_r = \frac{1}{2} I\omega^2$; Angular momentum, $L = I\omega$

$$\therefore \quad K_r = \frac{L^2}{2I}$$

Suppose K_A and K_B are the rotational kinetic energies of A and B respectively. Since the angular momentum L of the two bodies is the same,

$$\therefore \quad \frac{K_A}{K_B} = \frac{L^2/2I_A}{L^2/2I_B} = \frac{I_B}{I_A}$$

As $I_A > I_B$, therefore, $K_B > K_A$.

Q.10. How will you distinguish between a hard boiled egg and a raw egg by spinning each on table top?

Ans. The egg which spins at a lower rate will be the raw egg. In a raw egg, the liquid matter inside tries to get away from the axis of rotation, thereby increasing the moment of inertia. Since $\tau = I\alpha$, α decreases *i.e.* the raw egg will spin at a smaller rate. On the other hand, the hard boiled egg will rotate faster like a rigid body.

Q.11. If polar ice caps melt and spread uniformly, how will the length of the day be affected?

Ans. If the polar ice melts, the water formed will spread over the surface of earth. As a result, the moment of inertia of earth will increase. Since angular momentum ($=I\omega$) of earth remains constant, ω will decrease. This means the earth will complete one revolution about its own axis in a longer time instead of 24 hours. Hence the length of the day will increase.

Q.12. Show that the rotational kinetic energy of a ball rolling over a horizontal plane is 2/7 of its total kinetic energy.

Ans. Let M and R be the mass and radius of the ball respectively.

M.I. of the ball about symmetry axis of rotation, $I = \frac{2}{5}MR^2$

Rotational K.E. of ball, $K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\left(\frac{2}{5}MR^2\right)\times\left(\frac{v}{R}\right)^2 = \frac{Mv^2}{5}$

Total K.E. of ball, $K = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$

$$= \frac{Mv^2}{5} + \frac{Mv^2}{2} = \frac{7Mv^2}{10}$$

$$\therefore \quad \frac{K_r}{K} = \frac{Mv^2}{5} \div \frac{7Mv^2}{10} = \frac{2}{7}$$

or $$K_r = \frac{2}{7}\times K$$

Q.13. What is the moment of inertia of a uniform circular ring about a tangent in the plane of the ring?

Ans. Let M and R be the mass and radius of the ring respectively. The *M.I.* of a uniform circular ring about its diameter is

$$I = \frac{1}{2}MR^2$$

According to theorem of parallel axes, *M.I.* of the ring about a tangent in the plane of the ring is

$$I' = I + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Q.14. What is the moment of inertia of a uniform circular ring about a tangent perpendicular to the plane of the ring?

Ans. Let M and R be the mass and radius of the ring. The M.I. of the ring about an axis perpendicular to the plane of the ring and passing through the centre is

$$I = MR^2$$

According to theorem of parallel axes, M.I. of the ring about a tangent perpendicular to the plane of the ring is

$$I' = I + MR^2 = MR^2 + MR^2 = 2MR^2$$

Q.15. Why is cat not hurt while falling on the ground?

Ans. While falling, a cat stretches its body along with its tail. This increases the moment of inertia. Since no external torque is acting, the angular momentum ($= I\omega$) remains constant. The increase in I means ω will decrease and the cat lands gently on the ground.

Q.16. In a flywheel, most of the mass is concentrated at the rim. Explain why?

Ans. In a flywheel, most of the mass is concentrated at the rim. This increases the moment of inertia (I) of the rim. If the flywheel gains or loses some rotational K.E. $\left(=\frac{1}{2}I\omega^2\right)$, then due to large value of I, ω changes very slightly. The result is that the flywheel helps in maintaining uniform rotational motion.

Q.17. Why is the speed of a whirlwind in a tornado very high?

Ans. In a whirlwind, air from nearby regions gets concentrated in a small space. As a result, the moment of inertia decreases. Since angular momentum ($= I\omega$) remains constant, ω increases to a large value.

VERY SHORT ANSWER QUESTIONS

Q.1. What is moment of inertia?

Ans. The moment of inertia of a rigid body about an axis is a measure of the difficulty in starting or stopping or changing rotation of the body about that axis.

Q.2. How is moment of inertia of a rigid body measured?

Ans. The moment of inertia of a rigid body about a given axis of rotation is the sum of the products of the masses of its particles and the squares of their respective perpendicular distances from the axis of rotation.

Q.3. Why is moment of inertia also called rotational inertia?

Ans. The moment of inertia is also called rotational inertia because it is a measure of inertia of a rotating body.

Q.4. What are the units and dimensions of moment of inertia?

Ans. The units of M.I. are kg m^2 and its dimensional formula is $[ML^2T^0]$.

Q.5. Is moment of inertia a scalar or vector quantity?

Ans. It is a scalar quantity.

Q.6. What are the factors on which the moment of inertia of a body depends?

Ans. The M.I. of a body depends on (*i*) total mass of the body (*ii*) distribution of the mass (*iii*) axis of rotation.

Q.7. Is M.I. of a body a fixed quantity?

Ans. No. It depends on the location of axis about which the moment of inertia is calculated.

Q.8. Is radius of gyration of a body constant quantity?

Ans. No. Its value depends on the axis of rotation and distribution of mass of the body about this axis.

Q.9. What is the rotational analogue for (*i*) mass (*ii*) force (*iii*) linear momentum?

Ans. (*i*) Moment of inertia (*ii*) Torque (*iii*) Angular momentum.

Q.10. What is the moment of inertia of a (*i*) ring (*ii*) disc (*iii*) solid sphere about its diameter?

Ans. (*i*) $I = \frac{1}{2}MR^2$ (*ii*) $I = \frac{1}{4}MR^2$ (*iii*) $I = \frac{2}{5}MR^2$.

Q.11. There are two spheres of same mass and same radius, one is solid and other is hollow. Which has larger moment of inertia about its diameter?

Ans. The hollow sphere has larger M.I. It is because ($I = MR^2$) in case of hollow sphere, the mass is concentrated at the boundary and has large distance from the axis.

Q.12. What is the SI unit of radius of gyration?

Ans. The SI unit of radius of gyration is metre.

Q.13. What physical quantity is represented by (*i*) product of moment of inertia and angular velocity (*ii*) product of moment of inertia and angular acceleration.

Ans. (*i*) Angular momentum (*ii*) Torque.

Q.14. For a given mass and size, the moment of inertia of a solid disc is smaller than that of a ring. Why?

Ans. It is because the whole mass of the ring is at its periphery *i.e.*, at maximum distance from the centre. However, in case of disc, the mass is distributed from the centre of the rim.

Q.15. What is the physical meaning of radius of gyration?

Ans. The radius of gyration of a body is a measure of the distribution of mass of the body relative to a given axis of rotation.

SHORT ANSWER QUESTIONS

Q.1. A body has a large radius of gyration. What does it mean?

Ans. A large radius of gyration of a body means that, on the average, the mass of the body is relatively far from the given axis of rotation.

Q.2. Find the radius of gyration of a hollow sphere rotating about its diameter.

Ans. The M.I. of a hollow sphere about its diameter = $(2/3)MR^2$.

$$\therefore \quad \frac{2}{3}MR^2 = MK^2 \text{ or } K = R\sqrt{\frac{2}{3}}$$

Q.3. If no external torque acts on a body, will its angular velocity remain constant. Explain.

Ans. Torque, $\tau = \frac{dL}{dt}$. If $\tau = 0$, angular momentum L = constant. Now $L = I\omega$. Therefore, angular velocity (ω) of the body will remain constant if M.I. (I) of the body remains constant.

Q.4. Two satellites of equal masses which can be considered as particles are orbiting the earth at different heights. Will their moments of inertia be same or different?

Ans. The moments of inertia of the two satellites will be different. It is because I = Mass × (Distance)^2. Although the masses of the two satellites are the same, their distances from the centre of earth are different.

Q.5. Derive the expression for power of a rotating body.

Ans. Power, $P = \frac{d}{dt}$ (K.E. of rotation) $= \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = I\omega\frac{d\omega}{dt}$

But $d\omega/dt = \alpha$ so that $P = I\,\omega\alpha = \tau\omega$. $(\because \tau = I\alpha)$

Q.6. The M.I. of a circular disc of mass M and radius R about a diameter is (1/4) MR^2. What will be its M.I. about an axis passing through its centre and perpendicular to its plane?

Ans. If I is the M.I. of the disc about an axis passing through its centre and perpendicular to its plane, then according to theorem of perpendicular axes, we have,

$$I = I_X + I_Y = \frac{1}{4}MR^2 + \frac{1}{4}MR^2 = \frac{1}{2}MR^2$$

Q.7. The angular velocity of earth revolving around the sun increases when it comes closer to the sun. Why?

Ans. The earth revolves around the sun in elliptical orbit with the sun at one focus of the ellipse. Since the distance of earth from the sun keeps on changing, the M.I. of the earth about axis of rotation also changes. Now, no external torque acts on the earth so that angular momentum $L(=I\omega)$ of the earth around the sun remains constant. As earth comes closer to sun, the M.I. of earth (I) decreases and angular velocity (ω) of earth must increase to conserve angular momentum.

Q.8. An automobile engine develops 100 h.p. when rotating at a speed of 1800 r.p.m. What torque is acting?

Ans. Angular velocity, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60} = 60\,\pi$ rad/s

Now Power = $\tau\omega$ or $100 \times 746 = \tau \times 60\pi$ $\quad \therefore \tau = 395.8$ Nm.

Q.9. What is angular impulse? What are its units?

Ans. The effect of torque in a small interval of time is called angular impulse. It is equal to the change in angular momentum during that time interval. Its SI unit is kg m^2 s^{-1} (or Nms)

Q.10. Torque and work are both defined as force times distance. Explain how they differ?

Ans. In this regard, note the following points :

(*i*) Work, $W = \vec{F} \cdot \vec{S}$ and Torque, $\vec{\tau} = \vec{r} \times \vec{F}$. Therefore, work is a scalar quantity and torque is a vector quantity.

(*ii*) Work done is maximum when force acts in the direction of displacement. However, torque exerted is maximum when force and lever arm are perpendicular.

LONG ANSWER QUESTIONS

1. Define and explain moment of inertia of a body about an axis of rotation. **[Refer to Arts. 14.1 and 14.2]**
2. What is radius of gyration of a body? What is its significance? **[Refer to Art. 14.3]**
3. State and prove theorem of parallel axes. **[Refer to Art. 14.14]**
4. State and prove theorem of perpendicular axes. **[Refer to Art. 14.15]**
5. Derive an expression for the moment of inertia of a uniform thin rod about an axis through its centre and perpendicular to its length. **[Refer to Art. 14.19]**
6. Derive an expression for the moment of inertia of a uniform ring about an axis passing through the centre of the ring and perpendicular to the plane of the ring. **[Refer to Art. 14.17]**
7. Derive an expression for the moment of inertia of a uniform disc about an axis through the centre of the disc and perpendicular to the plane of the disc. **[Refer to Art. 14.18]**
8. Obtain an expression for the kinetic energy of a rotating body. **[Refer to Art. 14.4]**
9. Derive the relation between angular momentum and moment of inertia of a body about the axis of rotation. **[Refer to Art. 14.6]**
10. Prove the relation $\tau = I\alpha$ where the symbols have their usual meanings. **[Refer to Art. 14.5]**
11. State and explain principle of conservation of angular momentum. Give two practical examples. **[Refer to Arts. 14.11 and 14.12]**
12. A cylinder is rolling down (without slipping) along an inclined plane having rough surface and of inclination θ. Show that co-efficient of static friction is $\mu_s = \frac{1}{3}\tan\theta$. **[Refer to Art. 14.25]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The moment of inertia of a system of particles is given by :

$$I = \sum m_i r_i^2$$

The unit of I is kgm^2.

2. The radius of gyration (K) of an object is that length for which the object's moment of inertia can be written as :

$$I = MK^2$$

Here M is the total mass of the object.

3. If a rigid body rotates about a fixed axis with angular velocity ω, its kinetic energy is given by ;

$$\text{Kinetic energy, } K_r = \frac{1}{2} I\omega^2$$

where I is the moment of inertia about the axis of rotation.

4. The moment of inertia of a rigid body about a given axis of rotation is given by ;

$$I = \int r^2 \, dm$$

where r is the perpendicular distance from the mass element dm to the axis of rotation.

5. If a rigid body free to rotate about a fixed axis has a *net external torque* acting on it, the body will undergo angular acceleration α, where

$$\tau_{net} = I\alpha$$

This is Newton's second law applied to rotation. In using this relation, α must be expressed in rad/sec^2.

6. The net work done by external forces in rotating a rigid body about a fixed axis is equal to the change in the rotational kinetic energy of the body *i.e.*

$$\text{Work done, } W = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0^2$$

This is the work-energy theorem applied to rotational motion.

7. The angular momentum $\vec{L}$ of a symmetric rigid body rotating about a fixed axis of symmetry with angular velocity $\vec{\omega}$ is given by ;

$$\vec{L} = I\vec{\omega}$$

where I is the moment of inertia about the axis of rotation.

8. The net external torque acting on a rigid body is equal to the product of its moment of inertia about the given axis of rotation and its angular acceleration.

$$\Sigma \vec{\tau}_{ext} = I\vec{\alpha}$$

9. If the net external torque acting on a system is zero, the total angular momentum of the system is constant. Applying this conservation of angular momentum law to a body whose moment of inertia changes gives.

$$I_1 \omega_1 = I_2 \omega_2 = \text{Constant}$$

Moment of inertia can change in two ways. First, the total rotating mass may change. Secondly, the distribution of existing mass may change because of change of radius.

10. Rolling motion is a combination of rotational and translational motion. The axis of the rolling body is parallel to the plane on which the body rolls.

11. Consider a wheel rolling without slipping as shown in Fig. 14.33. Each little piece at material of the wheel is undergoing two types of motion at the same time. The centre of the wheel, which is the wheel's centre of mass, is moving horizontally with speed v_{CM}. However, at the same time, the wheel is rotating about its centre of mass with angular speed ω. Therefore, the wheel possesses both translational and rotational kinetic energy.

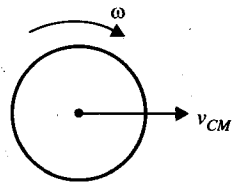


Fig. 14.33

12. An object which is both rotating and translating has a total K.E. given by :

$$\text{K.E.}_{total} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2$$

where M = mass of object

v_{CM} = velocity of object's centre of mass

I_{CM} = moment of inertia of the object about centre of mass

13. A uniform object of radius r rolling without slipping has the following relation between ω and v_{CM} ;

$$v_{CM} = r\omega$$

14. For rolling objects that have spherical or cylindrical symmetry (solid sphere, hollow sphere, disc, solid cylinder, hollow cylinder etc.), the speed and acceleration is independent of the mass and radius but depends only on moment of inertia about an axis perpendicular to the direction of motion, parallel to the surface, and through the centre of mass.

15. Consider four bodies, a sphere, disc, shell and ring of the same mass and radius resting at the top of an inclined plane at the same height above the ground. If all the bodies are released simultaneously, and pure rolling occurs, then velocity and acceleration of the bodies will be determined by their moment of inertia about centre of mass. The smaller the moment of inertia, the greater will be the velocity and acceleration of the body.

$$\therefore \quad (\text{Vel.})_{sphere} > (\text{Vel.})_{disc} > (\text{Vel.})_{shell} > (\text{Vel.})_{ring}$$

The time taken to reach the bottom of the inclined plane will be just the reverse.

$$(t)_{sphere} < (t)_{disc} < (t)_{shell} < (t)_{ring}$$

16. In all the above cases (point 15), the acceleration will be less than $g \sin\theta$, the value it would have if the inclined plane were frictionless and no rolling occurred.

17. For a body to roll down an inclined plane, there must be friction.

18. The table below gives the comparison of useful equations of linear (translational) and rotational motion :

Rotational motion About a fixed axis	**Linear Motion**
(*i*) Angular velocity, $\omega = d\theta/dt$	(*i*) Linear velocity, $v = dx/dt$
(*ii*) Angular acceleration, $\alpha = d\omega/dt$	(*ii*) Linear acceleration, $a = dv/dt$
(*iii*) Resultant torque, $\Sigma\tau = I\alpha$	(*iii*) Resultant force $\Sigma F = ma$

(*iv*) $\alpha = \text{constant} \begin{cases} \omega = \omega_0 + \alpha t \\ \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \end{cases}$

(*v*) Work, $W = \int_{\theta_0}^{\theta} \tau d\theta$

(*vi*) Kinetic energy, $K_r = \frac{1}{2} I\omega^2$

(*vii*) Power, $P = \tau\omega$

(*viii*) Angular momentum, $\vec{L} = I\,\vec{\omega}$

(*ix*) Resultant torque, $\vec{\tau} = \frac{d\vec{L}}{dt}$

(*iv*) $a = \text{constant} \begin{cases} v = v_0 + at \\ x - x_0 = v_0 t + \frac{1}{2}at^2 \\ v^2 = v_0^2 + 2a(x - x_0) \end{cases}$

(*v*) Work, $W = \int_{x_0}^{x} F_x dx$

(*vi*) Kinetic energy, $K = \frac{1}{2}mv^2$

(*vii*) Power, $P = F v$

(*viii*) Linear momentum, $\vec{P} = m\vec{v}$

(*ix*) Resultant Force, $\vec{F} = \frac{d\vec{p}}{dt}$

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The ratio of rotational and translational kinetic energies of a sphere is **[AFMC- Pune 2001]**
   (*a*) $\frac{2}{9}$ (*b*) $\frac{2}{7}$
   (*c*) $\frac{2}{5}$ (*d*) $\frac{2}{3}$
2. A disc is rotating with an angular speed ω. If a child sits on it, which of the following is conserved? **[CBSE-PMT 2002]**
   (*a*) angular momentum
   (*b*) kinetic energy
   (*c*) linear momentum
   (*d*) potential energy
3. A constant torque of 31.4 N-m is exerted on a pivoted wheel. If angular acceleration of wheel is 4π rad/sec^2, then moment of inertia of the wheel is **[AIIMS 2001]**
   (*a*) 3.5 kg m^2 (*b*) 2.5 kg m^2
   (*c*) 4.5 kg m^2 (*d*) 5.5 kg m^2
4. Moment of inertia of a thin circular disc of mass M and radius R about any diameter is **[Pb–PMT 1999]**
   (*a*) $\frac{MR^2}{4}$ (*b*) $\frac{MR^2}{2}$
   (*c*) MR^2 (*d*) $2MR^2$
5. The angular momentum of a body with moment of inertia I and angular velocity ω rad/sec is **[AFMC– Pune 1997]**
   (*a*) $I\omega^2$ (*b*) $I\omega$
   (*c*) I/ω (*d*) I/ω^2
6. The moment of inertia of a regular circular disc of mass 0.4 kg and radius 100 cm about an axis perpendicular to the plane of the disc and passing through its centre is **[AIIMS 1999]**
   (*a*) 0.2 kg m^2 (*b*) 0.02 kg m^2
   (*c*) 2 kg m^2 (*d*) 0.002 kg m^2
7. For a hollow cylinder and a solid cylinder of the same mass and radius rolling without slipping on an inclined plane, which of these reaches the ground earlier? **[CBSE–PMT 2000]**
   (*a*) hollow cylinder
   (*b*) solid cylinder
   (*c*) both simultaneously
   (*d*) cannot say anything
8. A thin circular ring of mass M and radius r is rotating about its axis with a constant angular velocity ω. If two objects, each of mass m are attached gently to opposite ends of a diameter of the ring, the ring will now rotate with an angular velocity given by : **[Pb–PMT 1998]**
   (*a*) $\frac{\omega M}{M + 2m}$ (*b*) $\frac{\omega M}{M - 2m}$
   (*c*) $\frac{\omega(M - 2m)}{M}$ (*d*) $\frac{2\omega M}{M - 2m}$
9. A solid spherical ball rolls on a table. Ratio of rotational kinetic energy to total kinetic energy is : **[CBSE PMT 1994]**
   (*a*) 7/10 (*b*) 2/7 (*c*) 1/2 (*d*) 1/6

10. A solid sphere, a hollow sphere and a solid cylinder, all of the same radius, roll down an inclined plane from the same height (starting from rest). Which of them takes the least time in reaching the bottom of the plane? **[Pb PMT 1993]**

(*a*) hollow sphere
(*b*) solid sphere
(*c*) solid cylinder
(*d*) All reach simultaneously

11. A thin uniform circular ring is rolling down an inclined plane of inclination 30° without slipping. Its linear acceleration along the inclined plane will be : **[CBSE PMT 1992]**

(*a*) $2g/3$ (*b*) $g/2$
(*c*) $g/3$ (*d*) $g/4$

12. The speed of a homogenous solid sphere after rolling down an inclined plane of vertical height h from rest without sliding is **[CBSE PMT 1992]**

(*a*) $\sqrt{\frac{10}{7}gh}$ (*b*) $\sqrt{gh}$
(*c*) $\sqrt{\frac{6}{5}gh}$ (*d*) $\sqrt{\left(\frac{4}{3}\right)gh}$

13. The rotational kinetic energy of a body is E and its moment of inertia is I. The angular momentum is **[MP PMT 1993]**

(*a*) EI (*b*) $\sqrt{EI}$
(*c*) $\sqrt{2EI}$ (*d*) E/I

14. A flywheel rotating about a fixed axis has a kinetic energy of 360 J when its angular speed is 30 rad/sec. The moment of inertia of the wheel about the axis of rotation is **[CBSE PMT 1999]**

(*a*) 0.6 kg m^2 (*b*) 0.15 kg m^2
(*c*) 0.8 kg m^2 (*d*) 0.75 kg m^2

15. When sand is poured on a rotating disc, its angular velocity will **[CPMT 1994]**

(*a*) increase (*b*) decrease
(*c*) remain constant (*d*) none of above

16. Two rings have their moments of inertia in the ratio 2 : 1 and their diameters are in the ratio 2 : 1. The ratio of their masses will be **[MP PMT 1998]**

(*a*) 2 : 1 (*b*) 1 : 2
(*c*) 1 : 4 (*d*) 1 : 1

17. A body rolls down an inclined plane. If its K.E. of rotational motion is 40% of its K.E. of translational motion, then the body is **[CBSE PMT 1997]**

(*a*) cylinder (*b*) ring
(*c*) solid disc (*d*) solid sphere

18. Moment of inertia of a uniform circular disc about a diameter is I. Its moment of inertia about an axis perpendicular to its plane passing through a point on its rim will be **[CBSE PMT 1990]**

(*a*) $5I$ (*b*) $3I$
(*c*) $6I$ (*d*) $4I$

19. Two discs have same mass and thickness. Their materials have densities d_1 and d_2. The ratio of their moments of inertia about the central axis will be **[CBSE PMT 1990]**

(*a*) $d_1 : d_2$ (*b*) $d_1 d_2 : 1$
(*c*) $1 : d_1 d_2$ (*d*) $d_2 : d_1$

20. A couple produces a **[CBSE PMT 1997]**

(*a*) pure linear motion
(*b*) pure rotational motion
(*c*) both linear and rotational motion
(*d*) no motion

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)
6. (*a*)	**7.** (*b*)	**8.** (*a*)	**9.** (*b*)	**10.** (*b*)
11. (*d*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*b*)
16. (*b*)	**17.** (*d*)	**18.** (*c*)	**19.** (*d*)	**20.** (*b*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Translational K.E. of sphere is

$$K_t = \frac{1}{2}Mv^2$$

Rotational K.E. of sphere is

$$K_2 = \frac{1}{2} I\omega^2 = \frac{1}{2} \times \left(\frac{2}{5} MR^2\right) \omega^2 = \frac{1}{5} Mv^2 \qquad \left(\because v^2 = \omega^2 R^2\right)$$

$$\therefore \quad \frac{K_2}{K_1} = \frac{1/5\, Mv^2}{1/2\, Mv^2} = \frac{2}{5}$$

2. Since no external torque acts, the angular momentum is conserved.

3. Torque, τ = 31.4 N-m. ; Angular acceleration, $\alpha = 4\pi$ rad/sec^2
Now $\tau = I\ \alpha$

$\therefore$ Moment of inertia, $I = \frac{\tau}{\alpha} = \frac{31.4}{4\pi} = 2.5$ kg m^2

4. The moment of inertia of a thin circular disc about an axis passing through the centre of the disc and perpendicular to the plane of the disc is (See Fig. 14.34)

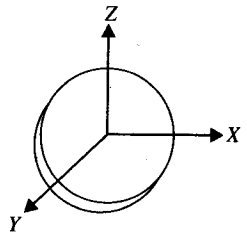


Fig. 14.34

$$I_Z = \frac{1}{2} M R^2$$

From theorem of perpendicular axes,we have,

$$I_X = I_Y = \frac{I_Z}{2} = \frac{1}{2}\left(\frac{1}{2} MR^2\right) = \frac{MR^2}{4}$$

5. Angular momentum, $L = m_1 v_1 r_1 + m_2 v_2 r_2 + \ldots\ldots\ldots\ldots$

$$= m_1 (\omega r_1) r_1 + m_2 (\omega r_2) r_2 + \ldots\ldots$$

$$= m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots\ldots\ldots$$

$$= \left(\sum m r^2\right) \omega$$

But $\sum m r^2 = I$, the moment of inertia of the body about a given axis of rotation.

$\therefore \quad L = I\,\omega$

6. The moment of inertia I of a thin circular disc about an axis perpendicular to the plane of the disc and passing through its centre is

$$I = \frac{1}{2} MR^2 = \frac{1}{2} \times 0.4 \times \left(\frac{100}{100}\right)^2 = 0.2 \text{ kg m}^2$$

7. The acceleration 'a' of a cylinder rolling down an inclined plane without slipping is given by ;

$$a = \frac{Mg \sin\theta}{M + \frac{I}{R^2}}$$

Here I is the moment of inertia of the cylinder about an axis through the centre of mass and perpendicular to the plane of motion. It is clear that acceleration of that cylinder will be more that has smaller moment of inertia.

For a solid cylinder, $I_{solid} = \frac{1}{2} MR^2$; For a hollow cylinder, $I_{hollow} = MR^2$

Since the acceleration of solid cylinder is more than that of hollow cylinder, the solid cylinder will reach the ground earlier.

8. Initial angular velocity = ω ; Initial mass = M

If ω_1 is the final velocity, then according to principle of conservation of angular momentum,

$$I\omega = I_1\,\omega_1$$

or $$MR^2\omega = (M+2m)R^2\omega_1 \qquad \therefore\ \omega_1 = \frac{\omega M}{M+2m}$$

9. M.I. of the ball, $I = \frac{2}{5}MR^2$

Rotational K.E. of ball, $K_1 = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\frac{2}{5}MR^2\times\omega^2 = \frac{1}{5}M\,R^2\omega^2$

Translational K.E. of ball, $K_2 = \frac{1}{2}Mv^2 = \frac{1}{2}M(\omega R)^2 = \frac{1}{2}M\,\omega^2R^2$

$\therefore$ Total K.E. of ball, $K = K_1 + K_2 = \frac{1}{5}MR^2\omega^2 + \frac{1}{2}M\,\omega^2R^2 = \frac{7}{10}M\,R^2\omega^2$

$\therefore$ $$\frac{K_1}{K} = \frac{(1/5)M\,R^2\omega^2}{(7/10)M\,R^2\omega^2} = \frac{2}{7}$$

10. The acceleration 'a' down the inclined plane for pure rolling of these objects is given by ;

$$a = \frac{Mg\sin\theta}{M + \frac{I}{R^2}}$$

Here I is the moment of inertia of the body about an axis through the centre of mass and perpendicular to the plane of motion. It is clear that acceleration of that object will be the greatest that has the smallest moment of inertia.

For solid sphere, $I_1 = \frac{2}{5}MR^2$; For hollow sphere, $I_2 = \frac{2}{3}MR^2$

For solid cylinder, $I_3 = \frac{1}{2}MR^2$

Since the acceleration of **solid sphere** is greatest, it will take the least time to reach the bottom of the inclined plane.

11. The linear acceleration 'a' of the ring is given by ;

$$a = \frac{Mg\sin\theta}{M + \frac{I}{R^2}}$$

Here $I = MR^2$; $\theta = 30°$ $\qquad \therefore\ a = \frac{Mg\sin 30°}{M + \frac{MR^2}{R^2}} = \frac{g}{2}\sin 30° = \frac{g}{4}$

12. Total K.E. $= \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\times\frac{2}{5}mr^2\times\left(\frac{v}{r}\right)^2 + \frac{1}{2}mv^2$

$$= \frac{1}{5}mv^2 + \frac{1}{2}mv^2 = \frac{7}{10}mv^2$$

Now P.E. = Total K.E.

or $$mgh = \frac{7}{10}mv^2 \qquad \therefore\ v = \sqrt{10\frac{gh}{7}}$$

13. Rotational K.E., $E = \frac{1}{2} I \omega^2$

$\therefore \quad 2EI = I^2\omega^2$ or $I\omega = \sqrt{2EI}$ or $L = \sqrt{2EI}$

14. Rotational K.E. $= \frac{1}{2} I\omega^2$

or $360 = \frac{1}{2} I \times (30)^2 \quad \therefore \quad I = \frac{2 \times 360}{(30)^2} = 0.8 \text{ kgm}^2$

15. When sand is poured on a rotating disc, its mass and hence moment of inertia (I) increases. Since angular momentum ($= I\omega$) is conserved, ω (angular velocity) decreases.

16. $\frac{I_1}{I_2} = \frac{M_1 R_1^2}{M_2 R_2^2}$ or $\frac{2}{1} = \frac{M_1}{M_2} \times \left(\frac{2}{1}\right)^2$

$\therefore \quad \frac{M_1}{M_2} = \frac{2}{1} \times \frac{1}{4} = \frac{1}{2}$ or $M_1 : M_2 = 1:2$

17. $\frac{\text{K.E. of rotation}}{\text{K.E. of translation}} = \frac{40}{100} = \frac{2}{5}$ or $\frac{1/2\, I\omega^2}{1/2\, Mv^2} = \frac{2}{5}$

or $\frac{I\omega^2}{M\omega^2 R^2} = \frac{2}{5} \quad \therefore \quad I = \frac{2}{5} MR^2$

This is the moment of inertia of a solid sphere.

18. M.I. of uniform circular disc about its diameter = I. According to theorem of perpendicular axes, its M.I. about its axis is $2I\left(= \frac{1}{2} MR^2\right)$. According to theorem of parallel axes, its M.I. about the given axes $= 2I + MR^2 = 2I + 4I = 6I$

19. Since $m_1 = m_2 \quad \therefore \quad \frac{\pi}{4} \times R_1^2 \times t \times d_1 = \frac{\pi}{4} R_2^2 \times t \times d_2$

$\therefore \quad \frac{R_1^2}{R_2^2} = \frac{d_2}{d_1}$. Now $\frac{I_1}{I_2} = \frac{(1/2) MR_1^2}{(1/2) MR_2^2} = \frac{R_1^2}{R_2^2} = \frac{d_2}{d_1}$

$\therefore \quad I_1 : I_2 = d_2 : d_1$

20. A couple produces only rotational motion.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A 2 kg mass is rotating on a circular path of radius 0.8 m with angular velocity of 44 rad/sec. If the radius of the path becomes 1 m, then angular velocity will be **[CET Karnataka 2000]**

(a) 8.12 rad/sec (b) 19.28 rad/sec

(c) 28.16 rad/sec (d) 35.26 rad/sec

2. The moment of inertia of a solid cylinder of mass M and radius R about a line parallel to the axis of the cylinder but lying on the surface of the cylinder is **[MP PET 1996]**

(a) $\frac{2}{5} MR^2$ (b) $\frac{3}{5} MR^2$

(c) $\frac{3}{2} MR^2$ (d) $\frac{5}{2} MR^2$

3. A person stands on a rotaing platform has his hands lowered. He suddenly outstretches his arms. The angular velocity **[CEET 1998]**

(a) becomes zero (b) decreases

(c) increases (d) remains same

4. The moment of inertia of a body about a given axis is 1.2 kg m^2. To produce a rotational kinetic energy of 1500 J, an angular acceleration of 25 rad/sec^2 must be applied about that axis for the duration of **[CET Karnataka 1998]**

(a) 4 sec (b) 8 sec

(c) 10 sec (d) 2 sec

5. A wheel of moment of inertia 5×10^{-3} kg m^2 is making 20 revolutions per sec. It is stopped in 20 seconds. The angular retardation is **[CET Haryana 1999]**

(a) 2π rad/sec^2 (b) 8π rad/sec^2

(c) 4π rad/sec^2 (d) π rad/sec^2

6. A thin wire of length L and uniform linear density ρ is bent into a circular loop with centre at O as shown in Fig. 14.35. The moment of inertia of the loop about the axis XX' is **[IIT 2000]**

(a) $\frac{\rho L^3}{8\pi^2}$ (b) $\frac{\rho L^3}{16\pi^2}$

(c) $\frac{5\rho L^3}{16\pi^2}$ (d) $\frac{3\rho L^3}{8\pi^2}$

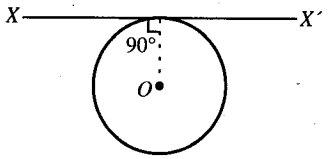


Fig. 14.35

7. A ring and a disc have the same mass and radius. The ratio of their moment of inertia about their axes is **[Pb CET 1995]**

(a) 1 : 1 (b) 2 : 1

(c) 4 : 1 (d) 1 : 2

8. One quarter sector is cut from a uniform circular disc of radius R. The sector has mass M. It is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc [See Fig. 14.36]. Its moment of inertia about the axis of rotation is **[IIT 2001]**

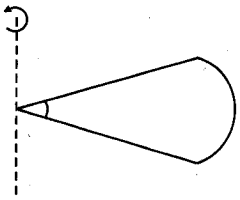


Fig. 14.36

(a) $\frac{1}{2}MR^2$ (b) $\frac{1}{4}MR^2$

(c) $\frac{1}{8}MR^2$ (d) $\sqrt{2}\,MR^2$

9. Three point masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a. The moment of inertia of the system about an axis along altitude of the triangle passing through m_1 is **[Haryana CET 1996]**

(a) $(m_1+m_2)\frac{a^2}{4}$ (b) $(m_1+m_2+m_3)a^2$

(c) $(m_1+m_2)\frac{a^2}{2}$ (d) $(m_2+m_3)\frac{a^2}{4}$

10. The least coefficient of friction for an inclined plane inclined at angle α with horizontal in order that a solid cylinder will roll down it without slipping is **[Pb CET 1998]**

(a) $\frac{2}{3}\tan\alpha$ (b) $\frac{2}{7}\tan\alpha$

(c) $\frac{1}{3}\tan\alpha$ (d) $\frac{5}{7}\tan\alpha$

11. The total K.E. of a sphere of mass M rolling with velocity V is **[AMU 1999]**

(a) $\frac{5}{6}MV^2$ (b) $\frac{7}{10}MV^2$

(c) $\frac{7}{5}MV^2$ (d) $\frac{10}{7}MV^2$

12. A rigid body is rotating about an axis. To stop the rotation, we have to apply **[Aligarh 2000]**

(a) pressure (b) force

(c) momentum (d) torque

13. The radius of gyration of a uniform rod of length L about an axis passing through its centre of mass is **[AMU 1999]**

(a) $L/\sqrt{12}$ (b) $L^2/12$

(c) $L/\sqrt{3}$ (d) $L/\sqrt{2}$

14. If the moment of inertia of a wheel having radius of gyration 60 cm is 360 kg m^2, then mass of the wheel is **[Karnataka CET 1997]**

(a) 200 kg (b) 500 kg

(c) 1000 kg (d) 800 kg

15. A solid cylinder of mass M and radius R has length L. Its moment of inertia about a generator is **[Rajasthan CET 2000]**

(a) $M\left(\frac{L^2}{12}+\frac{R^2}{4}\right)$ (b) $M\left(\frac{L^2}{3}+\frac{R^2}{4}\right)$

(c) $\frac{1}{2}MR^2$ (d) $\frac{3}{2}MR^2$

16. A uniform hollow cylinder of density ρ has length L, inner radius a and outer radius b. Its moment of inertia about the axis of the cylinder is **[CET Haryana 1993]**

(a) $2\pi\rho L(b^2+a^2)$ (b) $\frac{2\pi\rho L(b^4-a^4)}{4}$

(c) $\frac{2\pi\rho L(b^2-a^2)}{4}$ (d) $\frac{2\pi\rho L(b^4+a^4)}{2}$

17. A mass m is moving with a constant velocity along a line parallel to X-axis away from the origin. Its angular momentum *w.r.t.* origin **[IIT 1997]**

(a) is zero (b) is constant

(c) goes on decreasing

(d) goes on increasing

18. A wheel is rotating at 900 r.p.m. about its axis. When power is cut off, it comes to rest in 1minute. The angular retardation in rad/sec^2 is **[MP PET 1999]**

(a) $\pi/2$ (b) $\pi/4$

(c) $\pi/6$ (d) $\pi/8$

19. A dancer on ice spins faster when she folds hear arms. This is due to **[CET Haryana 1995]**

(a) decrease in M.I.

(b) increase in M.I.

(c) decrease in friction

(d) none of above

20. A thin hollow cylinder open at both ends (*i*) slides without rotating (*ii*) rolls with the same speed without slipping. The ratio of kinetic energies of the cylinder in the two cases is **[CET Haryana 1998]**

(a) 2 : 1 (b) 1 : 2

(c) 4 : 1 (d) 1 : 1

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (c)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (a)
6. (d)	**7.** (b)	**8.** (a)	**9.** (d)	**10.** (c)
11. (b)	**12.** (d)	**13.** (a)	**14.** (c)	**15.** (d)
16. (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (b)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. M.I. for the first case is $I_1 = MR_1^2 = (2)\times(0.8)^2 = 1.28$ kg m^2

M.I. for the second case is $I_2 = MR_2^2 = (2)\times(1)^2 = 2$ kg m^2

According to the law of conservation of angular momentum,

$$I_1\omega_1 = I_2\omega_2 \;\therefore\; \omega_2 = \frac{I_1\omega_1}{I_2} = \frac{1.28\times 44}{2} = 28.16 \text{ rad/sec}$$

2. M.I. of a solid cylinder about the central axis = $\frac{1}{2}MR^2$. Applying theorem of parallel axes, the M.I. about the given axis $= \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$

3. When the person outstretches his arms, moment of inertia (I) increases. As the angular momentum L ($= I\omega$) remains the same, therefore, angular velocity will decrease.

4. $I = 1.2$ kg m^2 ; $\alpha = 25$ rad/sec^2

$$\text{Rotational K.E.} = \frac{1}{2}I\omega^2 \text{ or } 1500 = \frac{1}{2}\times 1.2\times\omega^2$$

or
$$\omega^2 = \frac{2\times 1500}{1.2} = 2500 \;\therefore\; \omega = 50 \text{ rad/sec}$$

15. Generator is an axis touching the surface of cylinder and parallel to the axis of the cylinder. Therefore, using theorem of parallel axes, the M.I. of the cylinder about the given axis is

$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

16. M.I. of a shell of thickness dr and radius r is

$$dI = (\rho\, 2\,\pi r\, L dr) r^2$$

$$\therefore \quad I = \int_a^b dI = \int_a^b \rho\, 2\pi r\, L dr\, r^2 = 2\pi\rho L \int_a^b r^3 dr$$

$$= \frac{2\pi\rho L(b^4 - a^4)}{4}$$

17. Angular momentum = mvr

As m, v and r remain the same, the angular momentum is constant.

18. Angular retardation, $\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{2\pi(n_2 - n_1)}{t}$

$$= \frac{2\pi(0 - 900/60)}{60} = \frac{\pi}{2}\text{rad/sec}^2$$

Negative sign is omitted because only magnitude is required.

19. When the dancer folds her arms, **moment of inertia (M.I.) decreases**. Since angular momentum L $(=I\ \omega)$ remains the same, ω increases.

20. Let M be the mass and V be the speed of the cylinder.

K.E. of cylinder in case (i) is

$$K_1 = \frac{1}{2}MV^2$$

K.E. of cylinder in case (ii) is

$$K_2 = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MV^2 + \frac{1}{2}(MR^2)\omega^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}MV^2 = MV^2$$

$$\therefore \quad \frac{K_1}{K_2} = \frac{(1/2)MV^2}{MV^2} = 1 : 2$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. Four point masses shown in Fig. 14.37 are connected by a rod of negligible mass. Find the moment of inertia and radius of gyration for the system about (i) axis AA' (ii) axis BB'.

Hint. (i) M.I. of the system about the axis AA' is

$$I_{AA'} = \Sigma\, m_i r_i^2$$
$$= 0 + 2m\,(b^2) + m(4b^2) + 3m\,(9b^2)$$
$$= 33\, mb^2$$

Required duration of time is $t = \frac{\omega}{\alpha} = \frac{50}{25} = \mathbf{2\ sec}$

5. Angular retardation $= \frac{\text{Angular Velocity}}{\text{Time}} = \omega/t = 2\pi \times 20/20 = \mathbf{2\ \pi\ rad/sec^2}$

6. Mass of loop, $M = \rho L$; Radius of loop, $R = \frac{1}{2\pi}$

According to theorem of parallel axes, M.I. of loop about XX' axis is

$$I_{XX'} = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$
$$= \frac{3}{2} \times \rho\ L \times \frac{L^2}{4\pi^2} = \mathbf{\frac{3\rho L^3}{8\pi^2}}$$

7. $$\frac{I_{ring}}{I_{disc}} = \frac{MR^2}{\frac{1}{2}MR^2} = \mathbf{2:1}$$

8. The mass of complete disc $= 4M$ and its moment of inertia about the given axis $= \frac{1}{2}(4M)R^2 = 2MR^2$. By symmetry, for the given quarter disc,

$$\text{Moment of inertia} = \frac{1}{4} \times 2MR^2 = \frac{1}{2}MR^2$$

9. Perpendicular distance of m_2 and m_3 from the altitude of triangle passing through $m_1 = a \cos 60° = a/2$

$\therefore$ M.I. of the system about the given axis is

$$I = m_1 \times 0^2 + m_2 (a/2)^2 + m_3 (a/2)^2 = (m_2 + m_3)\frac{a^2}{4}$$

10. Friction must be present to make the cylinder roll. To prevent slipping, the coefficient of static friction must be at least as great as

$$\mu_s = \frac{f_s}{R} = \frac{\frac{1}{3}Mg \sin\alpha}{Mg \cos\alpha} = \mathbf{\frac{1}{3}\tan\alpha}$$

11. Total K.E. of a rolling sphere is

$$\text{K.E.}_{total} = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MV^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\omega^2$$
$$= \frac{1}{2}MV^2 + \frac{1}{5}MV^2 = \mathbf{\frac{7}{10}MV^2}$$

12. In order to stop the rotation of a rotating body, we will have to apply an **opposing torque.**

13. The M.I. of uniform rod about the given axis is

$$I = \frac{ML^2}{12}. \quad \text{Also } I = MK^2$$

$\therefore$ $$MK^2 = \frac{ML^2}{12} \text{ or } K = \mathbf{\frac{L}{\sqrt{12}}}$$

14. $K = 60$ cm $= 0.6$ m ; $I = 360$ kg m^2

Now $$I = MK^2 \quad \therefore \quad M = \frac{I}{K^2} = \frac{360}{(0.6)^2} = \mathbf{1000\ kg}$$

15. Generator is an axis touching the surface of cylinder and parallel to the axis of the cylinder. Therefore, using theorem of parallel axes, the M.I. of the cylinder about the given axis is

$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

16. M.I. of a shell of thickness dr and radius r is

$$dI = (\rho\, 2\,\pi r\, Ldr)r^2$$

$$\therefore \quad I = \int_a^b dI = \int_a^b \rho\, 2\pi r\, Ldr\, r^2 = 2\pi\rho L \int_a^b r^3 dr$$

$$= \frac{2\pi\rho L\left(b^4 - a^4\right)}{4}$$

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20. Let M be the mass and V be the speed of the cylinder.
K.E. of cylinder in case (i) is

$$K_1 = \frac{1}{2}MV^2$$

K.E. of cylinder in case (ii) is

$$K_2 = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MV^2 + \frac{1}{2}\left(MR^2\right)\omega^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}MV^2 = MV^2$$

$$\therefore \quad \frac{K_1}{K_2} = \frac{(1/2)MV^2}{MV^2} = 1 : 2$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. Four point masses shown in Fig. 14.37 are connected by a rod of negligible mass. Find the moment of inertia and radius of gyration for the system about (i) axis AA' (ii) axis BB'.

Hint. (i) M.I. of the system about the axis AA' is

$$I_{AA'} = \Sigma\, m_i r_i^2$$
$$= 0 + 2m\,(b^2) + m(4b^2) + 3m\,(9b^2)$$
$$= 33\ mb^2$$

If K is the radius of gyration, then,

$I_{AA'} = MK^2$

or $K = \sqrt{\frac{I_{AA'}}{M}} = \sqrt{\frac{33mb^2}{7m}} = 2.17\ b$

(*ii*) M.I. of the system about the axis BB' is

$I_{BB'} = m\ (4b^2) + 2m\ (b^2) + 0 + 3m\ (b^2) = 9mb^2$

$K = \sqrt{\frac{I_{BB'}}{M}} = \sqrt{\frac{9mb^2}{7m}} = 1.13\ b$

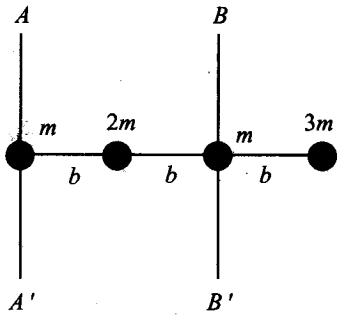

Fig. 14.37

2. An equilateral triangle is formed by three identical uniform rods each of length 40 cm and mass 2 kg. Find the moment of inertia of this system about an axis passing through one corner and perpendicular to the plane of the triangle.

Hint. Figure 14.38 shows the conditions of the problem.

Mass of each rod, M = 2 kg

Length of each rod, l = 0.4 m

Let us find M.I. of the system about an axis passing through one corner (say corner A).

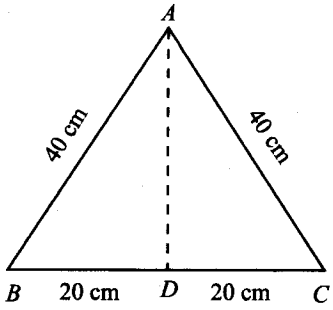

Fig. 14.38

M.I. of AB about an axis through A is

$$I_1 = \frac{Ml^2}{3}$$

M.I. of AC about an axis through A is

$$I_2 = \frac{Ml^2}{3}$$

M.I. of BC about an axis through A (using theorem of parallel axes) is

$$I_3 = I_{CM} + M \times AD^2 = \frac{Ml^2}{12} + M\left(\frac{\sqrt{3}}{2}l\right)^2 = \left(\frac{Ml^2}{12} + \frac{3Ml^2}{4}\right)$$

$\therefore$ M.I.of the system about an axis passing through A is

$$I = I_1 + I_2 + I_3 = \frac{Ml^2}{3} + \frac{Ml^2}{3} + \left(\frac{Ml^2}{12} + \frac{3Ml^2}{4}\right)$$

$$= \frac{3Ml^2}{2} = \frac{3 \times 2 \times (0.4)^2}{2} = 0.48 \text{ kg m}^2$$

3. A mass of 10 kg connected to the end of a rod of negligible mass is rotating in a circle of radius 30 cm with an angular velocity of 10 rad/s. If this mass is brought to rest in 10 s, what is the torque applied ?

Hint. M.I. of the mass about the symmetry axis of rotation is

$$I = MR^2 = 10 \times (0.3)^2 = 0.9 \text{ kg m}^2$$

Angular retardation, $\alpha = \frac{\Delta\omega}{\Delta t} = \frac{10}{10} = 1 \text{ rad/s}^2$

$\therefore$ Applied torque, $\tau = I\alpha = 0.9 \times 1 = 0.9$ Nm

4. A disc of mass 2 kg and radius 20 cm is free to rotate about an axis through its centre and perpendicular to the disc. If a force of 50 N is applied tangentially to the disc, calculate the angular acceleration.

Hint. M.I. of the disc about the given axis of rotation is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}\times 2\times\left(20\times10^{-2}\right)^2 = 4\times10^{-2}\text{ kg m}^2$$

$$\text{Torque, } \tau = F\times R = 50\times 20\times 10^{-2} = 10\text{ Nm}$$

$$\therefore \quad \text{Angular acceleration, } \alpha = \frac{\tau}{I} = \frac{10}{4\times10^{-2}} = \mathbf{250\ rad/s^2}$$

5. A grindstone has a moment of inertia of 1.6 $\times10^{-3}$ kg m^2. When a constant torque is applied, the flywheel reaches an angular velocity of 1200 rev./minute in 15 s. Assuming it started from rest, find (*i*) the angular acceleration (*ii*) the torque applied (*iii*) the angle turned through in 15 s (*iv*) the work done on the flywheel by the torque.

Hint. $\omega_0 = 0$; ω = 1200 r.p.m. = 20 r.p.s.= $2\pi\times 20 = 40\pi$ rad/s ; t= 15 s

(*i*) Angular acceleration, $\alpha = \dfrac{\omega-\omega_0}{t} = \dfrac{40\pi-0}{15} = \mathbf{8.38\ rad/s^2}$

(*ii*) Torque applied, $\tau = I\,\alpha = (1.6\times10^{-3})\times 8.38 = \mathbf{0.0134\ Nm}$

(*iii*) Average angular velocity, $\bar{\omega} = \dfrac{\omega_0+\omega}{2} = \dfrac{0+40\pi}{2} = 20\pi$ rad/s^2

$\therefore$ Angle turned, $\theta = \bar{\omega}\times t = 20\pi\times15 = \mathbf{942\ rad}$

(*iv*) Work done by torque, $W = \tau\theta = 0.0134\times942 = \mathbf{12.6\ J}$

Alternatively, according to work-energy principle,

Work done, W = Increase in K.E. $= \dfrac{1}{2}I\omega^2 - \dfrac{1}{2}I\,\omega_0^2$

$$= \frac{I}{2}\left(\omega^2-\omega_0^{\,2}\right) = \frac{1.6\times10^{-3}}{2}\left[(40\pi)^2-(0)^2\right] = \mathbf{12.6\ J}$$

6. Find (*i*) the angular momentum (*ii*) rotational kinetic energy of earth due to its daily rotation about its own axis. Assume earth to be a uniform sphere. Mass of earth, M = 5.98 $\times10^{24}$ kg and radius of earth = 6.37 $\times10^6$ m.

Hint. M.I. of earth about its axis of rotation is

$$I = \frac{2}{5}MR^2 = \frac{2}{5}\times(5.98\times10^{24})\times(6.37\times10^6)^2 = 9.71\times10^{37}\text{ kg m}^2$$

Angular velocity of earth is given by ;

$$\omega = \frac{1\text{ revolution}}{\text{seconds in a day}} = \frac{2\pi}{T} = \frac{2\pi}{24\times60\times60} = 7.27\times10^{-5}\text{ rad/s}$$

(*i*) Angular momentum of earth about the axis of rotation is

$$L = I\omega = (9.71\times10^{37})\times(7.27\times10^{-5}) = \mathbf{7.06\times10^{33}\ kg\ m^2s^{-1}}$$

(*ii*) Rotational K.E. $= \dfrac{1}{2}I\omega^2 = \dfrac{1}{2}\times(9.71\times10^{37})\times(7.27\times10^{-5})^2 = \mathbf{2.56\times10^{29}\ J}$

7. A cylinder is released from rest from the top of an incline of inclination θ and length l. If the cylinder rolls without slipping, what will be its speed when it reaches the bottom ?

Hint. Let v be the linear speed of the cylinder when it reaches the bottom. Since the cylinder rolls without slipping, its angular speed about its axis will be $\omega = v/r$.

Total K.E. at the bottom of the incline

$$= \frac{1}{2} M v^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} M v^2 + \frac{1}{2} \times \left(\frac{1}{2} M R^2\right)\left(\frac{v}{R}\right)^2 \qquad \left(\because I = \frac{1}{2} MR^2\right)$$

$$= \frac{1}{2} M v^2 + \frac{M v^2}{4} = \frac{3}{4} M v^2$$

Gain in K.E. = Loss of P.E. at the bottom

or $\frac{3}{4} M v^2 = M g l \sin\theta$ or $v = \sqrt{\frac{4}{3} g l \sin\theta}$

8. A sphere, a disc and a ring of the same radius are allowed to roll down on an inclined plane simultaneously from the same height without slipping. Give the order in which they reach the bottom of the plane.

Hint. For a body rolling down an inclined plane, the acceleration is given by;

$$a = \frac{g \sin\theta}{1 + \frac{K^2}{R^2}}, \qquad \text{where } K = \text{radius of gyration}$$

For a sphere, $K^2 = \frac{2}{5} R^2$ or $\frac{K^2}{R^2} = \frac{2}{5} = 0.4$ $\quad \therefore \quad a_1 = \frac{g \sin\theta}{1+0.4} = \frac{g \sin\theta}{1.4}$

For a disc, $K^2 = \frac{R^2}{2}$ or $\frac{K^2}{R^2} = \frac{1}{2} = 0.5$ $\quad \therefore \quad a_2 = \frac{g \sin\theta}{1+0.5} = \frac{g \sin\theta}{1.5}$

For a ring, $K^2 = R^2$ or $\frac{K^2}{R^2} = 1$ $\quad \therefore \quad a_3 = \frac{g \sin\theta}{1+1} = \frac{g \sin\theta}{2}$

From above results, it is clear that $a_1 > a_2 > a_3$. Therefore, the order in which they reach the bottom of the plane is **sphere, disc and ring.**

9. A thin hollow cylinder open at both ends and of mass 10 kg (*i*) slides with a speed of 10 ms^{-1} without rotating (*ii*) rolls with at a speed of 10 ms^{-1} without slipping. Calculate the kinetic energy in each case.

Hint. Here M = 10 kg ; v = 10ms^{-1}

(*i*) Kinetic energy = $\frac{1}{2} Mv^2 = \frac{1}{2} \times 10 \times (10)^2 = 500$ J

(*ii*) Kinetic energy = $\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2$

In case of a hollow cylinder, $I = MR^2$.

$\therefore$ K.E. = $\frac{1}{2} Mv^2 + \frac{1}{2} (MR)^2 \times \frac{v^2}{R^2}$

$$= \frac{1}{2} Mv^2 + \frac{1}{2} Mv^2 = Mv^2 = 10 \times (10)^2 = 1000 \text{ J}$$

Now $\vec{a} \cdot (\vec{b} \times \vec{c}) = a\,(bc) \cos 0° = abc$

Now abc is the volume of the parallelopiped.

$\therefore$ $\vec{a} \cdot (\vec{b} \times \vec{c})$ = Volume of parallelopiped in Fig. 7.02.

Q.6. Find the components along the x, y, z axes of the angular moment $\vec{L}$ of a particle, whose position vector is $\vec{r}$ with components x, y, z and momentum is $\vec{p}$ with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane, the angular momentum has only a z-component.

Ans. Refer to Art. 13.31.

Q.7. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two- particle system is the same whatever be the point about which the angular momentum is taken.

Ans. Fig. 7.03 shows the conditions of the problem. The two particles A and B, each of mass m move with speed v in opposite directions along the parallel lines X_1Y_1 and X_2Y_2 separated by a distance $AB = d$. Consider any point C on the line AB such that $AC = x$. Therefore, vector angular momentum of the two–particle system about point C is

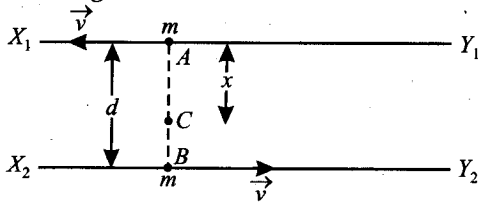

Fig. 7.03

$$\vec{L}_C = m\vec{v}(x) + m\vec{v}(d - x) = m\vec{v}\,d$$

Note that vector angular momentum of the two particles about point C is independent of x. Therefore, the vector angular momentum of the two particle–system is the same whatever may be the location of point C on the line AB about which the angular momentum is taken.

Q.8. A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig. 7.04. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.

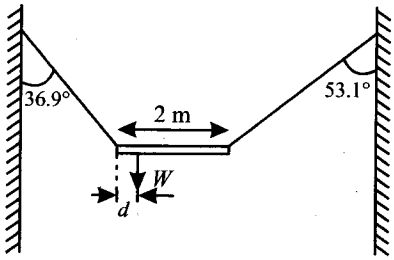

Fig. 7.04

Ans. Let T_1 and T_2 be the tensions in the two strings as shown in Fig. 7.05. The tensions in the two strings are resolved into rectangular components. Here $\theta_1 = 36.9°$ and $\theta_2 = 53.1°$.

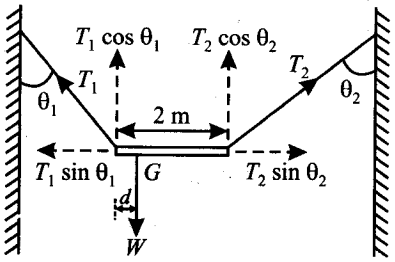

Fig. 7.05

For equilibrium of the rod along the horizontal, we have,

$$T_1 \sin \theta_1 = T_2 \sin \theta_2$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{\sin \theta_2}{\sin \theta_1} = \frac{\sin 53.1°}{\sin 36.9°} = 1.332$$

Let d be the distance of centre of gravity G of the bar from the left end. For the rotational equilibrium of the bar, the algebraic sum of moments of all the forces about point G should be zero *i.e.*,

$$T_1 \cos \theta_1 \times d - T_2 \cos \theta_2 \times (2 - d) = 0$$

or $$1.332\, T_2 \cos 36.9° \times d - T_2 \cos 53.1° \times (2 - d) = 0$$

or $$1.332 \times 0.8 \times d - 0.6 \times (2 - d) = 0$$

On solving, we have, $d = 0.721$ m

Q.9. A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Ans. Fig. 7.06 shows the conditions of the problem. Here A and B are the front and back axles of the car respectively and mg is the weight of the car acting vertically downward at its centre of gravity G. It is given that mass of the car is m = 1800 kg. Let R_1 be the force exerted by the level ground on front wheel and R_2 on the back wheel.

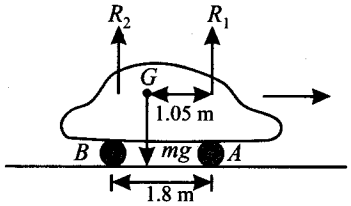

Fig. 7.06

For equilibrium of the car along the vertical, we have,

$$R_1 + R_2 = mg = 1800 \times 9.8$$

or $$R_1 + R_2 = 17640 \quad ...(i)$$

For rotational equilibrium of the car, the moments of all the forces about point G should be zero *i.e.*,

$$R_1 \times 1.05 - R_2 \times (1.8 - 1.05) = 0$$

or $$1.05\ R_1 - 0.75\ R_2 = 0 \quad ...(ii)$$

From eqs. (*i*) and (*ii*), we have, R_1 = **7350 N** ; R_2 = **10290 N**

Q.10. (*a*) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $2MR^2/5$, where M is the mass of the sphere and R is the radius of the sphere.

(*b*) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

Ans. (*a*) M.I. of sphere about any diameter = $\frac{2}{5}MR^2$

Applying theorem of parallel axes, we have,

M.I. of sphere about a tangent to sphere = $\frac{2}{5}MR^2 + M(R^2) = \frac{7}{5}MR^2$

(*b*) M.I. of disc about any of its diameter = $\frac{1}{4}MR^2$

Applying theorem of perpendicular axes, the M.I. of the disc about an axis passing through its centre and normal to the disc = $\frac{1}{4}MR^2 + \frac{1}{4}MR^2 = \frac{1}{2}MR^2$.

Again applying theorem of parallel axes, the M.I. of the disc passing through a point on its edge and normal to the disc is

$$I_{edg} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Q.11. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Ans. Let M be the mass and R the radius of the hollow cylinder and the solid sphere. Then,

M.I. of hollow cylinder about its axis of symmetry, $I_1 = MR^2$

M.I. of solid sphere about an axis through its centre, $I_2 = \frac{2}{5}MR^2$

Due to the applied torques, let α_1 be the angular acceleration produced in the hollow cylinder and α_2 in the solid sphere. Since the magnitude of applied torques are equal,

$$\therefore \quad I_1\alpha_1 = I_2\alpha_2 \text{ or } \frac{\alpha_2}{\alpha_1} = \frac{I_1}{I_2} = \frac{MR^2}{(2/5)\,MR^2} = \frac{5}{2} = 2.5$$

Therefore, $\alpha_2 = 2.5\ \alpha_1$. Since angular acceleration (α_2) produced in solid sphere is more, the sphere will acquire a greater angular speed in a given time ($\omega = \omega_0 + \alpha t$).

Q.12. A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of the angular momentum of the cylinder about its axis?

Ans. M = 20 kg ; ω = 100 s^{-1}; R = 0.25 m

The moment of inertia of the cylinder about given axis is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(20)\times(0.25)^2 = 0.625 \text{ kg m}^2$$

$\therefore$ K.E. associated with the rotation of the cylinder is

$$K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.625)\times(100)^2 = \mathbf{3125\ J}$$

Magnitude of angular momentum of the cylinder about its axis is

$$L = I\omega = (0.625)\times(100) = \mathbf{62.5\ kg\ m^2 s^{-1}}$$

Q.13. (*a*) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 r.p.m. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that turntable rotates without friction. (*b*) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

Ans. (*a*) Initial M.I., = I_1; Final M.I., I_2 = 2/5 I_1; Initial angular speed, ω_1 = 40 r.p.m.; Final angular speed, ω_2 = ?.

According to principle of conservation of angular momentum,

$$I_1\omega_1 = I_2\omega_2 \text{ or } \omega_2 = \frac{I_1\omega_1}{I_2} = \frac{I_1\times 40}{(2/5)I_1} = \mathbf{100\ r.p.m.}$$

(*b*) Initial K.E. of rotation, $E_1 = \frac{1}{2}I_1\omega_1^2$; Final K.E. of rotation, $E_2 = \frac{1}{2}I_2\omega_2^2$

$$\therefore \quad \frac{E_2}{E_1} = \frac{1/2\, I_2\, \omega_2^2}{1/2\, I_1\, \omega_1^2} = \left(\frac{I_2}{I_1}\right)\left(\frac{\omega_2}{\omega_1}\right)^2 = \left(\frac{2/5\, I_1}{I_1}\right)\left(\frac{100}{40}\right)^2 = 2.5$$

Since $E_2 = 2.5\ E_1$, the K.E. of rotation increases when the child folds his hands. This increase in K.E. of rotation is due to the fact that child spends energy in folding back his hands.

Q.14. A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

Ans. M= 3 kg ; R= 40 cm = 0.4 m ; T = 30 N

M.I. of the hollow cylinder about its axis of symmetry is

$$I = M R^2 = 3 \times (0.4)^2 = 0.48 \text{ kg m}^2$$

The tension T in the rope exerts torque τ on the cylinder.

$$\therefore \quad \tau = F R = 30 \times 0.4 = 12 \text{ Nm}$$

$\therefore$ Angular acceleration produced in the cylinder is

$$\alpha = \frac{\tau}{I} = \frac{12}{0.48} = \mathbf{25\ rad\ s^{-2}}$$

Linear acceleration of rope, $a = R\alpha = 0.4 \times 25 = \mathbf{10\ ms^{-2}}$

Q.15. To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 Nm. What is the power required by the engine? [Note: Uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque]. Assume that the engine is 100% efficient.

Ans. Here, ω = 200 rad s^{-1}; Torque, τ = 180 Nm ; Power, P = ?

Now $P = \tau\omega = 180 \times 200 = 36000$ W = **36 kW**

Q.16. From a uniform disc of radius R, a circular hole of radius $R/2$ is cut out. The centre of the hole is at $R/2$ from the centre of the original disc. Locate the centre of mass of the resulting flat body.

Ans. Fig. 7.07 shows the conditions of the problem. The centre O' of the hole lies at a distance $R/2$ from the centre O of the uniform disc. Let mass per unit area of the disc be m.

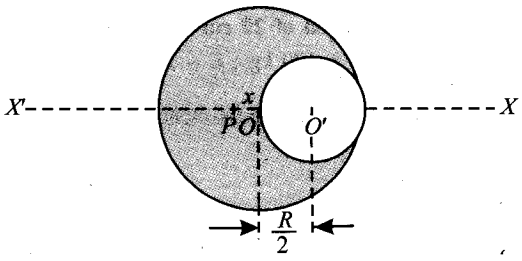

Fig. 7.07

Mass of original disc, $M = \pi R^2 \times m$

Mass of removed portion of disc,

$$M' = \pi(R/2)^2 \times m = \frac{\pi R^2 m}{4} = \frac{M}{4}$$

Let us consider the centre of the disc O as the origin. The masses M and M' $(= M/4)$ can be supposed to be concentrated at the centres of the disc (O) and the hole (O') respectively. Since $M'(= M/4)$ is removed, it will be considered negative. Therefore, after the circular disc of mass M' is removed, the remaining part of the disc can be considered as a system of two masses; M at O and $M'(= -M/4)$ at O'. If x is the distance of centre of mass (P) from O, then,

$$x = \frac{M \times 0 + (-M/4) \times R/2}{M + (-M/4)} = \frac{-MR/8}{3M/4} = -\frac{R}{6}$$

The negative sign shows that centre of mass P of the remaining part of the disc lies to the left of point O.

Q.17. A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 gm are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

Ans. Fig. 7.08 shows the conditions of the problem. Here AB is the metre stick and C is its centre of mass. The weight mg of the metre stick acts at point C (50 cm mark). The stick is balanced at point C' (45 cm mark) and coins of mass 10 gm are placed at point D (12 cm mark).

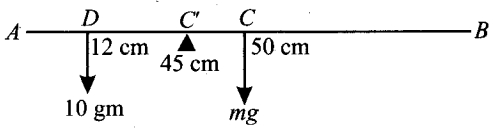

Fig. 7.08

Since the stick is balanced about point C',

$\therefore$ $10 \times g \times (45 - 12) = mg\ (50 - 45)$ or $10 \times (45 - 12) = m(50 - 45)$

$\therefore$ Mass of metre stick, $m =$ **66 gm**

Q.18. A solid sphere rolls down two different inclined planes at the same heights but different angles of inclination. (*a*) Will it reach the bottom with the same speed in each case? (*b*) Will it take longer to roll down one plane than the other? (*c*) If so, which one and why?

Ans. At the top of the inclined plane, the sphere has total energy = P.E. = mgh. As the sphere rolls down the inclined plane, it acquires K.E. of translation as well as K.E. of rotation. Let v $(= \omega r)$ be the linear speed of the solid sphere at the bottom of the incline. According to principle of conservation of energy, we have,

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

Here $I = \frac{2}{5}mr^2$; $\omega = v/r$

$\therefore$ $$\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right) \times \frac{v^2}{r^2} = mgh$$

or $$\frac{1}{2}mv^2 + \frac{1}{5}mv^2 = mgh$$

$\therefore$ $$v = \sqrt{\frac{10}{7}gh}$$

(*a*) Since h is same for both inclined planes, the sphere will reach the bottom with the same speed.

(*b*) As the height is the same, the time taken to reach the bottom will be the same in the two cases.

Q.19. A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?

Ans. Here M = 100 kg ; R = 2 m ; v = 20 cm/s = 0.2 m/s

$$\text{Total energy of hoop, } E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}(MR^2)\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = Mv^2$$

$$\text{Work done to stop hoop} = E = Mv^2 = 100 \times (0.2)^2 = \mathbf{4\ J}$$

Q.20. The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the line joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two–third of its kinetic energy of translation. Find the average angular velocity of the molecule.

Ans. Fig. 7.09 shows the conditions of the problem. The mass of oxygen molecule is m so that each oxygen atom has a mass of $m/2$. The distance between the two atoms is $2r$.

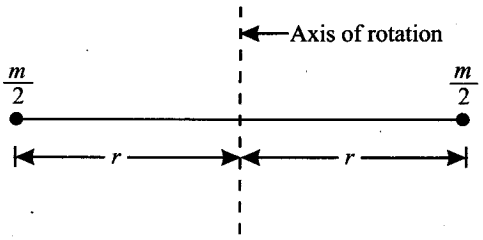

Fig. 7.09

M.I. of oxygen molecule,

$$I = \frac{m}{2} \times r^2 + \frac{m}{2} \times r^2 = mr^2$$

$$\text{or} \quad r = \sqrt{\frac{I}{m}} = \sqrt{\frac{1.94 \times 10^{-46}}{5.30 \times 10^{-26}}} = 0.61 \times 10^{-10} \text{ m}$$

Now K.E. of rotation = $\frac{2}{3}$ K.E. of translation ...given

$$\text{or} \quad \frac{1}{2}I\omega^2 = \frac{2}{3} \times \frac{1}{2}mv^2$$

$$\text{or} \quad \frac{1}{2}(mr^2)\omega^2 = \frac{1}{3}mv^2$$

$$\therefore \quad \omega = \sqrt{\frac{2}{3}} \times \frac{v}{r} = \sqrt{\frac{2}{3}} \times \frac{500}{0.61 \times 10^{-10}} = \mathbf{6.7 \times 10^{12}\ rad/s}$$

Q.21. A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane, the centre of mass of the cylinder has a speed of 5 m/s. (*a*) How far will the cylinder go up the plane? (*b*) How long will it take to return to the bottom?

Ans. Here θ = 30° ; v = 5 m/s ; $I = \frac{1}{2}mr^2$

(*a*) As the cylinder goes up the plane, it acquires potential energy at the expense of its K.E. of translational and rotational motion. Let the cylinder go up the plane upto height h. Then according to principle of conservation of energy, we have,

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

$$\text{or} \quad \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 = mgh$$

$$\text{or} \quad \frac{3}{4}mv^2 = mgh$$

$$\therefore \quad h = \frac{3v^2}{4g} = \frac{3 \times (5)^2}{4 \times 9.8} = 1.913 \text{ m}$$

Suppose that the cylinder covers a distance S along the inclined plane in reaching height h on the plane. Then,

$$\sin\theta = \frac{h}{S} \text{ or } S = \frac{h}{\sin\theta} = \frac{1.913}{\sin 30°} = \mathbf{3.826\ m}$$

(*b*) For a cylinder rolling down an inclined plane (See Art. 14.25),

$$\text{Linear acceleration, } a = \frac{2}{3}g\sin\theta = \frac{2}{3}\times 9.8\times\sin 30° = 3.27\ \text{ms}^{-2}$$

Now $$S = v_0 t + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2$$

$\therefore$ $$t = \sqrt{\frac{2S}{a}} = \mathbf{1.53\ s}$$

Q.22. As shown in Fig. 7.010, the two sides of a step ladder *BA* and *CA* are 1.6 m long and hinged at *A*. A rope *DE*, 0.5 m is tied half way up. A weight 40 kg is suspended from a point *F*, 1.2 m from *B* along the ladder *BA*. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take g = 9.8 m/s^2). [Hint: Consider the equilibrium of each side of the ladder separately.]

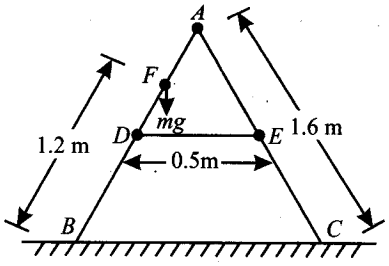

Fig. 7.010

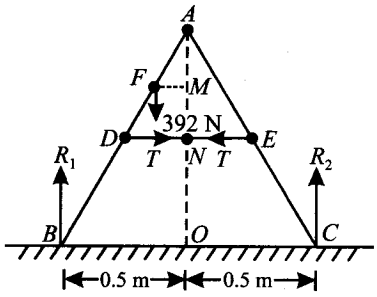

Fig. 7.011

Ans. Fig. 7.011 shows the various forces acting on the different parts of the ladder. Here $mg = 40 \times 9.8 = 392$ N, T is the tension in the rope DE and R_1 and R_2 are the normal reactions at points B and C respectively.

For equilibrium of the ladder along the vertical, we have,

$$R_1 + R_2 = mg \text{ or } R_1 + R_2 = 392 \quad ...(i)$$

For rotational equilibrium of the ladder, the algebraic sum of all the forces about point O should be zero, *i.e.*,

$$R_1 \times 0.5^* - R_2 \times 0.5 - 392 \times {}^{**}FM = 0$$

or $$0.5\,R_1 - 0.5\,R_2 - 392 \times 0.125 = 0$$

$\therefore$ $$R_1 - R_2 = 98 \quad ...(ii)$$

From eqs. (*i*) and (*ii*), R_1 = **245 N** and R_2 = **147 N**

In order to find tension T in the rope DE, consider the rotational equilibrium of side CA of the ladder about point O. For this, we have,

$$R_2 \times 0.5 = T \times ON$$

Now $$ON = \frac{1}{2}OA = \frac{1}{2}\times\sqrt{CA^2 - OC^2} = \frac{1}{2}\times\sqrt{(1.6)^2 - (0.5)^2} = 0.76\ \text{m}$$

$\therefore$ $$147 \times 0.5 = T \times 0.76 \quad \therefore \quad T = \mathbf{96.7\ N}$$

* Since DE is fixed half-way between the two sides of the ladder, $BC = 2DE = 2 \times 0.5 = 1$ m. Therefore, $OB = OC = 0.5$ m.

** $BF = 1.2$ m and F is the mid-point of AD.

$\therefore$ $$FM = \frac{1}{2}DN = \frac{1}{2}\times\frac{DE}{2} = \frac{1}{4}\times DE = \frac{1}{4}\times 0.5 = 0.125\ \text{m}$$

Q.23. **A man stands on a rotating platform, with his arms stretched horizontally holding 5 kg weight in each hand. The angular speed of the platform is 30 r.p.m. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90 cm to 20 cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2. (*a*) What is his new angular speed? (Neglect friction) (*b*) Is kinetic energy conserved in the process? If not, from where does the change come about?**

Ans. (*a*)Initial M.I., $I_1 = 7.6 + 2 \times 5 \times (0.9)^2 = 15.7$ kg m^2; $\omega_1 = 30$ r.p.m.

Final M.I., $I_2 = 7.6 + 2 \times 5 \times (0.2)^2 = 8$ kg m^2; $\omega_2 = ?$

According to principle of conservation of angular momentum, we have,

$$I_1\omega_1 = I_2\omega_2$$

$$\therefore \quad \omega_2 = \frac{I_1\omega_1}{I_2} = \frac{15.7 \times 30}{8} = 58.88 \text{ r.p.m.}$$

(*b*)The kinetic energy is not conserved in the process. The change in energy comes from the work done by the man in bringing the arms closer to his body.

Q.24. **A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it. [Hint: The moment of inertia of the door about the vertical axis at one end is $ML^2/3$.]**

Ans. Mass of bullet, $m = 10$ g $= 0.01$ kg ; Speed of bullet, $v = 500$ ms^{-1}

The distance from the axis where bullet is embedded in the door is

$$r = \frac{\text{Width of door}}{2} = \frac{1\text{ m}}{2} = 0.5 \text{ m}$$

$\therefore$ Angular momentum imparted by the bullet to the door is

$$L = mvr = 0.01 \times 500 \times 0.5 = 2.5 \text{ kg m}^2 \text{ s}^{-1}$$

M.I. of door about the vertical axis is

$$I = \frac{ML^2}{3} = \frac{12 \times 1^2}{3} = 4 \text{ kg m}^2$$

Now $$L = I\omega \quad \therefore \quad \omega = \frac{L}{I} = \frac{2.5}{4} = 0.625 \text{ rad s}^{-1}$$

Q.25. **Two discs of moment of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre) and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (*a*) What is the angular speed of the two-disc system? (*b*) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.**

Ans. (*a*) Total initial angular momentum of two discs, $L_1 = I_1\omega_1 + I_2\omega_2$

M.I. of the two disc system $= (I_1 + I_2)$

If ω is the angular speed of the combined system, then,

Final angular momentum of the system, $L_2 = (I_1 + I_2)\omega$

As no external torque is acting, therefore, according to principle of conservation of angular momentum, we have,

$$(I_1 + I_2)\omega = I_1\omega_1 + I_2\omega_2$$

or $$\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(*b*) Initial K.E. of two discs, $E_1 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$

Final K.E. of the system, $E_2 = \frac{1}{2}(I_1 + I_2)\omega^2$

$$\therefore \quad E_1 - E_2 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}(I_1 + I_2)\omega^2$$

Putting the value of ω from part (*a*) and on solving, we get,

$$E_1 - E_2 = \frac{I_1 I_2(\omega_1 - \omega_2)}{2(I_1 + I_2)} = \text{a positive quantity}$$

Hence the rotational K.E. of the combined system is less than the sum of the initial rotational K.E. of the two discs. Note that there is loss of K.E. in the process. This loss of energy is due to dissipation of energy in frictional contact of the two discs. It may be noted that angular momentum is conserved in the process as torque due to friction is only an internal torque.

Q.26. (*a*) **Prove the theorem of perpendicular axes. [Hint. Square of the distance of a point (x, y) in the x-y plane from an axis through the origin and perpendicular to the plane is $x^2 + y^2$.]**

(*b*) **Prove the theorem of parallel axes. [Hint: If the centre of mass is chosen to be the origin, $\Sigma mx = 0$.]**

Ans. (*a*) Refer to Art. 14.15.

(*b*) Refer to Art. 14.14.

Q.27. Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by $v^2 = \dfrac{2gh}{(1+k^2/R^2)}$ using dynamical considerations (*i.e.*, by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis and R is the radius of the body. The body starts from rest at the top of the plane.

Ans. Consider a body of mass M rolling down the inclined plane as shown in Fig. 7.012. At the start, the body has potential energy = Mgh. When it reaches the bottom of the incline, all P.E. is changed to rotational and translational kinetic energy. According to principle of conservation of energy, we have,

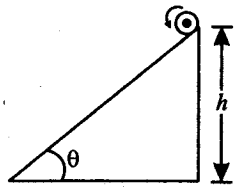

Fig. 7.012

$$\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh$$

As $\omega = v/R$ and $I = Mk^2$,

$$\therefore \quad \frac{1}{2}Mv^2 + \frac{1}{2}(Mk^2)\frac{v^2}{R^2} = Mgh$$

or $$v^2\left(1 + \frac{k^2}{R^2}\right) = 2gh$$

$$\therefore \quad v^2 = \frac{2gh}{1 + k^2/R^2} \qquad \text{...Q.E.D.}$$

Q.28. A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig. 7.013? Will the disc roll in the direction indicated?

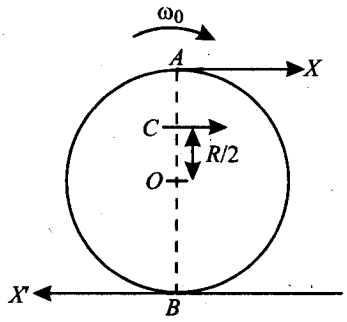

Fig. 7.013

Ans. We know that $v = R\omega$.

For point A : $v_A = R\omega_0$ along AX

For point B : $v_B = R\omega_0$ along BX'

For point C : $v_C = (R/2)\omega_0$ parallel to AX

The disc will not roll in the direction indicated. It is because the disc has been placed on a frictionless table. Without friction, rolling is not possible.

Q.29. Explain why friction is necessary to make the disc in Fig. 7.013 roll in the direction indicated. (*a*) Give the direction of frictional force at *B* and the sense of frictional torque before perfect rolling begins. (*b*) What is the force of friction after perfect rolling begins.

Ans. To make the disc roll, torque is required which can be provided only by a tangential force. Here only frictional force can provide the tangential force.

(*a*) The velocity at point *B* is along BX' (*i.e.*, towards left). Since frictional force opposes motion, it must act towards right. The direction of frictional torque will be perpendicular to the plane of the disc and in outward direction.

(*b*) Frictional force decreases the velocity of the point of contact *B*. The perfect rolling begins only when velocity of point *B* becomes zero. Once this is so, the force of friction is zero.

Q.30. A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously with initial angular speed equal to 10π rad s^{-1}. Which of the two will start to roll earlier? The co-efficient of kinetic friction is $\mu_k = 0.2$.

Ans. Initially the velocity of centre of mass (C.M.) of the body (disc or ring) is zero *i.e.*, $v_0 = 0$. The force of friction ($= \mu_k mg$) causes the C.M. to accelerate. If a is the acceleration produced, then,

$$\mu_k mg = ma \quad \therefore \; a = \mu_k g$$

Now $$v = v_0 + at \text{ or } v = 0 + \mu_k gt \quad \therefore \quad v = \mu_k gt$$

The frictional torque causes a retardation in the initial angular speed ω_0. If α is the angular retardation and I is the moment of inertia of the body (disc or ring), then,

$$\mu_k mg \times R = -I\alpha \quad \text{or} \quad \alpha = -\frac{\mu_k mgR}{I}$$

Now $$\omega = \omega_0 + \alpha t$$

$\therefore$ $$\omega = \omega_0 - \frac{\mu_k mgRt}{I}$$

The rolling begins, when $v = R\omega$.

$\therefore$ $$\mu_k gt = R\omega_0 - \frac{\mu_k mgR^2t}{I} \qquad (\because v = \mu_k gt)$$

For a ring. $$I = mR^2$$

$\therefore$ $$\mu_k gt = R\omega_0 - \frac{\mu_k mgR^2t}{mR^2} = R\omega_0 - \mu_k gt$$

or $$2\mu_k gt = R\omega_0 \quad \therefore \quad t = \frac{R\omega_0}{2\mu_k g} \qquad ...(i)$$

For a disc. $$I = \frac{1}{2}mR^2$$

$\therefore$ $$\mu_k gt = R\omega_0 - \frac{\mu_k mgR^2t}{mR^2/2} = R\omega_0 - 2\mu_k gt$$

$\therefore$ $$3\mu_k gt = R\omega_0 \quad \text{or} \quad t = \frac{\omega_0 R}{3\mu_k g} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), it is clear that the value of t for the disc is smaller than that for the ring. Therefore, the disc would begin to roll earlier than the ring.

Q.31. A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30° to the horizontal. The coefficient of static friction $\mu_s = 0.25$.

(*a*) How much is the force of friction acting on the cylinder?

(*b*) What is the work done against friction during rolling?

(*c*) If the angle θ of the plane is increased, at what value of θ does the cylinder skid and not roll perfectly?

Ans. $M = 10$ kg ; $R = 15$ cm $= 0.15$ m ; $\mu_s = 0.25$; $\theta = 30°$

(*a*) Force of friction, $f_s = \frac{1}{3}Mg \sin\theta = \frac{1}{3} \times 10 \times 9.8 \times \sin 30° =$ **16.33 N**

(*b*) Work done W against friction during rolling is

$$W = \mathbf{0\ J}$$

Note that friction involved is static and no work is done against it. For if we assume the cylinder is in contact with the incline at a point, then the force of friction acts parallel to the plane (Refer back to Fig. 14.26) and the point of contact of the cylinder at each point does not slide but moves perpendicular to the incline (upward) as the cylinder rolls. Since the force and motion are perpendicular to each other, no work is done.

(*c*) The maximum angle θ_m that will allow the cylinder to roll without slipping is

$$\mu_s = \frac{1}{3}\tan\theta_m \quad \text{or} \quad \tan\theta_m = 3\,\mu_s = 3 \times 0.25 = 0.75$$

$$\therefore \quad \theta_m = \tan^{-1}0.75 = \mathbf{36.87°}$$

Q.32. Read each statement below carefully and state with reasons if it is true or false. (*a*) During rolling, the force of friction acts in the same direction as the direction of motion of the C.M. of the body. (*b*) The instantaneous speed of the point of contact during rolling is zero. (*c*) The instantaneous acceleration of the point of contact during rolling is zero. (*d*) For perfect rolling motion, work done against friction is zero. (*e*) A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Ans. (*a*) False. During rolling, the force of friction acts in a direction opposite to the direction of motion of the C.M. of the body.

(*b*) True. It is because a rolling body can be imagined to be rotating about an axis passing through the point of contact of the body with the ground. Therefore, its instantaneous speed is zero.

(*c*) False. Since the body is rotating, its instantaneous acceleration cannot be zero.

(*d*) True. The reason is as follows. For if the rolling body is in contact with the surface at a point, then the force of friction acts parallel to the surface and the point of contact of the rolling body at each point does not slide but moves perpendicular to the surface (upward) as the body rolls. Since force and motion are perpendicular to each other, no work is done.

(*e*) True. For rolling motion, the point of contact should be momentarily at rest. This is achieved by force of friction. The inclined plane is perfectly frictionless. In the absence of friction, no rolling will take place. Hence the body will keep on slipping.

Unit VII

GRAVITATION

- Acceleration Due to Gravity
- Satellite and Planetary Motion
- Gravitational Field, Energy and Potential
- N.C.E.R.T. Textbook Exercises

15

Acceleration Due to Gravity

INTRODUCTION

Besides developing the three laws of motion, Sir Isaac Newton also examined the motion of the heavenly bodies—the planets and the moon. Newton recognised that a force of some kind must be acting on the planets to keep them in nearly circular orbits; otherwise their paths would be straight lines. What sort of force could act from one place to another without contact? A falling apple is attracted to the earth, but the apple attracts the earth as well (Newton's third law of motion). Extending this idea, Newton proposed that every body in this universe attracts every other body. This led to the discovery of the famous **law of universal gravitation** *i.e.* each object in this universe attracts every other object. Note that gravitational force is attractive. The force of attraction between any two bodies in the universe is called **gravitational force** or **gravitation**. The force of attraction exerted by earth on a body lying on or near the surface of earth has been given the special name *viz* **gravity** or **earth's gravitational pull**. In this chapter, we shall discuss the role of gravity *i.e.* earth's gravitational pull.

15.1. ACCELERATION DUE TO GRAVITY

The acceleration produced in a body when it falls freely under the effect of gravity alone is called **acceleration due to gravity.**

The acceleration due to gravity is denoted by the special symbol g and its value on or near the surface of earth is 9.8 m/s^2 *i.e.* g = 9.8 m/s^2. Its direction is towards the centre of earth which is called *vertically downward direction*. Note that bodies thrown upward or downward and those released from rest are all *freely falling bodies*. All freely falling objects have an acceleration of 9.8 m/s^2 vertically downward.

A freely falling body always has an acceleration of 9.8 m/s^2 vertically downward. It does not matter whether the body is rising or falling ; its acceleration is still g (= 9.8 m/s^2) vertically downward. Thus when a body is rising, its velocity decreases at the rate of 9.8 m/s every second until it reaches at the top of its motion. At this instant, the velocity of the body is zero and it starts downward journey. As the body is falling, its velocity increases at the rate of 9.8 m/s every second until it hits the ground. Note that during the entire journey of the body (upward or downward), acceleration due to gravity g (= 9.8 ms^{-2}) is vertically downward.

Discussion. The following points are worth noting :

(*i*) The value of g varies slightly from place to place on the earth's surface. However, it is normally sufficiently accurate to use a value of 9.8 ms^{-2} everywhere. We shall discuss the reasons for the variation of g later in this chapter.

(*ii*) Like all accelerations, the acceleration due to gravity is a vector. It is directed downward toward the centre of the earth.

(*iii*) *The vertical motion of a body under gravity alone is a case of uniformly accelerated motion in one dimension.*

(*v*) $$F = G\frac{m_1 m_2}{r^2}$$

If $m_1 = m_2 = 1$ and $r = 1$, then $F = G$.

Hence universal gravitational constant is equal to the force of attraction between two bodies each of unit mass placed unit distance (centre-to-centre) apart.

(*vi*) This law fails if the distance between the objects is less than 10^{-9} m *i.e.* of the order of intermolecular distances.

Example 15.1. The mass of planet Jupiter is 1.9×10^{27} kg and that of sun is 1.99×10^{30} kg. The mean distance of the sun from Jupiter is 7.8×10^{11}m. Calculate the gravitational force which sun exerts on Jupiter. Take $G = 6.67 \times 10^{-11}$Nm^2 kg^{-2}.

Solution. Mass of Jupiter, $m_1 = 1.9 \times 10^{27}$ kg

Mass of sun, $m_2 = 1.99 \times 10^{30}$ kg

Mean distance between them, $r = 7.8 \times 10^{11}$ m

The gravitational force exerted by sun on jupiter is

$$F = G\frac{m_1 m_2}{r^2}$$

$$= 6.67\times10^{-11} \times \frac{(1.9\times10^{27})\times(1.99\times10^{30})}{(7.8\times10^{11})^2} = 4.15 \times 10^{23} \text{ N}$$

Example 15.2. Assuming the orbit of the earth about the sun to be circular (it is actually slightly elliptical) with radius 1.5×10^{11} m, find the mass of the sun. The earth revolves around the sun in 3.15×10^7 seconds.

Solution. The centripetal force needed to hold the earth in an orbit of radius R is provided by the gravitational attraction of the sun. If m_e and m_s are the masses of earth and sun respectively, then,

Centripetal force = Gravitational force

or $$\frac{m_e v^2}{R} = G\frac{m_e m_s}{R^2} \quad \text{or} \quad m_s = \frac{v^2 R}{G}$$

Now v = speed of earth in its orbit around the sun

$$= \frac{2\pi R}{T} = \frac{2\pi\times(1.5\times10^{11})}{3.15\times10^7} = 3 \times 10^4 \text{ ms}^{-1}$$

$\therefore$ $$m_s = \frac{(3\times10^4)^2\times1.5\times10^{11}}{6.67\times10^{-11}} = 2 \times 10^{30} \text{ kg}$$

Example 15.3. A mass M is broken into two parts : m and $(M - m)$. How are m and M related so that force of gravitation between the two parts is maximum?

Solution. $$F = G\frac{m(M-m)}{r^2} = k\,m(M-m)$$

$$= k\left[mM - m^2\right] = k\left[\frac{M^2}{4} - \frac{M^2}{4} + mM - m^2\right]$$

$$= k\left[\frac{M^2}{4} - \left(\frac{M}{2} - m\right)^2\right]$$

Therefore, F will be maximum when $\left(\frac{M}{2} - m\right)^2 = 0$ or $m = M/2$

Example 15.4. **Two metal spheres of the same material and of equal radius R are touching each other. Show that force of attraction between them is proportional to R^4.**

Solution. $F = G\frac{m_1 m_2}{r^2}$

If d is the density of the material of the sphere, then,

$$m_1 = \frac{4}{3}\pi R^3 d \; ; \; m_2 = \frac{4}{3}\pi R^3 d \; ; \; r = 2R$$

$$\therefore \quad F = G\times\frac{\left(\frac{4}{3}\pi R^3 d\right)\times\left(\frac{4}{3}\pi R^3 d\right)}{(2R)^2} = \frac{4}{9}G\pi^2 d^2 R^4 \quad \text{or} \quad F \propto R^4$$

Example 15.5. **A mass m_1 = 1 kg weighs one-sixth as much on the surface of the moon as on the earth. Calculate the mass m_2 of the moon. The radius of the moon is 1.738 $\times 10^6$ m.**

Solution. On the moon, m_1 weighs 9.8/6 N.

$$\therefore \quad W_1 = G\frac{m_1 m_2}{r^2}$$

or $$\frac{9.8}{6} = 6.67\times10^{-11}\times\frac{1\times m_2}{\left(1.738\times10^6\right)^2}$$

$$\therefore \quad m_2 = \frac{9.8\times\left(1.738\times10^6\right)^2}{(6)\times\left(6.67\times10^{-11}\right)} = 7.4 \times 10^{22} \text{ kg}$$

PROBLEMS FOR PRACTICE

1. Two 20 kg spheres are placed with their centres 50 cm apart. What is the magnitude of gravitational force each exerts on the other? **[1.07 × 10^7 N]**
2. A sphere of mass 40 kg is attracted by another of mass 15 kg when their centres are 0.2 m apart, with a force of 9.8 × 10^{-7} N. Calculate the constant of gravitation. **[6.53 × 10^{-11} Nm^2 kg^{-2}]**
3. If earth-moon distance is 3.8 × 10^5 km, calculate the time (in days) it takes the moon to circle the earth. Mass of the earth = 5.98 × 10^{24} kg. **[27 days]**
4. Calculate the mass of Jupiter from the knowledge that its satellite *Io* orbits at an average distance of 4.22 × 10^5 km from its centre with an orbital period of 42.5 hours. **[1.9 × 10^{27} kg]**
5. A planet of mass m moves round the sun of mass M in a circular orbit of radius R with an angular velocity ω. Show that (*i*) ω is independent of mass m of the planet (*ii*) in a circular orbit of radius $4R$ round the sun, the angular velocity decreases to ω/8.

15.3. VECTOR FORM OF NEWTON'S LAW OF GRAVITATION

We can write Newton's law of universal gravitation in vector form as :

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{21} \qquad ...(i)$$

where $\vec{F}_{12}$ is the force on mass m_1 exerted by mass m_2 which is a distance r_{21} away ; $\hat{r}_{21}$ is a unit vector that points from m_2 to m_1 along the line joining their centres. Therefore, $\hat{r}_{21} = \vec{r}_{21}/r_{21}$ where

$\vec{r}_{21}$ is the displacement vector as shown in Fig. 15.2 (*i*). The minus sign in eq. (*i*) is necessary since the force on m_1 due to m_2 points in the direction opposite to $\hat{r}_{21}$. The displacement vector $\vec{r}_{12}$ is a vector of the same magnitude as $\vec{r}_{21}$ but it points in the opposite direction so that

$$\vec{r}_{12} = -\vec{r}_{21}$$

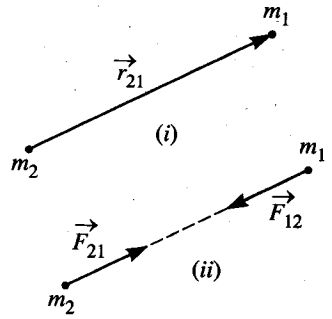

Fig. 15.2

By Newton's third law, the force $\vec{F}_{21}$ acting on m_2 due to m_1 must have the same magnitude as $\vec{F}_{12}$ but acts in the opposite direction [See Fig. 15.2 (*ii*)] so that

$$\vec{F}_{21} = -\vec{F}_{12} = G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{21}$$

$$\therefore \quad \vec{F}_{21} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12}$$

Note. The law of universal gravitation should not be confused with Newton's second law of motion ($\vec{F} = m\vec{a}$). The former describes a particular force — the gravitational force of attraction – and how its value varies with the distance and masses involved. On the other hand, Newton's second law relates the force on a body — it can be any force — to the mass and acceleration of that body.

Example 15.6. The mass *m* of one of the small spheres of a Cavendish balance is 0.001 kg, the mass *M* of one of the large spheres is 0.5 kg and the centre-to-centre distance between the spheres is 0.05 m. What is the gravitational force on each sphere?

Solution. The magnitude of gravitational force on each sphere is

$$F = G\frac{mM}{r^2}$$

Here $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} ; $m = 0.001$ kg ; $M = 0.5$ kg ; $r = 0.05$ m

$$\therefore \quad F = 6.67\times10^{-11}\times\frac{0.001\times0.5}{(0.05)^2} = 1.33 \times 10^{-11} \text{ N}$$

15.4. DIFFERENCE BETWEEN GRAVITATION AND GRAVITY

Gravitation is the force of attraction acting between any two bodies of the universe. On the other hand, the force of attraction between every body and earth is the earth's gravitational pull or gravity. Clearly, gravity is a special case of gravitation where one of the bodies is earth and the other body lying on earth or near the surface of earth. The following are the salient differences between gravitation and gravity :

(*i*) Gravitation is the force of attraction between any two bodies of the universe whereas gravity is the force of attraction exerted by earth on a body lying on or near the surface of earth.

(*ii*) The gravitational force between two masses m_1 and m_2 separated by a distance r is $F = Gm_1m_2/r^2$ where G is the universal gravitational constant. On the other hand, force of gravity on a body of mass m is $F = mg$ where g is acceleration due to gravity. Note that mg is called the weight of the body.

(*iii*) The force of gravitation between two bodies cannot be zero unless $r \to \infty$. However, force of gravity at the centre of earth is zero.

15.5. EXPRESSION FOR ACCELERATION DUE TO GRAVITY

Consider earth to be a sphere of mass M_E and radius R_E. Suppose a body of mass m is placed on the surface of the earth as shown in Fig. 15.3. It is *legitimate to consider that the whole mass M_E of the earth is concentrated at its centre. It is reasonable to assume that the distance between m and M_E is equal to radius R_E of the earth. According to the law of gravitation, the force of attraction acting on the body due to earth is given by ;

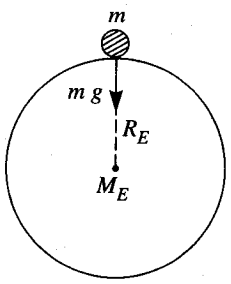

Fig. 15.3

$$F = G\frac{M_E m}{R_E^2} \qquad ...(i)$$

This attractive force which the earth exerts on the object is simply the weight (= mg) of the object *i.e.*

$$F = m\,g \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $$mg = \frac{GM_E m}{R_E^{\,2}}$$

or $$g = \frac{GM_E}{R_E^{\,2}} \qquad ...(iii)$$

Eq. (*iii*) gives the expression for acceleration due to gravity on the surface of the earth.

Discussion. The following points are worth noting :

(*i*) *Eq. (iii) shows that the value of g does not depend on the mass m of the body.* Thus if two bodies of different masses are allowed to fall *freely,* they will have the same acceleration. If they are allowed to fall from the same height, they will reach the earth simultaneously.

(*ii*) Both G and M_E are constant and R_E (= distance between the centres of body and earth) does not change appreciably for small variation in height near the surface of the earth. Therefore, the acceleration due to gravity of an object near the surface of earth is approximately constant and does not depend on the mass of the object.

(*iii*) *We can easily extend eq. (iii) to find the gravitational acceleration at the surface of any planet of mass M_P and radius R_P.*

$$g_P = \frac{GM_P}{R_P^2}$$

15.6. MASS AND DENSITY OF EARTH

Using the law of universal gravitation and the measured value of the acceleration due to gravity, we can find the mass and density of earth.

(*i*) **Mass of earth.** The acceleration due to gravity on the surface of the earth is given by ;

$$g = \frac{GM_E}{R_E^{\,2}}$$

$$\therefore \quad \text{Mass of earth, } M_E = \frac{g\,R_E^{\,2}}{G} \qquad ...(i)$$

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* It can be shown that the attractive force which a uniform spherical mass exerts on an object outside the sphere can be computed by assuming all the mass of the sphere to be concentrated at its centre. It involves fairly complex calculus.

Now $g = 9.8\ \text{m s}^{-2}$; $R_E = 6.37 \times 10^6\ \text{m}$; $G = 6.67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$

$$\therefore \text{Mass of Earth, } M_E = \frac{(9.8)\times(6.37\times10^6)^2}{6.67\times10^{-11}} = 6\times10^{24}\ \text{kg} \quad \text{(approximately)}$$

(*ii*) **Density of earth.** Let ρ be the average density of the earth. Earth is a sphere of radius R_E.

$\therefore$ Mass of earth, M_E = Volume × Density

$$= \left(\frac{4}{3}\pi R_E^{\,3}\right)\rho$$

$$\therefore \quad M_E = \frac{4}{3}\pi R_E^{\,3}\rho \qquad \ldots(ii)$$

From eqs. (*i*) and (*ii*), $\frac{4}{3}\pi R_E^{\,3}\rho = \frac{gR_E^{\,2}}{G}$

$$\therefore \quad \text{Average density of earth, } \rho = \frac{3g}{4\pi R_E G}$$

Now $g = 9.8\ \text{ms}^{-2}$; $R_E = 6.37 \times 10^6\ \text{m}$; $G = 6.67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$.

$$\therefore \text{Average density of earth, } \rho = \frac{3\times9.8}{4\pi\times(6.37\times10^6)\times6.67\times10^{-11}}$$

$$= 5.5 \times 10^3\ \text{kg m}^{-3}$$

This is the average density of the entire earth and is 5.5 times the density of water.

Example 15.7. What will be the acceleration due to gravity on the surface of the moon if its radius is 1/4th the radius of the earth and its mass 1/80th of the mass of the earth?

Solution. The acceleration due to gravity on the surface of earth is given by ;

$$g_E = \frac{GM_E}{R_E^{\,2}}$$

If M_m is the mass of the moon and R_m its radius, then acceleration due to gravity on the surface of the moon is given by ;

$$g_m = \frac{GM_m}{R_m^2}$$

$$\therefore \quad \frac{g_m}{g_E} = \frac{M_m}{M_E}\times\left(\frac{R_E}{R_m}\right)^2$$

Now $$\frac{M_m}{M_E} = \frac{1}{80} \text{ and } \frac{R_E}{R_m} = 4$$

$$\therefore \quad \frac{g_m}{g_E} = \frac{1}{80}\times\left(\frac{4}{1}\right)^2 = \frac{1}{5} \quad \text{or} \quad g_m = \frac{g_E}{5}$$

Example 15.8. A man can jump 1.5 m on the earth. Calculate the approximate height he might be able to jump on a planet whose density is one-quarter that of the earth and whose radius is one-third that of the earth.

Solution. Suppose a man of mass m leaps a height h on the earth and a height h_1 on the planet. Assuming he can give himself the same initial kinetic energy on the two planets, the potential energy is the same at the maximum height. Therefore,

$$mg_1h_1 = mgh$$

where g_1 and g are the respective accelerations due to gravity on the planet and earth.

$$\therefore \quad h_1 = \frac{g}{g_1}\times h$$

Now $$g = \frac{GM_E}{R_E^2} = G \times \frac{\frac{4}{3}\pi R_E^2 \rho_E}{R_E^2} = G \times \frac{4}{3}\pi R_E \rho_E$$

Also $$g_1 = G\frac{4}{3}\pi R_P \rho_P$$

$$\therefore \quad \frac{g}{g_1} = \frac{R_E \rho_E}{R_P \rho_P} = \left(\frac{R_E}{R_P}\right) \times \left(\frac{\rho_E}{\rho_P}\right) = 3 \times 4 = 12$$

$$\therefore \quad h_1 = 12 \times 1.5 = \mathbf{18\ m}$$

Example 15.9. What is the acceleration due to gravity on the surface of a planet that has a radius half that of earth and the same average density as earth?

Solution. The acceleration due to gravity on the surface of earth is

$$g_E = \frac{GM_E}{R_E^2}$$

The acceleration due to gravity on the surface of the planet is

$$\therefore \quad g_P = \frac{GM_P}{R_P^2}$$

$$\therefore \quad \frac{g_P}{g_E} = \left(\frac{M_P}{M_E}\right) \times \frac{R_E^2}{R_P^2}$$

Since both bodies have the same density, their masses are directly proportional to their volumes *i.e.* proportional to the cube of their radii.

$$\therefore \quad \frac{g_P}{g_E} = \left(\frac{R_P^3}{R_E^3}\right) \times \frac{R_E^2}{R_P^2} = \frac{R_P}{R_E} = \frac{1}{2}$$

$$\therefore \quad g_P = \frac{1}{2} \times g_E = \frac{1}{2} \times 9.8 = \mathbf{4.9\ m\ s^{-2}}$$

Example 15.10. If the radius of earth shrinks by 1.5% (mass remaining the same), then how would the value of acceleration due to gravity change?

Solution. We know that $g = \frac{GM}{R^2}$

$$\therefore \quad \log_e g = \log_e G + \log_e M - 2 \log_e R$$

Differentiating, we get, $\frac{dg}{g} = 0 + 0 - \frac{2dR}{R}$

or $$\frac{dg}{g} \times 100 = -2\left(\frac{dR}{R} \times 100\right) = -2(-1.5) = \mathbf{3\%}$$

Example 15.11. The mass of the planet Jupiter is 1.9×10^{27} kg and that of the sun is 1.99×10^{30} kg. The mean distance of the Jupiter from the sun is 7.8×10^{11} m. Calculate the gravitational force which the sun exerts on the Jupiter. Assuming that Jupiter moves in a circular orbit around the sun, calculate the speed of Jupiter.

Solution. Mass of sun, $M = 1.99 \times 10^{30}$ kg

Mass of Jupiter, $m = 1.9 \times 10^{27}$ kg

Jupiter–sun distance, $r = 7.8 \times 10^{11}$ m

The gravitational force that sun exerts on jupiter is

$$F = G\frac{Mm}{r^2} = 6.67 \times 10^{-11} \times \frac{(1.99 \times 10^{30}) \times (1.9 \times 10^{27})}{(7.8 \times 10^{11})^2} = \mathbf{4.15 \times 10^{23}\ N}$$

The gravitational force of attraction provides the necessary centripetal force to Jupiter to move in a circular orbit. If v is the speed of Jupiter, then,

$$F = \frac{mv^2}{r}$$

or $$v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{(4.15\times10^{23})\times(7.8\times10^{11})}{1.9\times10^{27}}} = 1.304 \times 10^4 \text{ ms}^{-1}$$

Example 15.12. If earth were made of lead of relative density 11.3, what would be the value of acceleration due to gravity on the surface of earth? Radius of earth = 6.4 $\times 10^6$ m and $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}.

Solution. Relative density of earth = $\frac{\text{Density of earth}}{\text{Density of water}}$

$\therefore$ Density of earth, ρ = Relative density of earth $\times$ Density of water = 11.3×10^3 kg /m^3

Mass of earth, $M_E = \frac{4}{3}\pi R_E^3 \rho = \frac{4}{3}\times\pi\times(6.4\times10^6)^3\times11.3\times10^3 = 1.24 \times 10^{25}$ kg

Acceleration due to gravity, $g = \frac{GM_E}{R_E^2} = \frac{(6.67\times10^{-11})\times1.24\times10^{25}}{(6.4\times10^6)^2} = 20.21 \text{ ms}^{-2}$

PROBLEMS FOR PRACTICE

1. Assuming the mean density of the earth is 5500 kg m^{-3}, that G is 6.67×10^{-11} N m^2 kg^{-2} and that radius of earth is 6400 km, find the value for the acceleration of free fall at the earth's surface. **[9.9 ms^{-2}]**
2. The acceleration due to gravity at the moon's surface is 1.67 ms^{-2}. If the radius of the moon is 1.74×10^6 m and $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, calculate the mass of the moon. **[7.58 $\times 10^{22}$ kg]**
3. If $G = 6.67 \times 10^{-11}$ SI units, radius of earth = 6.37×10^6 m and g = 9.8 ms^{-2}, calculate the density of earth. **[5570 kg m^{-3}]**
4. If the acceleration of free fall at the earth's surface is 9.8 ms^{-2} and the radius of the earth is 6400 km, calculate the mass of earth. **[6 $\times 10^{24}$ kg]**

15.7. VARIATIONS IN THE VALUE OF g

The value of acceleration due to gravity (g) varies as we go above or below the surface of the earth. It also varies from place to place on the surface of the earth. We shall discuss the following cases :

(*i*) Variation of g with altitude (*i.e.* height).
(*ii*) Variation of g with depth.
(*iii*) Variation of g due to shape of earth.
(*iv*) Variation of g due to rotation of earth.

15.8. VARIATION OF g WITH ALTITUDE

Consider earth to be a sphere of radius R and mass M. The acceleration due to gravity on the surface of earth (point Q in Fig. 15.4) is

$$g = \frac{GM}{R^2} \qquad ...(i)$$

Consider a point P at a height h above the surface of the earth. The accelration due to gravity at point P is

$$g' = \frac{GM}{(R+h)^2} \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we have,

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2}$$

or $$g' = \frac{gR^2}{(R+h)^2} \quad ...(iii)$$

Eq. (iii) gives the expression for variation of acceleration due to gravity with height h. It is clear that $g' < g$. Thus as we go above the surface of earth, acceleration due to gravity goes on decreasing. For example, at a height equal to radius of earth (*i.e.* $h = R = 6400$ km), we have,

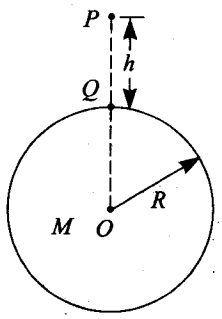

Fig. 15.4

$$g' = \frac{gR^2}{(R+h)^2} = \frac{gR^2}{(R+R)^2} = \frac{g}{4}$$

Expression for g' when $h << R$. When h is very small as compared to R (= radius of earth), the expression for g' can be found as under :

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{R^2\left(1+\frac{h}{R}\right)^2} = \left(1+\frac{h}{R}\right)^{-2}$$

$$\therefore \quad g' = g\left(1+\frac{h}{R}\right)^{-2}$$

$$= g\left(1-\frac{2h}{R} + \text{terms containing higher powers of } \frac{h}{R}\right)$$

Since $h << R$, the higher powers of h/R can be neglected.

$$\therefore \quad g' = g\left(1-\frac{2h}{R}\right) \quad ...(iv)$$

Eq. (iv) gives the expression for g' when $h << R$.

Percentage decrease in g. We can find the percentage decrease in the value of g with height h as follows. From eq. (*iv*), we have,

$$g' = g\left(1-\frac{2h}{R}\right)$$

or $$g - g' = \frac{2hg}{R}$$

or $$\frac{g-g'}{g} = \frac{2h}{R} \quad \text{...fractional decrease in } g$$

$\therefore$ %age decrease in the value of g with height h

$$= \frac{g-g'}{g} \times 100 = \frac{2h}{R} \times 100$$

15.9. VARIATION OF g WITH DEPTH

Consider the earth to be a sphere of radius R and mass M. The acceleration due to gravity at point Q on the surface of the earth (See Fig. 15.5) is

$$g = \frac{GM}{R^2}$$

If ρ is the density of the earth, then,

$$\text{Mass of earth, } M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore \quad g = \frac{G\left(\frac{4}{3}\pi R^3 \rho\right)}{R^2}$$

or $$g = \frac{4}{3}\pi R G \rho \qquad ...(i)$$

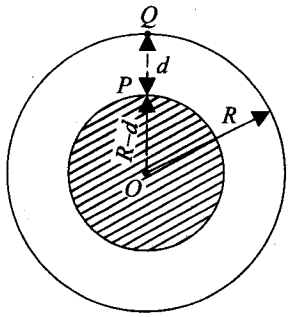

Fig. 15.5

Consider a point P which is *inside* the earth and at a depth d below the surface of the earth. Its distance from the centre O is $(R - d)$. Let a sphere be drawn with O as centre and $(R - d)$ as radius. The acceleration due to gravity g' at point P is *only due to the sphere of radius $(R - d)$.

$$\therefore \quad g' = \frac{GM'}{(R-d)^2}$$

where M' = mass of inner solid sphere (shaded portion)

$$= \frac{4}{3}\pi(R-d)^3\rho$$

$$\therefore \quad g' = \frac{G\left[\frac{4}{3}\pi(R-d)^3\right]\rho}{(R-d)^2}$$

or $$g' = \frac{4}{3}\pi(R-d)\ G\rho \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we have,

$$\frac{g'}{g} = \frac{(R-d)}{R}$$

or $$g' = g\left(1-\frac{d}{R}\right) \qquad ...(iii)$$

Eq. (iii) gives the expression for variation of g with depth d.

It is clear that $g' < g$. Therefore, as we go below the surface of the earth, the acceleration due to gravity goes on decreasing. At the centre of earth, it becomes zero *i.e.*

$$g' = g\left(1-\frac{d}{R}\right) = g\left(1-\frac{R}{R}\right) = 0$$

Therefore, acceleration due to gravity is zero at the centre of earth.

Thus the value of acceleration due to gravity is maximum at the surface of the earth and becomes zero at the centre of the earth. That is why the weight (= mg) of a body at the centre of the earth is zero though its mass is constant.

Percentage decrease in g. We can calculate the %age decrease in the value of g with depth d inside the earth as follows. From eq. (*iii*), we have,

$$g' = g\left(1-\frac{d}{R}\right)$$

or $$g - g' = \frac{dg}{R}$$

---

* Any body placed at point P is situated at the surface of inner solid sphere of radius $(R–d)$ and lies inside the outer spherical shell. The gravitational force on a body *inside* a spherical shell is always zero. Therefore, body at point P experiences gravitational force only due to the inner solid sphere.

or $$\frac{g-g'}{g} = \frac{d}{R}$$ ... fractional decrease in g

$\therefore$ % age decrease in the value of g with depth d inside the earth is

$$\frac{g-g'}{g} \times 100 = \frac{d}{R} \times 100$$

15.10. VARIATION OF g DUE TO SHAPE OF EARTH

The earth is not a perfect sphere. It flattens at the poles and bulges out at the equator as shown in Fig. 15.6. The value of g is given by :

$$g = \frac{GM}{R^2}$$

Since G and M are constant, $g \propto 1/R^2$ where R is radius of earth.

Thus the value of g at a place on the surface of earth varies inversely as the square of the radius of earth at that place. Where radius of earth is more, the value of g is less and *vice-versa*.

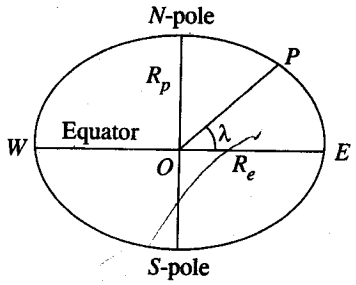

Fig. 15.6

Discussion. The following points may be noted :

(*i*) The radius of earth is least at the poles (R_p) and greatest at the equator (R_e). *Therefore, the value of g is maximum at the poles and least at the equator.*

(*ii*) *As we go from the equator towards the poles,* the radius of earth goes on decreasing (See Fig. 15.6) and hence *the value of g goes on increasing.*

(*iii*) The value of g at the equator is 9.78 m/s^2 whereas at the poles, it is 9.83 m/s^2.

15.11. VARIATION OF g DUE TO ROTATION OF EARTH

The earth rotates about the axis passing through its north and south poles. The value of g varies from place to place on the surface of earth due to rotation of the earth. An important parameter is the latitude at a place and is defined as under :

The **latitude at a place** *on the surface of earth is defined as the angle which the line joining the place to the centre of the earth makes with the equatorial plane.* It is denoted by λ.

Thus referring to Fig. 15.7, the latitude at place $P = \angle POE = \lambda$. It is clear that $\lambda = 90°$ at the poles and $\lambda = 0°$ at the equator.

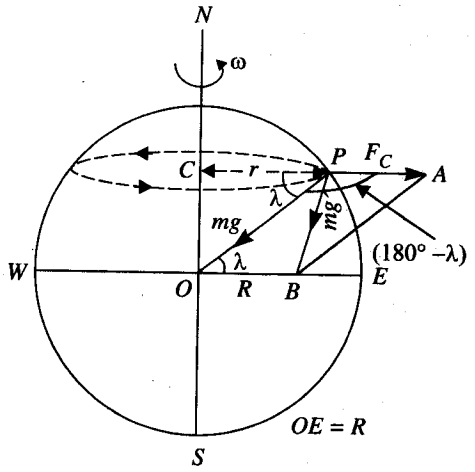

Fig. 15.7

Consider earth to be a sphere of mass M and radius R. Suppose a particle of mass m is situated at point P on the surface of the earth [See Fig. 15.7]. Let λ be the latitude of the point P *i.e.* $\angle POE = \lambda$. Suppose g is the acceleration due to gravity in the absence of rotational motion of the earth. In that case, the particle at P would have been attracted toward the centre O of the earth. Therefore true weight mg of the particle is directed toward O and is represented by the vector $\overrightarrow{PO}$.

Due to rotation of earth, the particle at P moves along a circular path whose centre is C and

radius r (= CP). Since $\angle OPC = \lambda$, $r = R\cos\lambda$. The centrifugal force on the particle due to rotational motion of the earth acts along the radius of the circular path and in outward direction. The magnitude of centrifugal force is given by ;

$$F_C = mr\omega^2 = mR\omega^2\cos\lambda$$

The centrifugal force is represented by the vector $\overrightarrow{PA}$. The apparent weight mg' of the particle is equal to the resultant of actual weight (= $m\,g$) and the centrifugal force $F_C(= mR\omega^2\cos\lambda)$. Complete the parallelogram $PABO$. Then diagonal $\overrightarrow{PB}$ of the parallelogram represents the apparent weight mg' of the particle.

Now $(PB)^2 = PO^2 + PA^2 + 2PO \times PA \times \cos(180° - \lambda)$

or $(mg')^2 = (mg)^2 + (mR\omega^2\cos\lambda)^2 + 2 \times mg \times mR\omega^2\cos\lambda(-\cos\lambda)$

or $$g' = g\left[1+\frac{R^2\omega^4\cos^2\lambda}{g^2}-\frac{2R\omega^2\cos^2\lambda}{g}\right]^{1/2}$$

Now $R = 6.4 \times 10^6$ m ; $\omega = \dfrac{2\pi}{24\times60\times60}$ rad/sec and $g = 9.8\ \text{ms}^{-2}$

$$\therefore \quad \frac{R\omega^2}{g} = \frac{6.4\times10^6}{9.8}\times\left(\frac{2\pi}{24\times60\times60}\right)^2 = 3.45\times10^{-3}$$

As $R\omega^2/g$ is very small, $R^4\omega^4/g^2$ will be still smaller. Therefore, neglecting the factor containing $R^2\omega^4/g^2$, we get,

$$g' = g\left[1-\frac{2R\omega^2\cos^2\lambda}{g}\right]^{1/2}$$

$$= g\left[1-\frac{1}{2}\times\frac{2R\omega^2\cos^2\lambda}{g}+\text{higher terms}\right]$$

Neglecting the terms containing higher powers of $R\omega^2\cos^2\lambda/g$, we have,

$$g' = g\left[1-\frac{R\omega^2}{g}\cos^2\lambda\right]$$

or $$g' = g - R\omega^2\cos^2\lambda \qquad ...(i)$$

From eq. (i), it is clear that acceleration due to gravity increases with the increase in the latitude of the place ($\because$ cos λ decreases as λ increases). This means that value of g increases as we go from equator to the poles.

At equator. At equator, $\lambda = 0°$ so that $\cos\lambda = \cos 0° = 1$.

$\therefore$ $g_e = g - R\omega^2$ ... *minimum*

Therefore, value of acceleration due to gravity is minimum at the equator. This is expected because the particle at the equator executes a circle of maximum radius. Therefore, the centrifugal force is maximum.

At poles. At poles, $\lambda = 90°$ so that $\cos\lambda = \cos 90° = 0$.

$\therefore$ $g_p = g$ ... *maximum*

Hence the value of g is maximum at the poles. This is expected because the particle at the pole moves in a circle of zero radius. Therefore, no centrifugal force acts on the particle. As a result, the value of g_P remains the same whether the earth is at rest or rotating.

Notes. (*i*) $g_P - g_e = g - (g - R\omega^2) = R\omega^2$

$\therefore$ $g_P - g_e = R\omega^2$

Note that $R = 6.4 \times 10^6$ m and $\omega = \dfrac{2\pi}{24\times60\times60}$ rad/s.

(*ii*) When a body of mass m is moved from equator to pole, the increase in weight

$$= m(g_P - g_e) = mR\omega^2$$

Example 15.13. Find the percentage decrease in weight of a body when taken 16 km below the surface of the earth. What happens to the weight of the body when it is taken to the centre of the earth? Radius of earth = 6400 km.

Solution. Here $d = 16$ km ; $R = 6400$ km

The acceleration due to gravity at a depth d is

$$g' = g\left(1-\frac{d}{R}\right)$$

or $$mg' = mg\left(1-\frac{d}{R}\right)$$

or $$\frac{mg'}{mg} = 1-\frac{d}{R}$$

$\therefore$ $$\frac{mg-mg'}{mg} = \frac{d}{R} = \frac{16}{6400} = 0.0025 = \mathbf{0.25\%}$$

At the centre of earth, $d = R$ so that :

$$\text{Weight of body} = mg' = mg\left(1-\frac{d}{R}\right) = mg\left(1-\frac{R}{R}\right) = \mathbf{0}$$

Example 15.14. Calculate that imaginary angular velocity of the earth for which the effective acceleration due to gravity at the equator becomes zero. In this condition, what will be the length (in hours) of the day? Radius of earth = 6400 km and $g = 10$ m s^{-2}.

Solution. Suppose g is the acceleration due to gravity in the absence of rotational motion of earth. The acceleration due to gravity at the equator in the presence of earth's rotation is given by ;

$$g_e = g - R\omega^2$$

Here ω is the angular velocity of earth. For $g_e = 0$, we have,

$$\omega = \sqrt{\frac{g}{R}} = \sqrt{\frac{10}{6400\times10^3}} = \mathbf{1.25 \times 10^{-3}\ rad\ s^{-1}}$$

Suppose in this situation, the time period of rotation of earth about its axis is T. Then,

$$T = \frac{2\pi}{\omega} = \frac{2\times3.14}{1.25\times10^{-3}} = 5024 \text{ s} = \mathbf{1.4\ h}$$

The earth would complete its rotation in 1.4 h instead of 24 h.

Example 15.15. How far above the earth-surface does the value of g become 16% of its value on the surface? Assume radius of earth to be 6400 km.

Solution. $R = 6400$ km ; $g' = \dfrac{16}{100}g$ so that $\dfrac{g'}{g} = \dfrac{16}{100}$; $h = ?$

Now $$g' = g\left[\frac{R^2}{(R+h)^2}\right] \quad \text{or} \quad \frac{g'}{g} = \left[\frac{R^2}{(R+h)^2}\right]$$

$\therefore$ $$\frac{16}{100} = \frac{R^2}{(R+h)^2} \quad \text{or} \quad \frac{4}{10} = \frac{R}{R+h}$$

$\therefore$ $$h = \frac{3R}{2} = \frac{3\times6400}{2} = \mathbf{9600\ km}$$

Example 15.16. **Find the value of acceleration due to gravity at a height of 400 km above the surface of the earth. Given radius of earth = 6400 km and acceleration due to gravity at the surface of earth is 9.8 m s^{-2}.**

Solution. $g = 9.8$ m s^{-2} ; $R = 6400$ km ; $h = 400$ km

Now $$g' = g\left(1-\frac{2h}{R}\right) = 9.8\left(1-\frac{2\times400}{6400}\right) = 8.575 \text{ ms}^{-2}$$

Example 15.17. **At what height above earth's surface, value of g is the same as in a mine 100 km deep?**

Solution. Suppose at a height h above the surface of earth, the acceleration due to gravity is the same as in a mine at a depth d (= 100 km) into the earth.

For a height h above earth's surface, $g' = g\left(1-\frac{2h}{R}\right)$

For a depth d into earth, $g' = g\left(1-\frac{d}{R}\right)$

As the acceleration due to gravity in the two cases is same,

$$\therefore \quad g\left(1-\frac{2h}{R}\right) = g\left(1-\frac{d}{R}\right)$$

or $$\frac{2h}{R} = \frac{d}{R} \quad \therefore \quad h = \frac{d}{2} = \frac{100}{2} = 50 \text{ km}$$

Example 15.18. **Determine the speed with which the earth would have to rotate on its axis so that a person on the equator would weigh 3/5th as much as at present. Take equatorial radius as 6400 km.**

Solution. At present, the weight of the person on the equator is nearly the same which would have been if earth were stationary. Suppose for weight to remain 3/5 th, the angular speed of earth is ω.

Now $$g_e = g - R\omega^2$$

or $$\frac{3}{5}mg = mg - mR\omega^2$$

$$\therefore \quad \omega = \sqrt{\frac{2g}{5R}} = \sqrt{\frac{2\times9.8}{5\times(6400\times10^3)}} = 7.8 \times 10^{-4} \text{ rad s}^{-1}$$

Example 15.19. **Calculate the effect of rotation of earth on the weight of the body at a place at latitude 45°. Take radius of earth = 6.37×10^6 m.**

Solution. $g' = g - R\omega^2 \cos^2 \lambda$

Here $\cos \lambda = \frac{1}{\sqrt{2}}$; $\omega = \frac{2\pi}{24\times60\times60} = 7.27 \times 10^{-5}$ rad s^{-1} ; $R = 6.37 \times 10^6$ m

$$\therefore \quad g - g' = R\omega^2 \cos^2 \lambda = (6.37 \times 10^6) \times (7.27 \times 10^{-5})^2 \times \frac{1}{2} = 0.0168 \text{ m s}^{-2}$$

It shows that the value of acceleration due to gravity at the poles is greater than at latitude 45° by 0.0168 ms^{-2}. Hence the weight of a body of 1 kg mass at the poles will be greater than at 45° altitude by 0.0168 N.

Example 15.20. **Find the percentage decrease in the weight of a body when taken to a depth of 32 km below the surface of earth. Radius of earth is 6400 km.**

Solution. Weight of the body at depth d below earth's surface

$$= mg' = mg\left(1-\frac{d}{R}\right)$$

or $$mg - mg' = \frac{mgd}{R}$$

or $$\frac{mg-mg'}{mg} = \frac{d}{R}$$

$\therefore$ % age decrease in weight $= \frac{mg-mg'}{mg}\times 100 = \frac{d}{R}\times 100 = \frac{32}{6400}\times 100 = \mathbf{0.5\%}$

Example 15.21. On a planet whose size is the same and mass 4 times as that of earth, find the amount of work done to lift 6 kg of mass vertically upward through a distance of 3 m on the planet. The value of g on the surface of earth is 10 ms^{-2}.

Solution. On the planet, $g_P = \frac{GM_P}{R_P^2}$; On the earth, $g_E = \frac{GM_E}{R_E^2}$

$\therefore$ $$\frac{g_P}{g_E} = \frac{M_P}{M_E}\times\frac{R_E^2}{R_p^2} = 4 \times 1^2 = 4$$

or $$g_P = 4g_E = 4 \times 10 = 40 \text{ ms}^{-2}$$

$\therefore$ work done $= mg_P h = 6 \times 40 \times 3 = \mathbf{720\ J}$

Example 15.22. A body of weight 72 N moves from the surface of earth at a height half of the radius of earth. What is the gravitational force on it due to the earth?

Solution. The acceleration due to gravity g' at given height h (= $R/2$) is

$$g' = \frac{GM}{(R+h)^2} = \frac{GM}{(R+R/2)^2} = \frac{GM}{R^2(1+1/2)^2}$$

$$= \frac{GM}{R^2}\times\left(\frac{2}{3}\right)^2 = g\times\left(\frac{2}{3}\right)^2$$

$\therefore$ Gravitational force, $F = mg\times\left(\frac{2}{3}\right)^2 = 72\times\frac{4}{9} = \mathbf{32\ N}$

Example 15.23. At what height in km over the earth's pole, the free fall acceleration decreases by 1%? Radius of earth = 6400 km.

Solution. Let g' be the acceleration due to gravity at height h from the surface of earth. Given that $g' = 0.99\ g$.

Now $$g' = g\left(1-\frac{2h}{R}\right) \quad \text{or} \quad \frac{g'}{g} = 1-\frac{2h}{R}$$

$\therefore$ $$0.99 = 1-\frac{2h}{6400} \quad \text{or} \quad h = \mathbf{32\ km}$$

Note. If we are to find the height (h) over earth's surface at which acceleration due to gravity becomes 1% of that on earth's surface, then use the formula :

$$g' = \frac{GM}{(R+h)^2}$$

PROBLEMS FOR PRACTICE

1. What will be the value of acceleration due to gravity at a height 3200 km above the surface of earth when its value at earth is 9.8 ms^{-2} ? Radius of earth is 6400 km. **[4.35 m s^{-2}]**
2. At what height from the surface of earth will the value of g be reduced by 36% from the value at the surface? Radius of earth = 6400 km. **[1600 km]**
3. Find the percentage decrease in the weight of a body when taken 16 km below the surface of the earth. Given that radius of earth is 6400 km. **[0.25%]**
4. Assuming that the whole variation of the weight of a body with its position on earth's surface is due to the rotation of the earth, find the difference in the weight of 1g as measured at the poles and at the equator. Radius of earth = 6.38×10^6 m. **[0.034 N]**

15.12. TORQUE DUE TO GRAVITY

Consider a rigid body of mass M placed in constant gravitational field of earth. The body can be supposed to be made up of a large number of particles. Suppose m_i is the mass of the ith particle and $\vec{r}_i$ its position vector *w.r.t.* the origin O of the reference frame XOY (See Fig. 15.8). If $\vec{g}$ is constant acceleration due to gravity, then,

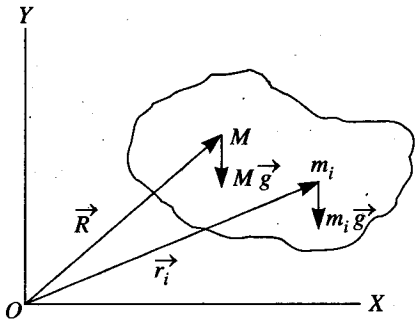

Fig. 15.8

Force acting on the ith particle due to gravity

$$= m_i \vec{g}$$

If $\vec{\tau}_i$ is the torque of $m_i \vec{g}$ about O, then,

$$\vec{\tau}_i = \vec{r}_i \times m_i \vec{g} = (m_i \vec{r}_i) \times \vec{g}$$

The total torque $\vec{\tau}$ on the body due to the forces on all the particles about point O is

$$\vec{\tau} = \sum \vec{\tau}_i = \sum (m_i \vec{r}_i) \times \vec{g}$$

Since $\vec{g}$ is constant, it can be taken out of the summation sign.

$$\therefore \quad \vec{\tau} = (\sum m_i \vec{r}_i) \times \vec{g} \qquad ...(i)$$

If $\vec{R}$ is the position vector of the centre of mass (C.M.) of the body, then by definition of centre of mass,

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M}$$

or $$\sum m_i \vec{r}_i = M \vec{R} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $\vec{\tau} = (M \vec{R}) \times \vec{g} = \vec{R} \times M \vec{g}$

or $$\vec{\tau} = \vec{R} \times M \vec{g} \qquad ...(iii)$$

Eq. (*iii*) reveals an interesting aspect. The torque on the body due to gravity acts as if the entire mass is concentrated at the centre of mass where the force $M \vec{g}$ appears to act. Of course, gravity pulls on all parts of the object. For torque purposes, however, the pull of gravity (the object's weight $M g$) appears to act at one point. We call this point the **centre of gravity** (C.G.) of the object.

We arrive at a very important conclusion that centre of gravity (C.G.) of a body coincides with its centre of mass (C.M.) in a uniform gravitational field such as found close to the surface of the earth.

Discussion. The following points may be noted :

(*i*) In case the point O about which torque is being considered coincides with centre of gravity, $R = 0$ and $\tau = 0$. Thus the torque due to gravity on a body about its centre of gravity is zero.

(*ii*) The centre of mass of a body is that point where the mass of the body can be thought to be concentrated. The centre of gravity is the point through which the total gravitational attraction can be considered to act on the body.

(*iii*) The centre of mass coincides with centre of gravity in a uniform gravitational field (*i.e.* constant $\vec{g}$). This is not the case with non-uniform gravitational fields such as those near a black hole where the gravitational intensity changes markedly with distance.

CONCEPTUAL QUESTIONS

Q.1. By how much does the gravitational force between two objects decrease when distance between them is doubled?

Ans. According to Newton's law of universal gravitation,

$$F = G\frac{m_1 m_2}{r^2}$$

If the distance between the objects is doubled, the gravitational force between the objects decreases to one-fourth.

Q.2. If there is an attraction force between all objects, why do we not feel ourselves gravitating towards massive buildings in our vicinity?

Ans. Gravity pulls us to massive buildings and everything else in the universe. The forces between us and the buildings are relatively small because masses are small compared to the mass of the earth. The forces due to stars are small because of their great distances. These tiny forces escape our notice when they are overwhelmed by the overpowering attraction to the earth.

Q.3. Consider an apple at the top of a tree that is pulled by earth gravity with a force of 1 N. If the tree were twice as tall, would the force of gravity be only 1/4 as strong?

Ans. No, because the twice-as-tall tree is not twice as far from the earth's centre. The tree's height would have to be equal to the radius of the earth (6370 km) before the force on apple reduces to (1/4) N. Before its weight reduces by 1%, the apple or any object must be raised 32 km above the surface of earth – nearly four times the height of Mount Everest. So as a practical matter, we disregard the effects of everday changes in elevation.

Q.4. Which pulls with greater force on the oceans of the earth, the sun or moon?

Ans. The sun, but it is so distant that it attracts all parts of the earth with almost equal strength. Hence its effectiveness in raising tides is less than that of the moon.

Q.5. The acceleration due to gravity is progressively less as we go toward the centre of earth. Why?

Ans. As you move toward the centre of earth, you are also being pulled up by the part of the earth above you. When you are at the centre of the earth, the pull down is balanced by the pull up so that $g = 0$.

Q.6. If you stepped into a hole bored clear through the centre of the earth and made no attempt to grab the edges at either end, what kind of motion you would experience?

Ans. You would oscillate back and forth, in what is called *simple harmonic motion* (S.H.M.). Each trip would take nearly 90 minutes. Interestingly enough, we will see in the next chapter that an earth satellite in close orbit about the earth also takes the same 90 minutes

to make a complete round trip. This is no coincidence, for if you study physics further you will learn that "up and down" simple harmonic motion is simply the vertical component of uniform circular motion.

Q.7. Halfway to the centre of the earth, would you weigh more or weigh less than you weigh at the surface of the earth?

Ans. You would weigh less because the part of earth's mass that pulls you "down" is counteracted by the mass above you that pulls you "up". If the density of earth were uniform, halfway to the centre, your weight would be exactly half your weight at the surface of the earth. But since earth's core is so dense (about seven times the density of surface rock), your weight would be somewhat more than half surface weight. Exactly how much depends upon how the earth's density varies with depth – information that is not known today.

Q.8. A body is taken from the centre of the earth to the moon. What will be the change in the weight of the body?

Ans. The weight of the body at the centre of earth will be zero ($g = 0$). As the body is moved from the centre to the earth surface, its weight will increase due to the increase in the value of g. At the surface of the earth, the weight of the body will be maximum. As the body is moved above the surface of the earth, its weight will decrease due to decrease in the value of g. At one place where the gravitational force of earth and moon are equal and opposite, the weight will become zero. Beyond this upto moon, gravitational force of the moon and hence weight of the body will go on increasing.

Q.9. The mass and diameter of a planet are twice those of the earth. What will be the time period of that pendulum on this planet which is a second pendulum on the earth?

Ans. $g = \dfrac{GM}{R^2}$

If mass M and diameter (or radius) are doubled, then g will reduce to half. Therefore, the time period of a pendulum will be $\sqrt{2}$ times to that on the earth. As a result, the time period of the second's pendulum of the earth will be $2\sqrt{2}$ seconds on the planet.

Q.10. There is no effect of the rotational motion of the earth on the value of g at the poles. Why?

Ans. Acceleration due to gravity at a latitude λ is given by ;

$$g' = g - R\omega^2 \cos^2 \lambda$$

At poles, $\lambda = 90°$ so that $g_P = g$. Hence there is no effect of rotational motion of earth on the value of g at the poles.

Q.11. The effect of rotation of earth on the value of g is greatest at the equator. Why?

Ans. Acceleration due to gravity at a latitude λ is given by ;

$$g' = g - R\omega^2 \cos^2 \lambda$$

The effect on acceleration due to gravity will be greatest when the factor $R\omega^2 \cos^2 \lambda$ is maximum. This will be so when $\cos \lambda = 1$ or $\lambda = 0°$. Since latitude at the equator = 0°, the effect of rotation of earth on the value of g is greatest at the equator.

Q.12. What are gravitation and gravity?

Ans. Gravitation is the force of attraction between any two bodies in the universe. However, gravity is the force of attraction exerted by earth towards its centre on a body lying on or near the surface of earth. Note that gravity is merely a special case of gravitation.

Q.14. What is the weight of a body at the centre of earth?

Ans. The weight (= mg) of a body becomes zero at the centre of earth because $g = 0$ at the centre of earth.

Q.15. When a body falls towards earth, earth moves towards the body. Why is motion of earth not noticed?

Ans. The earth attracts the body of mass m with a force = mg. The body also attracts earth with an equal force *i.e.*, mg. Since the mass of earth (M_E) is very large, acceleration produced in the earth is negligible ($a = F/M_E = mg/M_E$).

Q.16. If a person lands on moon, he has to tie a heavy weight at the back. Why?

Ans. The value of g on the moon is very small so that weight of the person will also be small.

Q.17. Where will the true weight of a body be zero?

Ans. The true weight of a body will be zero where gravitational force is zero *e.g.*, at the centre of earth or far away from any star or planet.

Q.18. If the radius of earth shrinks by 1.5% (mass remaining the same), by what percentage will acceleration due to gravity change?

Ans. 3% (See Example 15.10).

Q.19. Where does the body weigh more—at the surface of the earth or in a mine?

Ans. The body will weigh more at the surface of the earth.

Q.20. Where is the value of g maximum and minimum at the surface of earth?

Ans. The value of g is maximum at the poles and is minimum at the equator.

SHORT ANSWER QUESTIONS

Q.1. Why weight of a body becomes zero at the centre of earth?

Ans. $g' = g\left(1-\frac{d}{R}\right)$. Here g is acceleration due to gravity on earth's surface, d is depth inside the earth and R is radius of earth. At the centre of earth, $d = R$ so that $g' = 0$. Hence weight of the body is zero at the centre of earth.

Q.2. If a man goes from the surface of earth to a height equal to the radius of earth, then what will be its weight relative to that on earth?

Ans. $g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$. Here $h = R$ so that $g' = \frac{g}{4}$. Hence the weight of the man becomes one-fourth that on earth.

Q.3. A light body and a heavy body are allowed to fall from the same height. Which will reach the earth earlier?

Ans. Both bodies will reach the earth simultaneously. It is because acceleration due to gravity (g) is independent of mass of the body ($g = GM_E/R_E^2$).

Q.4. What would be the effect on the weight of a body if the earth stops rotating about its axis?

Ans. The weight of the body would increase because the effect of centrifugal force will be absent. Note that centrifugal force comes into picture because of the rotation of earth about its axis.

Q.5. If the radius of earth becomes two times the present value and its mass remains unchanged, then how would the weight of an object on the surface of the earth would be affected?

Ans. We know that $g = GM_E/R_E^2$.

Weight of object, $W = mg = \frac{mGM_E}{R_E^2}$

Since mass of earth M_E is unchanged, $W \propto 1/R_E^2$. If R_E is doubled, the weight of the body will become one-fourth (*i.e.*, $W/4$) of the present value

Q.6. Determine the fractional decrease in the value of g due to the increase in altitude by 8 km near the surface of earth, R_E = 6400 km.

Ans. $g' = g\left(1-\frac{2h}{R_E}\right)$ or $g - g' = \frac{2gh}{R_E}$

$\therefore$ Fractional decrease in $g = \frac{g-g'}{g} = \frac{2h}{R_E} = \frac{2\times 8\text{ km}}{6400\text{ km}} = 2.5 \times 10^{-3}$

Q.7. Assuming that the earth is a sphere of radius R, at what altitude will be the value of acceleration due to gravity be half its value on the earth's surface?

Ans. $g' = \frac{gR^2}{(R+h)^2}$ or $\frac{g}{2} = \frac{gR^2}{(R+h)^2}$ or $(R + h)^2 = 2R^2$

$\therefore R + h = \sqrt{2}R$ or $h = (\sqrt{2}-1)R$

Q.8. Determine the average density of earth.

Ans. Refer to Art. 15.6.

Q.9. Explain, why one can jump higher on the surface of the moon than on the earth?

Ans. Let v_0 be the speed of the man while taking a jump and h the height of the jump. Then,

$$v^2 - v_0^2 = 2gh \text{ or } 0^2 - v_0^2 = 2(-g)h \quad \therefore \quad h = v_0^2/2g.$$

Since acceleration due to gravity (g) on the moon is 1/6th of that on the earth, one can jump higher on the surface of moon than on the earth.

Q.10. If the earth stops rotating about its axis, by what value the acceleration due to gravity at the equator will change?

Ans. Acceleration due to gravity at the equator is $g_e = g - R\omega^2$ where R is the radius of earth and ω is the angular velocity of rotation of the earth. If earth stops rotating about its axis (*i.e.*, $\omega = 0$), then acceleration due to gravity will increase by $R\omega^2$.

Q.11. The acceleration due to gravity on the moon is 1/6th that on earth and mass of earth is 80 times that of the moon. What is the ratio of their radii?

Ans. $g = \frac{GM}{R^2}$ or $R = \sqrt{\frac{GM}{g}}$

For earth, $R_E = \sqrt{\frac{GM_E}{g_E}}$; For moon, $R_M = \sqrt{\frac{GM_M}{g_M}}$

$\therefore \quad \frac{R_E}{R_M} = \sqrt{\frac{M_E}{M_M}\times\frac{g_M}{g_E}} = \sqrt{80\times\frac{1}{6}} = 3.65$

Q.12. Why is the weight of a body at the equator less than that at the poles?

Ans. Acceleration due to gravity at a place on earth of latitude λ is $g' = g - R\omega^2\cos^2\lambda$.

At equator, $\lambda = 0°$ so that $g_e = g - R\omega^2$

At poles, $\lambda = 90°$ so that $g_p = g$

Since $g_p > g_e$, a body weighs less at the equator than at the poles.

Q.13. Distinguish between g and G.

Ans. (*i*) The acceleration due to gravity (g) is the acceleration produced in a freely falling body. The universal gravitational constant G is a constant of proportionality in the formula for force of attraction between two masses.

(*ii*) The value of g changes with height, depth and rotation of the earth. However, the value of G is constant at all locations.

(*iii*) The average value of g on the surface of earth is taken as 9.8 m/s^2. The value of $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Q.14. What will be the effect on the time period of a simple pendulum on taking it to a mountain?

Ans. Time period of simple pendulum, $T = 2\pi\sqrt{l/g}$. Therefore, $T \propto 1/\sqrt{g}$. Since the value of g is less at mountain than at plain, the time period of simple pendulum will be more at mountain than at plain.

Q.15. What is the difference between inertial mass and gravitational mass?

Ans. Refer to Chapter 3.

Q.16. How does the value of acceleration due to gravity change with (*i*) altitude (*ii*) depth (*iii*) rotation of earth?

Ans. In all the cases, the value of g decreases (Refer to Arts. 15.8, 15.9 and 15.11).

LONG ANSWER QUESTIONS

1. What do you understand by acceleration due to gravity? Write down equations of motion for a freely falling body. **[Refer to Art. 15.1]**
2. State and explain Newton's law of gravitation and hence define acceleration due to gravity. **[Refer to Arts. 15.2 and 15.5]**
3. Prove that acceleration due to gravity is maximum at the surface of earth. **[Refer to Arts. 15.8 and 15.9]**
4. Derive an expression for the variation of g with altitude. **[Refer to Art. 15.8]**
5. Derive an expression for the variation of g with depth. **[Refer to Art. 15.9]**
6. How does g change due to the rotation of earth about its own axis? **[Refer to Art. 15.11]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The gravitational force acting between two bodies is always attractive.
2. The gravitational force between two given bodies is
   (*i*) independent of nature of the intervening medium
   (*ii*) independent of the presence or absence of other bodies
   (*iii*) a central force *i.e.* it acts along the line joining the centres of the two bodies.
3. Gravitation is the force of attraction acting between any two bodies in the universe. Gravity is the force of attraction exerted by earth towards its centre on the body lying on or near the earth's surface.
4. The gravitational constant G is a scalar quantity and has the same value ($= 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$) throughout the universe.
5. The acceleration due to gravity on or near earth's suface is
$$g = \frac{GM}{R^2}$$
where M = mass of earth ; R = radius of earth
Note that the value of g is the same for all masses at earth's surface.
6. If ρ and V are the average density and volume of earth respectively, then,
$$\rho = \frac{\text{Mass of earth}}{\text{Volume of earth}} = \frac{M}{\frac{4}{3}\pi R^3}$$
Therefore, the value of g can be expressed in terms of density as :
$$g = \frac{GM}{R^2} = \frac{G\left(\frac{4}{3}\pi R^3 \rho\right)}{R^2} = \frac{4}{3}G\pi R\rho$$
7. The value of g (*i*) decreases with increase in height from the surface of earth (*ii*) decreases with increase in depth inside the earth.
8. Outside the earth, $g' = \frac{R^2}{r^2}g$; Inside the earth, $g' = \frac{r}{R}g$
where R = Radius of earth; r = Distance from centre of earth. It is clear from above relations that :
   (*i*) outside earth, $g' \propto \frac{1}{r^2}$; Inside earth, $g' \propto r$.
   (*ii*) value of acceleration due to gravity is maximum at earth's surface and zero at the centre of earth.
9. The value of acceleration due to gravity at the surface of planet of mass M_P and radius R_P is
$$g_P = \frac{GM_P}{R_P^2}$$
The value of g_p is minimum at Mercury and maximum at Jupiter.
10. The acceleration due to gravity decreases due to rotation of earth. The acceleration due to gravity at a place of latitude λ, due to rotation of earth is
$$g' = g - R\omega^2 \cos^2 \lambda$$
Here R = radius of earth;

ω = angular velocity of earth about its axis

At equator. $\lambda = 0°$ $\therefore$ $g_e = g - R\omega^2$ (minimum)

At poles. $\lambda = 90°$ $\therefore$ $g_p = g$ (maximum)

This is illustrated in Fig. 15.9.

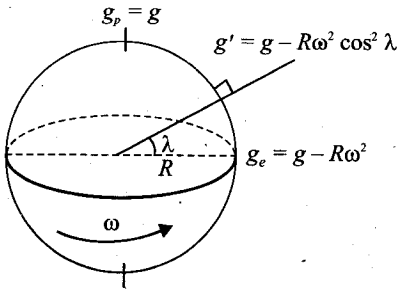

Fig. 15.9

11. If earth stops rotating ($\omega = 0$), the value of g' will increase by a factor $R\omega^2$ (maximum increase) at equator while there will be no change at poles.
12. The acceleration due to gravity is a vector quantity. It is always directed downward toward the centre of earth.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A body of weight 72 N moves from the surface of earth at a height half of the radius of the earth. Then gravitational force exerted on it will be **[CBSE PMT 2000]**
   (*a*) 36 N (*b*) 32 N
   (*c*) 14.4 N (*d*) 50 N
2. Assuming earth to be a sphere of uniform density, the acceleration due to gravity at a point 100 km below the earth's surface is (Given $R = 6380 \times 10^3$ m) **[Pb PMT 2000]**
   (*a*) 9.66 m/s^2 (*b*) 7.64 m/s^2
   (*c*) 5.06 m/s^2 (*d*) 3.10 m/s^2
3. The value of acceleration due to gravity on a planet having mass and radius twice those of earth is **[Pb PMT 2000]**
   (*a*) 19.6 m/s^2 (*b*) 9.8 m/s^2
   (*c*) 4.9 m/s^2 (*d*) zero
4. If the mass of earth is 80 times that of a planet and diameter double that of planet and g_E on earth is 9.8 m/s^2, then the value of g_p on that planet is **[AIIMS 1998]**
   (*a*) 4.9 m/s^2 (*b*) 0.98 m/s^2
   (*c*) 49 m/s^2 (*d*) 0.49 m/s^2
5. If radius of earth is 6.4×10^6 m and gravitational constant is 6.67×10^{-11} N-m^2/kg^2 then its mass is **[AFMC – Pune 1998]**
   (*a*) 6×10^{18} kg (*b*) 6×10^{24} kg
   (*c*) 6×10^{26} kg (*d*) 6×10^{31} kg
6. The radius of earth is about 6400 km and that of mars is 3200 km. The mass of earth is about 10 times mass of mars. An object weighs 200 N on the surface of earth. Its weight on the surface of mars will be **[CBSE PMT 1994]**
   (*a*) 80 N (*b*) 40 N
   (*c*) 20 N (*d*) 8 N
7. Two particles of equal masses go around a circle of radius R under the action of their mutual gravitational attraction. The speed v of each particle is **[CBSE PMT 1995]**
   (*a*) $\frac{1}{2R}\sqrt{\frac{1}{Gm}}$ (*b*) $\sqrt{\frac{Gm}{2R}}$
   (*c*) $\frac{1}{2}\sqrt{\frac{Gm}{R}}$ (*d*) $\sqrt{\frac{4Gm}{R}}$
8. If earth loses its gravity, then for a body **[Pb PMT 1998]**
   (*a*) both mass and weight become zero
   (*b*) neither mass nor weight becomes zero
   (*c*) mass becomes zero but not the weight
   (*d*) weight becomes zero but not the mass
9. At what height from the earth surface, the acceleration due to gravity will be half the value at the surface of earth? (R = 6400 km) **[AFMC–Pune 1997]**
   (*a*) 2650 km (*b*) 3200 km
   (*c*) 4800 km (*d*) 6400 km
10. If the mass of a body on the earth's surface is M, then its mass on the moon's surface is **[AIIMS 1997]**
    (*a*) $6M$ (*b*) $2M$
    (*c*) M (*d*) $M/6$
11. For moon, its mass is 1/81 of earth's mass and its diameter is 1/3.7 of earth's diameter. If acceleration due to gravity at earth's surface is 9.8 m/s^2, then at moon its value is **[CBSE PMT 1999]**
    (*a*) 2.86 m/s^2 (*b*) 1.65 m/s^2
    (*c*) 8.65 m/s^2 (*d*) 5.16 m/s^2

12. What will be the formula of mass of earth in terms of g, R and G? **[CBSE PMT 1996]**

(a) $g^2\frac{R}{G}$ (b) $G\frac{R}{g^2}$

(c) $G\frac{R}{g}$ (d) $g\frac{R^2}{G}$

13. If the radius of earth shrinks by 1% and its mass remains the same, then acceleration due to gravity on the earth's surface will **[AIIMS 1994]**

(a) decrease by 2% (b) increase by 2%

(c) decrease by 1% (d) increase by 1%

14. As we go from equator to poles, the value of g **[AFMC 1995]**

(a) remains the same (b) decreases

(c) increases

(d) decreases upto a latitude of 45°

15. Ratio of inertial mass to gravitational mass of a body is **[AFMC 2000]**

(a) 0.1 (b) 1

(c) 2 (d) no fixed number

16. The earth (mass = 6×10^{24} kg) revolves round the sun with angular velocity 2×10^{-7} rad/sec in a circular orbit of radius 1.5×10^8 km. The force exerted by the sun on earth is **[CBSE PMT 1995]**

(a) zero (b) 18×10^{25} N

(c) 27×10^{39} N (d) 36×10^{21} N

17. If the radius of earth contracted by 2% and its mass remains the same, then weight of the body at the earth's surface **[CPMT 1997]**

(a) will decrease (b) will increase

(c) will remain the same

(d) none of above

18. Mass of moon is 7.34×10^{22} kg. If acceleration due to gravity on moon is 1.4 m/s^2, the radius of the moon is ($G = 6.67 \times 10^{-11}$ Nm^2/kg^2). **[AFMC-Pune 1998]**

(a) 0.56×10^4 m (b) 1.92×10^8 m

(c) 1.87×10^6 m (d) 1.01×10^8 m

19. Height at which the value of g becomes one-fourth to that on earth is (R = radius of earth) **[BHU PMT 2000]**

(a) R (b) $2R$

(c) $(3/2)R$ (d) $4R$

20. Which of the following statement is true? **[Manipal 1995]**

(a) g is the same at all places on the surface of earth

(b) g has its maximum value at the equator

(c) g is less at earth's surface than at height above it or a depth below it

(d) g is greater at the poles than at the equator

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (b)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (b)
6. (a)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (c)
11. (b)	**12.** (d)	**13.** (b)	**14.** (c)	**15.** (b)
16. (d)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (d)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The acceleration due to gravity g' at given height h ($= R/2$) is

$$g' = \frac{GM}{(R+h)^2} = \frac{GM}{(R+R/2)^2} = \frac{GM}{R^2(1+1/2)^2}$$

$$= \frac{GM}{R^2} \times \left(\frac{2}{3}\right)^2 = g \times \left(\frac{2}{3}\right)^2$$

$\therefore$ Gravitational force, $F = mg \times \left(\frac{2}{3}\right)^2 = 72 \times \frac{4}{9} = 32$ N

2. The acceleration due to gravity below the surface of earth is

$$g' = g\left(1 - \frac{d}{R}\right)$$

Here $g = 9.8\ \text{m/s}^2$; $d = 100\ \text{km}$; $R = 6380 \times 10^3\ \text{m}$

$$\therefore \quad g' = 9.8\left(1 - \frac{100\times10^3}{6380\times10^3}\right) = \mathbf{9.66\ m/s^2}$$

3. Let us use the subscript P and E for planet and earth respectively.

$$g_p = \frac{GM_P}{R_P^2};\ g_E = \frac{GM_E}{R_E^2}$$

$$\therefore \quad \frac{g_P}{g_E} = \frac{M_P}{M_E}\times\left(\frac{R_E}{R_p}\right)^2 = 2\times\left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\therefore \quad g_P = \frac{g_E}{2} = \frac{9.8}{2} = \mathbf{4.9\ m/s^2}$$

4.
$$g_P = \frac{GM_P}{R_P^2};\ g_E = \frac{GM_E}{R_E^2}$$

$$\therefore \quad \frac{g_p}{g_E} = \frac{M_P}{M_E}\times\left(\frac{R_E}{R_P}\right)^2 = \frac{1}{80}\times(2)^2 = \frac{1}{20}$$

$$\therefore \quad g_p = \frac{g_E}{20} = \frac{9.8}{20} = \mathbf{0.49\ m/s^2}$$

5.
$$g = \frac{GM}{R^2} \quad \text{or} \quad M = \frac{gR^2}{G} = \frac{9.8\times(6.4\times10^6)^2}{6.67\times10^{-11}} = \mathbf{6\times10^{24}\ kg}$$

6.
$$\frac{g_M}{g_E} = \frac{M_M}{M_E}\times\left(\frac{R_E}{R_M}\right)^2 = \frac{1}{10}\times\left(\frac{6400}{3200}\right)^2 = \frac{4}{10}$$

$$\therefore \quad g_M = g_e\times\frac{4}{10}$$

$$\therefore \quad \text{Weight on mars} = mg_M = m\left(g_E\times\frac{4}{10}\right)$$

$$= mg_E\times\frac{4}{10} = 200\times\frac{4}{10} = \mathbf{80\ N}$$

7. Centripetal force $= \dfrac{mv^2}{R}$; Gravitational force $= \dfrac{Gmm}{4R^2}$

The gravitational force provides the necessary centripetal force. Therefore, referring to Fig. 15.10, we have,

$$\frac{mv^2}{R} = \frac{Gmm}{4R^2} \quad \text{or} \quad v^2 = \frac{Gm}{4R}$$

$$\therefore \quad v = \sqrt{\frac{Gm}{4R}} = \frac{1}{2}\sqrt{\frac{Gm}{R}}$$

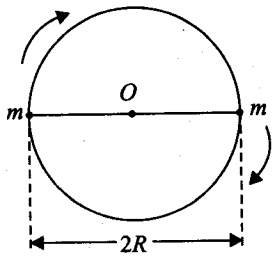

Fig. 15.10

8. Weight of body = mg. When earth loses its gravity, $g = 0$. Therefore weight of body = 0 but the mass does not change.

9. Let h be the height above earth's surface at which the acceleration due to gravity becomes one-half its value at earth's surface.

$\therefore \quad g' = \frac{gR^2}{(R+h)^2}$ or $\frac{g}{2} = \frac{gR^2}{(R+h)^2}$ or $(R+h)^2 = 2R^2$

$\therefore \quad R + h = \sqrt{2}R$ or $h = (\sqrt{2}-1)R = (\sqrt{2}-1)6400 = 2650$ km

10. The mass of a body remains constant irrespective of the place where the body is taken. Therefore, mass of the body on the surface of the moon is M.

11. $g_E = \frac{GM_E}{R_E^2}$; $g_M = \frac{GM_M}{R_M^2}$

$\therefore \quad \frac{g_M}{g_E} = \frac{M_M}{M_E} \times \left(\frac{R_E}{R_M}\right)^2 = \frac{1}{81} \times (3.7)^2$

$\therefore \quad g_M = g_E \times \frac{1}{81} \times (3.7)^2 = 9.8 \times \frac{1}{81} \times (3.7)^2 = 1.65 \text{ m/s}^2$

12. $g = \frac{GM}{R^2}$ $\therefore$ Mass of earth, $M = \frac{gR^2}{G}$

13. Decrease in radius, $\frac{dR}{R} = -1\%$. Minus sign is given due to decrease of radius.

Now $g = \frac{GM}{R^2}$

Taking log on both sides, we have, $\log_e g = \log_e G + \log_e M - 2\log_e R$.

Differentiating both sides, we get,

$$\frac{dg}{g} = 0 + 0 - \frac{2dR}{R} = -2(-1\%) = 2\%$$

Therefore, the value of acceleration due to gravity will increase by 2%.

14. $g' = g - R\omega^2 \cos^2 \lambda$

At equator, $\lambda = 0$ and $g_e = g - R\omega^2$ (minimum). As we go toward pole of earth, λ increases and $\cos^2 \lambda$ decreases. Therefore the velue of g' increases.

15. The ratio of inertial mass to gravitational mass of a body is 1.

16. The gravitational force between sun and earth provides the necessary centripetal force to cause the earth to rotate about the sun.

$\therefore$ Force exerted by sun on earth is

$$F = \frac{mv^2}{R} = mR\omega^2 = (6\times10^{24}) \times (1.5\times10^{11}) \times (2\times10^{-7})^2 = 36 \times 10^{21} \text{ N}$$

17. Weight of body $W = mg = \frac{GMm}{R^2}$

$\therefore \quad W \propto \frac{1}{R^2}$

If R decreases, the weight of the body will increase.

18. $g_m = \frac{GM_M}{R_M^2}$

$\therefore$ Radius of moon, $R_M = \sqrt{\frac{GM_M}{g_M}} = \sqrt{\frac{6.67\times10^{-11}\times7.34\times10^{22}}{1.4}} = 1.87 \times 10^6$ m

19. $\frac{g'}{g} = \frac{R^2}{(R+h)^2}$ or $\frac{1}{4} = \frac{R^2}{(R+h)^2}$

$\therefore$ $\frac{1}{2} = \frac{R}{R+h}$ or $h = R$

20. The value of g is maximum at poles and is minimum at the equator.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Acceleration due to gravity is 'g' on the surface of earth. The value of acceleration due to gravity at a height of 32 km above earth's surface is (Radius of earth = 6400 km) **[CET Karnataka 1999]**

(*a*) 0.9 g (*b*) 0.99 g

(*c*) 0.8 g (*d*) 1.01 g

2. At what height in km over the earth's pole the free fall acceleration decreases by 1%? (Radius of earth = 6400 km) **[KCET 1994]**

(*a*) 64 (*b*) 80

(*c*) 32 (*d*) 1.25

3. A spring balance is graduated on sea level. If a body is weighed with this balance at consecutively increasing heights from earth's surface, the weight indicated by the balance **[MP PET 1995]**

(*a*) will go on decreasing

(*b*) will go on increasing

(*c*) will remain same

(*d*) will increase first and then decrease

4. Where will it be profitable to purchase 1 kg sugar? **[Rajastan PET 1996]**

(*a*) at poles (*b*) at equator

(*c*) at 45° latitude (*d*) at 40° latitude

5. If radius of earth shrinks by 1.5% (mass remaining same), then the value of acceleration due to gravity changes by **[BHU 1997]**

(*a*) 1 % (*b*) 2 %

(*c*) 3% (*d*) 4 %

6. If R is the radius of a planet, g is the acceleration due to gravity, then the mean density of the planet is given by; **[BHU 1998]**

(*a*) $\frac{3g}{4\pi GR}$ (*b*) $\frac{4\pi GR}{3g}$

(*c*) $\frac{4\pi gG}{3G}$ (*d*) $\frac{3G}{4\pi gR}$

7. If the earth suddenly shrinks to half of its present radius, the acceleration due to gravity will be **[Pb CET 1998]**

(*a*) $g/2$ (*b*) $4g$

(*c*) $g/4$ (*d*) $2g$

8. Determine the speed at which the earth will have to rotate on its axis so that a person on its equator weighs (3/5) as much as present. Take equatorial radius = 6400 km **[Roorkee 1998]**

(*a*) 7.83×10^{-4} rad/sec

(*b*) 2.4×10^{-2} rad/sec

(*c*) 5.74×10^{-3} rad/sec

(*d*) 4.12×10^{-2} rad/sec

9. What should be the angular speed of earth so that an object on equator may appear weightless? (g = 10 ms^{-2}; radius of earth = 6400 km) **[AMU 1997]**

(*a*) 1.56 rad/s (*b*) 1.25×10^{-1} rad/s

(*c*) 1.56×10^{-3} rad/s (*d*) 1.25×10^{-3} rad/s

10. The mass of moon is 7.34×10^{22} kg and the radius is 1.74×10^6 m. The value of acceleration due to gravity is **[MNR 1999]**

(*a*) 1.45 m/s^2 (*b*) 1.55 m/s^2

(*c*) 1.75 m/s^2 (*d*) 1.62 m/s^2

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS				
1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*b*)	**5.** (*c*)
6. (*a*)	**7.** (*b*)	**8.** (*a*)	**9.** (*d*)	**10.** (*d*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. The acceleration due to gravity at a height '*h*' above earth's surface is

$$g' = g\left(1-\frac{2h}{R}\right) = g\left(1-\frac{2\times32}{6400}\right) = 0.99\ g$$

2.
$$g' = g\left(1-\frac{2h}{R}\right) \quad \text{or} \quad \frac{g'}{g} = 1-\frac{2h}{R}$$

$$\therefore \quad \frac{99}{100} = 1-\frac{2h}{6400} \quad \text{or} \quad h = 32 \text{ km}$$

3. Weight of body = mg. As the height above earth's surface increases, the value of g decreases.

4. The shopkeeper normally uses spring balance for weighing purposes. Weight = mg. Since g is minimum at the equator, we will get more mass at equator as compared to the other places.

5. $g = \dfrac{GM}{R^2} \quad \therefore \dfrac{\Delta g}{g} = -\dfrac{2\Delta R}{R} = -2(-1.5\%) = 3\%$

6. Mass of planet, $M = \dfrac{4}{3}\pi R^3 \times \rho$

Now,
$$g = \frac{GM}{R^2} = \frac{G\left(\frac{4}{3}\pi R^3 \times \rho\right)}{R^2} \quad \therefore \rho = \frac{3g}{4\pi GR}$$

7.
$$g = \frac{GM}{R^2} \text{ or } g \propto \frac{1}{R^2}$$

Now
$$g' \propto \frac{1}{(R/2)^2} \quad \therefore \quad \frac{g'}{g} = 4 \text{ or } g' = 4g$$

8. Weight at equator = $mg - mR\omega^2$

or
$$\frac{3}{5}mg = mg - mR\omega^2 \quad \text{or} \quad mR\omega^2 = \frac{2}{5}mg$$

$$\therefore \quad \omega = \sqrt{\frac{2g}{5R}} = \sqrt{\frac{2\times9.8}{5\times6.4\times10^6}} = 7.83 \times 10^{-4} \text{ rad/s}$$

9. Weight at equator $= mg - mR\omega^2$ or $0 = mg - mR\omega^2$

$$\therefore \quad \omega = \sqrt{\frac{g}{R}} = \sqrt{\frac{10}{6400\times10^3}} = 1.25 \times 10^{-3} \text{ rad/s}$$

10. $g = \dfrac{GM}{R^2} = \dfrac{6.67\times10^{-11}\times7.34\times10^{22}}{(1.74\times10^6)^2} = 1.62 \text{ m/s}^2$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **The earth's radius is about 6370 km. An object of mass 20 kg is taken to a height of 160 km above the earth's surface.**
   (*i*) What is the object's mass at this height?
   (*ii*) How much does the object weigh at this height?

Hint. (*i*) The mass of the object is the same as on the earth's surface.

(*ii*) Outside the earth's surface, the weight varies inversely as the square of the distance from the centre of earth. Indeed,

$$W = G\frac{mM}{r^2}$$

Here m and M are the masses of the object and earth respectively. And r is the distance of the object from the centre of earth.

$$\therefore \quad \frac{W_2}{W_1} = \frac{r_1^2}{r_2^2} \quad \text{or} \quad W_2 = W_1 \times \frac{r_1^2}{r_2^2}$$

Here $W_1 = 20 \times 9.8 = 196$ N ; $r_1 = 6370$ km ; $r_2 = 6370 + 160 = 6530$ km

$$\therefore \quad W_2 = 196 \times \left(\frac{6370}{6530}\right)^2 = \mathbf{186.5\ N}$$

2. **Determine the magnitude of acceleration due to gravity at an altitude of 500 km. By what percentage is the weight of a body reduced at this altitude? Mass of earth, $M = 5.98 \times 10^{24}$ kg, Radius of earth = 6.38×10^6 m.**

Hint. If g (= 9.8 ms^{-2}) be the acceleration due to gravity on the surface of the earth, then its value at a height h above the surface of earth is given by ;

$$g' = \frac{GM}{(R+h)^2} = \frac{(6.67\times10^{-11})\times(5.98\times10^{24})}{(6.38\times10^6+0.5\times10^6)^2} = \mathbf{8.43\ ms^{-2}}$$

% decrease in the weight of the body

$$= \frac{mg - mg'}{mg}\times100 = \frac{g-g'}{g}\times100 = \frac{9.8-8.43}{9.8}\times100 = \mathbf{14\%}$$

3. **The acceleration due to gravity on the moon is 1.7 ms^{-2} and its radius is 0.27 times that of earth. Compute the ratio of the mass of the earth to that of the moon. The acceleration due to gravity on earth's surface is 9.8 ms^{-2}.**

Hint. Acceleration due to gravity on the earth's surface is given by ;

$$g_E = \frac{GM_E}{R_E^2} \qquad ...(i)$$

Acceleration due to gravity on the moon is given by;

$$g_M = \frac{GM_M}{R_M^2} \qquad ...(ii)$$

$$\therefore \quad \frac{g_E}{g_M} = \left(\frac{M_E}{M_M}\right)\times\left(\frac{R_M^2}{R_E^2}\right) \quad \text{or} \quad \left(\frac{M_E}{M_M}\right) = \frac{g_E}{g_M}\times\left(\frac{R_E}{R_M}\right)^2$$

Here $g_E = 9.8\ ms^{-2}$; $g_M = 1.7\ ms^{-2}$; $R_M = 0.27\ R_E$

$$\therefore \quad \frac{M_E}{M_M} = \left(\frac{9.8}{1.7}\right)\times\left(\frac{1}{0.27}\right)^2 = 79$$

$$\therefore \quad M_E = 79M_M$$

That is the mass of the earth is 79 times the mass of the moon.

4. A man weighs 60 kg at earth's surface. At what height above the earth's surface, his weight becomes 30 kg? Radius of earth, R = 6400 km.

Hint. If g is the acceleration due to gravity on the surface of the earth, then its value at a height h above the surface of earth is given by ;

$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \quad \text{or} \quad \frac{g}{2} = \frac{g}{\left(1+\frac{h}{R}\right)^2} \quad \therefore \quad 1+\frac{h}{R} = \sqrt{2} = 1.41$$

or $$h = R(1.41-1) = R\times 0.41 = 6400\times 0.41 = \mathbf{2624\ km}$$

5. A body of mass m is raised through a height h from a point on the surface of earth where the acceleration due to gravity is g. Prove that loss of weight due to variation of g is approximately $2mgh/R$, where R is the radius of earth.

Hint. $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} = g\left(1+\frac{h}{R}\right)^{-2} = g\left(1-\frac{2h}{R}+\text{terms containing higher powers of } \frac{h}{R}\right)$$

If h is small compared to R, higher powers of h/R can be neglected.

$$\therefore \quad g' = g\left(1-\frac{2h}{R}\right) \quad \text{or} \quad mg' = mg\left(1-\frac{2h}{R}\right)$$

or $$mg - mg' = \frac{2mgh}{R} \quad \therefore \quad \text{Loss of weight} = \frac{2mgh}{R}$$

6. The radius of earth is about 6370 km and that of Mars is about 3440 km. If an object weighs 200 N on earth, what would it weigh and what would be the acceleration due to gravity on Mars? Mars has a mass 0.11 that of earth.

Hint. According to Newton's law of gravitation,

$$\text{Weight, } W = \frac{GmM}{r^2}$$

where m and M are the masses of the object and the planet respectively and r is the radius of the planet. Letting 1 refer to earth and 2 to Mars, we have,

$$\frac{W_2}{W_1} = \frac{M_2}{M_1}\times\frac{r_1^2}{r_2^2} = 0.11\times\left(\frac{6370}{3440}\right)^2$$

$$\therefore \quad W_2 = W_1\times 0.11\times\left(\frac{6370}{3440}\right)^2 = 200\times 0.11\times\left(\frac{6370}{3440}\right)^2 = \mathbf{75\ N}$$

Also $$\frac{W_2}{W_1} = \frac{g_2}{g_1} \quad \text{or} \quad g_2 = \frac{W_2}{W_1}\times g_1 = \frac{75}{200}\times 9.8 = \mathbf{3.7\ ms^{-2}}$$

16

Satellite and Planetary Motion

INTRODUCTION

The **universe** is the totality of all matter, space and time. The smallest basic unit of universe is the star just as the smallest unit in the army is the platoon. A **star** is a luminous gaseous heavenly body that generates energy by means of nuclear fusion reactions. The sun is the nearest star to earth. Around some stars are planets. A **planet** is a heavenly body that orbits around the sun or another star and shines only by the light that it reflects. We know that earth is one of the smaller planets around our star — the sun. It shines by reflecting sunlight. A heavenly body that orbits the planet is called a **natural satellite** because it is created by nature. The natural satellites of planets are also not self-luminous but shine only by reflecting sun's (or star's) light. A planet may or may not have natural satellites. For example, moon is the only natural satellite of the earth and it shines by the reflected sunlight. Mercury as well as Venus has no natural satellite. However, Mars has two natural satellites while Jupiter has 16 natural satellites. A man-made object that orbits around the earth or some other heavenly body is called an **artificial satellite**. In this chapter, we shall discuss the various aspects of satellites and planetary motion.

16.1. NATURAL AND ARTIFICIAL SATELLITES

A heavenly body that revolves around a planet in a stable orbit is called a **natural satellite.** For example, moon is the natural satellite of the earth. It goes round the earth in about 27.3 days in a nearly circular orbit of radius 3.84×10^5 km. *A man-made satellite that orbits around the earth or some other heavenly body is called an* **artificial satellite.** Artificial satellites circling the earth are now quite common. They are called earth satellites. For example, communication satellites are used routinely to transmit information around the globe.

For the sake of simplicity, we shall consider that satellites move in circular orbits. The centripetal force required by a satellite to move in a circular orbit is provided by the gravitational force of attraction between the satellite and the body around which it revolves. For example, in case of earth satellite, the gravitational attraction of earth provides the necessary centripetal force required to move the satellite in a circular path around the earth. Thus we use the concept of circular motion and Newton's law of universal gravitation in the study of satellites and planetary motion.

16.2. PRINCIPLE OF LAUNCHING EARTH SATELLITE

To understand the principle of launching earth satellite, consider a ball thrown horizontally from the top of a high mountain as shown in Fig. 16.1. If the gravity did not act on the ball, the ball would follow a straight line path shown by the dotted line. But there is gravity so that the ball

follows a parabolic path and hits the surface of earth at point A_1. If the horizontal velocity is increased, the ball will travel a greater horizontal distance before hitting the surface of earth at point A_2. As the horizontal velocity of the ball is increased, the horizontal range of the ball also increases. Finally, a stage is reached when the horizontal velocity is large enough that ball's path follows the curvature of the earth. This is the launching velocity which places the ball (or any other object) in a circular orbit around the earth. Thus the object in the circular orbit may be regarded as falling, but as it falls, its path is concentric with the earth's spherical surface so that the object maintains a fixed distance from the centre of the earth. The velocity required to put the object in its orbit is called the *orbital velocity.*

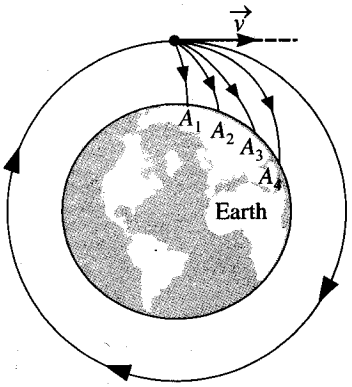

Fig. 16.1

For a satellite close to earth's surface, the required orbital velocity is about 8 km/s. At this high speed, the atmospheric friction would probably burn the satellite. Therefore, the satellite is taken by a rocket to a suitable height above the surface of earth (*150 km or so) and then given the suitable horizontal velocity so as to put it in a circular path around the earth.

16.3. ORBITAL VELOCITY OF SATELLITE

The velocity required to put a satellite into a given orbit around the earth is called **orbital velocity** *of the satellite in that orbit.*

Suppose a satellite of mass m is to be put into circular orbit around the earth at a height h above its surface as shown in Fig. 16.2. The mass and radius of earth are M and R. Then the radius of the orbit of the satellite is $(R + h)$. Let v be the orbital velocity of the satellite in this orbit. *Note that the direction of the orbital velocity of the satellite at any instant is along the tangent to the orbit of satellite at that instant* (See Fig. 16.2). The centripetal force required by the satellite to move in the circular orbit is provided by the gravitational pull of earth acting on the satellite.

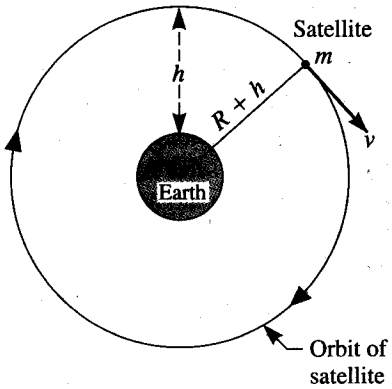

Fig. 16.2

$$\text{Centripetal force required by satellite} = \frac{mv^2}{R+h}$$

$$\text{Gravitational pull of earth on satellite} = \frac{GMm}{(R+h)^2}$$

$$\therefore \quad \frac{mv^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$\therefore \quad \text{Orbital velocity, } v = \sqrt{\frac{GM}{(R+h)}} \qquad ...(i)$$

Discussion. The following points are worth noting :

(*i*) *The orbital velocity of the satellite is independent of the mass of the satellite.*

(*ii*) *The orbital velocity of the satellite decreases as the height (h) of the satellite increases.* Therefore, the value of orbital velocity is different for different orbits around the earth.

* 100 km above the surface of earth, the atmosphere is only one-millionth as dense as it is at sea level. So friction at that height should be negligible.

(*iii*) The orbital velocity of a satellite depends upon the mass (M) and radius (R) of the earth/ planet around which it revolves.

For satellite close to earth. When the satellite orbits very close to earth's surface, then h can be neglected as compared to R in eq. (*i*) above.

$$\therefore \quad \text{Orbital velocity, } v = \sqrt{\frac{GM}{R}} \qquad \text{...For satellite close to earth}$$

Alternative expression for v. Now $v = \sqrt{\dfrac{GM}{(R+h)}}$

The acceleration due to gravity on earth's surface is

$$g = \frac{GM}{R^2} \quad \text{or} \quad GM = gR^2$$

Putting the value of $GM = gR^2$ in the above expression,

$$\text{Orbital velocity, } v = R\sqrt{\frac{g}{(R+h)}}$$

For satellite close to earth Neglecting h compared to R, we have,

$$\text{Orbital velocity, } v = R\sqrt{\frac{g}{R}} = \sqrt{gR}$$

$$\text{Now,} \quad R = 6.4 \times 10^6 \text{ m} \; ; \; g = 9.8 \text{ m s}^{-2}$$

$$\therefore \quad v = \sqrt{9.8 \times 6.4 \times 10^6} \simeq 8 \times 10^3 \text{ m s}^{-1} = 8 \text{ km s}^{-1} \text{ (approx.)}$$

Thus the orbital speed of the satellite close to earth's surface is about 8 km s^{-1}. In practice, the satellite is carried by a rocket to the height of the orbit and then given an impulse, by firing jets, to deflect it in a direction parallel to the tangent of the orbit. Its velocity is boosted to 8 km s^{-1} so that it stays in the orbit. Since this motion may continue indefinitely, we may say that the orbit is *stable.*

Note. The orbital velocity of a satellite revolving around any planet is

$$v_P = \sqrt{\frac{GM_P}{(R_P + h)}}$$

Here M_P = mass of the planet ; R_P = radius of the planet.

Note that v_P depends upon M_P, R_P and h (= height of satellite above the planet's surface).

16.4. TIME PERIOD OF SATELLITE

The time taken by the satellite to complete one revolution around the earth is called **time period** *of the satellite.* It is denoted by T.

Suppose a satellite of mass m is to be put into circular orbit around the earth at a height h above its surface. The mass and radius of earth are M and R. Then the radius of the orbit of the satellite is $(R + h)$. Let v be the orbital velocity of the satellite in this orbit.

$$\therefore \quad \text{Time period of satellite, } T = \frac{\text{Circumference of orbit}}{\text{Orbital velocity}} = \frac{2\pi(R+h)}{v} \qquad \text{...(i)}$$

We shall derive three equivalent expressions for the time period T of the satellite.

(*i*) **First expression for T.** Now $v = \sqrt{\dfrac{GM}{R+h}}$. Putting this value of v in eq. (*i*), we have,

$$T = \frac{2\pi(R+h)}{[GM/R+h]^{1/2}} = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} \qquad ...(ii)$$

It is clear that the time period (T) of the satellite depends only upon its height (h) above the earth's surface. The greater is the height of a satellite above the earth's surface, the greater is its period of revolution. It is for this reason that the moon, which is at a height of 3.84×10^5 km above earth, completes one revolution in about 27 days while an artificial satellite circling close to the earth's surface completes 10 to 20 revolutions in a day.

(*ii*) **Second expression for *T*.** Also $v = R\sqrt{\frac{g}{R+h}}$. Putting this value of v in eq. (*i*), we have,

$$T = \frac{2\pi(R+h)}{R\sqrt{\frac{g}{R+h}}} = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}}$$

$$\therefore \quad T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}} \qquad ...(iii)$$

Note again that the time period (T) of the satellite depends *only* upon its height (h) above the earth's surface.

(*iii*) **Third expression for *T*.** Now $T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$.

Mass of earth, M = Volume × Density = $\frac{4}{3}\pi R^3 \times \rho$

Here ρ is the mean density of earth. Putting the value of M in the above equation, we have,

$$T = 2\pi = \sqrt{\frac{(R+h)^3}{G} \times \frac{1}{\frac{4}{3}\pi R^3\rho}} = \sqrt{\frac{3\pi(R+h)^3}{G\rho R^3}}$$

$$\therefore \quad T = \sqrt{\frac{3\pi(R+h)^3}{G\rho R^3}} \qquad ...(iv)$$

Again, the time period (T) of the satellite depends *only* upon its height (h) above the earth's surface.

For satellite close to earth. If the satellite orbits very close to the surface of the earth, then h can be neglected as compared to R. Therefore, eqs. (*ii*), (*iii*) and (*iv*) for the time period (T) of the satellite become :

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \; ; \; T = 2\pi\sqrt{\frac{R}{g}} \; ; \; T = \sqrt{\frac{3\pi}{G\rho}}$$

Illustration. Now $T = 2\pi\sqrt{\frac{R}{g}}$

Here $R = 6.4 \times 10^6$ m ; $g = 9.8$ m s^{-2}

$$\therefore \quad T = 2\pi\sqrt{\frac{6.4\times10^6}{9.8}} = 5075 \text{ seconds} \simeq 84 \text{ minutes}$$

Thus the orbital speed of a satellite revolving very near to the earth's surface is about 8 km s^{-1} and its period of revolution is nearly 84 minutes.

16.5. HEIGHT OF SATELLITE ABOVE EARTH'S SURFACE

Let us now see how the height of the satellite above earth's surface can be determined. Consider a satellite of mass m orbiting the earth at a height h above its surface. The mass and radius of earth are M and R. Then the radius of the orbit of the satellite is $(R + h)$. The time period (T) of the satellite is given by ;

$$T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}}$$

Squaring both sides, we get, $T^2 = \dfrac{4\pi^2(R+h)^3}{R^2 g}$

or $$(R+h)^3 = \frac{gR^2T^2}{4\pi^2}$$

or $$R+h = \left(\frac{gR^2T^2}{4\pi^2}\right)^{1/3}$$

or $$h = \left(\frac{gR^2T^2}{4\pi^2}\right)^{1/3} - R$$

If we know the time period T of the satellite, radius R of earth and acceleration due to gravity (g) at earth's surface, the height h of the satellite above earth's surface can be calculated.

16.6. ENERGY OF SATELLITE

A satellite orbiting the earth has both kinetic energy (K.E.) and potential energy (P.E.). Therefore, the total mechanical energy of the satellite is the sum of its K.E. and P.E.

Consider earth to be a sphere of radius R and mass M. Suppose a satellite of mass m revolves around the earth in a circular orbit at a height h above the surface of earth. The radius of the orbit of the satellite is $(R + h)$. Let v be the orbital velocity of the satellite in this orbit.

$$\therefore \quad \text{K.E. of satellite} = \frac{1}{2}mv^2$$

The gravitational force exerted by the earth on the satellite is

$$F = \frac{GMm}{(R+h)^2}$$

This gravitational force provides the necessary centripetal force $(= m\,v^2/R + h)$ to the satellite to move in the circular orbit of radius $(R + h)$.

$$\therefore \quad \frac{mv^2}{R+h} = \frac{GMm}{(R+h)^2}$$

or $$m\,v^2 = \frac{GMm}{R+h}$$

or $$\frac{1}{2}mv^2 = \frac{GMm}{2(R+h)}$$

$$\therefore \quad \text{K.E. of satellite, } K = \frac{GMm}{2(R+h)} \qquad ...(i)$$

Eq. (i) gives the K.E. of the satellite at a height h above the surface of the earth.

The potential energy of the satellite at a height h above the earth's surface is

$$\text{P.E. of satellite} = -\frac{*GMm}{(R+h)} \qquad ...(ii)$$

$$\therefore \quad \text{Total energy of satellite, } E = \frac{GMm}{2(R+h)} + \left(\frac{-GMm}{R+h}\right)$$

or
$$E = \frac{-GMm}{2(R+h)}$$

Discussion. The following points may be noted :

(*i*) P.E. of satellite = −2 × K.E. of satellite

(*ii*) *The total energy of the satellite in the orbit is* **negative**. It has a special meaning. At infinity ($R + h = \infty$), the potential energy as well as kinetic energy is zero. Therefore, total energy is zero. The kinetic energy can never be negative. Therefore, a negative total energy means that in order to send the satellite to infinity, we have to give energy to the satellite *i.e.,* positive work must be done to free the satellite from the earth's gravitational force. Unless a revolving satellite gets extra energy, it would not leave its orbit, *i.e.,* it will go on revolving in a *closed* orbit. We say that the satellite is *bound* to the earth.

Binding energy of satellite. *It is the energy required to remove the satellite from its orbit around the earth to infinity.* At infinity, the total energy of the satellite is zero.

$$\therefore \quad \text{Binding energy of satellite} = 0 - (\text{Total energy of satellite})$$

$$= -\left(\frac{-GMm}{2(R+h)}\right) = \frac{GMm}{2(R+h)}$$

16.7. ANGULAR MOMENTUM OF SATELLITE

Consider a satellite of mass m orbiting the earth at a height h above its surface. The mass and radius of earth are M and R. Then the radius of the orbit of the satellite is $(R + h)$. Let v be the orbital velocity of the satellite in this orbit. Therefore, the magnitude of angular momentum (L) is given by ;

$$L = mv \times \text{Radius of satellite's orbit}$$

$$= mv \times (R + h)$$

$$\text{Now, orbital velocity, } v = \sqrt{\frac{GM}{R+h}}$$

$$\therefore \quad L = m \times \sqrt{\frac{GM}{R+h}} \times (R+h)$$

or
$$L = m\sqrt{GM(R+h)}$$

Thus the angular momentum of a satellite depends on :

(*i*) the mass (m) of the satellite and the mass (M) of the earth/planet.

(*ii*) the radius of the orbit ($R + h$) of the satellite.

---

* For proof, the reader may refer to Art. 17.5 in Chapter 17.

16.8. GEOSTATIONARY SATELLITES

A satellite that appears to be at a fixed position in the sky to an observer on earth is called a **geostationary satellite** *or* **stationary satellite.**

A geostationary satellite revolves around the earth with the same angular speed in the same direction (west to east) as is done by the earth. Therefore, velocity of such a satellite relative to earth is zero. For this reason, the satellite appears to be stationary to an observer on earth. Since the angular speed of geostationary satellite is the same as that of earth, the period of revolution of this satellite is 24 hours *i.e.*, T = 24 hours. Fig. 16.3 shows the orbit of geostationary satellite. Note that the geostationary satellite revolves around the earth from west to east in a close circular orbit and coplanar with the equatorial plane. Most of communication satellites are geostationary satellites so that information can be transmitted from one part of the world to another.

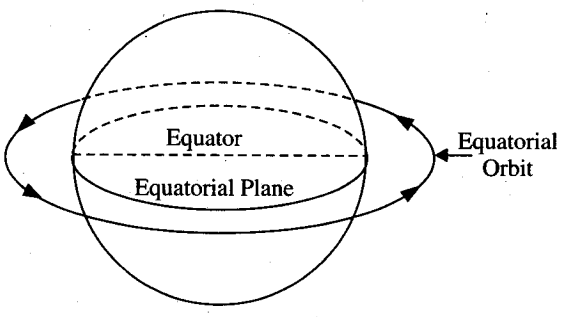

Fig. 16.3

1. Height of geostationary satellite above earth's surface. We know that the time period T of an earth satellite is given by ;

$$T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}} \qquad ...(i)$$

Here R = Radius of earth = 6400 km

T = Time period = 24 hours = 24 × 60 × 60 s

h = Height of the artificial satellite above earth's surface, corresponding to T = 24 hours

g = 9.8 ms^{-2} = 0.0098 km s^{-2}

$$\therefore \quad 24\times60\times60 = \frac{2\pi}{6400}\sqrt{\frac{(6400+h)^3}{0.0098}}$$

or $h \simeq 36000\,\text{km}$

Discussion. The following points are worth noting :

(*i*) A geostationary satellite orbits around the earth at a height of 36000 km above the surface of the earth.

(*ii*) *The height of the orbit of geostationary satellite is independent of the mass of the satellite.*

(*iii*) The orbit of the geostationary satellite is sometimes called **parking orbit.**

2. Orbital speed of geostationary satellite. The radius of the parking orbit is $R_P = R + h$ = 6400 + 36000 = 42400 km. Therefore, orbital speed v of the satellite in the parking orbit is

$$v = \frac{\text{Circumference of parking orbit}}{\text{Time period of satellite}}$$

or $$v = \frac{2\pi R_P}{24\,\text{hours}} = \frac{2\pi\times42400}{24\times60\times60} = 3.1\ \text{km s}^{-1}$$

Thus a geostationary satellite revolves around the earth at a height of 36000 km above earth's surface with an orbital speed of 3.1 km s^{-1}.

3. Placing satellite in the parking orbit. When a satellite is to be placed in the parking orbit, it is first carried to a height of 36000 km above earth's surface. It is then given the necessary tangential velocity v (= 3.1 km s^{-1}) by firing rocket engines which are aligned parallel to earth's surface. *In order to put a satellite in the parking orbit, the following conditions must be satisfied :*

(*i*) It should revolve in an orbit concentric and coplanar with the equatorial plane.

(*ii*) Its motion should be synchronous with the axial rotation of the earth. In other words, the time period of the satellite should be 24 hours.

(*iii*) It should rotate in the same direction as the earth is rotating.

(*iv*) The height of the parking orbit should be 36000 km above the earth's surface.

16.9. POLAR SATELLITE

The satellite that revolves around the earth in polar orbit is called **polar satellite.**

A polar orbit is that orbit which is at right angles to the equatorial plane as shown in Fig. 16.4. Note that the polar satellite passes over both geographical north and south poles in one revolution. Since the polar satellite passes over different parts of the earth during its each revolution, it can survey and scan the entire surface of the earth.

Polar satellites are used to record the land and sea temperatures, take picture of clouds and make forecasting of climatic changes. It may be noted that a polar satellite is more reliable than a weather satellite in the geostationary orbit. It is because a polar satellite can survey and scan earth's surface more closely.

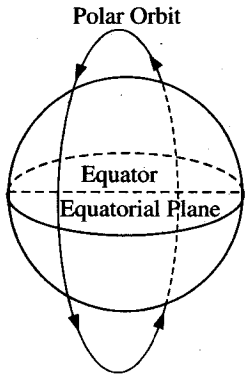

Fig. 16.4

16.10. USES OF ARTIFICIAL SATELLITES

When a satellite is to be placed in the orbit, it is first carried to the desired height by a rocket. It is then given the necessary tangential velocity by firing rocket engines which are aligned parallel to the earth's surface. The satellite then turns into the required orbit. The earth satellites have several uses.

(*i*) They are used to learn about the atmosphere near the earth.

(*ii*) They are used to forecast weather.

(*iii*) They are used to study radiation from the sun and the outer space.

(*iv*) They are used to receive and transmit various radio and television signals.

(*v*) They are used to know the exact shape and dimensions of the earth.

(*vi*) Space flights are possible due to artificial satellites.

16.11. WEIGHTLESSNESS

When the effective weight of a body becomes zero, it is said to be in a state of **weightlessness.**

The weight $W(= mg)$ of a body becomes zero where $g = 0$. Under this condition, the body is in a weightlessness state. The state of weightlessness can be observed in the following situations :

(*i*) **Body at the centre of earth.** At the centre of earth, $g = 0$. Therefore, if a body is taken to the centre of the earth, its effective weight becomes zero.

(*ii*) **Body at null points in space.** As we go up, the gravitational pull of earth goes on decreasing but the gravitational pull of moon goes on increasing. At a particular position in space, the two gravitational pulls are equal and opposite and effective value of $g = 0$. Such points in space are called **null points**. If a body is taken to null points, its effective weight becomes zero.

(*iii*) **Body in a freely falling lift.** When a body is lying in a lift and falls freely, the effective acceleration is $g' = (g - a) = (g - g) = 0$. Therefore, the effective weight of a body in the freely falling lift is zero.

(*iv*) **Body in a satellite orbiting earth.** The effective weight of body inside a satellite orbiting around the earth becomes zero. This is explained in detail in Art. 16.12.

The condition of weightlessness in a satellite poses many problems to astronauts including *difficulty in controlling movements, difficulty in eating and drinking* and *adverse effects on human organism.*

16.12. WEIGHTLESSNESS IN SATELLITES

People and other objects in a satellite circling the earth are said to experience apparent weightlessness. Before tackling the case of a satellite, let us first look at the simpler case of a falling elevator. We defined the weight of an object as the force it exerts against the supporting floor. The object appears weightless in a freely falling elevator. If we think about it a little, this is not strange at all. Since the elevator and everything in it are accelerating with the acceleration of free fall, by the very definition of free fall, there can be no force supporting the objects (elevator and everything in it) or in any way retarding their free-fall. Hence all support forces on the elevator and everything in it must be zero. This is called state of weightlessness.

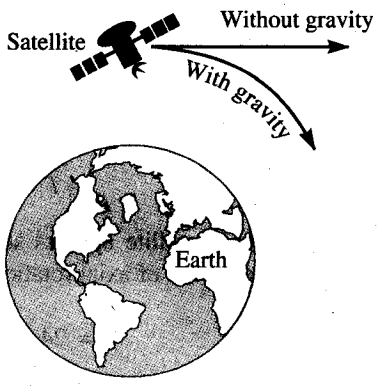

Fig. 16.5

The weightlessness experienced by people in a satellite orbiting close to the earth is the same apparent weightlessness experienced in a freely falling elevator. It may seem strange, at first, to think of a satellite as freely falling. But a satellite is indeed falling toward the earth as shown in Fig. 16.5; the force of gravity causes it to "fall" out of its natural straight-line path. The acceleration of the satellite must be the acceleration due to gravity since the only force acting on it is gravity. Thus, although the force of gravity acts on objects within the satellite, the objects experience an apparent weightlessness because they and the satellite are accelerating as in free fall.

Example 16.1. The period of moon around the earth is 27.3 days and the radius of its orbit is 3.9×10^5 km. If $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}, find the mass of the earth.

Solution. Time period of moon, $T = 27.3$ days $= 27.3 \times 24 \times 60 \times 60$ s

Radius of moon's orbit $= R + h = 3.9 \times 10^5$ km $= 3.9 \times 10^8$ m

The time period of a satellite is given by ;

$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

or
$$M = \frac{4\pi^2(R+h)^3}{GT^2} = \frac{4\pi^2 \times (3.9\times10^8)^3}{(6.67\times10^{-11})\times(27.3\times24\times60\times60)^2}$$

$$= 6.31\times 10^{24} \text{ kg}$$

Example 16.2. A satellite is revolving in a circular orbit at a height of 3400 km above earth's surface. Calculate the orbital velocity and the period of revolution. Radius of earth = 6400 km and $g = 9.8$ m s^{-2}.

Solution. Radius of earth, $R = 6400$ km $= 6.4 \times 10^6$ m

Radius of satellite's orbit $= R + h = 6400 + 3400 = 9800$ km $= 9.8 \times 10^6$ m

The orbital velocity of the satellite is

$$v = R\sqrt{\frac{g}{R+h}} = 6.4\times10^6\sqrt{\frac{9.8}{9.8\times10^6}} = \mathbf{6.4 \times 10^3\ ms^{-1}}$$

Period of revolution of the satellite is

$$T = \frac{2\pi(R+h)}{v} = \frac{2\pi\times9.8\times10^6}{6.4\times10^3} = 9616\text{ s} = \mathbf{2.67\ hours}$$

Example 16.3. **An artificial satellite revolving coplanar with the equator around the earth, appears stationary to an observer on the earth. Calculate the height of the satellite above earth's surface. Given that g = 9.8 ms^{-2} and radius of earth R = 6.37 × 10^6 m.**

Solution. Let h be the height of the satellite above the surface of the earth. The period of revolution of a satellite is given by ;

$$T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}} = \frac{2\pi(R+h)^{3/2}}{R\sqrt{g}}$$

Since the satellite appears stationary to an observer on the earth, its period of revolution is equal to the period of axial rotation (24 hours) of the earth (also the satellite is revolving from west to east) *i.e.*,

$$T = 24\text{ hours} = 24 \times 60 \times 60\text{ s} = 24 \times 3600\text{ s}$$

$$\therefore \quad 24 \times 3600 = \frac{2\pi(R+h)^{3/2}}{R\sqrt{g}}$$

or
$$(R+h)^3 = \frac{(24\times3600)^2 \times R^2 \times g}{4\pi^2}$$

$$= \frac{(24\times3600)^2\times(6.37\times10^6)^2\times9.8}{4\pi^2} \simeq 75\times10^{21}\text{ m}^3$$

$$\therefore \quad R + h \simeq (75 \times 10^{21})^{1/3} \simeq 4.2 \times 10^7\text{ m} = 42 \times 10^6\text{ m}$$

$$\therefore \quad h = 42 \times 10^6 - R = 42 \times 10^6 - 6.37 \times 10^6 \simeq 36 \times 10^6\text{ m} \simeq \mathbf{36000\ km}$$

Example 16.4. **A small satellite revolves around a planet in an orbit just above the surface of the planet. Taking G = 6.67 × 10^{-11} N m^2 kg^{-2} and mean density of the planet as 8 × 10^3 kg m^{-3}, find the time period of the satellite.**

Solution. The time period of a satellite orbiting very close to the surface of a planet is given by ;

$$T = \sqrt{\frac{3\pi}{G\rho}}$$

where ρ = mean density of the planet

$$\therefore \quad T = \sqrt{\frac{3\pi}{(6.67\times10^{-11})\times8\times10^3}} = \mathbf{4202\ s}$$

Example 16.5. An artificial satellite of mass 100 kg is in a circular orbit of 500 km above the earth's surface. Radius of earth = 6.5 × 10^6 m. Find (*i*) the acceleration due to gravity at any point along the satellite orbit (*ii*) the centripetal acceleration of the satellite.

Solution. Mass of satellite, m = 100 kg ; Radius of earth, R = 6.5 × 10^6 m

Height of satellite, h = 500 km = 0.5 × 10^6 m

Radius of satellite orbit, $r = R + h$ = 6.5 × 10^6 + 0.5 × 10^6 = 7 ×10^6 m

(*i*) $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} = \frac{9.8}{\left(1+\frac{0.5\times10^6}{6.5\times10^6}\right)^2} = \mathbf{8.45\ ms^{-2}}$$

(*ii*) Orbital velocity, $v = R\sqrt{\frac{g}{R+h}} = R\sqrt{\frac{g}{r}} = 6.5\times10^6\sqrt{\frac{9.8}{7\times10^6}} = 7691\ ms^{-1}$

$\therefore$ Centripetal acceleration $= \frac{v^2}{r} = \frac{(7691)^2}{7\times10^6} = \mathbf{8.45\ ms^{-2}}$

Example 16.6. Prove that the angular momentum of a satellite of mass M_s revolving round the earth of mass M_e in an orbit of radius r is equal to $[GM_eM_s^2r]^{1/2}$.

Solution. The gravitational force between the earth and satellite provides the necessary centripetal force to move the satellite in the circular orbit. If v is the orbital velocity of the satellite, then,

Centripetal force required $= \frac{M_sv^2}{r}$

Gravitational force between earth and satellite $= \frac{GM_eM_s}{r^2}$

$\therefore$ $$\frac{M_sv^2}{r} = \frac{GM_eM_s}{r^2} \quad \text{or} \quad v = \sqrt{\frac{GM_e}{r}}$$

Angular momentum of satellite $= M_svr = M_s\times\sqrt{\frac{GM_e}{r}}\times r = \sqrt{GM_eM_s^2r}$

Example 16.7. How much work must be done to lift an artificial satellite of mass m from the surface of earth (mass M, radius R) and put in a circular orbit with a radius equal to twice the earth's radius?

Solution. Initial K.E. of satellite, $K_0 = 0$; Initial P.E. of satellite, $U_0 = -\frac{GMm}{R}$

Total initial energy of satellite, $E_0 = K_0 + U_0 = 0 + \left(-\frac{GMm}{R}\right) = -\frac{GMm}{R}$

Total final energy of satellite, $E = -\frac{GMm}{2(R+h)} = -\frac{GMm}{2(R+R)} = -\frac{GMm}{4R}$

$\therefore$ Work done, $W = E - E_0 = -\frac{GMm}{4R} - \left(-\frac{GMm}{R}\right) = \mathbf{\frac{3GMm}{4R}}$

Example 16.8. A satellite orbits the earth at a height of 500 km from its surface. Compute its (*i*) kinetic energy (*ii*) potential energy and (*iii*) total energy. Mass of the satellite = 300 kg ; mass of the earth = 6.0×10^{24} kg ; radius of earth = 6.4×10^6 m and $G = 6.67\times10^{-11}$ N m^2 kg^{-2}.

Solution. h = 500 km = 5×10^5 m ; m = 300 kg ; $M = 6.0\times10^{24}$ kg ; $R = 6.4\times10^6$ m; $G = 6.67\times10^{-11}$ N m^2 kg^{-2}.

(*i*) K.E. of satellite $= \frac{GMm}{2(R+h)} = \frac{(6.67\times10^{-11})\times(6.0\times10^{24})\times300}{2(6.4\times10^6+5\times10^5)}$

$= \mathbf{8.7\times10^9\ J}$

(*ii*) P.E. of satellite = –2 × K.E. of the satellite

$= -2\times8.7\times10^9 = \mathbf{-17.4\times10^9\ J}$

(*iii*) Total energy of satellite = K.E. + P.E.

$= (8.7\times10^9) + (-17.4\times10^9) = \mathbf{-8.7\times10^9\ J}$

The total energy is negative for the satellite in the orbit. This indicates that the satellite is held in the orbit. Positive work must be done to free the satellite from the gravitational force.

Example 16.9. **A satellite orbits the earth at a height of 400 km above earth's surface. How much energy must be expended to rocket the satellite out of the earth's gravitational influence? Mass of the satellite = 200 kg ; mass of the earth = 6.0 × 10^{24} kg ; G = 6.67 × 10^{-11} Nm^2 kg^{-2}; radius of earth = 6.4 × 10^6 m.**

Solution. The total energy of the satellite in the orbit is

$$E = -\frac{GMm}{2(R+h)}$$

Here $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} ; $M = 6.0 \times 10^{24}$ kg ; $m = 200$ kg ; $R = 6.4 \times 10^6$ m; $h = 400$ km $= 4 \times 10^5$ m

$$\therefore \quad E = -\frac{(6.67\times10^{-11})\times(6.0\times10^{24})\times200}{2(6.4\times10^6+4\times10^5)} = -5.89\times10^9 \text{ J}$$

The total energy of the satellite at infinity is zero. Therefore, the energy required for the satellite to leave its orbit around the earth and escape to infinity

$$= 0 - E = 0 - (-5.89 \times 10^9) = 5.89 \times 10^9 \text{ J}$$

This is also the binding energy of the satellite.

PROBLEMS FOR PRACTICE

1. A satellite is revolving in a circular orbit at a distance of 2620 km from the surface of the earth. Calculate the orbital velocity and the period of revolution of the satellite. Radius of the earth = 6380 km, mass of earth = 6 × 10^{24} kg and G = 6.67 × 10^{-11} Nm^2 kg^{-2}. **[6.67 km s^{-1} ; 2.35 hours]**
2. An artificial satellite is revolving at a height of 500 km above earth's surface in a circular orbit completing one revolution in 98 minutes. Calculate the mass of the earth. Given G = 6.67 × 10^{-11} Nm^2 kg^{-2}, radius of earth = 6.37 ×10^6 m. **[5.54 × 10^{24} kg]**
3. The distance of the moon from the earth is 3.8 × 10^5 km. Calculate the speed of the moon revolving around the earth. Mass of earth = 6.1 × 10^{24} kg and G = 6.67 ×10^{-11} Nm^2 kg^{-2}. **[1.03 km s^{-1}]**
4. An earth satellite is revolving at a height of 1800 km from the earth's surface. Radius of earth is 6300 km and acceleration due to gravity at earth's surface is 10 ms^{-2}. Find (*i*) the orbital velocity of the satellite (*ii*) radial acceleration of the satellite (*iii*) period of revolution of the satellite. **[(*i*) 7 km s^{-1} (*ii*) 6 m s^{-2} (*iii*) 7267 s]**
5. An earth's satellite makes a circle around the earth in 90 minutes. Calculate the height of the satellite above the surface of the earth. Given that radius of earth = 6400 km and g = 9.8 m s^{-2}. **[268 km]**

16.13. ESCAPE VELOCITY

If a ball is thrown upwards from the surface of the earth, its speed decreases from the moment it is projected due to the retarding effect of earth's gravitational field. It would simply rise to a certain height, reverse direction and then fall back to earth. The height which the ball ultimately attains depends upon the speed with which it is projected—the greater the speed, the greater the height. Ultimately, at a certain velocity of projection, the body will go out of the gravitational field and will never return to the earth. The minimum projection velocity that achieves this is known as the escape velocity.

The **escape velocity** *of a body is the minimum velocity with which it is to be projected so that it just overcomes the gravitational pull of the earth (or any other planet).*

Since earth's gravitational field extends to infinity (however weak it may be at large distances), escape velocity is obviously the velocity that must be given to an object for it to escape all the way to infinity.

16.14. EXPRESSION FOR ESCAPE VELOCITY

In order to project a body with escape velocity, we give kinetic energy to it. Let us calculate this energy. Consider earth to be a sphere of mass M and radius R. Suppose a body of mass m is projected upward with escape velocity V_e. When the body is at point P at a distance x from the centre of the earth, the gravitational force of attraction exerted by the earth on the body [See Fig. 16.6] is

$$F = G\frac{Mm}{x^2}$$

In moving a small distance Δx against this gravitational force, the small work done at the expense of the kinetic energy of the body is given by ;

$$\Delta W = F\Delta x = G\frac{Mm}{x^2}\Delta x$$

Therefore, total work done (W) in moving the body from earth's surface (where $x = R$) to infinity (where $x = \infty$) is given by :

$$W = \int_R^\infty \frac{GMm}{x^2}dx = GMm\int_R^\infty x^{-2}\,dx$$

$$= GMm\left[-\frac{1}{x}\right]_R^\infty = \frac{GMm}{R}$$

$$\therefore \qquad W = \frac{GMm}{R}$$

If the body is to be able to do this amount of work (and so escape), it needs to have atleast this amount of kinetic energy at the moment it is projected. Therefore, escape velocity V_e is given by ;

$$\frac{1}{2}mV_e^2 = \frac{GMm}{R}$$

or
$$V_e = \sqrt{\frac{2GM}{R}} \qquad ...(i)$$

Discussion. The following points are worth noting :

(*i*) *The escape velocity does not depend on the mass (m) of the body.* It is the same for all masses for a given planet.

(*ii*) *The escape velocity does not depend upon the direction of projection from the surface of earth/planet.* It is because the kinetic energy a body loses in reaching any particular height depends *only* on the height concerned and not the path taken to reach it.

(*iii*) The escape velocity depends upon the mass (M) and radius (R) of the planet from the surface of which the body is projected.

V_e in terms of g and R. We know that $V_e = \sqrt{\frac{2GM}{R}}$

Now $g = \frac{GM}{R^2}$ or $GM = gR^2$

Putting $GM = gR^2$ in the expression for V_e, we have,

$$V_e = \sqrt{\frac{2gR^2}{R}} = \sqrt{2gR}$$

Discussion. The following points may be noted :

(*i*) The orbital velocity of a satellite close to earth surface is $v = \sqrt{gR}$.

Now escape velocity, $V_e = \sqrt{2gR} = \sqrt{2}v = 1.41\ v$

Thus the escape velocity is only 41% greater than the orbital velocity of a satellite close to the surface of earth.

(*ii*) Since escape velocity depends upon mass ($g = GM/R^2$) and radius of the planet, it is different for different planets.

(*iii*) The escape velocity from the earth's surface is

$$V_e = \sqrt{2gR}$$

Now $g = 9.8\ \text{m s}^{-2}$; $R = 6.37 \times 10^6\ \text{m}$

$\therefore$ $V_e = \sqrt{2\times9.8\times6.37\times10^6} = 11.2 \times10^3\ \text{m s}^{-1} = 11.2\ \text{km s}^{-1}$

Therefore, with an initial velocity of about 11.2 km s^{-1}, a rocket will completely escape from the gravitational attraction of the earth. For example, it can be made to travel toward the moon so that eventually it comes under the gravitational attraction of the moon. At present, "soft landings" on the moon have been made by firing retarding retro rockets.

V_e in terms of ρ and R. We can also express escape velocity (V_e) in terms of density (ρ) and radius (R) of earth.

$$\text{Escape velocity, } V_e = \sqrt{\frac{2GM}{R}}$$

$$\text{Mass of earth, } M = \frac{4}{3}\pi R^3 \rho$$

Putting the value of M in the expression for V_e, we have,

$$V_e = \sqrt{\frac{2G}{R}\times\frac{4}{3}\pi R^3 \rho} = \sqrt{\frac{8}{3}\pi \rho G R^2} = D\sqrt{\frac{2\pi\rho G}{3}}$$

$\therefore$ $$V_e = D\sqrt{\frac{2\pi\rho G}{3}}$$

Here $D = 2R$ = Diameter of earth.

Hence denser planets have greater escape velocities.

Notes. (*i*) If the velocity of projection (V) of the body from the surface of a planet is greater than the escape velocity (V_e) of the planet, the body will escape out from the gravitational field of that planet and will move in the interstellar space with velocity V'. We can find the value of V' by applying law of conservation of energy *i.e.,*

$$\frac{1}{2}mV'^2 = \frac{1}{2}mV^2 - \frac{1}{2}mV_e^2$$

or $$V'^2 = V^2 - V_e^2$$

$\therefore$ $$V' = \sqrt{V^2 - V_e^2}$$

(*ii*) The escape velocity of the sun is 618 km/s.

Example 16.10. What is the escape velocity for a rocket on the surface of Mars? Mass of Mars = 6.58×10^{23} kg and radius of Mars = 3.38×10^6 m.

Solution. The escape velocity on the surface of Mars is

$$V_e = \sqrt{\frac{2GM_M}{R_M}}$$

Now $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2} ; $M_M = 6.58 \times 10^{23}$ kg ; $R_M = 3.38 \times 10^6$ m

$$\therefore \quad V_e = \sqrt{\frac{2\times(6.67\times10^{-11})\times(6.58\times10^{23})}{3.38\times10^6}}$$

$$= 5.1 \times 10^3 \text{ m s}^{-1} = \mathbf{5.1 \text{ km s}^{-1}}$$

Note that escape velocity from the surface of Mars is less than one half of the escape velocity (= 11.2 km s^{-1}) from the earth's surface.

Example 16.11. **Find the velocity of escape at the moon given that its radius is 1.7×10^6 m and the value of g at its surface is 1.63 m s^{-2}.**

Solution. The escape velocity on the surface of moon is

$$V_e = \sqrt{2gR}$$

Here g= 1.63 m s^{-2} ; $R = 1.7 \times 10^6$ m

$$\therefore \quad V_e = \sqrt{2\times1.63\times1.7\times10^6} = 2.354\times10^3 \text{ ms}^{-1} = \mathbf{2.354 \text{ km s}^{-1}}$$

Example 16.12. **A body is at a height equal to the radius of the earth from the surface of the earth. With what velocity be it thrown so that it goes out of the gravitational field of the earth? Given mass of earth = 6×10^{24} kg, radius of earth = 6.4×10^6 m and G = 6.67×10^{-11} N m^2 kg^{-2}.**

Solution. The escape velocity of a body on the surface of earth is

$$V_e = \sqrt{\frac{2GM}{R}}$$

where R is the radius of earth *i.e.*, distance of the body from the centre of the earth. If the body is at height h (= R = radius of earth) above the surface of earth, then escape velocity of the body is

$$V_e = \sqrt{\frac{2GM}{R+h}} = \sqrt{\frac{2GM}{R+R}} = \sqrt{\frac{2GM}{2R}}$$

$$= \sqrt{\frac{2\times(6.67\times10^{-11})\times(6\times10^{24})}{2\times(6.4\times10^6)}}$$

$$= 7.9 \times10^3 \text{ m s}^{-1} = \mathbf{7.9 \text{ km s}^{-1}}$$

Example 16.13. **Jupiter has a mass 318 times that of the earth and its radius is 11.2 times the earth's radius. Estimate the escape velocity of a body from Jupiter's surface, given the escape velocity from earth's surface is 11.2 km s^{-1}.**

Solution. Escape velocity on the surface of earth is

$$V_E = \sqrt{\frac{2GM_E}{R_E}}$$

Escape velocity on the surface of Jupiter is

$$V_J = \sqrt{\frac{2GM_J}{R_J}}$$

$$\therefore \quad \frac{V_J}{V_E} = \sqrt{\frac{2GM_J}{R_J}\times\frac{R_E}{2GM_E}} = \sqrt{\frac{R_E}{R_J}\times\frac{M_J}{M_E}}$$

Now $\frac{R_J}{R_E} = 11.2$; $\frac{M_J}{M_E} = 318$; $V_E = 11.2$ km s^{-1}

$$\therefore \quad \frac{V_J}{V_E} = \sqrt{\frac{1}{11.2}\times 318}$$

or $$V_J = V_E\sqrt{\frac{318}{11.2}} = 11.2\sqrt{\frac{318}{11.2}} = \mathbf{59.68\ km\ s^{-1}}$$

Example 16.14. Show that moon would depart forever if its speed were increased by 42%.

Solution. Mass of earth = M_E ; Mass of moon = M_M

If the distance between the earth and moon is r, then necessary centripetal force ($= M_M v^2/r$) is provided by the gravitational attraction ($= GM_E M_M/r^2$) between earth and moon.

$$\therefore \quad \frac{M_M v^2}{r} = \frac{GM_E M_M}{r^2}$$

or $$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{gr^2}{r}} = \sqrt{gr}$$

Escape velocity for moon is $V_e = \sqrt{2gr}$

$\therefore$ Percentage increase in velocity of moon

$$= \frac{V_e - v}{v}\times 100 = \frac{\sqrt{2gr} - \sqrt{gr}}{\sqrt{gr}}\times 100 = 42\%$$

Example 16.15. Calculate the escape velocity for an atmospheric particle 1600 km above the earth's surface, given that the radius of earth is 6400 km and acceleration due to gravity on earth's surface is 9.8 m/s^2.

Solution. If g' is the acceleration due to gravity at height h above earth's surface, then,

$$g' = \frac{gR^2}{(R+h)^2}$$

Here R = 6400 km = 6.4 × 10^6 m ; $R + h$ = 6400 + 1600 = 8000 km = 8 × 10^6 m

$$\therefore \quad g' = \frac{9.8\times(6.4\times10^6)^2}{(8\times10^6)^2} = 6.27\ \text{ms}^{-2}$$

$$\therefore \quad \text{Escape velocity, } V_e = \sqrt{2g'(R+h)} = \sqrt{2\times6.27\times(8\times10^6)} = \mathbf{10.02\times10^3\ ms^{-1}}$$

PROBLEMS FOR PRACTICE

1. Mass of the moon is 7.34 × 10^{22} kg and mean radius is 1.74 × 10^6 m. If G = 6.67 × 10^{-11} Nm^2 kg^{-2}, calculate escape velocity on moon's surface. **[2.37 km s^{-1}]**
2. The escape velocity of a body on the surface of earth is 11.2 km s^{-1}. If the radius of a planet be half the radius of the earth and its mass be one fourth that of the earth, then what will be the escape velocity from the planet? **[7.92 km s^{-1}]**
3. What will be the escape velocity on the surface of the moon if it is 11.2 kms^{-1} on the surface of the earth? Assume that value of g at the surface of moon is 1/5th of that at the earth and the radius of moon is 1/4th the radius of earth. **[2.5 km s^{-1}]**
4. The speed of revolution of an artificial satellite revolving very close to the surface of the earth is 8 kms^{-1}. What will be the escape velocity for a body on the surface of the earth? **[11.3 km s^{-1}]**
5. The radius of Uranus is approximately 3.69 times the radius of the earth. If the escape velocity from Uranus is 22 km s^{-1}, what is the ratio of acceleration of gravity on Uranus to its value on earth? **[1.05]**

16.15. PLANETARY MOTION AND KEPLER'S LAWS

The study of planetary motion played a pivotal role in the development of physics. The German astronomer Johannes Kepler (1571–1630) made a detailed study of the motion of planets about the sun. He worked out three empirical laws that govern the motion of planets about the sun. These are now known as **Kepler's laws of planetary motion.** These laws were all empirical and had no theoretical basis. So Kepler knew how the planets moved in their orbits, but not why they did so. We now discuss each law in turn. Although we apply the laws here to planets orbiting the sun, they hold equally well for satellites (natural or artificial) orbiting the earth or any other massive central body.

Kepler's First Law. *The path of each planet about the sun is an *ellipse with sun at the one focus (See Fig. 16.7) of the ellipse.*

In other words, each planet moves around the sun in an elliptical path with sun at one focus. By using law of universal gravitation, Newton showed that the general path of a planet under the influence of an inverse-square law of force is an ellipse with the centre of force at one focus if the path is closed. Kepler's first law is also known as **orbital rule**.

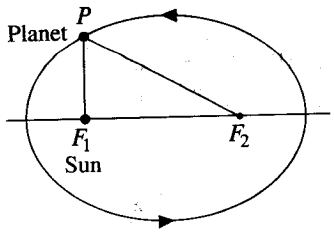

Fig. 16.7

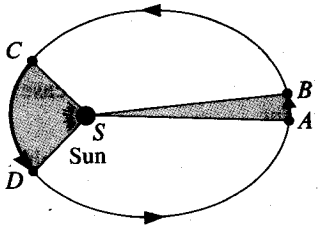

Fig. 16.8

Kepler's second law. *Each planet moves in such a way that an imaginary line drawn from the sun to the planet sweeps out equal areas in equal periods of time* (See Fig. 16.8).

The second law can be used to predict the speed of a planet in one part of its orbit if we know its speed in another part. For example, suppose a planet takes one month to go from A to B (See Fig. 16.8). If we draw pie-shaped sectors ASB and CSD so that each sector has the same area, then according to second law, the planet will also take one month to go from C to D. Since arc CD is longer than the arc AB and the times of travel are the same, the speed along CD is greater than the speed along AB. In other words, a planet must be moving faster when it is closer to the sun and *vice-versa.* Kepler's second law is also known as **area rule**.

Proof. We have already discussed the relation between Kepler's second law and angular momentum in chapter 13. The time rate of area (dA/dt) swept out by a planet is directly proportional to the angular momentum of the planet about the sun *i.e.*,

$$\frac{dA}{dt} \propto \text{Angular momentum of planet about the sun}$$

The dominant force on each planet is the gravitational attraction of the sun which is a radially directed force that exerts no torque on the planet. Thus the torque on the planets in their orbits is zero and the angular momentum of any planet about the sun is constant.

$$\therefore \qquad \frac{dA}{dt} = \text{Constant}$$

---

* An ellipse is a closed curve such that the sum of the distances from any point P on the curve to two fixed points (called foci F_1 and F_2) remains constant. That is the sum of the distances $F_1P + F_2P$ (See Fig. 16.7) is the same for all points P on the curve.

This means that for a given planet, *dA/dt* is constant *i.e.*, the time rate at which the planet sweeps out an area is constant. This is Kepler's second law (equal areas in equal times).

Kepler's third law. *The square of the period of any planet (time needed for one revolution about the sun) is directly proportional to the cube of the planet's average distance from the sun.*

If T is the period of the planet and r is its average distance from the sun, then,

$$T^2 \propto r^3 \quad \text{or} \quad \frac{T^2}{r^3} = \text{Constant}$$

Thus T^2/r^3 is the same for all planets. Hence smaller the orbit of the planet around the sun, the shorter the time it takes to complete one revolution. If T_1 and T_2 are the periods of two planets and r_1 and r_2 are their respective average distances from the sun, then,

$$\frac{T_1^2}{r_1^3} = \frac{T_2^2}{r_2^3} \quad \text{or} \quad \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

Kepler's third law is also known as **period rule.**

16.16. DEDUCTION OF KEPLER'S THIRD LAW

We now show that Kepler's third law follows from the law of universal gravitation. We make the approximation that the orbits of the planets are circles (Most of the planetary orbits are fairly close to a circle, which is a special case of an ellipse). It is also approximately true that the orbital speed is constant.

Consider a planet of mass m revolving around the sun of mass M in a circular orbit of radius r as shown in Fig. 16.9. Suppose v is the orbital speed of the planet and T is its time period. The sun's gravitational force F on the planet is

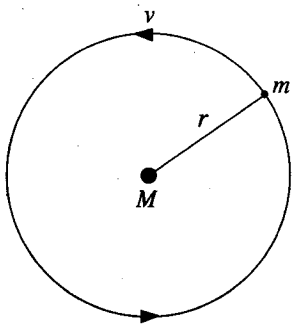

Fig. 16.9

$$F = \frac{GmM}{r^2}$$

Because the mass of the sun is so much larger than the mass of the planet, we can assume (as Kepler did) that the sun lies at the centre of the orbit. Therefore, the centripetal force F_C acting on the planet is

$$F_C = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 mr}{T^2}$$

This centripetal force is provided by the sun's gravitational force on the planet.

$$\therefore \quad \frac{GmM}{r^2} = \frac{4\pi^2 mr}{T^2} \quad \text{or} \quad \frac{T^2}{r^3} = \frac{4\pi^2}{GM}$$

The quantities $4\pi^2$, G and M are constant.

$$\therefore \quad \frac{T^2}{r^3} = \text{Constant}$$

This is Kepler's third law. We have derived Kepler's third law only for uniform circular motion of the planet but the result is true for elliptical orbits if we use the average distance from the sun for r.

16.17. LAW OF GRAVITATION FROM KEPLER'S THIRD LAW

Newton was able to show that Kepler's laws could be derived mathematically from his law of universal gravitation and his laws of motion. Indeed, he used Kepler's third law as evidence in favour of his law of universal gravitation.

We now show that Newton's law of universal gravitation follows from Kepler's third law. We make the approximation that orbits of the planets are circles. Consider a planet of mass m revolving

around the sun of mass M in a circular orbit of radius r (Refer back to Fig. 16.9). Suppose v is the orbital speed of the planet and T is its time period. In time T, the planet travels a distance $2\pi r$.

$$\therefore \quad T = \frac{2\pi r}{v} \quad \text{or} \quad v = \frac{2\pi r}{T}$$

The centripetal force F required to keep the planet in the circular orbit is

$$F = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 mr}{T^2}$$

or $$F = \frac{4\pi^2 mr}{T^2} \qquad ...(i)$$

This centripetal force is provided by the sun's gravitational force on the planet.

According to Kepler's third law,

$$\frac{T^2}{r^3} = \text{Constant} = k \quad \text{or} \quad T^2 = kr^3$$

Putting $T^2 = k r^3$ in eq. (i), we have,

$$F = \frac{4\pi^2 mr}{kr^3} \quad \text{or} \quad F = \frac{4\pi^2}{k} \times \frac{m}{r^2} \qquad ...(ii)$$

Note that F is the force exerted on the planet by the sun. We see that force on the planet is directly proportional to the mass of the planet and inversely proportional to the square of distance of the planet from the sun. By the law of action and reaction, this equals in magnitude to the force exerted on the sun by the planet. Since the force exerted by the planet on the sun is proportional to the mass of the planet, the symmetry suggests that the force exerted by the sun on the planet is proportional to the mass M of the sun *i.e.*,

$$\frac{4\pi^2}{k} \propto M \quad \text{or} \quad \frac{4\pi^2}{k} = GM \qquad ...(iii)$$

where G is a constant of proportionality independent of the mass of the sun or planet. It is called *universal gravitational constant.* From eqs. (ii) and (iii), we have,

$$F = G\frac{mM}{r^2}$$

This is Newton's law of universal gravitation.

Example 16.16. The distances of the two planets from the sun are 10^{13} m and 10^{12} m respectively. Find the ratio of the time periods and speeds of the two planets.

Solution. According to Kepler's third law of planetary motion,

$$\frac{T^2}{r^3} = \text{Constant} \quad \therefore \quad \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

or $$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2}. \text{ Here } r_1 = 10^{13} \text{ m} ; \; r_2 = 10^{12} \text{ m}$$

$$\therefore \quad \frac{T_1}{T_2} = \left(\frac{10^{13}}{10^{12}}\right)^{3/2} = \mathbf{10\sqrt{10}}$$

Now $$v_1 = \frac{2\pi r_1}{T_1} \quad \text{and} \quad v_2 = \frac{2\pi r_2}{T_2}$$

$$\therefore \quad \frac{v_1}{v_2} = \frac{r_1}{r_2} \times \frac{T_2}{T_1} = \frac{10^{13}}{10^{12}} \times \frac{1}{10\sqrt{10}} = \frac{1}{\sqrt{10}}$$

Example 16.17. A stationary satellite orbits the earth in the plane of the equator with a period of 24 hours. What is the orbital radius for this stationary satellite? Use the fact that a satellite close to the earth has an orbital radius of 1.025 R where R is the earth's mean radius and time period of 88 minutes.

Solution. Fig. 16.10 shows a satellite that orbits from west to east at the equator with a period of 24 hours so that its motion is synchronous with the earth's spin. Such a satellite is called stationary satellite and can be used as a relay station for telephone, television and other signals carried by radio waves between two points on earth that are not on a direct line of sight.

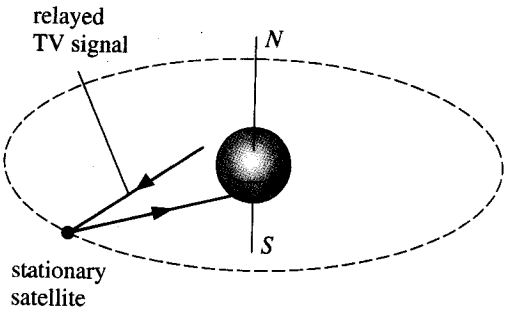

Fig. 16.10

Kepler's third law relates the period T of a satellite to the radius r for the satellite as :

$$T^2 \propto r^3 \quad \text{or} \quad \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

or
$$r_2 = r_1\left(\frac{T_2}{T_1}\right)^{2/3}$$

Here $r_1 = 1.025\ R$; $T_1 = 88$ min $= 88/60$ hours ; $T_2 = 24$ hours

$$\therefore \quad r_2 = 1.025R \times \left(\frac{24\times60}{88}\right)^{2/3} = \mathbf{6.6\ R}$$

The orbital radius is 6.6 times the earth's mean radius.

Example 16.18. The moon has a period of 28 days and an orbital radius 3.8×10^5 km. What is the orbital radius of a satellite that has a period of one day?

Solution. According to Kepler's third law,

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3} \quad \text{or} \quad r_2 = r_1\left(\frac{T_2}{T_1}\right)^{2/3}$$

Here $r_1 = 3.8 \times 10^5$ km ; $T_1 = 28$ days ; $T_2 = 1$ day

$$\therefore \quad r_2 = 3.8\times10^5 \times \left(\frac{1}{28}\right)^{2/3} = \mathbf{4.1 \times 10^4\ km}$$

Example 16.19. In an imaginary planetary system, the central star has the same mass as our sun but is much brighter so that only a planet twice the distance between the earth and the sun can support life. Assuming biological evolution (including ageing processes etc.) on that planet similar to ours, what would be the average life span of a human on that planet in terms of its natural year? The average life span of a human on earth may be taken as 70 years.

Solution. The natural year of a planet means the time taken by the planet to complete one revolution around the star. For example, natural year of earth is 1 year. Let us find the natural year of the imaginary planet.

Suppose R_1 is the orbital radius of earth. Then orbital radius of imaginary planet is $R_2 = 2R_1$. If T_1 and T_2 are respectively the periods of earth and the imaginary planet, then according to Kepler's third law,

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$$

$$\therefore \quad T_2 = T_1 \times \left(\frac{R_2}{R_1}\right)^{3/2} = 1 \times \left(\frac{2R_1}{R_1}\right)^{3/2} = 2.828 \text{ years} \qquad (\because T_1 = 1 \text{ year})$$

Thus we find that one natural year of imaginary planet corresponds to 2.828 earth years.

$\therefore$ Average life span of a human on the planet

$$= \frac{70}{2.828} = \textbf{24.75 natural years of the planet}$$

Example 16.20. How fast (in m^2/s) is area swept out by (*i*) the radius from sun to earth (*ii*) the radius from earth to moon? Sun to earth distance = 1.496×10^{11}m, earth to moon distance = 3.845×10^8 m and period of revolution of moon = $27\frac{1}{3}$ days.

Solution. (*i*) $R = 1.496 \times 10^{11}$m ; $T = 365$ days $= 365 \times 24 \times 60 \times 60$ s

Area swept out by radius from sun to earth, $\frac{dA}{dt} = \frac{\pi R^2}{T}$

$$= \frac{\pi \times (1.496 \times 10^{11})^2}{365 \times 24 \times 60 \times 60} = \textbf{2.23}\times\textbf{10}^{\textbf{15}} \textbf{ m}^2\textbf{s}^{-1}$$

(*ii*) $R = 3.845 \times 10^8$ m ; $T = 27\frac{1}{3}$ days $= \frac{82}{3} \times 24 \times 60 \times 60$ s

Area swept out from earth to moon, $\frac{dA}{dt} = \frac{\pi R^2}{T}$

$$= \frac{\pi \times (3.845 \times 10^8)^2}{\frac{82}{3} \times 24 \times 60 \times 60} = \textbf{1.97}\times\textbf{10}^{\textbf{11}} \textbf{ m}^2\textbf{s}^{-1}$$

PROBLEMS FOR PRACTICE

1. Compare the period of rotation of the planet Mars about the sun with that of the earth about it. The mean distance of the Mars from the sun is 1.52 A.U. **[1.874]**
2. Venus is at an average distance of 1.08×10^8 km from the sun. Estimate the length of venus year using the fact that the average distance of the earth from the sun is 1.49×10^8 km. **[225 days]**
3. Calculate the period of Neptune around the sun. Given that diameter of its orbit is 30 times the diameter of earth's orbit around the sun; both orbits assumed circular. **[164.3 years]**
4. Kepler's third law is $T^2 = kR^3$. What is the value of k for the earth if the period is expressed in years and the distance in A.U. What is the value if the period is expressed in seconds and the distance expressed in metres? **[1 year^2/(A.U.)^3; 3.0×10^{-19} s^2 m^{-3}]**
5. An amateur astronomer claims that a new planet is seen beyond the orbit of Pluto with a period of 230 years. Can this claim be true? Assume the orbit to be nearly circular. **[Untrue]**

CONCEPTUAL QUESTIONS

Q.1. An artificial satellite revolves in its orbit around the earth without using any fuel. But an aeroplane requires fuel to fly at a certain height. Why?

Ans. Air is necessary for the aeroplane to fly. Therefore, aeroplane requires fuel against the frictional force due to air. However, a satellite revolves at a greater height where air is

very much rare and, therefore, frictional force due to air is negligible. For example, at 100 km above the surface of earth, the atmosphere is only one-millionth as dense as it is at sea level. So air friction at that height should be negligible.

Q.2. What is a parking orbit?

Ans. The orbit in which the period of revolution of a satellite is equal to the period of earth revolution (24 hours) about its axis is known as parking orbit. This happens at a height of 36000 km above the earth's surface. The orbital velocity of a satellite in the parking orbit is 3.1 km s^{-1}. When in a parking orbit, a satellite stays at the same place *w.r.t.* earth.

Q.3. Why is the speed of a satellite in a circular orbit not changed by earth's gravitational field?

Ans. In a circular orbit, the speed of a satellite is not changed by gravity; only direction changes. It is because a satellite in circular orbit always moves perpendicular to earth's gravitational field. With no component of motion along or against the earth's field, no change in speed occurs; only the direction changes.

Q.4. Satellites remain in orbit instead of falling to the earth because they are beyond the main pull of earth's gravity. Is this satement correct?

Ans. No, not at all. If any object were beyond the pull of gravity, it would move in a straight line and would not curve around the earth. Satellites remain in the orbit because they are being pulled by gravity, not because they are beyond it. For the altitudes of most earth satellites, the earth's gravitational field is only a few percent weaker than at the earth's surface.

Q.5. A person sitting in an artificial satellite revolving around the earth feels weightlessness. Why?

Ans. A person feels weight only when the surface on which he is standing exerts a reactionary force on him. Since the acceleration of the astronaut and that of the satellite revolving around the earth are equal (each equal to acceleration due to gravity), the acceleration of the astronaut *w.r.t.* the satellite is zero. Therefore, the person is weightless *w.r.t.* the satellite.

Q.6. A satellite orbiting close to earth's surface in a circular path falls about 4.9 m each second. Why does not this distance accumulate and send satellite crashing to earth?

Ans. In each second, the satellite falls about 4.9 m below the straight-line tangent it would have taken if there were no gravity. The earth's surface also curves 4.9 m beneath a straight-line 8 km tangent. The process of falling with the curvature of the earth continues from tangent line to tangent line, so the curved path of the satellite and the curve of earth's surface "match" all the way around the earth.

Q.7. The escape velocity from earth for a piece of 1 g is 11.2 km s^{-1}. What would it be for a piece of 10 g?

Ans. The escape velocity from earth is given by ;

$V_e = \sqrt{2gR}$ where R = Earth's radius.

It is clear that escape velocity does not depend upon the mass of the body to be projected. It is the same for all masses.

Q.8. Why does the force of gravity change the speed of a satellite when it is in an elliptical orbit but not when it is in a circular orbit?

Ans. At any point on its path, the direction of motion of a satellite is always tangent to its path. If a component of force exists along this tangent, then the acceleration of the satellite will involve a change in speed as well as direction. In circular orbit, the gravitational force is always perpendicular to the direction of motion of the satellite. So there is no component

of gravitational force along the tangent, and only the direction of motion changes; not the speed. But when the satellite moves in directions that are not perpendicular to the force of gravity, as in an elliptical path, there is a component of force along the direction of motion that changes the speed of the satellite.

Q.9. Is it possible to put an artificial satellite in an orbit in such a way that it always remains visible directly over New Delhi?

Ans. No. In order to put a satellite in the parking orbit, its time period should be the same as that of the earth in the equatorial plane. Since New Delhi does not lie in the equatorial plane, a geostationary satellite cannot be seen over New Delhi.

Q.10. Why are space rockets usually launched from west to east?

Ans. We know that earth rotates about its axis from west to east. Therefore, any point on the earth's surface has linear velocity from west to east. When a rocket is launched from west to east, the linear velcoity of earth is added to the launching velocity of the rocket.

Q.11. The gravitational force acting on a rocket at a height h from the earth's surface is one-third of the force acting on a body of same mass at sea level. What is the relation between h and R_e (radius of earth)?

Ans. $\frac{GM_e m}{(R_e+h)^2}=\frac{1}{3}\frac{GM_e m}{R_e^2}$ or $\frac{1}{(R_e+h)^2}=\frac{1}{3}\frac{1}{R_e^2}$

On solving, $h = 0.732\ R_e$

Q.12. Can a pendulum vibrate in an artificial satellite?

Ans. No. The reason is simple. The time period of a pendulum is given by ;

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Inside the artificial satellite, $g = 0$ so that $T \to \infty$. Hence the pendulum will not vibrate.

Q.13. If the kinetic energy of a satellite revolving around earth in any orbit is doubled, what will happen to it?

Ans. Total energy of satellite $= -K$

where K = kinetic energy of satellite in that orbit. If kinetic energy is doubled, it means additional energy equal to K is given. Then total energy $= -K + K = 0$. Therefore, the satellite will leave its orbit and go to infinity.

VERY SHORT ANSWER QUESTIONS

Q.1. Does the orbital velocity of a satellite around the earth depend on its mass?

Ans. No, it is independent of the mass of the satellite because orbital velocity of a satellite is $v = \sqrt{GM/(R+h)}$. Here M is the mass of earth.

Q.2. From where does a satellite revolving around a planet get the required centripetal force?

Ans. The satellite gets the required centripetal force from the gravitational attraction of planet exerted on the satellite.

Q.3. Does the speed of a satellite remain constant in a particular orbit?

Ans. Yes. It is because orbital velocity $v = \sqrt{GM/(R+h)}$ and all the terms on the R.H.S. remain constant.

Q.4. What is the time period of a geostationary satellite?

Ans. The time period of a geostationary satellite is 24 hrs.

Q.5. What is the height of geostationary satellite above the surface of the earth?

Ans. The height of geostationary satellite above the surface of earth is 35930 km.

SHORT ANSWER QUESTIONS

Q.1. What is the difference between geostationary and ordinary satellite?

Ans. A geostationary satellite orbits around the earth at a height of about 36000 km above earth's surface and its motion is made synchronous with earth's rotation about its axis. Therefore, it has a time period of 24 hours and stays over the same place relative to the observer on the earth. An ordinary satellite has a time period other than 24 hours and hence appears to be moving *w.r.t.* earth.

Q.2. An earth satellite S has an orbit radius which is 4 times that of a communication satellite C. What is the time period of S?

Ans. According to Kepler's third law of planetary motion,

$$\left(\frac{T_S}{T_C}\right)^2 = \left(\frac{R_S}{R_C}\right)^3 \quad \therefore T_S = T_C \times \left(\frac{R_S}{R_C}\right)^{3/2} = 1 \times (4)^{3/2} = 8 \text{ days}$$

Note that time period T_C of communication satellite is 1 day (= 24 hours).

Q.3. The escape velocity of a body on the surface of earth is 11.2 km/s. If the earth's mass increases to twice its present value and radius of earth becomes half, what will be the escape velocity ?

Ans. Escape velocity, $V_e = \sqrt{\frac{2GM}{R}}$ or $V_e \propto \sqrt{\frac{M}{R}}$

$$\therefore \quad \frac{V_e'}{V_e} = \sqrt{\frac{M'}{R'} \times \frac{R}{M}} = \sqrt{\frac{M'}{M} \times \frac{R}{R'}} = \sqrt{2 \times 2} = 2$$

$$\therefore \quad V_e' = 2 \times V_e = 2 \times 11.2 = 22.4 \text{ km/s}$$

Q.4. What is the reason for no atmosphere on the moon?

Ans. Since the value of acceleration due to gravity on the surface of the moon is small, the escape velocity on moon's surface $\left(=\sqrt{2g\,R_m}\right)$ is also small. The molecules of atmospheric gases on the moon's surface have thermal velocities greater than its escape velocity. Due to this reason, all the molecules of gases have escaped and there is no atmosphere.

Q.5. The linear speed of a planet is not constant in its orbit. Comment.

Ans. The areal velocity (= dA/dt = time rate at which the planet sweeps out an area) around the sun is constant but not the linear velocity. When the planet passes close to the sun, its linear velocity increases and *vice-versa.*

Q.6. The escape velocity of a body when projected from earth's surface is 11.2 km/s. If it is projected at an angle of 50° from the horizontal, what will be the escape velocity?

Ans. It will be unchanged *i.e.,* 11.2 km/s. It is because escape velocity ($V_e = \sqrt{2GM/R}$) does not depend upon the direction of projection. The physical reason is that the kinetic energy a body loses in reaching any particular height depends only on the height concerned and not the path taken to reach it.

Q.7. Two satellites A and B are orbiting around the earth in circular orbits of the same radius. The mass of A is 8 times that of B. What is the ratio of their periods of revolution?

Ans. The time period of a satellite is $T = 2\pi r/v$ where r is the radius of the orbital path. Since the speed of the satellite ($v = \sqrt{GM/r}$) is independent of its mass, their time periods will be the same *i.e.,* $T_A/T_B = 1 : 1$.

Q.8. What are the essential conditions for a satellite to be stationary (geostationary satellite)?

Ans. It should satisfy the following three conditions :

(*i*) It should revolve in an orbit coplanar and concentric with the equatorial plane.

(*ii*) Its sense of rotation should be the same as that of the earth about its own axis *i.e.,* anticlockwise direction from west to east.

(*iii*) Its time period should be the same as that of earth about its own axis *i.e.,* 24 hours.

Q.9. **If V_0 is the orbital velocity of a satellite in a circular orbit close to earth's surface and V_e is the escape velocity from earth, what is the relation between the two?**

Ans. The orbital velocity of a satellite close to earth's surface is $V_0 = \sqrt{gR}$ where R is earth's radius.

Now $V_e = \sqrt{2gR} = \sqrt{2} \times \sqrt{gR} = \sqrt{2}\, V_0 \quad \therefore \quad V_e = \sqrt{2}\, V_0$

Q.10. **Air friction increases the velocity of the satellite. Why?**

Ans. The centripetal force ($= mv^2/r$) for the satellite is provided by the gravitational pull ($= GMm/r^2$) of the earth *i.e.*,

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \quad \text{or} \quad mv^2 r = GMm \quad \text{or} \quad (mvr)v = \text{constant}$$

But $mvr = L$ = angular momentum of the satellite. Therefore, Lv = constant. Since air friction provides the retarding torque, the angular momentum of the satellite decreases. Hence the velocity of the satellite increases.

Q.11. **What are the salient features of an earth satellite?**

Ans. (*i*) The orbital speed of an earth satellite is independent of the mass of the satellite and depends *only* upon its height above the earth's surface. The greater the height of the satellite above the earth's surface, the smaller is the orbital speed of the satellite and *vice-versa.*

(*ii*) The time period of an earth satellite depends *only* upon its height above the earth's surface. The greater is the height of a satellite above earth's surface, the greater is the time period of the satellite and *vice-versa.*

(*iii*) The orbital speed of a satellite close to earth's surface is about 8 km s^{-1}.

(*iv*) The relation between escape velocity (V_e) from earth's surface and the orbital velocity (V_0) of earth satellite close to the earth's surface is $V_e = \sqrt{2}\, V_0$.

Q.12. **What are the salient features of a geostationary satellite?**

Ans. (*i*) The time period of a geostationary satellite is 24 hours *i.e.*, the same as that for the earth about its own axis.

(*ii*) The height of geostationary satellite is about 36000 km above the earth's surface and is independent of mass of the satellite.

(*iii*) The geostationary satellite always appears to be over the same place relative to an observer on the earth.

(*iv*) The orbital speed of a geostationary satellite is 3.1 km s^{-1}.

(*v*) It rotates in the same direction (*i.e.*, anticlockwise direction) as the earth is rotating.

Q.13. **What are the salient features of escape velocity?**

Ans. (*i*) The escape velocity from the surface of a planet (*e.g.*, earth) does not depend upon the mass of the body.

(*ii*) The escape velocity does not depend upon the angle of projection..

(*iii*) The escape velocity from a planet depends upon its density. The denser planets have greater escape velocities and *vice-versa.*

(*iv*) The escape velocity from earth's surface is about 11.2 kms^{-1}.

(*v*) The escape velocity depends upon the mass and radius of the planet.

Q.14. **Does the speed of a satellite remain constant in an orbit. Explain.**

Ans. If a satellite orbits in a circular path, then its orbital speed is constant. However, if a satellite orbits in an elliptical path, then its speed in its orbit is not constant. In this case, its areal velocity is constant.

Q.15. **A satellite of mass m is revolving in a circular orbit of radius r about the earth of mass M. What is the angular momentum of the satellite about the centre of the orbit in terms of M and G?**

Ans. The angular momentum of the satellite about centre of orbit is $L = mvr$.

Now $\quad v = \sqrt{\frac{GM}{r}} \quad \therefore \quad L = mr\sqrt{\left(\frac{GM}{r}\right)} = \sqrt{m^2\, GM\, r}$

LONG ANSWER QUESTIONS

1. What is a satellite? Derive expressions for orbital velocity and period of revolution of satellite above the surface of earth. **[Refer to Arts. 16.1, 16.3 and 16.4]**
2. What is the principle of launching an earth satellite? Derive an expression for the height of the orbit above the surface of earth. **[Refer to Arts. 16.2 and 16.5]**
3. What is a geostationary satellite? Prove that height of geostationary satellite above the surface of earth is about 36000 km. **[Refer to Art. 16.8]**
4. What is escape velocity? Derive an expression of escape velocity from the surface of earth. **[Refer to Arts. 16.13 and 16.14]**
5. Explain Kepler's laws of planetary motion. **[Refer to Art. 16.15]**
6. Derive Newton's law of gravitation from Kepler's third law of planetary motion. **[Refer to Art. 16.17]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The orbital speed v of an earth satellite at a height h above the earth's surface is given the following formulas :

$$v = \sqrt{\frac{GM}{R+h}}\ ;\quad v = R\sqrt{\frac{g}{R+h}}$$

where R = Radius of earth ; M = Mass of earth ;
g = acceleration due to gravity.
If the satellite is close to earth's surface, $h = 0$ and

$$v = \sqrt{\frac{GM}{R}}\ ;\quad v = R\sqrt{\frac{g}{R}} = \sqrt{gR}$$

2. The orbital speed of earth satellite close to earth's surface is about 8 km/sec.
3. The orbital speed of a satellite is independent of the mass of satellite but depends upon the mass and radius of the planet about which the satellite rotates.
4. The greater the height of the satellite above earth/planet surface, the smaller is its orbital speed and *vice-versa.*
5. For stationary earth satellite, the orbital speed increases with the increase in height from earth's surface.
6. When a satellite is revolving in its orbit, no energy is required to keep it in its orbit. In other words, the work done by a satellite in one cycle of rotation is zero.
7. The time taken by a satellite to complete one revolution around the earth/planet is called time period of the satellite.
8. The time period of earth satellite orbiting close to earth surface is about 84 minutes.
9. The greater the height of a satellite above earth/planet surface, the greater is its time period and *vice-versa.*
10. The time period of a stationary satellite is 24 hours.
11. The orbit of stationary satellite is called parking orbit.
12. The height of parking orbit above earth's surface is about 36000 km.
13. The orbital speed of stationary satellite is about 3.1 km/sec.
14. The escape velocity V_e for the surface of earth is given by ;

$$V_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} = D\sqrt{\frac{2\pi\rho G}{3}}$$

where M = Mass of earth ; R = Radius of earth ; $D = 2R$
ρ = Average density of earth ; g = Acceleration due to gravity.
Note that denser planets have greater escape velocities.

15. The value of escape velocity is independent of (*i*) mass of the body (*ii*) direction of projection.
16. The value of escape velocity depends upon the mass and radius of earth/planet from which the body is projected.
17. Escape velocity V_e is related to orbital velocity v close to the surface of earth as :

$$V_e = \sqrt{2}\, v$$

18. The escape velocity from earth's surface is 11.2 km/s.
19. When a body falls from infinite height, it will strike earth's surface with a velocity of 11.2 km/s.
20. The escape velocity is only 41% greater than the orbital speed of a satellite close to the surface of earth.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The period of revolution of a certain planet in an orbit of radius R is T. Its period of revolution in an orbit of radius $4R$ will be **[CBSE PMT 2002]**

(a) $2T$ (b) T

(c) $4T$ (d) $8T$

2. The escape velocity of earth is **[CBSE PMT 2001]**

(a) 1.12 km/sec (b) 11.2 km/sec

(c) 112 km/sec (d) 7.92 km/sec

3. The escape velocity of a sphere of mass m is given by (G = Universal gravitational constant, M = mass of earth, R = radius of earth) **[Pb PMT 2001]**

(a) $\sqrt{\frac{GM}{R}}$ (b) $\sqrt{\frac{2GM}{R}}$

(c) $\sqrt{\frac{2GMm}{R}}$ (d) $\sqrt{\frac{2GM+R}{R}}$

4. The distance of a geostationary satellite from the centre of earth (Radius R = 6400 km) is nearest to **[AFMC-Pune 2001]**

(a) $7R$ (b) $5R$

(c) $10R$ (d) $18R$

5. The escape velocity from earth is 11.2 km/s. The escape velocity from a planet having twice the radius and the same mean density is **[AIIMS 2001]**

(a) 11.2 km/s (b) 5.6 km/s

(c) 22.4 km/s (d) 15.8 km/s

6. Rohini satellite is at a height of 500 km and Insat-B at a height 3600 km from the surface of earth. The relation between their velocities (V_R , V_I) is **[CBSE PMT 1999]**

(a) $V_R > V_I$ (b) $V_R < V_I$

(c) $V_R = V_I$ (d) no relation

7. Two satellites of masses m_1 and m_2 ($m_1 > m_2$) are going around the earth in orbits of radii r_1 and r_2 ($r_1 > r_2$). Which statement about velocities is correct? **[Pb PMT 1997]**

(a) $v_1 > v_2$ (b) $v_1 = v_2$

(c) $v_1 < v_2$ (d) $v_1/r_1 = v_2/r_2$

8. Mass of moon is 1/81 times that of earth and radius is 1/4 the earth's radius. If escape velocity at surface of earth is 11.2 km/sec, then its value at the surface of moon is **[AIIMS 2000]**

(a) 2.5 km/sec (b) 0.14 km/sec

(c) 5 km/sec (d) 0.5 km/sec

9. Escape velocity of a body when projected from earth's surface is 11.2 km/s. If it is projected at an angle of 50° from the horizontal, then escape velocity will be **[AIIMS 1999]**

(a) 11.6 km/s (b) 12.8 km/s

(c) 16.2 km/s (d) 11.2 km/s

10. The orbital velocity for an earth satellite near the surface of earth is 7 km/sec. If the radius of the orbit is 4 times the radius of the earth, its orbit velocity would be **[AFMC Pune 1997]**

(a) 7 km/sec (b) 3.5 km/sec

(c) $7\sqrt{2}$ km/sec (d) 14 km/sec

11. The period of revolution of a planet A around the sun is 8 times that of planet B. The distance of A is how many times greater than that of B from the sun. **[CBSE PMT 1997]**

(a) 2 (b) 3

(c) 4 (d) 5

12. The escape velocity of a body on the surface of earth is 11.2 km/s. If the earth's mass increases to twice its present value and radius of earth becomes half, the escape velocity becomes **[CBSE PMT 1997]**

(a) 5.6 km/sec (b) 11.2 km/sec

(c) 22.4 km/sec (d) 44.8 km/sec

13. The earth rotates about the sun (See Fig. 16.11) in an elliptical orbit. At which point is velocity maximum? **[CPMT 1997]**

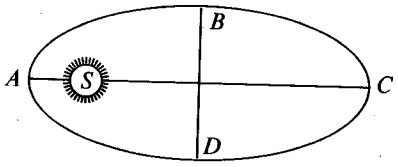

Fig. 16.11

(a) At A (b) At B

(c) At C (d) At D

14. Orbital velocity of earth satellite does not depend on **[MP PMT 1996]**

(a) mass of earth (b) mass of satellite

(c) radius of earth

(d) acceleration due to gravity

15. An earth satellite S has an orbit radius which is 4 times that of a communication satellite C. The period of revolution of S is **[MP PMT 1994]**

5. Escape velocity , $V_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2G}{R} \times \left(\frac{4}{3}\pi R^3 \rho\right)} = R\sqrt{\frac{8\pi G\rho}{3}}$

For the same mean density, $V_e \propto R \quad \therefore \quad \frac{V_P}{V_E} = \frac{R_P}{R_E} = \frac{2}{1}$

or $V_P = 2V_E = 2 \times 11.2 =$ **22.4 km/s**

6. Orbital velocity $= \sqrt{\frac{GM}{R+h}}$. Since h for Rohini is less, $V_R > V_I$.

7. Orbital velocity, $v = \sqrt{\frac{GM}{r}}$

Here M = mass of earth ; r = radius of orbit

$\therefore \quad v \propto \sqrt{\frac{1}{r}} \quad \therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{r_1}{r_2}}$

Since $r_1 > r_2$, $v_1 < v_2$.

8. Escape velocity, $V_e = \sqrt{\frac{2GM}{R}}$ or $V_e \propto \sqrt{\frac{M}{R}}$

$\therefore \quad \frac{V_M}{V_E} = \sqrt{\frac{M_M}{M_E} \times \frac{R_E}{R_M}} = \sqrt{\frac{1}{81} \times 4} = \frac{2}{9}$

or $V_M = \frac{2}{9} \times V_E = \frac{2}{9} \times 11.2 =$ **2.5 km/sec**

9. The escape velocity does not depend upon the direction of projection. It is because the kinetic energy a body loses in reaching any particular height depends only on the height concerned and not the path taken to reach it.

Escape velocity, $V_e = \sqrt{\frac{2GM}{R}}$

It is clear that escape velocity is independent of direction of projection.

10. Orbital velocity, $v = \sqrt{\frac{GM}{r}}$ or $v \propto \sqrt{\frac{1}{r}}$

$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{r_1}{r_2}}$ or $\frac{v_2}{7} = \sqrt{\frac{R}{4R}} = \frac{1}{2}$

$\therefore \quad v_2 = \frac{1}{2} \times 7 =$ **3.5 km/sec**

11. According to Kepler's third law of planetary motion,

$$T^2 \propto R^3 \text{ or } \left(\frac{T_A}{T_B}\right)^2 = \left(\frac{R_A}{R_B}\right)^3$$

$\therefore \quad (8)^2 = \left(\frac{R_A}{R_B}\right)^3$ or $\boldsymbol{R_A = 4R_B}$

12. Escape velocity, $V_e = \sqrt{\frac{2GM}{R}}$ or $V_e \propto \sqrt{\frac{M}{R}}$

$\therefore \quad \frac{V_e'}{V_e} = \sqrt{\frac{M'}{R'} \times \frac{R}{M}} = \sqrt{\frac{M'}{M} \times \frac{R}{R'}} = \sqrt{2 \times 2} = 2$

$\therefore \quad V_e' = 2V_e = 2 \times 11.2 =$ **22.4 km/sec**

13. According to Kepler's second law of planetary motion, the velocity of a planet is maximum when its distance from the sun is least. Therefore, the velocity of planet is maximum at A.

14. Orbital velocity, $v=\sqrt{\frac{GM}{R+h}}=R\sqrt{\frac{g}{R+h}}$

The orbital velocity is independent of mass of the satellite.

15. According to Kepler's third law of planetary motion,

$$\left(\frac{T_S}{T_C}\right)^2=\left(\frac{R_S}{R_C}\right)^3 \quad \therefore T_S=T_C\times\left(\frac{R_S}{R_C}\right)^{3/2}=1\times(4)^{3/2} = 8 \text{ days}$$

The time period T_C of a communication satellite is 1 day.

16. $\left(\frac{T_2}{T_1}\right)^2=\left(\frac{r_2}{r_1}\right)^3$ or $T_2=T_1\left(\frac{R+3R}{R}\right)^{3/2}=83(4)^{3/2}$ = 664 minutes

17. $\frac{T_A^2}{T_B^2}=\frac{r_1^3}{r_2^3}=\frac{(4r_2)^3}{r_2^3}$ $\quad \therefore \frac{T_A}{T_B}=(4)^{3/2}$ = 8

18. Escape velocity, $V_e=\sqrt{\frac{2GM}{R}}$ or $V_e\propto\sqrt{\frac{M}{R}}$

$$\therefore \quad \frac{V_E}{V_M}=\sqrt{\frac{M_E}{R_E}\times\frac{R_M}{M_M}}=\sqrt{\frac{M_E}{M_M}\times\frac{R_M}{R_E}}=\sqrt{81\times\frac{1}{3.5}} = 4.81$$

19. $\left(\frac{R_1}{R_2}\right)^3=\left(\frac{T_1}{T_2}\right)^2$ Here T_1 = 3 hr ; T_2 = 24 hr ; R_1= R ; R_2 = 36000 km

$$\therefore \quad R=R_2\left(\frac{T_1}{T_2}\right)^{2/3}=36000\left(\frac{3}{24}\right)^{2/3} = 9000 \text{ km}$$

20. $V_e=\sqrt{\frac{2GM}{R}}$ $\quad \therefore V_p=\sqrt{\frac{2G\times3M}{3R}}=\sqrt{\frac{2GM}{R}}=V_e$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. If V_0 be the orbital velocity of a satellite in a circular orbit close to earth's surface and V_e is the escape velocity from earth, then relation between the two is **[CET Karnataka 1999]**
   (a) $V_0=V_e$ (b) $V_e=\sqrt{2}\,V_0$
   (c) $V_e=\sqrt{3}\,V_0$ (d) $V_e=2V_0$
2. The orbital speed of Jupiter is **[CET Haryana 2000]**
   (a) less than the orbital speed of earth
   (b) equal to the orbital speed of earth
   (c) greater than the orbital speed of earth
   (d) zero
3. The earth retains its atmosphere because **[CET Haryana 2000]**
   (a) the earth has population
   (b) the escape velocity is less than the mean speed of molecules
   (c) the escape velocity is more than the mean speed of molecules
   (d) the earth is sphere
4. A satellite in a circular orbit of radius R has a time period of 4 hours. A satellite with orbital radius of $4R$ around the same planet will have a time period of **[CET Karnataka 1996]**
   (a) 8 hours (b) 16 hours
   (c) 32 hours (d) 64 hours
5. A satellite is launched into a circular orbit of radius R around the earth. A second satellite is launched into an orbit of radius 1.01 R. The

period of second satellite is larger than the first one by approximately **[IIT 1995]**

(a) 1.5% (b) 0.5%

(c) 3% (d) 1%

6. A satellite is moving around the earth with a speed v in a circular orbit of radius r. If the orbit radius is decreased by 1%, its speed will **[MP PET 1996]**

(a) increase by 1% (b) increase by 0.5%

(c) decrease by 1% (d) decrease by 0.5%

7. If the distance between earth and the sun were half of its present value, the number of days in a year would have been **[IIT 1996]**

(a) 64.5 (b) 182.5

(c) 730 (d) 129

8. What will be the duration of day and night (in hours) if the diameter of the earth is suddenly reduced to half its original value, the mass remaining constant? **[CET Haryana 1996]**

(a) 3 (b) 6

(c) 2 (d) 12

9. The escape velocity of a projectile from the earth is approximately **[BHU 1998]**

(a) 11.2 km/s (b) 7 km/s

(c) 112 km/s (d) 1.1 km/s

10. Two satellites are moving in orbits $R_1 > R_2$. The velocities associated with them will be **[BHU 1998]**

(a) $V_1 < V_2$ (b) $V_1 > V_2$

(c) $V_1 = V_2$ (d) $V_1 = 2V_2$

11. A satellite is placed in a circular orbit around earth at such a height that it always remains stationary *w.r.t.* earth surface. In such case, its height from earth's surface is **[AMU 1999]**

(a) 32000 km (b) 6400 km

(c) 36000 km (d) 4800 km

12. A satellite is moving around the earth (close to earth surface) with a speed v in a circular orbit of radius r. If the orbital radius is decreased by 1%, the orbital speed of the satellite will **[MP PET 1999]**

(a) increase by 1% (b) increase by 0.5%

(c) decrease by 1% (d) decrease by 0.5%

13. For any rocket on earth, the escape velocity is 11.2 km/sec. Its value at the planet where acceleration due to gravity is two times that on earth and diameter is two times that of earth will be (in km/s) **[MNR 1999]**

(a) 11.2 (b) 22.4

(c) 33.6 (d) 5.6

14. A small satellite is revolving near earth's surface. Its orbital velocity will be nearly **[CET Haryana 1995]**

(a) 6 km/sec (b) 4 km/sec

(c) 11.2 km/sec (d) 8 km/sec

15. Two planets A and B have the same material density. If the radius of A is twice that of B, then the ratio of their escape velocities V_A/V_B is **[CSEP 1999]**

(a) 2 (b) $\sqrt{2}$

(c) $1/\sqrt{2}$ (d) 1/2

16. The earth completes one revolution around the sun in one year. If the distance between them becomes double, the new period of revolution will be **[MP PET 2000]**

(a) 1/2 year (b) $2\sqrt{2}$ years

(c) 4 years (d) 8 years

17. A satellite is moving in a circular orbit at certain height above the earth's surface. It takes 5.26×10^3 sec to complete one revolution with a centripetal acceleration equal to 9.32 m/s^2. The height of the satellite orbit above earth's surface is nearly (radius of earth = 6.37×10^6 m) **[CET Haryana 1994]**

(a) 190 km (b) 220 km

(c) 160 km (d) 70 km

18. The largest and shortest distances of the earth from the sun are r_1 and r_2. Its distance from the sun when it is perpendicular to the major axis of the orbit drawn from the sun is **[CET Haryana 1998]**

(a) $\frac{r_1 + r_2}{r_1 - r_2}$ (b) $\frac{2r_1 r_2}{r_1 + r_2}$

(c) $\frac{r_1 + r_2}{3}$ (d) $\frac{r_1 + r_2}{4}$

19. The orbital velocity of an artificial satellite in a circular orbit just above earth's surface is v. For a satellite orbiting at an altitude of half the earth's radius, the orbital velocity is **[MNR 1994]**

(a) $\frac{3v}{2}$ (b) $\sqrt{\frac{3}{2}}\,v$

(c) $\sqrt{\frac{2}{3}}\,v$ (d) $\frac{2}{3}v$

20. If the radius of earth's orbit is made 1/4th, then duration of year will become **[BHU 1998]**

(a) 8 times (b) 4 times

(c) 1/8 times (d) 1/4 times

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*c*)	**5.** (*a*)
6. (*b*)	**7.** (*d*)	**8.** (*b*)	**9.** (*a*)	**10.** (*a*)
11. (*c*)	**12.** (*d*)	**13.** (*b*)	**14.** (*d*)	**15.** (*a*)
16. (*b*)	**17.** (*c*)	**18.** (*b*)	**19.** (*c*)	**20.** (*c*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Orbital velocity of satellite close to earth's surface is

$V_0 = \sqrt{gR}$; Escape velocity, $V_e = \sqrt{2gR} = \sqrt{2} \times \sqrt{gR}$

$\therefore \quad V_e = \sqrt{2}\, V_0$

2. Orbital speed, $v \propto 1/\sqrt{r}$. As the radius of jupiter orbit is more than that of earth orbit, the jupiter has **less orbital speed than that of earth.**

3. The **velocity of escape at earth is more than the mean speed of molecules.** Therefore, earth is able to retain its atmosphere.

4. $\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3$ or $T_2 = T_1\left(\frac{R_2}{R_1}\right)^{3/2} = 4 \times \left(\frac{4R}{R}\right)^{3/2}$ = **32 hours**

5. $\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3$ or $T_2 = T_1\left(\frac{R_2}{R_1}\right)^{3/2} = T_1\left(\frac{1.01R}{R}\right)^{3/2} = 1.015\, T_1$

$\therefore$ %age increase in the period of second satellite

$$= \frac{T_2 - T_1}{T_1} \times 100 = \frac{1.015T_1 - T_1}{T_1} \times 100 = \textbf{1.5\%}$$

6. Orbital speed, $v = R\sqrt{\frac{g}{R+h}} = R\sqrt{\frac{g}{r}} \quad (r = R + h)$

$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{r_1}{r_2}} = \sqrt{\frac{r_1}{0.99\, r_1}} = \sqrt{\frac{1}{0.99}} = 1.005 \quad \therefore v_2 = 1.005 v_1$

$\therefore$ %age increase in speed

$$= \frac{v_2 - v_1}{v_1} \times 100 = \frac{1.005 v_1 - v_1}{v_1} \times 100 = \textbf{0.5\%}$$

7. $\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{r_2}{r_1}\right)^3$ or $T_2 = T_1\left(\frac{r_2}{r_1}\right)^{3/2} = 365\left(\frac{r/2}{r}\right)^{3/2}$ = **129 days**

8. Since no external torque is acting, the angular momentum L will be constant.

Now $\quad L = I\omega = \frac{2}{5}MR^2\omega$

$\therefore \quad \frac{2}{5}MR_1^2\omega_1 = \frac{2}{5}MR_2^2\omega_2 \quad$ or $\quad R_1^2\omega_1 = \frac{R_1^2}{4}\omega_2$

$\therefore \quad \omega_2 = 4\omega_1$

Now $\quad \frac{T_2}{T_1} = \frac{2\pi/\omega_2}{2\pi/\omega_1} = \frac{\omega_1}{\omega_2} = \frac{1}{4}$

$\therefore \quad T_2 = \frac{1}{4} \times T_1 = \frac{1}{4} \times 24$ = **6 hours**

9. Escape velocity, $V_e = \sqrt{2gR} = \sqrt{2\times9.8\times6.4\times10^6} = 11.2\times10^3$ m/s = **11.2 km/s**

10. Orbital velocity $= \sqrt{\dfrac{GM}{R+h}} = \sqrt{\dfrac{GM}{r}}$ $(\because r = R+h)$

$\therefore$ Orbital velocity $\propto \dfrac{1}{\sqrt{r}}$. Since $R_1 > R_2$, $V_1 < V_2$

11. $h = \left(\dfrac{gR^2T^2}{4\pi^2}\right)^{1/3} - R = \left[\dfrac{9.8\times(6.4\times10^6)^2\times(24\times60\times60)^2}{4\pi^2}\right]^{1/3} - 6.4\times10^6$

$\simeq 36000\times10^3$ m = **36000 km**

12. If r is the radius of the orbit of the satellite, then,

Orbital speed $v = \sqrt{gr}$ $\therefore \dfrac{dv}{v} = \dfrac{1}{2}\dfrac{\Delta r}{r} = \dfrac{1}{2}(-1\%)$ = **0.5% decrease**

13. Escape velocity on earth = $\sqrt{2gR}$ = 11.2 km/sec

Escape velocity on planet $= \sqrt{2\times(2g)\times(2R)} = 2\sqrt{2gR} = 2\times11.2 =$ **22.4 km/sec**

14. The orbital velocity v of a satellite orbiting close to earth's surface is

$$v = \sqrt{gR} = \sqrt{9.8\times6.4\times10^6} \simeq 8\times10^3 \text{ m/sec} = \textbf{8 km/sec}$$

15. Escape velocity, $V_e = \sqrt{\dfrac{2GM}{R}} = \sqrt{\dfrac{2G}{R}\times\dfrac{4}{3}\pi R^3\rho}$

For the same density, we have, $V_e \propto R$ $\therefore$ $\dfrac{V_A}{V_B} = \dfrac{R_A}{R_B} = 2$

16. $\left(\dfrac{T_2}{T_1}\right)^2 = \left(\dfrac{r_2}{r_1}\right)^3$ $\therefore T_2 = T_1\left(\dfrac{r_2}{r_1}\right)^{3/2} = 1(2)^{3/2} =$ **$2\sqrt{2}$ years**

17. Let h metres be the height of the satellite above earth's surface.

Time period, $T = 2\pi\sqrt{\dfrac{(R+h)^3}{GM}}$ or $\left(\dfrac{T}{2\pi}\right)^2 = \dfrac{(R+h)^3}{GM}$ ...(*i*)

Centripetal acceleration, $a_c = \dfrac{GM}{(R+h)^2}$ or $\dfrac{1}{a_c} = \dfrac{(R+h)^2}{GM}$ ...(*ii*)

Dividing eq. (*i*) by eq. (*ii*), we get, $R+h = \left(\dfrac{T}{2\pi}\right)^2 \times a_c$

or $6.37\times10^6 + h = \left(\dfrac{5.26\times10^3}{2\pi}\right)^2 \times 9.32$ On solving, $h \simeq$ **160 km**

18. Using the properties of ellipse, the required distance d is

$$\frac{2}{d} = \frac{1}{r_1} + \frac{1}{r_2} \quad \therefore \quad d = \frac{2r_1r_2}{r_1+r_2}$$

19. $v = \sqrt{\dfrac{GM}{R}}$ $\therefore v' = \sqrt{\dfrac{GM}{R+R/2}} = \sqrt{\dfrac{2}{3}\dfrac{GM}{R}} = \sqrt{\dfrac{2}{3}}\,v$

20. $\left(\dfrac{T_2}{T_1}\right)^2 = \left(\dfrac{r_2}{r_1}\right)^3$ $\therefore T_2 = T_1\left(\dfrac{r_2}{r_1}\right)^{3/2} = T_1\left(\dfrac{1}{4}\right)^{3/2} = \dfrac{T_1}{8}$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **With what velocity should a body be projected horizontally at a height of 30 km from the ground so that it becomes the satellite of the earth? Neglect friction. Also calculate the time period of revolution of the satellite. Radius of earth = 6370 km.**

Hint. Suppose the mass of the body is m. Let the radius of earth be R and the projection velocity be v. Then,

$$\frac{mv^2}{R+h} = mg' \quad \text{where} \quad g' = g\left(1-\frac{2h}{R}\right)$$

or $$v^2 = g'(R+h) = g(R+h)\left(1-\frac{2h}{R}\right)$$

$$\therefore \quad v = \sqrt{gR\left(1-\frac{h}{R}\right)} = \sqrt{9.8\times6370\times10^3\left(1-\frac{30}{6370}\right)}$$

$$= 7882 \text{ ms}^{-1} = 7.882 \text{ kms}^{-1}$$

Time period, $$T = \frac{2\pi(R+h)}{v} = \frac{2\pi(6370+30)}{7.882} = 5099 \text{ s} = 1.416 \text{ h}$$

2. **Calculate the mass of sun given that the distance between the sun and earth is 1.49×10^{11} m and $G = 6.67 \times 10^{-11}$ SI units. 1 year = 365 days.**

Hint. Let m and M be the masses of the earth and sun in SI units

$$\therefore \quad \frac{mv^2}{r} = \frac{GMm}{r^2} \quad \text{or} \quad v^2 = \frac{GM}{r}$$

or $$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

or $$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2}{G}\times\left(\frac{r^3}{T^2}\right)$$

Here $r = 1.49 \times 10^{11}$ m ; $G = 6.67 \times 10^{-11}$ SI units ; $T = 365 \times 86400$ s

$$\therefore \quad M = \frac{4\pi^2}{6.67\times10^{-11}}\times\frac{(1.49\times10^{11})^3}{(365\times86400)^2} = 1.972 \times 10^{30} \text{ kg}$$

3. **Calculate the limiting velocity required by an artificial satellite orbiting round the earth. Given that radius of earth = 6.4×10^6 m.**

Hint. The orbital velocity of earth satellite is given by ;

$$v = \sqrt{\frac{gR^2}{R+h}}$$

When the satellite orbits very close to the surface of the earth, $h \simeq 0$. Therefore, limiting velocity of the satellite is

$$v = \sqrt{gR} = \sqrt{9.8\times6.4\times10^6} = 7.92\times10^3 \text{ ms}^{-1} = 7.92 \text{ kms}^{-1}$$

4. **The moon orbits the earth in an approximately circular orbit of radius 3.8×10^8 m. It takes about 27 days to complete one orbit. What is the mass of earth as obtained from these data?**

Hint. The gravitational attraction between the earth and moon provides the centripetal force. If m and M are the masses of the moon and earth respectively in SI units, then,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

or $$M = \frac{v^2 r}{G} = \frac{\omega^2 r^3}{G} \qquad (\because v = \omega r)$$

Here ω = 1 rev/27 days = $\frac{2\pi}{27\times24\times60\times60} = 2.7\times10^{-6}$ rad/s ;

$r = 3.8 \times 10^8$ m ; $G = 6.67 \times 10^{-11}$ in SI units

$\therefore$ $$M = \frac{(2.7\times10^{-6})^2\times(3.8\times10^8)^3}{6.67\times10^{-11}} = \mathbf{6 \times 10^{24}\ kg}$$

5. Calculate the escape velocity from earth for a 5000 kg spacecraft and determine the kinetic energy it must have at earth's surface in order to escape earth's field. Mass of earth M= 5.98 $\times10^{24}$ kg and radius of earth R = 6.37 $\times$ 10^6 m.

Hint. The escape velocity V_e of a body on the earth's surface is

$$V_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2\times(6.67\times10^{-11})\times(5.98\times10^{24})}{6.37\times10^6}} = \mathbf{1.12 \times 10^4\ ms^{-1}}$$

The K.E. of the spacecraft is given by ;

$$K = \frac{1}{2}mV_e^2 = \frac{1}{2}\times5000\times(1.12\times10^4)^2 = \mathbf{3.14 \times 10^{11}\ J}$$

6. A spaceship is launched into a circular orbit close to earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull ? Radius of earth = 6400 km; g = 9.8 ms^{-2}.

Hint. Orbital velocity of spaceship in the circular orbit is

$$v = \sqrt{\frac{gR^2}{R+h}}$$

When the satellite orbits close to the earth, $h \simeq 0$.

$\therefore$ $$v = \sqrt{gR} = \sqrt{9.8\times6400\times1000} = 7.9195\times10^3\ ms^{-1} = 7.9195\ kms^{-1}$$

Escape velocity, $V_e = \sqrt{2Rg} = \sqrt{2}\times\sqrt{Rg} = \sqrt{2}\times7.9195 = 11.2000\ kms^{-1}$

Additional velocity required = 11.2000 – 7.9195 = **3.2805 kms^{-1}**

7. A body is raised to a height of 500 km. What is the acceleration due to gravity at this point? With what velocity should this body be projected in a direction perpendicular to the line joining the centres of the earth and body so that it describes a circle about the earth? What will be the period of revolution of the body about the earth? Radius of earth, R = 6500 km; g = 9.8 ms^{-2}. **[I.I.T.]**

Hint. The acceleration due to gravity on earth's surface is g (= 9.8 ms^{-2}). Let g' be the acceleration due to gravity at height h (= 500 km). Then (M = mass of earth),

$$g = \frac{GM}{R^2}\ ;\quad g' = \frac{GM}{(R+h)^2}$$

$\therefore$ $$\frac{g'}{g} = \frac{R^2}{(R+h)^2}$$

or $$g' = g\times\frac{R^2}{(R+h)^2} = 9.8\times\frac{(65\times10^5)^2}{[(65+5)\times10^5]^2} = \mathbf{8.45\ ms^{-2}}$$

Let v be the required projection velocity for the body to become an artificial satellite of earth. Then (m = mass of body),

$$mg' = \frac{mv^2}{R+h}$$

or $v = \sqrt{g'(R+h)} = \sqrt{8.45[(65+5)\times10^5]} = 7.69\times10^3 \text{ ms}^{-1} = 7.69 \text{ kms}^{-1}$

Period of revolution, $T = \dfrac{2\pi(R+h)}{v} = \dfrac{2\pi[(65+5)\times10^5]}{7.69\times10^3}$

$= 5719 \text{ s} = 1 \text{ hour } 35 \text{ min}$

8. Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their period of revolutions are 1 hour and 8 hours respectively. The radius of the orbit of $S_1 = 10^4$ km. When S_2 is closest to S_1, find (*i*) the speed of S_2 relative to S_1 (*ii*) the angular speed of S_2 actually observed by an astronaut in S_1.

[I.I.T.]

Hint. Figure 16.12 shows the conditions of the problem. If r_1 and r_2 are the radii of orbits of satellites S_1 and S_2 and T_1 and T_2 are their respective periods of revolution, then according to Kepler's third law,

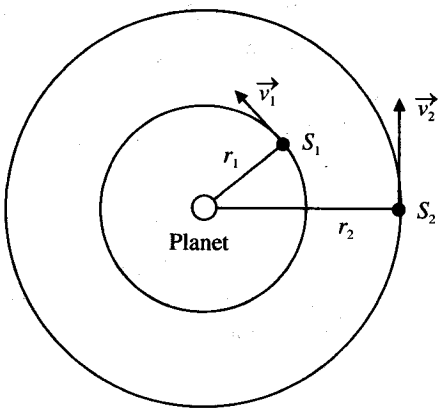

Fig. 16.12

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

or $$r_2^3 = r_1^3 \times \left(\frac{T_2}{T_1}\right)^2 = (10^4)^3 \times \left(\frac{8}{1}\right)^2$$

or $$r_2 = 4 \times 10^4 \text{ km}$$

(*i*) If orbital speeds of S_1 and S_2 are v_1 and v_2 respectively, then,

$$v_1 = \frac{2\pi r_1}{T_1} = \frac{2\pi\times10^4}{1} = 2\pi\times10^4 \text{ km/h}$$

$$v_2 = \frac{2\pi r_2}{T_2} = \frac{2\pi\times4\times10^4}{8} = \pi\times10^4 \text{ km/h}$$

$\therefore$ Speed of S_2 relative to S_1 is

$$|v_2 - v_1| = (\pi\times10^4) - (2\pi\times10^4) = \mathbf{-3.14 \times 10^4 \text{ km/h}}$$

(*ii*) Angular speed of S_2 relative to S_1

$$= \frac{|v_2 - v_1|}{r_2 - r_1} = \frac{-3.14\times10^4}{4\times10^4 - 10^4} \text{ rad/h}$$

$$= -1.046 \text{ rad/h} = -\frac{1.046}{3600} \text{ rad/s} = \mathbf{-2.907 \times 10^{-4} \text{ rad/s}}$$

9. The revolution period of earth is 365.3 days and its mean distance from the sun is 1.5×10^8 km. The revolution periods of Venus and Mars are 224.7 days and 687 days respectively. Estimate the mean distances of these planets from the sun. Neglect the effect of other planets.

Hint. Revolution period of earth, T= 365.3 days ; Distance of earth from sun = 1.5×10^8 km

Revolution period of Venus, T_1 = 224.7 days ; Revolution period of Mars = 687 days

According to Kepler's third law, we have,

$$\frac{T^2}{R^3} = \text{Constant} \quad \text{or} \quad \frac{T^2}{R^3} = \frac{(365.3)^2}{(1.5\times10^8)^3} = 3.95\times10^{-20} \text{ days}^2 \text{ km}^{-3}$$

For Venus $\dfrac{T_1^2}{R_1^3} = \dfrac{T^2}{R^3} = 3.95\times10^{-20}$

$\therefore \quad R_1^3 = \dfrac{T_1^2}{3.95\times10^{-20}} = \dfrac{(224.7)^2}{3.95\times10^{-20}}$ or $R_1 = 1.085 \times 10^8$ km

For Mars $\dfrac{T_2^2}{R_2^3} = \dfrac{T^2}{R^3} = 3.95\times10^{-20}$

$\therefore \quad R_2^3 = \dfrac{T_2^2}{3.95\times10^{-20}} = \dfrac{(687)^2}{3.95\times10^{-20}}$ or $R_2 = 2.28 \times 10^8$ km

10. A satellite goes round the earth in 90 minutes in a circular orbit. Calculate the height of the satellite above the earth, taking earth to be a sphere of radius 6370 km. The value of g at the orbit of the satellite = 9.8 ms^{-2}.

Hint. If m, v and r are the mass, orbital velocity and radius of orbit of the satellite respectively, then,

$$\frac{mv^2}{r} = mg \quad \text{or} \quad \frac{v^2}{r} = g$$

But $v = \omega r$ and $\omega = 2\pi/T$. $\therefore \quad g = \dfrac{(\omega r)^2}{r} = \dfrac{\omega^2 r^2}{r} = \omega^2 r = \dfrac{4\pi^2 r}{T^2}$

or $r = \dfrac{gT^2}{4\pi^2}$. Here $T = 90 \times 60 = 5400$ s ; $g = 9.8$ ms^{-2}

$\therefore \quad r = \dfrac{9.8\times(5400)^2}{4\pi^2} = 7238\times10^3 \text{ m} = 7238 \text{ km}$

Since the radius of earth is 6370 km, the height of the satellite above earth's surface is

$h = r -$ earth's radius $= 7238 - 6370 = 868$ km

17

Gravitational Field, Energy and Potential

INTRODUCTION

Most of the forces we meet in everyday life are contact forces. For example, a tennis racket exerts a force on a tennis ball when they make contact or a ball exerts a force on a window when they make contact. But gravitational force of attraction exists between two masses even if they are not in contact. In order to explain this gravitational interaction, we introduce the concept of *gravitational field.* According to the field concept, every mass is a source of gravitational field. If another mass is placed in this field, it will experience a gravitational force of attraction due to interaction of gravitational field on the mass. In this chapter, we shall discuss the important concept of gravitational field, gravitational potential energy and gravitational potential.

17.1. GRAVITATIONAL FIELD

Consider an isolated body of mass M as shown in Fig. 17.1. If a small test mass m_0 is placed near it (say at point P), the mass m_0 experiences a force of attraction. The question arises, how does M exert force on m_0? The answer is that M sets up gravitational field in the space surrounding it. The gravitational field of M exerts a force on m_0 placed at point P.

The **gravitational field** *due to a material body is the space around the body in which any other mass experiences a force of attraction.*

Theoretically, the gravitational field due to a material body extends up to infinity. However, the effect of gravitational field decreases as the distance from the body is increased. We represent

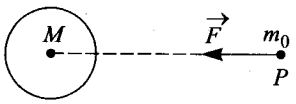

Fig. 17.1

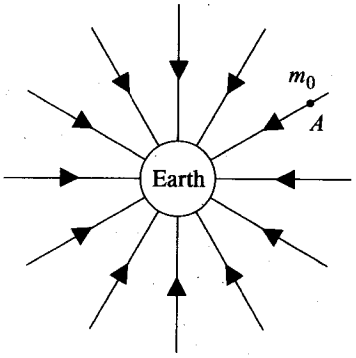

Fig. 17.2

the gravitational field of a mass by *field lines.* Fig. 17.2 shows the gravitational field of the earth. The lines with arrows are called field lines. Their direction is determined in the following way.

If a small mass m_0 (called the **test mass)* is placed at a point such as A, it will be attracted toward the earth's centre as shown in Fig. 17.2. The field lines show us the direction of gravitational force on the test mass. Since we know masses are always attracted toward the earth's centre, the gravitational field lines must be directed radially inward. The field lines reveal an important feature. *Note that field lines are closer together near the earth than they are at more distant points.* Since we know that earth's gravitational force is strongest close to the earth, the closeness of the gravitational field lines is a measure of the strength of the gravitational field.

17.2. INTENSITY OF GRAVITATIONAL FIELD

The **intensity of gravitational field** (or **gravitational field strength**) *at a point in a gravitational field is defined as the force per unit mass acting on a test mass placed at that point.* It is a vector quantity and is denoted by $\vec{E}$.

Consider an isolated body of mass M as shown in Fig. 17.3. The mass M sets up gravitational field in the space surrounding it. If a small test mass m_0 placed at point P experiences an attractive force $\vec{F}$, then gravitational field strength at point P is

$$\vec{E} = \frac{\vec{F}}{m_0}$$

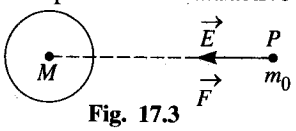

Fig. 17.3

Note that *intensity of gravitational field* ($\vec{E}$) *at a point is just the acceleration that a unit mass would experience when placed at that point.* Therefore, it is a vector quantity possessing both magnitude and direction. Its unit is that of acceleration *i.e.*, m/s^2 or N/kg. The direction of $\vec{E}$ is that in which the test mass (m_0) would move under the influence of the field.

$$\text{Dimensional formula of } E = \frac{F}{m_0} = \frac{[MLT^{-2}]}{[M]} = [M^0LT^{-2}]$$

Note that intensity of gravitational field is directed towards the centre of gravity of the body. Clearly, for a spherical body, it will be directed towards the geometric centre.

17.3. INTENSITY OF GRAVITATIONAL FIELD DUE TO EARTH

The **intensity of earth's gravitational field** *at a point is equal to the acceleration that a unit mass would experience when placed at that point.*

Thus the intensity of earth's gravitational field on the surface of earth is $g = 9.8$ m/s^2. Consider earth to be a sphere of radius R and mass M as shown in Fig. 17.4. Suppose we want to find the intensity of gravitational field due to earth at a point P located at a height h above the surface of earth. If a test mass m_0 is placed at point P, then gravitational force exerted by earth on m_0 is

$$F = \frac{GMm_0}{(R+h)^2} \qquad \text{...Law of gravitation.}$$

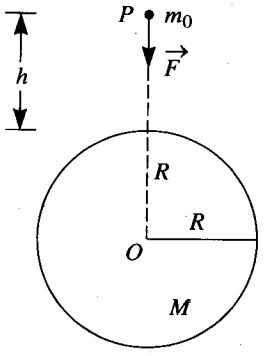

Fig. 17.4

By definition, the magnitude of the intensity of gravitational field due to earth at point P is

$$E = \frac{F}{m_0} = \frac{GMm_0}{(R+h)^2} \times \frac{1}{m_0}$$

$$\therefore \quad E = \frac{GM}{(R+h)^2} \qquad \text{...}(i)$$

* The test mass should be so small that its gravitational field does not affect the gravitational field of the body under consideration.

The direction of E is along PO *i.e.* gravitational field due to earth is directed toward the centre of the earth. We have already seen that acceleration due to earth's gravity g' (g is acceleration due to gravity on earth's surface) at any point P at a height h above the surface of earth is

$$g' = \frac{GM}{(R+h)^2} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $E = g' = \frac{GM}{(R+h)^2}$.

Thus intensity of earth's gravitational field at any point simply means acceleration due to gravity at that point. On the surface of earth, $E = g = 9.8\ \text{ms}^{-2}$.

Fig. 17.5 shows the graph between gravitational field strength of earth versus distance from the centre of the earth. It is assumed that the density of the earth is uniform. Inside the earth, g' is directly proportional to the distance from the centre. Therefore, g' increases linearly from zero at the centre to g ($= GM/R^2$) at the surface. Outside earth's surface, g' is inversely proportional to the square of distance from the centre of earth. Note that g' ($= g = G\,M/R^2$) is maximum at the surface of earth.

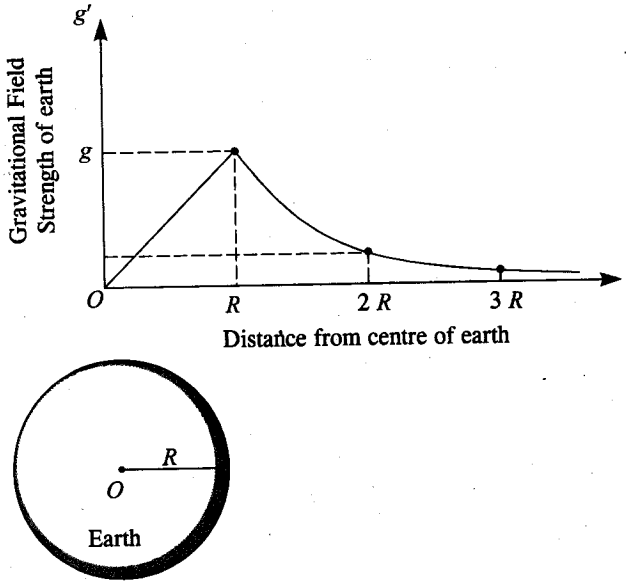

Fig. 17.5

Now $$g' = E = \frac{GM}{(R+h)^2}$$

As $h \to \infty$; $g' \to 0$. Clearly, the intensity of earth's gravitational field is zero at infinity. In other words, acceleration due to gravity is zero at infinity.

17.4. GRAVITATIONAL POTENTIAL ENERGY

In chapter 12, we considered the gravitational potential energy of a body near the surface of the earth. It was proved that gravitational potential energy of a body of mass m raised through a height h is

$$\text{Gravitational P.E.} = mgh \qquad ...(i)$$

This expression mgh for gravitational potential energy is an approximation based on the assumption that g is constant—an assumption that is approximately valid near the surface of earth.

If we wish to determine the gravitational potential energy of an object that is far removed from the earth's surface, then eq. (*i*) is inadequate. In eq. (*i*), we arbitrarily assume that gravitational potential energy of a body is zero at the earth's surface. Since the gravitational force is attractive, the gravitational potential energy will be zero at infinity. In other words, gravitational potential energy of a body near the earth's surface is negative and becomes less negative (*i.e.* greater) as it moves away from the earth. The maximum potential energy is zero, the value obtained at infinite distance from the earth's centre.

17.5. GENERAL EXPRESSION FOR GRAVITATIONAL POTENTIAL ENERGY

The gravitational potential energy of a body at a point in the gravitational field is defined as the amount of work done in bringing the body from infinity to that point. Because work is obtained (not done) in bringing the body from infinity to the desired point, *the gravitational potential energy is always negative.*

We now derive the general expression for gravitational potential energy of a body in the gravitational field of the earth. Consider the earth to be a sphere of radius R and mass M. Suppose a body of mass m is situated outside the earth at point A at a distance r from the centre of earth [See Fig. 17.6]. It is desired to find the *potential energy of the body at point A ($OA = r$). By definition, the gravitational potential energy U_A of the body at point A is

U_A = W.D. in bringing the body from infinity to point A.

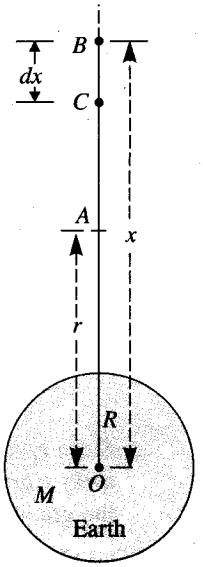

Fig. 17.6

Suppose at any instant the body is at point B at a distance x from the centre of the earth (*i.e.*, $OB = x$). The gravitational force exerted by earth on the body at B is

$$F = \frac{GMm}{x^2}$$

Small amount of work done when body moves from B to C (where $BC = dx$) is

$$dW = Fdx = \frac{GMm}{x^2}dx$$

Total work done by the gravitational force when the body of mass m moves from infinity to point A is

$$W = \int_{\infty}^{r} \frac{GMm}{x^2}dx = GMm\int_{\infty}^{r} \frac{1}{x^2}dx$$

$$= GMm\left|\frac{x^{-1}}{-1}\right|_{\infty}^{r} = -GMm\left[\frac{1}{r} - \frac{1}{\infty}\right] = -\frac{GMm}{r}$$

or $$W = -\frac{GMm}{r}$$

Therefore, gravitational potential energy (U_A) of a body of mass m at a distance r from the centre of the earth is

$$U_A = -\frac{GMm}{r} \qquad ...(i)$$

---

* The potential energy is the property of a system of two bodies (M and m in this case) rather than of a single body. There is no way to divide the energy and say so much belongs to one body and so much to the other. But if $M >> m$ (as in our case), we often speak of "the potential energy" of m. It is because as mass m moves in the vicinity of the earth, changes in the potential energy of earth-body system appear almost entirely as changes in the kinetic energy of the body (m) since the changes in the kinetic energy of the earth are too small to be measured.

Discussion. The following points are worth noting :

(*i*) Eq. (*i*) reveals that a body's gravitational potential energy is negative near the earth's surface and becomes less negative (*i.e.*, greater) as it moves away from the earth. The maximum gravitational potential energy of a body is zero at $r = \infty$. This is in agreement with our previous observation that objects have more gravitational potential energy as they move away from the earth.

(*ii*) Since the force between the body and earth is attractive, an external agent must do positive work to increase the separation of the body from the earth. The external work done increases the potential energy as the body is taken away from the earth.

(*iii*) The gravitational potential energy of a body of mass m at a distance r_1 from the centre of earth is [See Fig. 17.7]

$$U_1 = -\frac{GMm}{r_1}$$

The gravitational potential energy of the body at a distance r_2 from the centre of earth ($r_2 > r_1$) is

$$U_2 = -\frac{GMm}{r_2}$$

Change in potential energy is

$$\Delta P.E. = U_2 - U_1 = \left(-\frac{GMm}{r_2}\right) - \left(-\frac{GMm}{r_1}\right)$$

$$= GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

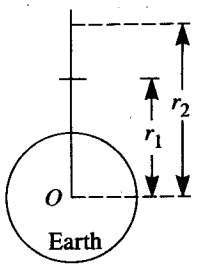

Fig. 17.7

Since $r_1 < r_2$, the change in potential energy is positive. It means that if the body is moved away from the earth, the gravitational potential energy of the body increases. Conversely, the gravitational potential energy of a body decreases as it is brought closer to the earth.

17.6. GRAVITATIONAL P.E. NEAR EARTH'S SURFACE

We now prove that near the surface of earth, the gravitational potential energy of a body of mass m is mgh where h is the height of the body above the surface of earth.

Consider a body of mass m at rest on the surface of earth (point A in Fig. 17.8). The gravitational potential energy of the body at the surface of earth is

$$U_A = -\frac{GMm}{R} \quad \text{...}R \text{ is earth's radius}$$

We then lift it to a height h (point B) above the surface of the earth. The gravitational potential energy of the body now is

$$U_B = -\frac{GMm}{(R+h)}$$

$\therefore$ Increase in potential energy of the body is

$$\Delta P.E. = U_B - U_A = \left(-\frac{GMm}{R+h}\right) - \left(-\frac{GMm}{R}\right)$$

$$= GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$\therefore \quad \Delta P.E. = GMm\left(\frac{h}{R(R+h)}\right) \qquad \text{...(i)}$$

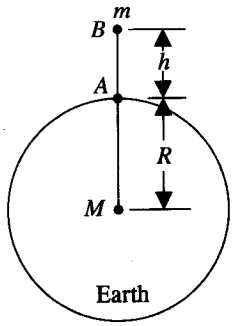

Fig. 17.8

If $R >> h$ (*i.e.*, the body is close to the surface of earth), h can be neglected as compared to R.

$$\therefore \quad \Delta P.E. = \frac{GMmh}{R^2}$$

We know that $g = \frac{GM}{R^2}$

Therefore, the above expression becomes :

$$\Delta P.E. = mgh \qquad ...(ii)$$

Thus for changes in position near earth's surface, eqs. (*i*) and (*ii*) both give the same answer. For motion over distances that are not small compared with the earth's radius, you must use eq. (*i*).

17.7. ESCAPE VELOCITY FROM ENERGY PRINCIPLE

If you fire a projectile upward, it slows down, stops momentarily and returns to earth. However, there is a certain initial velocity that will cause it to move upward forever, theoretically coming to rest only at infinity. This initial velocity is called *escape velocity.* Consider a body of mass m leaving earth with escape velocity V_e.

$$\text{Initial K.E. of body} = \frac{1}{2}mV_e^2$$

$$\text{Initial P.E. of body} = -\frac{GM*m}{R} \qquad (\because r = R)$$

$$\therefore \quad \text{Total initial energy} = \left(\frac{1}{2}mV_e^2\right) + \left(-\frac{GMm}{R}\right)$$

When the projectile reaches infinity, it stops and thus has no kinetic energy. It also has zero potential energy because potential energy of a body is zero at infinity. From the principle of conservation of energy, its total energy at the earth's surface must also have been zero.

$$\therefore \quad \left(\frac{1}{2}mV_e^2\right) + \left(-\frac{GMm}{R}\right) = 0$$

$$\text{or} \quad \frac{1}{2}mV_e^2 = \frac{GMm}{R}$$

$$\text{or} \quad \text{Escape velocity, } V_e = \sqrt{\frac{2GM}{R}} \qquad ...(i)$$

As $g = \frac{GM}{R^2}$, therefore, $GM = gR^2$. Putting $GM = gR^2$ in eq. (*i*), we have,

$$V_e = \sqrt{\frac{2}{R} \times gR^2} = \sqrt{2gR} \qquad ...(ii)$$

The reader may note that eqs. (*i*) and (*ii*) represent the same equations for escape velocity derived earlier. Here the derivation is based on energy principle and gives more insight to the problem.

Example 17.1. **With what velocity must a body be thrown upward from the surface of the earth so that it reaches a height of 10 R ? Earth's radius $R = 6.4 \times 10^6$ m ; Earth's mass $M = 6 \times 10^{24}$ kg ; $G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}.**

---

* M is the mass of the earth and R is its radius.

Solution. Suppose m is the mass of the body. The gravitational potential energy of the body at earth's surface is

$$U = -\frac{GMm}{R}$$

The gravitational potential energy of the body at a height $10\,R$ above the surface of earth is

$$U' = -\frac{GMm}{R+10R} = -\frac{GMm}{11R}$$

$$\text{Increase in potential energy} = U' - U = -\frac{GMm}{11R} - \left(\frac{-GMm}{R}\right) = \frac{10GMm}{11R}$$

If v_0 is the initial speed of the body, then kinetic energy of the body is $mv_0^2/2$. According to conservation of energy principle,

$$\frac{1}{2}mv_0^2 = \frac{10GMm}{11R}$$

or
$$v_0 = \sqrt{\frac{20GM}{11R}} = \sqrt{\frac{20\times(6.67\times10^{-11})\times6\times10^{24}}{11\times(6.4\times10^6)}} = \mathbf{1.07 \times 10^4\ ms^{-1}}$$

Example 17.2. A 90 kg satellite is shot from earth with an initial speed of 12 km s^{-1}. Can the satellite attain a circular orbit around the earth? Given that radius of earth = 6.37 × 10^6 m ; mass of earth = 5.97 × 10^{24} kg ; G = 6.67 × 10^{-11}N m^2 kg^{-2}.

Solution. If the total energy is negative, the satellite is bound and may go into the orbit. If the total energy is positive, the satellite will escape earth's gravitational field.

v_0 = 12 km s^{-1} = 12 × 10^3 ms^{-1} ; m = 90 kg; R = 6.37 × 10^6 m ; M = 5.97 × 10^{24} kg ; G = 6.67 × 10^{-11} Nm^2 kg^{-2}

$$\text{Initial P.E. of the satellite} = -\frac{GMm}{R}$$

$$= -\frac{(6.67\times10^{-11})\times(5.97\times10^{24})\times90}{6.37\times10^6} = -5.62\times10^9\ \text{J}$$

$$\text{Initial K.E. of the satellite} = \frac{1}{2}mv_0^2 = \frac{1}{2}\times(90)\times(12\times10^3)^2 = 6.48\times10^9\ \text{J}$$

Total energy of the satellite at take off is

$$E = \text{P.E.} + \text{K.E.}$$

$$= (-5.62 \times 10^9) + (6.48 \times 10^9) = 0.86\times10^9\ \text{J}$$

If atmospheric effects are neglected, the total energy of the satellite in the space will be the same as that at take off. Since the total energy of the satellite *w.r.t.* earth is positive, the satellite will not orbit the earth.

Example 17.3. Two masses 800 kg and 600 kg are at a distance of 0.25 m apart. Compute the magnitude of intensity of gravitational field at a point distant 0.2 m from 800 kg mass and 0.15 m from the 600 kg mass.

Solution. Fig. 17.9 shows the conditions of the problem. It is desired to find the magnitude of the resultant gravitational field at point P.

Gravitational intensity at P due to mass at A is

$$E_A = \frac{Gm_1}{r^2} = \frac{G\times 800}{(0.2)^2} = 20{,}000G \text{ along } PA$$

Gravitational intensity at P due to mass at B is

$$E_B = \frac{Gm_2}{r^2} = \frac{G\times 600}{(0.15)^2} = \frac{80{,}000\,G}{3} \text{ along } PB$$

In $\Delta APB, PA^2 + PB^2 = AB^2 \quad \therefore \angle APB = 90^\circ$

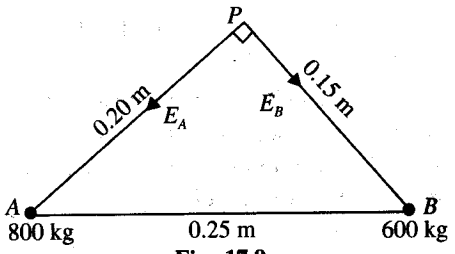

Fig. 17.9

$\therefore$ Magnitude of resultant gravitational field at P is

$$E = \sqrt{E_A^2 + E_B^2} = G\sqrt{(20{,}000)^2 + \left(\frac{80{,}000}{3}\right)^2}$$

$$= 6.67\times 10^{11} \times \frac{10{,}0000}{3} = \mathbf{2.22 \times 10^{-6}\ N\ kg^{-1}}$$

Example 17.4. Three masses are in the configuration as shown in Fig. 17.10. What is the total gravitational potential energy of the configuration?

Solution. Using superposition principle, the total gravitational potential energy of the system is

$$U = \left(-\frac{Gm_1m_2}{r_{12}}\right) + \left(-\frac{Gm_2m_3}{r_{23}}\right) + \left(-\frac{Gm_3m_1}{r_{31}}\right)$$

$$= -G\left(\frac{m_1m_2}{r_{12}} + \frac{m_2m_3}{r_{23}} + \frac{m_3m_1}{r_{31}}\right)$$

$$= -6.67\times 10^{-11} \times \left(\frac{1\times 2}{4} + \frac{2\times 2}{5} + \frac{2\times 1}{3}\right)$$

$$= \mathbf{-1.31 \times 10^{-10}\ J}$$

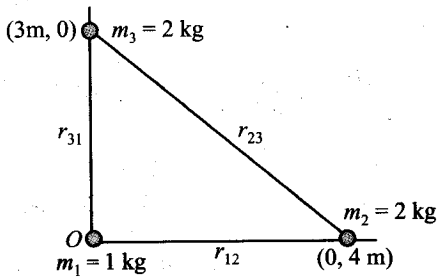

Fig. 17.10

PROBLEMS FOR PRACTICE

1. With what velocity must a body be thrown from earth's surface so that it may reach a height $4R$ above the earth's surface? Radius of earth, R = 6400 km ; g = 9.8 m s^{-2}. **[10 km s^{-1}]**
2. How much energy would be needed for a 100 kg body to escape from the earth? Take g = 10 m s^{-2} and radius of earth = 6.4 $\times 10^6$ m. **[6.4 $\times$ 10^9 J]**
3. A satellite of mass 1000 kg moves in a circular orbit of radius 7000 km round the earth, assumed to be a sphere of radius 6400 km. Calculate the total energy needed to place the satellite in the orbit from the earth. Assume g = 10 m s^{-2} at the earth's surface. **[3.5 $\times$ 10^{10} J]**
4. A rocket is fired vertically with a speed of 5 km s^{-1} from the earth's surface. How far from the earth does the rocket go before returning to the earth? Mass of earth = 6.0 $\times$ 10^{24} kg ; mean radius of earth = 6.4 $\times$ 10^6 m ; G = 6.67 $\times$ 10^{-11} N m^2 kg^{-2}. **[1.6 $\times$ 10^6 m]**

17.8. GRAVITATIONAL POTENTIAL

The **gravitational potential** *at a point in a gravitational field is defined as the amount of work done in bringing a body of unit mass from infinity (where the potential is zero) to that point i.e.,*

$$V = \frac{W}{m}$$

where V = gravitational potential at some point.

W = W.D. in bringing mass m from infinity to that point.

The SI unit of gravitational potential is J kg^{-1} and its dimensional formula is $\left[M^0L^2T^{-2}\right]$. Suppose we want to find the gravitational potential at a point P (outside the earth) at a distance r from the centre of the earth. We know that the amount of work done in bringing a body of mass m from inifinity to point P is

$$W = -\frac{GMm}{r}$$

$\therefore$ Gravitational potential at point P in earth's gravitational field is

$$V_P = \frac{GMm}{r} \times \frac{1}{m} = \frac{GM}{r}$$

or $$V_P = \frac{GM}{r}$$

Note that gravitational potential is always negative; it is zero at infinity. It follows that a body at infinity would 'fall' towards the earth ; a body on the earth does not 'fall' to infinity.

Discussion. The following points may be noted :

(*i*) The gravitational potential (like gravitational potential energy) is a scalar quantity.

(*ii*) The gravitational potential at earth's surface is

$$V = -\frac{GM}{R} \qquad (r = R = \text{radius of earth})$$

(*iii*) The gravitational potential energy of a body of mass m at a point where the gravitational potential is V is given by ;

$$\text{Gravitational P.E.} = mV = -\frac{GMm}{r}$$

where $r = R + h$ = Distance of mass m from the centre of earth

$$\text{P.E. at earth surface} = -\frac{GMm}{R} \qquad (\because h = 0)$$

$$\text{P.E. at centre of earth} = -\frac{3}{2}\frac{G^*Mm}{R}$$

(*iv*) For large distances from the earth (*e.g.* when a rocket travels from earth to moon), the change in potential energy of the rocket can *only* be calculated by using the following relation ;

Change in gravitational P.E. = mass × change in gravitational potential

For small distances above the earth, however, the gravitational force on a mass is fairly constant. So the change in potential energy in this case can be calculated using *force* × *distance* or *mgh*.

Example 17.5. Calculate the gravitational potential and potential energy of a body of mass 0.1 kg (*i*) at earth's surface (*ii*) at a height above the earth's surface equal to its mean radius R. Mass of earth, $M = 5.98 \times 10^{24}$ kg, $R = 6.37 \times 10^6$ m and $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}. Assume the earth to be a perfect sphere of uniform mass density.

Solution. $M = 5.98 \times 10^{24}$ kg ; $m = 0.1$ kg ; $R = 6.37 \times 10^6$ m ; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}

(*i*) The gravitational potential of the body at earth's surface is

$$V_E = -\frac{GM}{R} = -\frac{(6.67\times10^{-11})\times(5.98\times10^{24})}{6.37\times10^6} = -6.26 \times 10^7 \text{ J/kg}$$

The gravitational P.E. of the body at earth's surface is

$$U_E = mV_E = 0.1 \times (-6.26 \times 10^7) = -6.26 \times 10^6 \text{ J}$$

* Potential at the centre of earth = 3/2 × Potential at earth surface.

(*ii*) The gravitational potential of the body at $r = 2R$ is

$$V = -\frac{GM}{2R} = \frac{1}{2}\left(\frac{-GM}{R}\right) = \frac{1}{2}\left(-6.26 \times 10^7\right) = -3.13 \times 10^7 \text{ J/kg}$$

The gravitational P.E. of the body at $r = 2R$ is

$$U = mV = 0.1 \times (-3.13 \times 10^7) = -3.13 \times 10^6 \text{ J}$$

Example 17.6. **Three point masses, each of mass m are placed at the vertices of an equilateral triangle of length l. What is the gravitational potential due to three masses at the centroid of the triangle?**

Solution. Fig. 17.11 shows the conditions of the problem. If O is the centroid of the triangle, then,

$OA = OB = OC$

In right angled triangle OPB,

$$\frac{BP}{OB} = \cos 30°$$

or $$\frac{l/2}{OB} = \frac{\sqrt{3}}{2}$$

$\therefore$ $$OB = \frac{l}{2} \times \frac{2}{\sqrt{3}} = \frac{l}{\sqrt{3}}$$

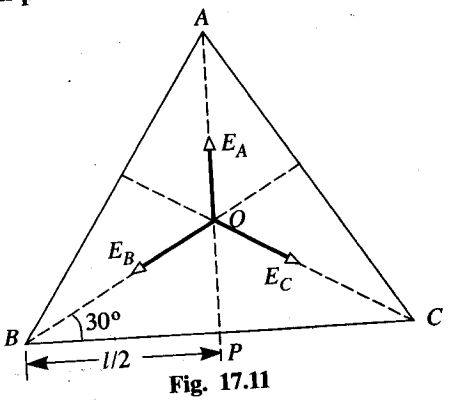

Fig. 17.11

The gravitational potential at O due to the three masses (each equal to m) at the corners of the triangle is

$$V_O = \left(-\frac{GM}{OA}\right) + \left(-\frac{GM}{OB}\right) + \left(-\frac{GM}{OC}\right)$$

$$= \left(-\frac{GM}{l/\sqrt{3}}\right) + \left(-\frac{GM}{l/\sqrt{3}}\right) + \left(-\frac{GM}{l/\sqrt{3}}\right)$$

$$= -3\sqrt{3}\frac{GM}{l}$$

Example 17.7. **At a point above the surface of earth, the gravitational potential is -5.12×10^7 J kg^{-1} and the acceleration due to gravity is 6.4 ms^{-2}. Assuming the mean radius of earth to be 6400 km, calculate the height of this point from the surface of the earth.**

Solution. Let r be the distance of the given point from the centre of the earth. Then,

Gravitational potential at this point is

$$V = -\frac{GM}{r} = -5.12 \times 10^7 \quad \ldots(i)$$

Acceleration due to gravity at this point is

$$g = \frac{GM}{r^2} = 6.4 \quad \ldots(ii)$$

Dividing eq. (*i*) by eq. (*ii*), we have (neglecting –ve sign),

$$r = \frac{5.12 \times 10^7}{6.4} = 8 \times 10^6 \text{ m} = 8000 \text{ km}$$

$\therefore$ Height of the point from earth's surface = 8000 – 6400 = **1600 km**

17.9. RELATION BETWEEN GRAVITATIONAL FIELD AND POTENTIAL

Suppose that a particle of mass m is moved by a force F from A to B in the gravitational field E (See Fig. 17.12). Suppose that $AB = \Delta r$ where Δr is so small that F can be considered constant between A and B.

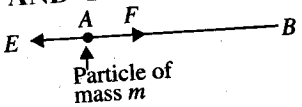

Fig. 17.12

Work done ΔW in going from A to B is

$$\Delta W = F\ \Delta r$$

By definition, $E = \dfrac{-F}{m}$

The minus sign is necessary because E and F are oppositely directed.

$$\therefore \quad \Delta W = -m\ E\ \Delta r$$

By definition, gravitational potential $\Delta V = \dfrac{\Delta W}{m}$

$$\therefore \quad \frac{\Delta W}{m} = -E\Delta r$$

or $$\Delta V = -E\Delta r$$

or $$E = -\frac{\Delta V}{\Delta r}$$

In the limit, $E = -\dfrac{dV}{dr}$

The quantity dV/dr is called potential gradient i.e., rate of change of gravitational potential with distance.

CONCEPTUAL QUESTIONS

Q.1. The gravitational potential energy of a body on the surface of earth is -6.5×10^6 J. What do you mean by this statement?

Ans. It means that 6.5×10^6 J of energy is required on the surface of earth to send the body outside the gravitational field of the earth.

Q.2. In a circular orbit, an earth satellite has the same P.E. and K.E. everywhere in the orbit. Why?

Ans. In a circular orbit, the distance between the centre of earth and satellite does not change which means P.E. of the satellite is the same everywhere in the orbit. The force of gravity on the satellite is always toward the centre of the earth. For a satellite in circular orbit, no component of force acts along the direction of motion. Therefore, the speed and the K.E. of the satellite do not change.

Q.3. An earth satellite moves in an elliptical path. How are P.E. and K.E. of the satellite affected?

Ans. In an elliptical orbit, both speed and distance of the satellite change. Now P.E. is greatest when the satellite is farthest away (at the apogee) and least when the satellite is closest (at the perigee). Note that K.E. is the least when the P.E. is most and the K.E. will be most when the P.E. is least. At every point in the orbit, the sum of K.E. and P.E. is the same.

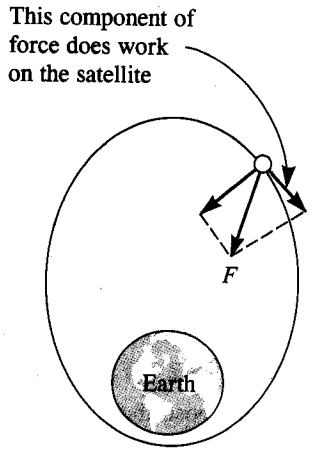

Fig. 17.13

Q.4. Why does the speed of an earth satellite change in the elliptical orbit?

Ans. In an elliptical orbit, at all points there is a component of gravitational force in the direction of motion (with two exceptions, the apogee and perigee, where the force is perpendicular only). This component changes the speed and thus K.E. of the satellite. Note that the perpendicular component changes only the direction [See Fig. 17.13].

Q.5. The escape velocity of a body from earth's surface is 11.2 kms^{-1}. What does it mean?

Ans. It means that if a body is projected at a speed of 11.2 km s^{-1} from earth's surface (neglecting air resistance), the body will leave the earth and will never return to earth. Interestingly enough, this might well be called the *maximum falling speed.* Any object, however far from earth, released from rest and allowed to fall to earth only under the influence of the earth's gravity would not exceed a speed of 11.2 km s^{-1}.

Q.6. What is the gravitational potential energy in case of three-particle system?

Ans. The total potential energy of the system is the sum over all *pairs* of particles.

$$U_{total} = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$

This represents the total work done by an external agent against the gravitational force in assembling the system of particles from an infinite separation.

Q.7. Why does earth not retain hydrogen and helium molecules in its atmosphere?

Ans. The average velocity of a gas molecule depends upon its mass and temperature. Lighter molecules such as hydrogen and helium have a high average velocity than the heavier molecules. Therefore, lighter molecules have a greater chance to escape from the planet. This explains why the earth does not retain hydrogen and helium molecules in its atmosphere while much heavier molecules, such as oxygen and nitrogen, do not escape.

Q.8. The gravitational potential energy of a body at the surface of earth is negative. What does it mean?

Ans. It means that the gravitational potential energy of the body at the surface of earth is less than at infinity (P.E. of the body at *infinity is zero). It follows that a body at infinity would 'fall' towards the earth ; a body on the earth does not 'fall' to infinity.

VERY SHORT ANSWER QUESTIONS

Q.1. What is meant by gravitational field?

Ans. The gravitational field due to a material body is the space around the body in which any other mass experiences a force of attraction.

Q.2. What do you mean by gravitational field strength?

Ans. The gravitational field strength at a point in a gravitational field is the force per unit mass acting on a test mass placed at that point. Note that gravitational field strength is also called gravitational field intensity or simply gravitational field.

Q.3. Is gravitational field strength scalar or vector?

Ans. Gravitational field strength is a vector quantity. Its direction is that in which a test mass would move under the influence of the gravitational field.

Q.4. What are the units and dimensional formula of gravitational field strength?

Ans. If a small test mass m_o placed at point P in the gravitational field of a body experiences an attractive force $\vec{F}$, then gravitational field strength $(\vec{E})$ at P is

$$\vec{E} = \frac{\vec{F}}{m_o}$$

Clearly, the SI unit of gravitational field strength is N/kg or ms^{-2} and its dimensional formula is $[M^0LT^{-2}]$.

Q.5. What is the value of gravitational field on the surface of earth?

Ans. The gravitational field on the surface of earth = $GM/R^2 = g$.

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* We choose that point as of zero potential energy where the force on the body is zero.

Q.6. What is the gravitational field strength of a planet where the weight of a 60 kg body is 300 N?

Ans. Gravitational field strength = $\frac{F}{m} = \frac{300\text{ N}}{60\text{ kg}} = 5\text{ N kg}^{-1}$.

Q.7. Name two factors which determine whether a planet has atmosphere or not?

Ans. (*i*) Acceleration due to gravity.

(*ii*) The surface temperature of the planet.

Q.8. What is more fundamental, mass or weight?

Ans. Mass is more fundamental because the mass of a body remains constant. However, the weight (= mg) of a body changes due to the change in the value of g.

Q.9. In case of earth, where is gravitational field zero?

Ans. The gravitational field is zero at the centre of earth and at infinity from the centre of earth.

Q.10. Is the gravitational potential energy of a body near earth's surface positive or negative?

Ans. The gravitational potential energy of a body near earth's surface is negative.

Q.11. Where is the gravitational potential energy of a body maximum?

Ans. It is maximum at infinity.

Q.12. Is the kinetic energy of a body moving near earth's surface positive or negative?

Ans. The kinetic energy of a body is always positive ($\because$ K.E. = $mv^2/2$).

Q.13. If a body is moved away from the earth, does its gravitational P.E. increase or decrease?

Ans. If a body is moved away from earth, its gravitational potential energy (P.E.) increases.

Q.14. The gravitational potential energy of a body near earth's surface is negative. What does it mean?

Ans. It means that work is done by the gravitational force as the body moves towards the earth.

Q.15. Is the total energy of a satellite orbiting around the earth positive or negative?

Ans. It is negative.

Q.16. What is the relation between P.E. and K.E. of earth satellite?

Ans. P.E. of satellite = –2 × K.E. of satellite

Q.17. Can gravitational potential have positive value?

Ans. No. It is always negative.

Q.18. In case of earth, where is gravitational potential zero?

Ans. The gravitational potential is zero at infinity.

Q.19. Give the SI units and dimensional formula of gravitational potential.

Ans. The SI unit of gravitational potential is J kg^{-1} and its dimensional formula is $[M^0L^2T^{-2}]$.

Q.20. Is gravitational potential scalar or vector?

Ans. Gravitational potential is a scalar quantity.

SHORT ANSWER QUESTIONS

Q.1. The acceleration due to gravity at a place is a measure of gravitational field strength at that point. Explain.

Ans. By definition, the gravitational field strength at a point P distant r from the centre of earth is

$$E = \frac{F}{m} = \frac{GMm}{r^2} \times \frac{1}{m} = \frac{GM}{r^2}$$

But $GM/r^2 = g'$ = acceleration due to gravity at point P. Therefore, acceleration due to gravity at a place is a measure of gravitational field strength at that place.

Q.2. How does gravitational field strength vary inside and outside the earth?

Ans. Inside the earth, the gravitational field is directly proportional to the distance from the centre of earth. Outside earth's surface, gravitational field is inversely proportional to the square of distance from the centre of earth.

Q.3. Prove that gravitational field due to earth is zero at infinity.

Ans. The magnitude of gravitational field at a point distant r from the centre of earth is $E = GM/r^2$. Clearly, gravitational field (strength or intensity) is zero at infinity.

Q.4. What would happen if gravity suddenly disappears?

Ans. If gravity suddenly disappears, then,

(*i*) all bodies will have zero weight.

(*ii*) human beings will be thrown out of the surface of the earth due to centrifugal force.

Q.5. If a body is projected from a planet with a velocity V smaller than the escape velocity V_e of the planet, what will happen?

Ans. If the projection velocity is smaller than the escape velocity, then the body after reaching a certain height may either move in an orbit around the planet or fall back to the planet.

Q.6. If a body is projected from a planet with a velocity greater than the escape velocity V_e of the planet, what will happen?

Ans. If the projection velocity is greater than the escape velocity, then the body will escape from the gravitational field of the planet and will move in the interstellar space.

Q.7. A rocket is fired upward from earth's surface with a speed of $V = 2\sqrt{gR}$. Find its speed in interstellar space.

Ans. Suppose V_0 is the velocity of rocket in the interstellar space. If m is the mass of the rocket and V_e is the escape velocity $(=\sqrt{2gR})$ of earth, then according to principle of conservation of energy, we have,

$$\frac{1}{2}mV_0^2 = \frac{1}{2}mV^2 - \frac{1}{2}mV_e^2$$

or $$V_0^2 = V^2 - V_e^2 = (2\sqrt{gR})^2 - (\sqrt{2gR})^2 = 4gR - 2gR = 2gR$$

Therefore, $V_0 = \sqrt{2gR}$.

Q.8. Show that total energy of a satellite revolving around the earth is negative.

Ans. Consider a satellite of mass m revolving close to earth of mass M and radius R. Then,

$$\text{P.E. of satellite} = -\frac{GMm}{R}; \quad \text{K.E. of satellite} = \frac{GMm}{2R}$$

$$\therefore \text{ Total energy of satellite} = \text{P.E.} + \text{K.E.} = -\frac{GMm}{2R}$$

It is clear that total energy of the satellite in the orbit is negative.

Q.9. What is the binding energy of a satellite orbiting close to earth?

Ans. It is the minimum energy required to remove the satellite from its orbit around the earth to infinity. At infinity, the total energy of the satellite is zero.

$$\therefore \quad \text{Binding energy} = 0 - (\text{Total energy of the satellite})$$

$$= 0 - \left(-\frac{GMm}{2R}\right) = \frac{GMm}{2R}$$

Q.10. If K.E. of a satellite revolving in an orbit close to the earth is doubled, will the satellite escape?

Ans. For a satellite orbiting close to earth's surface,

$$\text{P.E.} = -\frac{GMm}{R} \quad \text{and} \quad \text{K.E.} = \frac{GMm}{2R}$$

If K.E. is doubled, the total energy of the satellite (= P.E. + K.E.) becomes zero and hence the satellite will escape from earth.

Q.11. Why are satellites launched with multistage rockets?

Ans. The orbital speed required to put the satellite in an orbit close to earth is about 8 km s^{-1}. This *high speed cannot be achieved easily by a single stage rocket. Therefore, the satellite is put in the desired orbit with the help of a three-stage rocket.

Q.12. Does the change in gravitational potential energy of a body between two points depend upon the nature of path followed?

Ans. No. It is because the gravitational force is a conservative force. We know that the work done to move a body between two points under the action of a conservative force is independent of the path followed. Therefore, change in gravitational potential energy between two points is independent of the path followed.

LONG ANSWER QUESTIONS

1. What do you mean by gravitational field? Prove that gravitational field at a point is equal to the acceleration due to gravity at that point. **[Refer to Arts. 17.1 and 17.3]**
2. Derive the general expression for the gravitational potential energy of a body of mass m at a distance (outside the earth) r form the centre of the earth. **[Refer to Art. 17.5]**
3. Derive an expression for the escape velocity from the surface of earth using energy principle. **[Refer to Art. 17.7]**
4. Write short note on (*i*) earth's gravitational field (*ii*) gravitational potential energy and (*iii*) gravitational potential. **[Refer to Arts. 17.3, 17.4, 17.5 and 17.8]**

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* At this high speed on earth's surface, the atmospheric friction would probably burn the satellite.

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The gravitational field due to a material body is the space around the body in which any other mass experiences a force of attraction.
2. The gravitational field strength at a point in a gravitational field is defined as the force per unit mass acting on a test mass placed at that point.
3. The value of g at a location gives the value of gravitational field strength due to earth at that point.
4. The gravitational field strength at a point P due to earth depends upon the distance of the point P from the centre of earth.
   (*i*) If point P is at a distance h from the surface of earth,
   $$g' = \frac{GM}{(R+h)^2} \; ; \; R = \text{Radius of earth.}$$
   (*ii*) If point P is on the surface of earth,
   $$g' = g = \frac{GM}{R^2}$$
   (*iii*) Inside the earth (assuming density of earth is uniform), g' is directly proportional to the distance from the centre. Therefore, g' increases linearly from zero at the centre to g $(= GM/R^2)$ at the surface.
5. The gravitational potential energy U of a body of mass m at a distance r from the centre of earth is
   $$U = -\frac{GMm}{r}$$
   Note that gravitational potential energy of a body near earth's surface is negative and becomes less negative (*i.e.* greater) as it moves away from earth.
6. The maximum gravitational potential energy of a body is zero and occurs at $r = \infty$.
7. A satellite in orbit around the earth has both kinetic energy (K.E.) and potential energy (P.E.).
8. The K.E. and P.E. of a satellite at a height h above the surface of earth are :
   $$\text{K.E. of satellite} = \frac{1}{2}mv^2 = \frac{GMm}{2(R+h)}$$
   $$\text{P.E. of satellite} = -\frac{GMm}{R+h}$$
   Note that P.E. of satellite = –2 × K.E. of satellite
   Here M = mass of earth ; m = mass of satellite ; R = radius of earth.
9. The total energy of a satellite is negative.
10. When the height of a satellite is increased, its potential energy increases and kinetic energy decreases.
11. When the total energy of a satellite is zero, it will escape away from its orbit.
12. The K.E. of a satellite (or any body) can never be negative.
13. For a satellite orbiting in a circular path, its P.E. is always greater than its K.E.
14. The binding energy of a satellite is the energy that has to be given to the satellite so that it may escape from the gravitational field of earth/planet.
    Binding energy of earth satellite is
    $$E = +\frac{GMm}{2(R+h)}$$
15. When the total energy of a satellite is negative, it will be moving either in circular or elliptical orbit.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Potential energy of a satellite of mass m rotating at a height of 6.4×10^6 m from the earth's centre is **[AIIMS 2000]**

(a) $-0.5\ mg\ R_e$ (b) $-mg\ R_e$

(c) $-2\ mg\ R_e$ (d) $4\ mg\ R_e$

2. A body of mass m is placed on the surface of earth. It is taken from earth's surface to a height $h = 3R$. The change in gravitational potential energy of the body is **[CBSE-PMT 2002]**

(a) $\frac{2}{3}mgR$ (b) $\frac{3}{4}mgR$

(c) $\frac{mgR}{2}$ (d) $\frac{mgR}{4}$

3. The mass of the earth is 6×10^{24} kg and that of moon is 7.4×10^{22} kg. The constant of gravitation $G = 6.67 \times 10^{-11}$ Nm^2/kg^2. The potential energy of the system is -7.79×10^{28} joules. The mean distance between earth and moon is **[MP PMT 1995]**

(a) 3.8×10^8 metres (b) 3.37×10^6 metres

(c) 7.6×10^4 metres (d) 1.9×10^2 metres

4. A body of mass m is taken from earth's surface to a height equal to radius of earth. The change (increase) in potential energy will be **[CBSE PMT 1991]**

(a) mgR (b) $2\ mgR$

(c) $0.25\ mgR$ (d) $0.5\ mgR$

5. The change in potential energy when a body of mass m is raised to a height nR from the earth's surface (R = radius of earth) **[MP PMT 1996]**

(a) $mgR\ \frac{n}{n-1}$ (b) $nmgR$

(c) $mgR\ \frac{n}{n+1}$ (d) $mgR\ \frac{n^2}{n^2+1}$

6. In a missile launched with a velocity less than escape velocity, the sum of its K.E. and P.E. is always **[CPMT 1997]**

(a) positive (b) zero

(c) negative (d) none of these

7. A satellite is in an orbit around the earth. If its kinetic energy is doubled, then, **[AIIMS 2000]**

(a) it will rotate with greater speed

(b) it will fall on earth

(c) it will maintain its path

(d) it will escape out of earth's gravitational field

8. Intensity of gravitational field of earth is maximum **[CPMT 1996]**

(a) at poles (b) at equator

(c) at centre of earth (d) same everywhere

9. The gravitational potential difference between the surface of a planet and a point 100 m above is 10 J kg^{-1}. If the gravitational field over this range is uniform, then work done to raise a body of 5 kg from the surface to a height 40 m is **[MP PMT 1997]**

(a) 10 J (b) 20 J

(c) 40 J (d) 80 J

10. The gravitational intensity at any point due to earth inside the earth is **[CPMT 1995]**

(a) same as at surface

(b) increased as we go down

(c) same as at the centre of earth

(d) decreased as we go down

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (a)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (c)
6. (c)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (d)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. R_e = radius of earth = 6.4×10^6 m = 6400 km

$$\text{P.E. of satellite} = -\frac{GMm}{R_e+h} = -\frac{GMm}{R_e+R_e} = -\frac{GMm}{2R_e}$$

$$= -\frac{GMm}{2R_e^2} \times R_e = -\frac{gmR_e}{2} = \mathbf{-0.5\ mg\ R_e} \qquad \left(\because\ g = \frac{GM}{R_e^2}\right)$$

2. R = radius of earth. P.E. of body at earth's surface is

$$U_1 = -\frac{GMm}{R} = -\frac{gR^2m}{R} = -mgR$$

P.E. of body at height h ($= 3R + R = 4R$) is

$$U_2 = -\frac{GMm}{4R} = -\frac{gR^2m}{R} = -\frac{mgR}{4}$$

$$\text{Change in P.E.} = U_2 - U_1 = -\frac{mgR}{4} - (mgR) = \frac{3}{4}mgR$$

3. The P.E. of earth-moon system is

$$U = -\frac{GMm}{r}$$

$\therefore$ Mean distance between earth and moon is

$$r = -\frac{GMm}{U} = -\frac{6.67\times10^{-11}\times6\times10^{24}\times7.4\times10^{22}}{-7.79\times10^{28}} = \mathbf{3.8 \times 10^8\ m}$$

4. P.E. of body at earth's surface is

$$U_1 = -\frac{GMm}{R} = -\frac{gR^2m}{R} = -mgR \qquad (\because GM = gR^2)$$

P.E. body at height h $(= R)$ is

$$U_2 = -\frac{GMm}{R+R} = -\frac{gR^2m}{2R} = -\frac{mgR}{2}$$

$\therefore$ Change in P.E. $= U_2 - U_1 = -\frac{mgR}{2} - (-mgR) = \mathbf{0.5\ mgR}$

5. P.E. of the body at earth's surface is

$$U_1 = -\frac{GMm}{R} = -\frac{gR^2m}{R} = -mgR$$

P.E. of the body at height h $(= R + nR)$ is

$$U_2 = -\frac{GMm}{R+nR} = -\frac{gR^2m}{R(n+1)} = -\frac{mgR}{n+1}$$

$$\text{Change in P.E.} = U_2 - U_1 = -\frac{mgR}{n+1} - (-mgR)$$

$$= mgR\left(1 - \frac{1}{n+1}\right) = \mathbf{mgR\frac{n}{n+1}}$$

6. P.E. of satellite $= -\frac{GMm}{R+h}$; K.E. of satellite $= \frac{1}{2}mv^2$

The P.E. of satellite is negative for finite values of h while the K.E. of satellite is always positive. For velocity v less than escape velocity, P.E. is greater than K.E. Therefore, for velocity less than escape velocity, the sum of K.E. and P.E. of a satellite is always **negative**.

7. Orbital velocity of satellite close to earth is

$$v = \sqrt{\frac{GM}{R}}$$

K.E. of satellite, $K = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{GM}{R}$

When K.E. of satellite is doubled, new K.E. is

$$K' = 2K = \frac{mGM}{R} \qquad ...(i)$$

K.E. of satellite for escape velocity

$$= \frac{1}{2}mV_e^2 = \frac{1}{2}m\left(\sqrt{\frac{2GM}{R}}\right)^2 = \frac{mGM}{R} \quad ...(ii)$$

It is evident from eqs. (*i*) and (*ii*) that if K.E. of satellite is doubled, **it will escape from earth's gravitational field.**

8. Intensity of gravitational field due to earth at a point means acceleration due to gravity at that point. Its value is maximum **at poles.**

9. Acceleration due to gravity is

g = Rate of change of gravitational potential = 10/100 = 1/10

$\therefore$ W.D. = $mgh = 5 \times \frac{1}{10} \times 40$ = **20 J**

10. The gravitational intensity inside the earth decreases linearly from g at the surface of earth to zero at the centre of earth.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. The ratio of K.E. required to be given to the satellite to escape earth's gravitational field to the K.E. required to be given so that the satellite moves in a circular orbit close to earth's surface is **[CET Haryana 1997]**

(*a*) 1 : 1 (*b*) 2 : 1

(*c*) 2 : 0.5 (*d*) 4 : 1

2. In a satellite if the time period is T, then K.E. is proportional to **[BHU 1995]**

(*a*) $\frac{1}{T^{2/3}}$ (*b*) $\frac{1}{T}$

(*c*) $\frac{1}{T^2}$ (*d*) $\frac{1}{T^3}$

3. How much energy will be necessary for making a body of mass 500 kg escape from the earth? (g = 9.8 m/s^2, radius of earth = 6.4 × 10^6 m) **[MP PET 1999]**

(*a*) about 9.8 × 10^6 J (*b*) about 6.4 × 10^8 J

(*c*) about 3.1 × 10^{10} J (*d*) about 27.4 × 10^{12}J

4. At what distance from the centre of earth, the intensity of gravitational field will be zero? Take mass of moon and earth as 7.35 × 10^{22} kg and 5.98 × 10^{24} kg respectively and the distance between moon and earth is 3.85 × 10^8 km.. **[CET Haryana 1993]**

(*a*) 3.46 × 10^8 m (*b*) 3.85 × 10^6 m

(*c*) 8 × 10^8 m (*d*) zero

5. If V denotes the maximum velocity of the bob and h the maximum height to which the bob rises in case of a simple pendulum, then, **[Pb CET 1998]**

(*a*) $V = 2gh$ (*b*) $V = gh$

(*c*) $V = 4gh$ (*d*) $V = \sqrt{2gh}$

6. In a gravitational field, the work done in transporting mass from one point to another **[Pb CET 1992]**

(*a*) depends on end positions

(*b*) depends on distance between them

(*c*) depends on actual point of motion

(*d*) depends on velocity of transport.

7. A satellite of mass m is revolving around the earth at a height R (R = radius of earth). If g is the gravitational field intensity, K.E. of the satellite is **[Rajasthan PET 1995]**

(*a*) $mgR/2$ (*b*) mgR

(*c*) $mgR/4$ (*d*) 2 mgR

8. The kinetic energy required to make a body to move to infinity from earth's surface is **[MP PET 1997]**

(*a*) 2 mgR (*b*) mgR

(*c*) 0.5 mgR (*d*) infinite

9. An artificial satellite revolves around the earth in a circular orbit with a speed v. If m is the mass of the satellite, its total energy is **[CET Karnataka 1995]**

(*a*) $\frac{1}{2}mv^2$ (*b*) $-\frac{1}{2}mv^2$

(*c*) $-mv^2$ (*d*) $1.5mv^2$

10. A mass m is placed at a point B in the gravitational field of mass M. When mass m is brought from B to near point A, its gravitational potential energy will **[CET Haryana 1994]**

(*a*) remain unchanged (*b*) increase

(*c*) decrease (*d*) become zero

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*a*)	**5.** (*d*)
6. (*a*)	**7.** (*c*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Escape velocity from earth, $V_e = \sqrt{2gR}$

Orbital velocity close to earth, $v = \sqrt{gR}$

K.E. required by satellite to escape earth's gravitational field is

$$K_1 = \frac{1}{2}mV_e^2 = \frac{1}{2}m \times 2gR = mgR$$

K.E. required so that satellite moves in a circular orbit close to earth's surface is

$$K_2 = \frac{1}{2}mv^2 = \frac{1}{2}m \times gR = \frac{1}{2}mgR$$

$$\therefore \quad \frac{K_1}{K_2} = 2 \quad \text{or} \quad K_1 : K_2 = 2 : 1$$

2. Orbital velocity, $v \propto \frac{1}{\sqrt{r}}$ or $v^2 \propto \frac{1}{r}$. (where $r = R + h$)

Now $T^2 \propto r^3$ or $\frac{1}{T^2} \propto \frac{1}{r^3}$

Now K.E. $\propto v^2$ or K.E $\propto \frac{1}{T^{2/3}}$

3. Escape velocity, $V_e = \sqrt{2gR}$

K.E. required to escape from earth is

$$K = \frac{1}{2}mV_e^2 = \frac{1}{2}m \times 2gR = mgR$$

$$= 500 \times 9.8 \times 6.4 \times 10^6 = 3.1 \times 10^{10} \text{ J}$$

4. Suppose P is the required point between the earth and the moon. Let the distance of P from the centre of earth be x_e and the distance of P from the centre of moon be x_m. At point P, the gravitational forces on the mass m due to earth and moon will be equal and opposite,

$$\therefore \quad \frac{GM_e m}{x_e^2} = \frac{GM_m m}{x_m^2} \quad \text{or} \quad \frac{x_e}{x_m} = \sqrt{\frac{M_e}{M_m}} = \sqrt{\frac{5.98 \times 10^{24}}{7.35 \times 10^{22}}} = 9$$

$$\therefore \quad x_e = 9\, x_m \quad \text{Also} \quad x_e + x_m = 3.85 \times 10^8 \text{ m}$$

On solving, $x_e = \mathbf{3.46 \times 10^8}$ **m.**

5. According to principle of conservation of energy,

$$\frac{1}{2}mV^2 = mgh \quad \therefore \; \boldsymbol{V = \sqrt{2gh}}$$

6. Since gravitational force is a conservative force, the work done in transporting mass from one point to another in the gravitational field depends on the initial and final positions and not on the path followed.

7. Orbital velocity, $v = \sqrt{\frac{GM}{R+h}}$

$$\text{K.E. of satellite} = \frac{1}{2}mv^2 = \frac{1}{2}m \times \frac{GM}{R+h} = \frac{mgR}{4} \quad (\because h = R)$$

8. Escape velocity, $V_e = \sqrt{2gR}$

$$\text{K.E. required} = \frac{1}{2}mV_e^2 = \frac{1}{2}\times 2gR = mgR$$

9. $$\text{K.E. of satellite} = \frac{1}{2}mv^2$$

Now $$v = \sqrt{\frac{GM}{r}} \quad \text{or} \quad v^2 = \frac{GM}{r}$$

$$\text{P.E. of satellite} = -\frac{GMm}{r} = -mv^2$$

$\therefore$ Total energy of satellite is given by ;

$$E = K.E. + P.E. = \left(\frac{1}{2}mv^2\right) + \left(-mv^2\right) = -\frac{1}{2}mv^2$$

10. Gravitational P.E. of body $= -\left(\frac{GMm}{r}\right)$

As r decreases, the negative value of P.E. increases. This means P.E. decreases.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **The mass of earth is $M = 6 \times 10^{24}$ kg and its radius $R = 6.4 \times 10^6$ m. How much work will have to be done in taking a 10 kg body from the surface of the earth to infinity? What is the gravitational potential energy of the body on the earth's surface? If this body falls from infinity to earth, what will be its velocity when striking the earth?**

Hint. The work done in taking a body of mass m from the surface of the earth to infinity is

$$W = \frac{GMm}{R} = \frac{(6.67\times10^{-11})\times(6\times10^{24})\times10}{6.4\times10^6} = 6.25 \times 10^8 \text{ J}$$

Gravitational P.E. of the body on earth's surface is

$$U = -\frac{GMm}{R} = -6.25 \times 10^8 \text{ J}$$

When the body falls back from infinity, the work done W (= 6.25×10^8 J) will be obtained in the form of kinetic energy. If the body strikes the earth with a velocity v, then,

$$\frac{1}{2}mv^2 = W = 6.25\times10^8$$

$\therefore$ $$v = \sqrt{\frac{2(6.25\times10^8)}{10}} = 11.18\times10^3 \text{ ms}^{-1} = 11.18 \text{ kms}^{-1}$$

2. **A satellite of mass 1000 kg moves in a circular orbit of radius 7000 km around the earth. Calculate the total energy required to place the satellite in the orbit from earth's surface.**
Take $g = 10$ ms^{-2} and earth's radius $R = 6400$ km.

Hint. Suppose a satellite of mass m is launched from the surface of earth (radius R, mass M) into an orbit of radius r (= $R + h$= 7000 km). If v is the orbital speed of the satellite in the orbit, then,

Total energy required, W = Gain in K.E.+ Gain in P.E.

$$= \frac{1}{2}mv^2 + \left[-\frac{GMm}{r} - \left(-\frac{GMm}{R}\right)\right]$$

$$= \frac{1}{2}mv^2 + \frac{GMm}{R} - \frac{GMm}{r}$$

For the satellite to be in circular orbit of radius r, we have,

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \text{ or } \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$$\therefore \quad W = \frac{GMm}{2r} + \frac{GMm}{R} - \frac{GMm}{r} = GMm\left[\frac{1}{R} - \frac{1}{2r}\right]$$

$$= R^2 gm\left[\frac{1}{R} - \frac{1}{2r}\right] \qquad (\because GM = R^2 g)$$

$$= mg\left[R - \frac{R^2}{2r}\right] = 1000 \times 9.8\left[6.4\times10^6 - \frac{(6.4\times10^6)^2}{2\times7\times10^6}\right] = 3.4 \times 10^{10} \text{ J}$$

3. A particle is projected vertically upward from the surface of the earth (radius R) with a kinetic energy equal to half of the minimum value needed for it to escape. Calculate the height through which it rises above the surface of earth. [I.I.T.]

Hint. Escape energy $= \frac{GMm}{R}$

Here M is the mass of earth and m is the mass of projected particle.

$$\text{K.E. of projection} = \frac{1}{2} \times \text{Escape Energy} = \frac{1}{2} \times \frac{GMm}{R}$$

Gravitational P.E. of the particle at the earth's surface is

$$U_0 = -\frac{GMm}{R}$$

Gravitational P.E. of the particle at height h from earth's surface is

$$U_h = -\frac{GMm}{(R+h)}$$

$\therefore$ Increase in gravitational P.E.

$$= U_h - U_0 = -\frac{GMm}{(R+h)} - \left(-\frac{GMm}{R}\right) = -\frac{GMm}{(R+h)} + \frac{GMm}{R}$$

But increase in gravitational P.E. of the particle is equal to K.E. of projection *i.e.*,

$$-\frac{GMm}{(R+h)} + \frac{GMm}{R} = \frac{1}{2}\frac{GMm}{R}$$

On solving, $h = R$

4. A rocket is launched vertically upward from the surface of the earth with an initial velocity of v_0. Show that its velocity v at a height h is given by ;

$$v_0^2 - v^2 = \frac{2gh}{1+h/R}$$

where R is the radius of the earth and g is the acceleration due to gravity at earth's surface. Hence, find the maximum height reached by a rocket fired with 90% of escape velocity.

Hint. Let m and M be the masses of the rocket and earth respectively.

Gravitational P.E. of rocket at earth's surface $= -\frac{GMm}{R}$

Kinetic energy of rocket at earth's surface $= \frac{1}{2}mv_0^2$

Gravitational P.E. of rocket at height $h = -\frac{GMm}{(R+h)}$

Kinetic energy of rocket at height $h = \frac{1}{2}mv^2$

According to principle of conservation of energy, we have,

(P.E. + K.E.) at earth = (P.E. + K.E.) at height h

$$-\frac{GMm}{R}+\frac{1}{2}mv_0^2 = -\frac{GMm}{(R+h)}+\frac{1}{2}mv^2$$

or $$v_0^2 - v^2 = \frac{2GM}{R} - \frac{2GM}{(R+h)}$$

But $$GM = R^2 g$$

$$\therefore \quad v_0^2 - v^2 = 2R^2 g\left[\frac{1}{R} - \frac{1}{R+h}\right] = 2R^2 g\left[\frac{h}{R(R+h)}\right]$$

or $$v_0^2 - v^2 = \frac{2gh}{1+\frac{h}{R}}$$

Now v_0 = 90% of escape velocity $= \frac{90}{100}\times\sqrt{2gR}$. When the rocket reaches maximum height, $v = 0$.

$$\therefore \quad \left(\frac{90}{100}\right)^2 \times 2gR - (0)^2 = \frac{2gh}{1+\frac{h}{R}}$$

On solving, $h = \frac{81}{19}R$

5. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from earth.

(*i*) Determine the height of the satellite above the earth's surface.

(*ii*) If the satellite is stopped suddenly in its orbit and allowed to fall freely on to the earth, find the speed with which it hits the surface of earth.

Take g = 9.8 ms^{-2}; Radius of earth, R = 6400 km. **[I.I.T.]**

Hint. (*i*) Orbital velocity of the satellite at height h above earth's surface is

$$v = R\sqrt{\frac{g}{R+h}}$$

It is given that $v = \frac{1}{2}V_e = \frac{1}{2}\sqrt{2gR}$

$$\therefore \quad \frac{1}{2}\sqrt{2gR} = R\sqrt{\frac{g}{R+h}}$$

On solving, $h = R$ = **6400 km**

(*ii*) Gravitational P.E. of satellite at a height $2R$ $(= R + h)$ $= -\frac{GMm}{2R}$

Gravitational P.E. of satellite when it hits the ground $= -\frac{GMm}{R}$

$\therefore$ Decrease in gravitational P.E. of satellite

$$= -\frac{GMm}{R} - \left(-\frac{GMm}{2R}\right) = -\frac{GMm}{2R}$$

According to law of conservation of energy, the decrease in P.E. of satellite is equal to the increase in its K.E. If v is the speed of the satellite when it hits the surface of earth, then,

$$\frac{1}{2}mv^2 = -\frac{GMm}{2R}$$

or $$v^2 = \frac{GM}{R} = \frac{R^2 g}{R} = Rg \qquad (\because GM = R^2 g)$$

$\therefore$ $$v = \sqrt{Rg} = \sqrt{6400\times10^3\times9.8} = 7.92\times10^3 \text{ ms}^{-1} = \mathbf{7.92\ kms^{-1}}$$

6. A satellite with kinetic energy E_k is revolving around the earth in a circular orbit. How much more kinetic energy should be given to it so that it may just escape into outer space?

Hint. Let m and v be the mass and orbital velocity of the satellite respectively.

$\therefore$ K.E. of satellite, $E_k = \frac{1}{2}mv^2$

If V_e is the escape velocity, then, $V_e = \sqrt{2}\,v$.

$\therefore$ Kinetic energy required for escape is $E_k' = \frac{1}{2}mV_e^2 = \frac{1}{2}m\,(\sqrt{2}v)^2 = mv^2 = 2E_k$

Additional K.E. required for satellite to escape

$$= E_k' - E_k = 2E_k - E_k = \boldsymbol{E_k}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 8 : GRAVITATION

Q.1. Answer the following :

(*a*) You can shield a charge from electrical forces by putting it inside a hollow conductor. Can you shield a body from the gravitational influence of nearby matter by putting it inside a hollow sphere or by some other means?

(*b*) An astronaut inside a small spaceship orbiting around the earth cannot detect gravity. If the space station orbiting around the earth has a large size, can he hope to detect gravity?

(*c*) If you compare the gravitational force on the earth due to the sun to that due to the moon, you would find that the sun's pull is greater than the moon's pull (you can check this yourself using the data available in the succeeding exercises). However, the tidal effect of the moon's pull is greater than the tidal effect of sun. Why?

Ans. (*a*) You cannot shield a body from the gravitational influence of nearby matter by any means. It is because the gravitational force on a body due to nearby matter is not altered due to the presence of other bodies. In other words, gravitational field cannot be shielded by any means.

(*b*) Yes. If the size of the spaceship orbiting earth is very large, an astronaut in it can detect the gravity.

(*c*) The tidal effect varies inversely as the cube of distance and is not governed by inverse square law. Since sun is very far off from earth compared to the moon, the tidal effect of moon's pull is greater than the tidal effect of sun.

Q.2. Choose the correct alternative :

(*a*) Acceleration due to gravity increases/decreases with increasing altitude.

(*b*) Acceleration due to gravity increases/decreases with increasing depth (assume the earth to be a sphere of uniform density).

(*c*) Acceleration due to gravity is independent of mass of the earth/mass of the body.

(*d*) The formula $-GMm\ (1/r_2 - 1/r_1)$ is more/less accurate than the formula $mg(r_2 - r_1)$ for the difference of potential energy between two points r_2 and r_1 distance away from the centre of the earth.

Ans. (*a*) decreases (*b*) decreases (*c*) mass of the body (*d*) more.

Q.3. Suppose there existed a planet that went around the sun twice as fast as the earth. What would be its orbital size as compared to that of the earth?

Ans. Let r_e and r_p be the radii of orbit of earth and planet respectively.

Now $T_e = 1$ year ; $T_p = T_e/2 = 1/2 = 0.5$ year

According to Kepler's third law, we have,

$$\left(\frac{r_p}{r_e}\right)^3 = \left(\frac{T_p}{T_e}\right)^2 \quad \therefore \quad r_p = r_e\left(\frac{T_p}{T_e}\right)^{2/3} = r_e \times \left(\frac{0.5}{1}\right)^{2/3} = 0.63\ r_e$$

Q.4. I_O, one of the satellites of Jupiter, has an orbital period of 1.769 days and the radius of the orbit is 4.22×10^8 m. Show that the mass of Jupiter is about one-thousandth that of the sun.

Ans. When a satellite revolves around a planet of mass M in an orbit of radius r, the time period of the satellite is given by :

$$T = 2\pi\sqrt{\frac{r^3}{GM}} \quad \text{or} \quad M = \frac{4\pi^2 r^3}{GT^2} \qquad ...(i)$$

Mass of Jupiter M_J : The time period of satellite of Jupiter is $T_1 = 1.769$ days and its orbit radius is $r_1 = 4.22 \times 10^8$ m.

$\therefore$ From eq. (*i*), $M_J = \dfrac{4\pi^2 \times (4.22 \times 10^8)^3}{G \times (1.769)^2} = \dfrac{4\pi^2}{G} \times 2.4 \times 10^{25}$

Mass of Sun M_S : The time period of earth around the sun is T_2 = 365 days and its orbit radius, r_2 = 1 A.U. = 1.496×10^{11} m.

$\therefore$ From eq. (*i*), $M_S = \dfrac{4\pi^2 \times (1.496\times10^{11})^3}{G\times(365)^2} = \dfrac{4\pi^2}{G} \times 2.5\times10^{28}$

$\therefore$ $\dfrac{M_J}{M_S} = \dfrac{2.4\times10^{25}}{2.5\times10^{28}} \simeq \dfrac{1}{1000}$

Q.5. Let us assume that our galaxy consists of 2.5×10^{11} stars each of one solar mass. How long will a star at a distance of 50,000 ly from the galactic centre take to complete one revolution? Take the diameter of the Milky Way to be 10^5 ly.

Ans. Mass of galaxy, $M = 2.5 \times 10^{11}$ solar mass = $2.5 \times 10^{11} \times (2 \times 10^{30}) = 5 \times 10^{41}$ kg

Radius of orbit of star, r = 50000 ly = $50000 \times 9.46 \times 10^{15}$ m = 4.73×10^{20} m

$\therefore$ Time period of star, $T = 2\pi \times \sqrt{\dfrac{r^3}{GM}} = 2\pi \times \sqrt{\dfrac{(4.73\times10^{20})^3}{6.67\times10^{-11}\times5\times10^{41}}} = 1.12 \times 10^{16}$ s

Q.6. Choose the correct alternative :

(*a*) If the zero of potential energy is at infinity, the total energy of an orbiting satellite is negative of its kinetic/potential energy.

(*b*) The energy required to launch an orbiting satellite out of earth's gravitational influence is more/less than the energy required to project a stationary object at the same height (as the satellite) out of earth's influence.

Ans. (*a*) Kinetic energy (*b*) less.

Q.7. Does the escape speed of a body from the earth depend upon (*a*) the mass of the body (*b*) the location from where it is projected (*c*) the direction of projection (*d*) the height of the location from where the body is launched?

Ans. (*a*) The escape speed of a body from earth does not depend on the mass of the body because $V_e = \sqrt{2GM/R}$.

(*b*) Yes. The value of g depends on latitude which is different at different locations on earth. We know that escape speed is also given by $V_e = \sqrt{2gR}$.

(*c*) It does not depend on the direction of projection.

(*d*) Yes. It is because $V_e = \sqrt{2gR}$ and the value of g depends upon the height of location from where the body is projected.

Q.8. A comet orbits the sun in a highly elliptical orbit. Does the comet have a constant (*a*) linear speed (*b*) angular speed (*c*) angular momentum (*d*) kinetic energy (*e*) potential energy (*f*) total energy throughout its orbit? Neglect any mass loss of the comet when it comes very close to the sun.

Ans. (*a*) The linear speed is not constant.

(*b*) The angular speed is not constant.

(*c*) The angular momentum remains constant.

(*d*) Since the linear speed varies, the kinetic energy (= $mv^2/2$) is not constant.

(*e*) The potential energy is not constant because the distance of the comet from the sun changes.

(*f*) The total energy remains constant. Although P.E. and K.E. change but total energy (P.E. + K.E.) at every point in the orbit remains the same.

Q.9. Which of the following symptoms is likely to affect an astronaut in space (*a*) swollen feet (*b*) swollen face (*c*) headache (*d*) orientational problem?

Ans. (*a*) On earth, the weight of the body of a person is carried by his legs. In space, the astronaut is in a state of weightlessness. Therefore, the feet of the astronaut do not get swollen.

(*b*) In the gravity free space, the face of the astronaut is expected to get more blood supply. Therefore, the face of the astronaut may get swollen.

(*c*) In the gravity free space, the swollen face of the astronaut may cause headache.

(*d*) In the gravity free space, the orientational problem may affect the astronaut.

If r_2 is the distance of the saturn from sun, then according to Kepler's third law,

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

or $$r_2 = r_1\left(\frac{T_2}{T_1}\right)^{2/3} = 1.5\times10^8\times\left(\frac{29.5}{1}\right)^{2/3} = 14.32 \times 10^{11} \text{ m}$$

Q.15. A body weighs 63 N on the surface of earth. What is the gravitational force on it due to the earth at a height equal to half the radius of the earth? Given that radius of earth is 6400 km.

Ans. Weight of body = mg = 63 N ; $h = R/2$

The value of acceleration due to gravity (g') at height h is

$$g' = \frac{gR^2}{(R+h)^2} = \frac{gR^2}{(R+R/2)^2} = \frac{4}{9}g$$

$\therefore$ Gravitational force on the body at height h is

$$F = mg' = m\times\frac{4}{9}g = \frac{4}{9}\times mg = \frac{4}{9}\times 63 = \mathbf{28\ N}$$

Q.16. Assuming the earth to be a sphere of uniform mass density, how much would a body weigh half way down to the centre of the earth if it weighed 250 N on the surface?

Ans. mg = 250 N ; $d = R/2$

Now $$g' = g\left(1-\frac{d}{R}\right)$$

or $$mg' = mg\left(1-\frac{d}{R}\right) = 250\left(1-\frac{R/2}{R}\right) = 250\times\frac{1}{2} = \mathbf{125\ N}$$

Q.17. A rocket is fired vertically upward with a speed of 5 kms^{-1} from earth's surface. How far from the earth does the rocket go before returning to the earth? Mass of earth = 6 ×10^{24} kg ; mean radius of earth = 6.4 × 10^6 m ; G = 6.67 × 10^{-11} Nm^2 kg^{-2}.

Ans. Here v = 5 km s^{-1} = 5000 ms^{-1} ; M = 6 × 10^{24} kg ; R= 6.4 × 10^6 m

Suppose the rocket goes to a height h before returning to the earth. Clearly, at this height the kinetic energy of the rocket is zero. According to principle of conservation of energy,

(K.E. + P.E) at earth = (K.E. + P.E.) at height h

or $$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 + \left(-\frac{GMm}{(R+h)}\right)$$

or $$v^2 - \frac{2GM}{R} = -\frac{2GM}{R+h}$$

or $$\frac{2GM}{R+h} = \frac{2GM}{R} - v^2 = \frac{2GM - v^2R}{R}$$

$\therefore$ $$R + h = \frac{2GMR}{2GM - v^2R}$$

$$= \frac{2\times(6.67\times10^{-11})\times(6\times10^{24})\times6.4\times10^6}{2\times6.67\times10^{-11}\times6\times10^{24} - (5000)^2\times6.4\times10^6} = 8\times10^6 \text{ m}$$

$\therefore$ $$h = 8\times10^6 - R = 8\times10^6 - 6.4\times10^6 = \mathbf{1.6 \times 10^6\ m}$$

Q.18. The escape velocity of a projectile on the earth's surface is 11.2 km s^{-1}. A body is projected out with thrice this speed. What is the speed of the body far from earth? Ignore the presence of the sun and other planets.

Ans. Escape velocity, V_e = 11.2 km s^{-1} = 11.2 × 10^3 m s^{-1}

Velocity of projection of projectile, $v = 3\ V_e$

Suppose M is the mass of the projectile and v_0 is its velocity after escaping the gravitational pull. Applying the law of conservation of energy, we have,

$$\frac{1}{2}Mv_0^2 = \frac{1}{2}Mv^2\ \frac{1}{2}MV_e^2$$

Q.21. Two heavy spheres each of mass 100 kg and radius 0.10 m are placed 1.0 m apart on a horizontal table. What is the gravitational force and potential at the mid point of the line joining the centres of the sphere? Is an object placed at that point in equilibrium? If so, is the equilibrium stable or unstable?

Ans. Fig. 8.02 shows the conditions of the problem. Here $CA = CB = 0.5$ m $= r$.

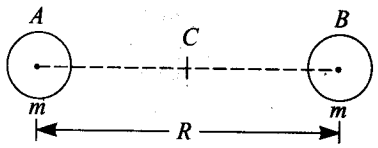


Fig. 8.02

Force of gravitational field at C due to sphere A

$$= \frac{Gm}{(0.5)^2} \text{ along } CA$$

Force of gravitational field at C due to sphere B

$$= \frac{Gm}{(0.5)^2} \text{ along } CB$$

Since the gravitational forces at C due to the two spheres are equal in magnitude and opposite in direction, the net gravitational force at C is zero.

Gravitational potential at C due to the two spheres is

$$V_C = \left(-\frac{Gm}{r}\right)+\left(\frac{-Gm}{r}\right) = -\frac{2Gm}{r}$$

$$= -\frac{2\times\left(6.67\times10^{-11}\right)\times100}{0.5} = -2.7 \times 10^{-8} \text{ J kg}^{-1}$$

Since the net force on the object placed at the mid-point C is zero, the object is in equilibrium. If the object is displaced from mid-point C towards either sphere, the object will not return to its initial position of equilibrium. Therefore, the object is in unstable equilibrium.

Q.22. As you have learnt in the text, a geostationary satellite orbits the earth at a height of nearly 36000 km from the surface of the earth. What is the potential due to earth's gravity at the site of the satellite? (Take the gravitational potential energy at infinity to be zero). Mass of earth = 6.0×10^{24} kg, radius of earth = 6400 km.

Ans. $M = 6.0 \times 10^{24}$ kg ; $R = 6400$ km ; $h = 36000$ km ; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}

Distance of the satellite from the centre of earth is

$$r = R + h = 6400 + 36000 = 42400 \text{ km} = 4.24 \times 10^7 \text{ m}$$

$\therefore$ Gravitational potential due to gravity at the site of the satellite is

$$V = -\frac{GM}{r} = -\frac{(6.67\times10^{-11})\times(6.0\times10^{24})}{4.24\times10^7} = -9.44 \times 10^6 \text{ J kg}^{-1}$$

Q.23. A star 2.5 times the mass of the sun and collapsed to a size of 12 km rotates with a speed of 1.5 revolutions per second. (Extremely compact stars of this kind are known as neutron stars. Certain stellar objects called pulsars belong to this category). Will an object placed on its equator remain stuck to its surface due to gravity? Mass of sun = 2×10^{30} kg.

Ans. Mass of star, $M = 2.5 \times$ Mass of sun $= 2.5 \times (2 \times 10^{30}) = 5 \times 10^{30}$ kg

Radius of star, $R = 12$ km $= 12 \times 10^3$ m

Angular speed of star, $\omega = 1.5 \times 2\pi = 3\pi$ rad s^{-1}

An object of mass m placed at the equator of the star will experience (*i*) inward force due to gravity and (*ii*) outward centrifugal force. Clearly, the object will remain stuck to the star's surface if inward force of gravity is greater than the outward centrifugal force.

Force on the object due to gravity is

$$F_g = \frac{GMm}{R^2} = \frac{(6.67\times10^{-11})\times(5\times10^{30})\times m}{(12\times10^3)^2} = 2.32\times10^{12} m \text{ N}$$

Centrifugal force on the object is

$$F_C = mR\omega^2 = m\times(12\times10^3)\times(3\pi)^2 = 1.1\times10^6 m \text{ N}$$

Since $F_g > F_C$, the object will remain stuck to the surface of the star.

Q.24. A spaceship is stationed on Mars. How much energy must be expended on the spaceship to launch it out of the solar system? Mass of the spaceship = 1000 kg, mass of sun = 2×10^{30} kg, mass of Mars = 6.4×10^{23} kg, radius of Mars = 3395 km, radius of the orbit of Mars = 2.28×10^8 km, $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}.

Ans. Mass of spaceship, m = 1000 kg ; Mass of Mars, $M_M = 6.4 \times 10^{23}$ kg ;
Radius of Mars, R_M = 3395 km = 3.395×10^6 m ; Radius of orbit of Mars, $r = 2.28 \times 10^{11}$ m ;
Mass of sun, $M_S = 2 \times 10^{30}$ kg ; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}

Potential energy of spaceship-Mars system is

$$U_1 = -\frac{GM_M m}{R_M} = -\frac{6.67\times10^{-11}\times6.4\times10^{23}\times1000}{3.395\times10^6} = -1.257 \times 10^{10} \text{ J}$$

Potential energy of spaceship-sun system is

$$U_2 = -\frac{GM_S m}{r} = -\frac{6.67\times10^{-11}\times2\times10^{30}\times1000}{2.28\times10^{11}} = -5.85 \times 10^{11} \text{ J}$$

Now kinetic energy (K.E.) of spaceship is zero.

$\therefore$ Total energy of the spaceship is

$$E = U_1 + U_2 = (-1.257 \times 10^{10}) + (-5.85 \times 10^{11}) = -5.98 \times 10^{11} \text{ J}$$

$\therefore$ Energy required to rocket out the spaceship from solar system

$$= 0 - E = 0 - (-5.98 \times 10^{11} \text{ J}) = \mathbf{5.98 \times 10^{11} \text{ J}}$$

Q.25. A rocket is fired vertically from the surface of Mars with a speed of 2 kms^{-1}. If 20 % of its initial energy is lost due to Martian atmospheric resistance, how far will the rocket go from the surface of Mars before returning to it? Mass of Mars = 6.4×10^{23} kg ; radius of Mars = 3395 km ; $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}.

Ans. Let m be the mass of the rocket.

Initial K.E. of the rocket = $\frac{1}{2}mv^2 = \frac{1}{2}\times(m)\times\left(2\times10^3\right)^2 = 2\times10^6 m$ J

Since 20% of energy is lost in Martian atmosphere,

$\therefore$ Net K.E. of the rocket = $2\times10^6 m\times\frac{80}{100} = 1.6\times10^6 m$ J

As the rocket rises, its K.E. decreases and P.E. increases. Suppose at a height R' from the centre of Mars, K.E. of the rocket becomes zero. The change in K.E. of the rocket will appear as increase in its P.E.

$\therefore$ Increase in P.E. of rocket = $\left(\frac{-GMm}{R'}\right)-\left(\frac{-GMm}{R}\right) = GMm\left(\frac{1}{R}-\frac{1}{R'}\right)$

According to law of conservation of energy,

$$GMm\left(\frac{1}{R}-\frac{1}{R'}\right) = 1.6 \times 10^6 \ m$$

or $$\frac{GM}{R}\left(1-\frac{R}{R'}\right) = 1.6 \times 10^6$$

or $$\frac{R}{R'} = 1-\frac{R\times1.6\times10^6}{GM}$$

Here $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} ; R = 3395 km = 3395×10^3 m ; $M = 6.4 \times 10^{23}$ kg

$\therefore$ $$\frac{R}{R'} = 1-\frac{(3395\times10^3)\times1.6\times10^6}{(6.67\times10^{-11})\times(6.4\times10^{23})} = 1 - 0.127 = 0.873$$

or $$R' = \frac{R}{0.873} = \frac{3395}{0.873} = 3888.9 \text{ km}$$

$\therefore$ Height upto which the rocket will go = 3888.9 − 3395 = **493.9 km**

Unit VIII

PROPERTIES OF BULK MATTER

- Elasticity
- Hydrostatics (Fluids at rest)
- Hydrodynamics (Fluids in motion)
- Thermal Properties of Matter
- Transfer of Heat
- N.C.E.R.T. Textbook Exercises

18

Elasticity

INTRODUCTION

Most of the substances that exist in the world can be classified into one of *three phases :* solid, liquid or gas. *These three phases of matter are governed by* **interatomic and intermolecular forces** *under ordinary conditions of temperature and pressure.* When acted on by outside forces, solids tend to keep their volume and shape, liquids tend to keep their volume but not their shape and gases tend to keep neither their volume nor their shape. To properly describe a substance, we should provide a description of motion of each atom of which it is composed. Such a description would be worthless for most purposes. It would be far too complicated and detailed for the everyday uses to which the materials are put. The engineer who wishes to use a certain type of steel in construction neither wants nor requires an atomic description of the material. We are generally interested in the overall properties of a material rather than in its atomic description. In this unit, we shall discuss some of the *mechanical properties* of matter *e.g.,* elasticity, fluid pressure etc.

18.1. INTERATOMIC AND INTERMOLECULAR FORCES

The forces which bind two or more atoms together to make a molecule are called **interatomic forces.** When molecules are formed as a result of interatomic forces between the atoms, there must be some intermolecular forces (*i.e.*, forces between molecules) which bind the molecules together. The interatomic and intermolecular forces are electrical in nature, arising from interactions of the electrically charged fundamental particles that make up the atoms and molecules. The gravitational forces between atoms/molecules are so weak compared with electrical forces that they are completely negligible.

Consider two atoms or molecules exerting forces of attraction on each other as shown in Fig. 18.1. If the force F on A moves it a small distance Δr (so that F can be considered constant) to the right, then work done ΔW on A is given by ;

$$\Delta W = F\,\Delta r$$

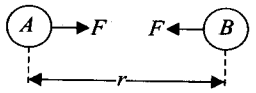


Fig. 18.1

If ΔU is the resulting change in the potential energy, then,

$$\Delta U = -\Delta W$$

The minus sign is present because the force is attractive and the potential energy decreases.

$$\therefore \qquad \Delta U = -F\,\Delta r$$

$$\therefore \qquad \text{In the limit, } F = -\frac{dU}{dr}$$

1. Interatomic forces. An atom has positively charged nucleus and negatively charged electrons. The total positive charge of the nucleus is equal to the total negative charge of electrons. Therefore, an atom is electrically neutral. However, when two atoms are brought close together, there is electrical interaction between the electrons and the nuclei of the atoms. As a result, the force

between the atoms and the potential energy of the system of two atoms undergo change. Fig. 18.2 shows how the potential energy of two atoms and the force between them changes with their separation (r). *Note that interatomic forces may be attractive or repulsive depending upon the separation between the two atoms.* At large distances, the force is small and attractive and the potential energy of the system is negative. As the separation between the atoms decreases, the potential energy decreases (more negative) and the force of attraction becomes larger. The attractive force becomes maximum and then decreases to zero at an equilibrium separation r_0. At $r = r_0$, there is no net force between the atoms and their potential energy has its minimum value. When the separation is less than r_0, the force becomes repulsive and increases quite rapidly.

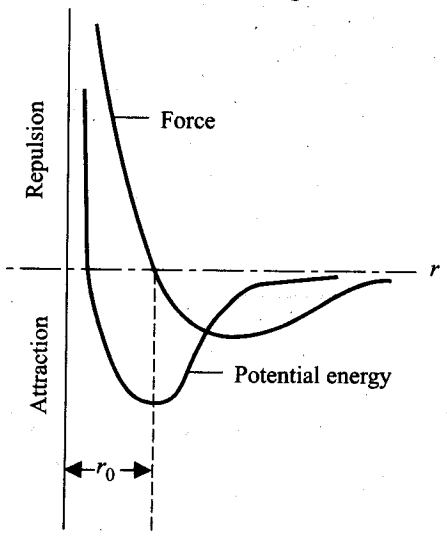


Fig. 18.2

If the two atoms have a separation of r_0, they are in their equilibrium separation. Any increase or decrease in their separation would require energy, since work would have to be done against the net attraction or the net repulsion respectively. The equilibrium is stable because an increase in r leads to an attractive force which restores r to r_0. Similarly, a decrease in r produces a repulsive force which again restores r to r_0. For NaCl, r_0 is of the order of 10^{-10} m — the same order as the distance from the centre of the atom to its outermost electrons. For small displacement from equilibrium, the force curve is linear and resembles the force exerted by a spring when compressed or extended. In solid state, NaCl exists in a regular array of Na^+ and Cl^- ions. When we try to deform the solid by pushing or pulling on it in a bid to change the separation distances between the ions, the strong forces between the ions resist compression or extension.

Characteristics of interatomic forces

(*i*) They are electrical in origin.
(*ii*) They are active over short distances.
(*iii*) They are attractive upto a certain distance between the atoms and become repulsive if distance between atoms is less than this value.
(*iv*) For two atoms attracting each other, potential energy is negative.
(*v*) They are about 50 to 100 times stronger than intermolecular forces.

2. Intermolecular forces. *Like interatomic forces, the origin of intermolecular forces is also electrical in nature. These arise due to interaction of the charged particles making up the molecules.*

The potential energy–separation and force – separation graphs for intermolecular forces are similar to those for interatomic forces. In other words, the force between two molecules changes from an attraction when the separation is large to a repulsion when the separation is small. At an equilibrium separation r_0, the intermolecular force becomes zero and the potential energy of the molecules is minimum (maximum negative). It may be noted that intermolecular forces are much weaker than interatomic forces ; say 100 times. One reason for small intermolecular attraction is that a molecule attracts many more molecules around it whereas interatomic attraction are not affected by other atoms and molecules.

In view of the attractive molecular forces, why do not all molecules eventually coalesce into matter in the liquid or solid phase? It is because the molecules are always in *motion* and have

kinetic energy associated with this motion. In general, the kinetic energy increases with temperature. At very low temperatures, the average kinetic energy of a molecule may be much *less* than the maximum magnitude of potential energy—the depth of the potential well in Fig. 18.2. The molecules then condense into a liquid or solid phase with average intermolecular spacing of about r_0. But at higher temperatures, the average kinetic energy becomes larger than the depth of the potential well. The molecules can then escape the intermolecular forces and become free to move independently as in the gaseous phase of matter.

Characteristics of intermolecular forces

(*i*) They are electrical in nature.

(*ii*) They are active over short distances.

(*iii*) They are attractive upto certain distance between the molecules and become repulsive if the distance between the molecules is less than this value.

(*iv*) For two molecules attracting each other, potential energy is negative.

(*v*) They are much weaker than interatomic forces.

18.2. STATES OF MATTER

The world around us is composed of three types of materials ; solids, liquids and gases. We call these types the three *states of matter.* Bricks, wood, stone, metals are solids. Milk, water and oils are liquids. Hydrogen, oxygen and air are gases at room temperature. The fundamental difference between these three states of matter lies in the way the forces act between their atoms or molecules. In other words, under ordinary conditions of temperature and pressure, the interatomic and intermolecular forces decide the state of matter.

(*i*) **Gas.** *In gases, the interatomic/intermolecular forces are practically nonexistent, allowing the atoms or molecules to move independently* except when they collide with another molecule or the walls of the container. The average separation between the molecules of a gas is large compared with their own size. Consequently, we can compress or expand a gas to fill a container of any shape or size. For this reason, a gas has neither a definite shape nor a definite volume.

(*ii*) **Liquid.** *In a liquid, the average separation between the molecules is comparable to their own diameter.* Individual molecules are free to move about because of the forces between them, they move so that the average separation between neighbours remains constant. As a result, a liquid is virtually incompressible and has a definite volume, although its shape can change to match the shape of its container.

(*iii*) **Solid.** In solids, the separations between molecules are comparable to the separations between the molecules of liquids. *However, the intermolecular forces are so strong that atoms in a solid are not free to move about.* Instead, the atoms of a solid are confined to small oscillations about fixed positions. Thus a solid has not only a definite volume but a definite shape as well.

The state in which a particular substance exists depends upon the temperature of the substance and on the external pressure surrounding it. As an example, water can have the form of ice (solid), water (liquid) or steam (gas). The change from one state to another, such as the melting of ice is usually caused by a transfer of heat.

18.3. CLASSIFICATION OF SOLIDS

In a liquid or gas, atoms and/or molecules are relatively free and can move around at will. However, in solids, they almost completely lack mobility ; all they can do is to vibrate about a mean position when energy is added to the solid. It is due to this lack of mobility that solids have definite shape and volume. The way in which a solid behaves depends upon its internal structure. Accordingly, solids can be divided into two categories *viz.*

(*i*) Crystalline solids (*ii*) Amorphous solids

18.5. ELASTIC BEHAVIOUR OF SOLIDS

So far we have assumed that the rigid bodies or solids remain undeformed when external forces act on them. In actual practice, all bodies are deformable. In other words, it is possible to change the size or shape or both of a body by the application of external forces. This deformation may not be visible to the naked eye in some cases.

Illustration. Consider a steel wire (solid) fixed to a rigid support at one end and supporting a small mass m at the other end [See Fig. 18.4]. The steel wire is under stress due to the weight of mass m. The stress pulls the molecules of the steel wire further apart, thus slightly increasing the length of the wire. When the stress is removed (*i.e.*, mass m is removed), the molecular forces restore the molecules of the steel wire to their normal spacing. Consequently, the wire regains its original length. This tendency of a solid to regain shape and size after the removal of deforming force is called its *elastic property.* The elastic property of a solid determines its behaviour under the action of deforming forces. On the removal of the deforming force, a solid body will regain its original state (*i.e.*, shape and size) if the deforming force does not exceed a certain limit called *elastic limit.* In the above case, for example, if a sufficiently large load is suspended from the wire, it is found that the wire does not regain its original length after the removal of the load. The wire is said to have permanent deformation. Elastic limit differs widely for different solid materials. It is high for a material like steel and low for a material like lead.

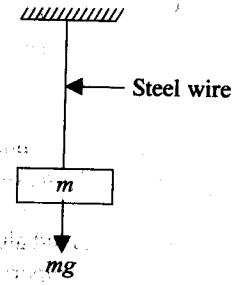


Fig. 18.4

Note. In the above case, if elastic limit is exceeded, the length of the wire will continue to increase till at a particular load, it will finally break.

18.6. ELASTICITY

The property of a body by virtue of which it tends to regain its original shape and size when deforming force is removed is called **elasticity**.

All solids show the property of elasticity. The greater the elasticity of a body, the greater is its tendency to regain its original shape and size after the removal of the deforming force/forces. The elastic property varies from solid to solid. For example, steel is very elastic. It means that steel closely returns to its original dimensions even after being subjected to relatively large forces. From this point of view, rubber is not highly elastic, even though it may be readily stretched.

(*i*) **Perfectly elastic body.** *A body which regains its exact original shape and size immediately after the removal of the deforming force is called a perfectly elastic body.* Quartz and phosphor bronze are the examples of nearly perfectly elastic bodies.

No body in nature is perfectly elastic. In other words, a deforming force does produce permanent effect (even if elastic limit is not exceeded) in shape or size in every body to some or more extent. In certain cases, it may not be visible to the naked eye.

(*ii*) **Perfectly inelastic body.** *A body which does not have any tendency at all to regain its original shape and size on the removal of the slightest deforming force is called a perfectly inelastic or plastic body.* Wet clay and paraffin wax are the examples of nearly perfect plastic bodies. When wet clay is deformed by a force, it remains deformed. Again, no body in nature is perfectly plastic or inelastic body.

18.7. TYPES OF SOLID DEFORMATIONS

Although a solid can be deformed in many ways, there are only three basic types of deformations. All others are combinations of these three types. Fig. 18.5 shows the three basic types of deformations in solids.

(*i*) In Fig. 18.5 (*i*), the deforming force causes a **change in length** of the object. On removal of the deforming force, the object regains its length. This is an illustration of *elasticity of extension* (or compression).

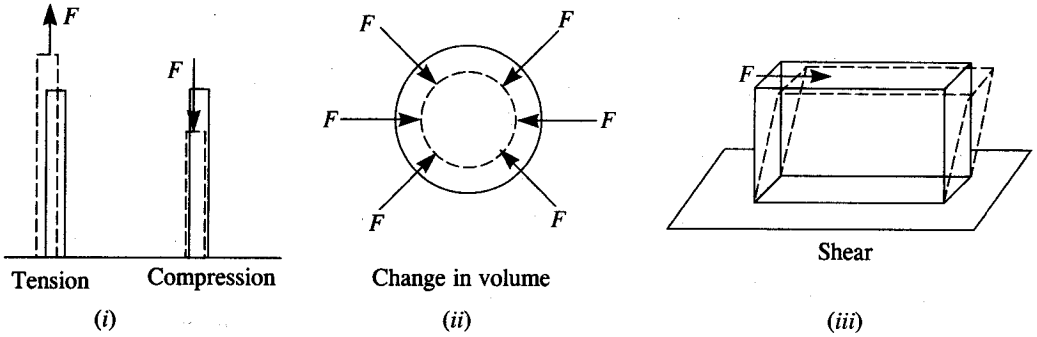


Fig. 18.5

(*ii*) In Fig. 18.5 (*ii*), the deforming force produces a **change in volume**. Since the rubber ball is pressed from all sides, its volume decreases. When pressure is removed, the ball regains its original volume. This is an example of *bulk* or *volume elasticity.*

(*iii*) In Fig. 18.5 (*iii*), the deforming force produces a **change in the shape** of the body. The box is made of cardboard fixed at its lower face. A tangential force F is applied at the upper surface. This force causes the consecutive horizontal layers of the box to be slightly displaced or sheared relative to one another. Note that shearing force is causing no change in the dimensions of the body ; only the shape of the body changes. When the deforming force is removed, the body regains its original shape. This is an example of *elasticity of rigidity.*

In each of the above cases, the deforming force changes the distances between the molecules of the body, either pulling them further apart or pushing them close together. When the deforming force is removed, the molecular forces of the body restore the molecules to their normal spacing. Consequently, the body regains its original state.

Note. Other types of deformation are either a special example of one of these deformations or a combination of two or three of them. For example, bending of a beam is a combination of a stretch and a compression. The outer surface is stretched and the inner surface is compressed.

18.8. STRESS

Stress and strain are two quantities used to describe the nature of the applied force and the resulting deformation respectively. In this section, we shall deal with stress. When an external force applied to a body changes the size or shape of the body, an internal restoring force is developed in the body. *The internal restoring force acting per unit area of the deformed body is called* **stress**. Since in equilibrium, the restoring force is equal in magnitude to the external force, *the stress is measured as the external force acting on a unit area of the body i.e.,*

$$\text{Stress} = \frac{\text{External deforming force}}{\text{Area}} = \frac{F}{A}$$

$$\text{Dimensional formula of stress} = \frac{F}{A} = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$$

The SI unit of force is 1 N and that of area is 1 m^2. Therefore, SI unit of stress is 1 N/m^2. Thus referring back to Fig. 18.4, if the applied force is 50 N and the area of X-section of the wire is 0.01 m^2, then stress = F/A = 50/0.01 = 5000 N/m^2.

which a line originally perpendicular to the fixed face is turned due to the application of the tangential force. Consider a reactangular block fixed at its lower face *EFGH* as shown in Fig. 18.7. If a tangential force F is applied on its upper face *ABCD* in the direction shown, then each layer of the block parallel to face *EFGH* will be displaced laterally in the direction of force. Note that the four vertical sides have been displaced through angle θ so that there is only change in the shape of the body. The face *ABCD* is said to be sheared through an angle θ.

$$\text{Shearing strain} = \theta \simeq \tan\theta = \frac{AA'}{AF} = \frac{\Delta L}{L}$$

Although the expression looks the same as for longitudinal strain, you should understand that the two strains refer to different physical characteristics.

18.10. ELASTIC LIMIT

On the removal of the deforming force, an elastic body will regain its original state only if the deforming force does not exceed a certain limit called elastic limit. For example, if a sufficiently large load is suspended from a copper wire, it is found that the wire does not regain its original length after removal of the load. The wire is said to have permanent deformation. This is true of volumetric or shearing strain also.

The maximum stress from which an elastic body will recover its original state after the removal of the deforming force is called **elastic limit**.

Every solid has a certain range through which it may be deformed and yet return to its original condition, before its elastic limit is reached. Elastic limit differs widely for different materials. It is very high for a substance like steel and low for a substance like lead.

18.11. HOOKE'S LAW AND MODULUS OF ELASTICITY

Robert Hooke studied the elastic behaviour of such objects as coiled springs, metal rods, metallic wires etc. He summed up his findings in a rule now known as **Hooke's law**. It is stated belows :

Provided the elastic limit is not exceeded, the extension (x) produced in a wire is directly proportional to the applied force (F) i.e.,

$$\text{Force} \propto \text{Extension}$$

or $$F \propto x \quad \text{... within elastic limit}$$

For example, if a force of magnitude F causes an extension x in the wire, then a force of $2F$ will cause an extension of $2x$ in the wire.

Now stress is a measure of the deforming force and strain is a measure of the distortion. Therefore, we can restate Hooke's law as under :

Within the elastic limit, the strain produced in an elastic body is directly proportional to the stress i.e.,

$$\text{Stress} \propto \text{strain}$$

or $$\frac{\text{Stress}}{\text{Strain}} = \text{Constant} = \text{Modulus of elasticity}$$

The constant of proportionality is called modulus of elasticity of the material. Its value depends upon the material and on the type of deformation involved. Since strain has no units, modulus of elasticity has the same units as of stress *i.e.*, N/m^2. Note that if the modulus of elasticity of a material is large, it means a large stress will produce only a small strain. *Thus the greater the modulus of elasticity of a material body, the harder it is to change its size or shape and vice-versa.*

Note. Modulus of elasticity is also measured in Pascal (1 Pa = 1 N/m^2). The dimensional formula of modulus of elasticity is $[ML^{-1}T^{-2}]$.

18.12. TYPES OF MODULI OF ELASTICITY

Corresponding to three types of strains, there are three moduli of elasticity *viz.*

(*i*) Young's modulus of elasticity, $Y = \dfrac{\text{Normal stress}}{\text{Longitudinal strain}}$

(*ii*) Bulk modulus of elasticity, $K = \dfrac{\text{Normal stress}}{\text{Volumetric strain}}$

(*iii*) Modulus of rigidity, $\eta = \dfrac{\text{Shearing stress}}{\text{Shearing strain}}$

We shall discuss each in turn.

18.13. YOUNG'S MODULUS (OR ELASTICITY IN LENGTH)

It is defined as the ratio of normal stress to the longitudinal strain. It is denoted by the symbol *Y*.

$$\text{Young's modulus, } Y = \frac{\text{Normal stress}}{\text{Longitudinal strain}}$$

Consider a wire of length L and area of X-section A fixed at one end to a rigid support as shown in Fig. 18.8. If a normal force F is applied to the free end, the length of the wire will change. Suppose the length increases by ΔL. Then,

$$\text{Normal stress} = \frac{F}{A}$$

$$\text{Longitudinal strain} = \frac{\Delta L}{L}$$

$$\therefore \quad Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta L}$$

If r is the radius of the wire, then $A = \pi r^2$.

$$\therefore \quad Y = \frac{FL}{\pi r^2\,\Delta L}$$

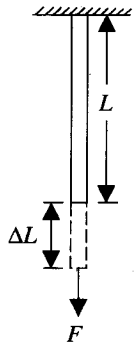


Fig. 18.8

S.No.	Material	Y(N/m^2)
1.	Aluminium	7.1×10^{10}
2.	Steel	20×10^{10}
3.	Glass	7×10^{10}
4.	Brass	7×10^{10}
5.	Rubber	1.0×10^{6}

As mentioned earlier, the SI unit of Y is Nm^{-2}. The value of Y is different for different materials. However, it is reasonably the same for a given solid even though the material may be fashioned in different shapes. Some representative values of Y for different materials are listed in the given table. We see that Young's modulus for steel is approximately 10^5 times greater than that for rubber *i.e.*, steel is 10^5 times harder to stretch than the rubber.

18.14 STRESS-STRAIN CURVE FOR AN ELASTIC MATERIAL

Suppose a metallic wire of uniform cross-section is suspended from a rigid support. When the load on the other end is gradually increased, the length of the wire goes on increasing. If graph between stress and strain is plotted, the shape of the curve will be as shown in Fig. 18.9.

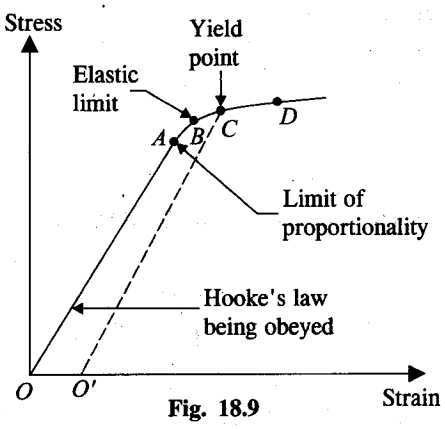


Fig. 18.9

(*i*) **Portion *OA*.** The portion *OA* of the graph is a straight line showing that upto point *A*, strain produced in the wire is directly proportional to the stress *i.e.*, strain $\propto$ stress. In this portion, the material of the wire obeys Hooke's law. The point *A* is called the **limit of proportionality**. The proportionality limit is the greatest stress a material can sustain without the departure from a linear stress-strain relation. If the applied force is removed at any point between *O* and *A*, the wire regains its original length.

(*ii*) **Portion *AB*.** The portion *AB* of the graph is not a straight line showing that in this region, strain is not proportional to the stress. Note that the slope of the graph is decreased ; this means that strain increases more rapidly with stress. Nevertheless, if load is removed at any point between *O* and *B*, the wire will return to its original length. The point *B* is called the **elastic limit**. *The elastic limit is the maximum stress which a body can sustain and still regain its original size and shape once the load has been removed.*

(*iii*) **Portion *BC*.** If the stress is increased beyond the elastic limit, a point *C* is reached at which there is marked increase in extension. This point is called **yield point**. Between *B* and *C*, the material becomes **plastic** *i.e.* if the wire is unloaded at any point between *B* and *C*, the wire does not quite come back to its original length. *The extension not recoverable after removing the load is known as* **permanent set**. However, this permanent deformation is not serious enough to be important. In practice, we must keep the stress below the yield point. Here *OO'* is the permanent set.

(*iv*) **Portion *CD*.** If the stress is increased beyond point *C*, the wire lengthens rapidly until we reach point *D* at the top of the curve. The point *D* is called the **utlimate strength** or **breaking stress**. Beyond point *D*, even a stress smaller than at *C* may continue to stretch the wire until it breaks.

Note. Elastic limit and limit of proportionality are very close to each other so that Hooke's law is nearly applicable upto elastic limit.

18.15. STRESS–STRAIN CURVE FOR RUBBER

Fig. 18.10 shows a stress-strain curve for a typical sample of vulcanised rubber that has been stretched to over seven times its original length. During *no* portion of this curve is the stress proportional to the strain *i.e.*, stress-strain curve for rubber does not obey Hooke's law. However, the substance is elastic in the sense that when the load is removed, the rubber is restored to its original length. On decreasing the load, the stress-strain curve is *not* retraced but follows the thin curve as shown in Fig. 18.10. *The lack of coincidence of the curves for increasing and decreasing stress is known as elastic hysteresis.* When stress-strain relation exhibits this behaviour, the associated forces are not conservative since the work

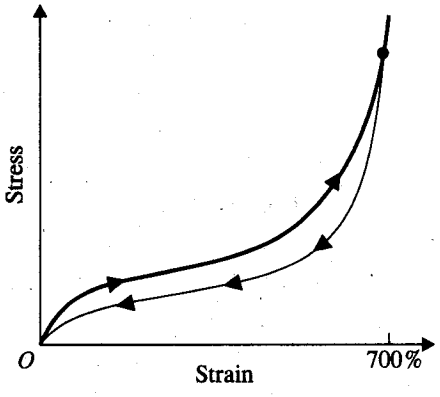


Fig. 18.10

done by the material in returning to its original length is *less* than the work required to deform it. It can be shown that the area bounded by the two curves (*i.e.*, the area of the *hysteresis loop*) is proportional to the energy dissipated within the elastic material.

The large elastic hysteresis of some kinds of rubber makes these materials very valuable as vibration absorbers. If a block of such material is placed between a piece of machinery and the floor, elastic hysteresis takes place during every cycle of vibration. Mechanical energy is converted to a form known as internal energy which evidences itself by a rise in temperature. As a result, only a small amount of energy of vibration is transmitted to the floor. *Materials like rubber which can undergo large elongation under load at room temperature and return to their original shapes when the load is removed are called* **elastomers**.

18.16. CLASSIFICATION OF MATERIALS ON ELASTIC PROPERTIES

On the basis of elastic properties, the materials can be classified as (*i*) Ductile materials (*ii*) Brittle materials and (*iii*) Elastomers.

(*i*) **Ductile materials.** *Those materials which show large plastic range beyond elastic limit are called ductile materials e.g.* copper, silver, iron etc. If you investigate the stress-strain curve of a ductile material, the breaking point is widely separated from the elastic limit. A ductile material can be permanently stretched. The curve shown in Fig. 18.11 is for a ductile material (copper).

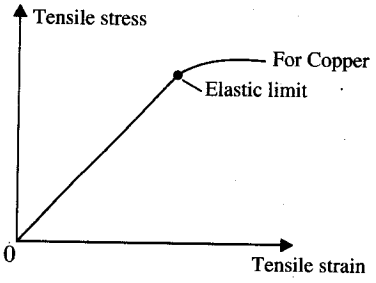


Fig. 18.11

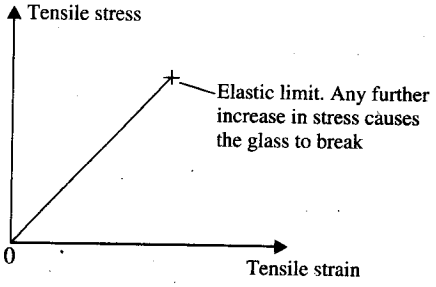


Fig. 18.12

(*ii*) **Brittle materials.** *Those materials which show small plastic range beyond elastic limit are called brittle materials e.g.* glass, cast iron etc. A brittle material cannot be permanently stretched ; it breaks soon after the elastic limit is reached. In other words, in the stress-strain curve of a brittle material, the elastic limit and the breaking point are close together. Brittle materials are often very strong in compression. Fig. 18.12 shows the stress-strain curve of glass.

(*iii*) **Elastomers.** *Those materials which do not obey Hooke's law within elastic limit (i.e. within elastic limit, stress-strain curve is not a straight line) are called elastomers e.g.* rubber. Such materials can be stretched considerably and still return to their original lengths when the stresses are removed. For example, rubber can be stretched as much as 10 times its original length before elastic limit is exceeded. On the other hand, a metallic wire can be subjected to only about 1/10,000 of this extension before its elastic limit is exceeded. Fig. 18.10 shows the stress-strain curve for an elastomer (rubber).

Note. Breaking force = Breaking stress × Cross-sectional area of wire.

Note that breaking stress has a fixed value for a material.

18.17. ELASTIC FATIGUE

If a material is subjected to repeated stresses over long periods of time, it becomes weaker *i.e.* the strain produced by a given amount of stress increases.

The loss of strength of the material because of repeated stresses is known as **fatigue**.

Each time the stress is applied, the internal structure of the material is changed. Each time the stress is removed, there is slight permanent change in the structure of the material. As this process

continues, some portions of the material are weakened and the material may fracture. The failure of a material under these circumstances is called **fatigue failure** or **fatigue fracture**. It has been estimated that about 90% of the failures which occur in an aircraft components are due to fatigue.

18.18. CREEP

The gradual increase in strain which occurs when a material is subjected to stress for a long period of time is known as **creep**.

Unlike fatigue, it occurs even when the stress is constant. It is most marked at elevated temperatures and may be so severe that the material eventually fractures. The greater the stress, the more rapidly it happens. The turbine blades of jet engines are likely to fracture because of creep since they are under high stress and are at high temperatures. Soft metals (*e.g.* lead) and most plastics show considerable creep even at room temperature.

Example 18.1. A spring 60 cm long is stretched by 2 cm by the application of a load of 200 g. What will be the length when a load of 500 g is applied?

Solution. Young's modulus of elasticity, $Y = \dfrac{FL}{A\,\Delta L}$

$$\text{For the first case, } Y = \frac{(0.2\times 9.8)\times 0.6}{A\times 0.02} \qquad ...(i)$$

$$\text{For the second case, } Y = \frac{(0.5\times 9.8)\times 0.6}{A\times \Delta L_2} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), we have,

$$\frac{(0.2\times 9.8)\times 0.6}{A\times 0.02} = \frac{(0.5\times 9.8)\times 0.6}{A\times \Delta L_2}$$

$$\therefore \quad \Delta L_2 = 0.05 \text{ m} = 5 \text{ cm}$$

$$\therefore \quad \text{Total length} = 60 + 5 = \mathbf{65 \text{ cm}}$$

Example 18.2. Calculate the percentage increase in length of a wire of diameter 2.2 mm stretched by a load of 100 kg. Young's modulus of wire is 12.5×10^{10} N/m^2.

Solution. $F = 100 \times 9.8$ N ; $Y = 12.5 \times 10^{10}$ N/m^2 ;

$$A = \frac{\pi}{4}d^2 = \left(\frac{\pi}{4}\right)\times(2.2\times 10^{-3})^2 = 3.8\times 10^{-6} \text{ m}^2$$

Now $$Y = \frac{FL}{A\,\Delta L}$$

or $$\frac{\Delta L}{L} = \frac{F}{AY} = \frac{100\times 9.8}{(3.8\times 10^{-6})\times(12.5\times 10^{10})} = 0.0021$$

$$\therefore \text{ \%age increase in length} = \frac{\Delta L}{L}\times 100 = 0.0021\times 100 = \mathbf{0.21\%}$$

Example 18.3. The breaking stress of steel is 8×10^9 N/m^2. Find the greatest length of steel wire that can hang vertically without breaking. Density of steel is 10^4 kg/m^3.

Solution. Suppose L is the greatest length of wire in metres that can hang vertically without breaking.

$$\text{Weight of wire} = m\,g = (\text{Volume} \times \text{density}) \times g$$

$$= (L \times A \times 10^4) \times 9.8 = 9.8 \times 10^4\ L\,A$$

Breaking force = Breaking stress × area of wire = $(8 \times 10^9) \times A$

Now $9.8 \times 10^4 \, LA = (8 \times 10^9) \times A$

$$\therefore \quad L = \frac{8 \times 10^9}{9.8 \times 10^4} = 81632.6 \text{ m}$$

Example 18.4. Two exactly similar wires of steel and copper are stretched by equal forces. If the total elongation is 1 cm, find how much each wire is elongated. Given Y for steel $= 20 \times 10^{10}$ N/m^2 and Y for copper $= 12 \times 10^{10}$ N/m^2.

Solution. Let L and A be the length and area of X-section of each wire respectively.

$$\text{For steel, } Y_s = \frac{FL}{A\,\Delta L_s} \text{ ; For copper, } Y_c = \frac{FL}{A\Delta L_c}$$

$$\therefore \quad \frac{Y_s}{Y_c} = \frac{\Delta L_c}{\Delta L_s}$$

$$\text{or} \quad \frac{\Delta L_c}{\Delta L_s} = \frac{Y_s}{Y_c} = \frac{20 \times 10^{10}}{12 \times 10^{10}} = \frac{5}{3} \qquad ...(i)$$

$$\text{Also} \quad \Delta L_c + \Delta L_s = 1 \text{ cm (given)} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $\Delta L_s = 0.375$ cm ; $\Delta L_c = 0.625$ cm

Example 18.5. The Young's modulus for steel is 2×10^{11} N/m^2. If the interatomic spacing for the metal is 2.8 Å, find the increase in interatomic spacing for a stress of 10^9 N/m^2.

Solution. $$Y = \frac{FL}{A\Delta L}$$

$$\text{or} \quad \Delta L = \frac{FL}{YA}$$

Here $F/A = 10^9$ N/m^2 ; $L = 2.8$ Å $= 2.8 \times 10^{-10}$ m ; $Y = 2 \times 10^{11}$ N/m^2

$$\therefore \quad \Delta L = \frac{10^9 \times 2.8 \times 10^{-10}}{2 \times 10^{11}} = 1.4 \times 10^{-12} \text{ m} = 0.014 \text{ Å}$$

Example 18.6. A composite wire of diameter 1 cm consists of copper and steel wires of lengths 2.2 m and 2 m respectively. Total extension of the wire when stretched by a force is 1.2 mm. Calculate the force, given that Young's modulus for copper is 1.1×10^{11} Pa and for steel is 2×10^{11} Pa.

Solution. Young's modulus, $Y = \dfrac{F/A}{\Delta L/L} = \dfrac{F}{A} \times \dfrac{L}{\Delta L}$

Here L is the length of the wire and ΔL its extension. The values of F and A are the same in the two cases.

For copper wire : $$Y_c = \frac{F}{A} \times \frac{L_c}{\Delta L_c} \qquad ...(i)$$

For steel wire : $$Y_s = \frac{F}{A} \times \frac{L_s}{\Delta L_s} \qquad ...(ii)$$

$$\therefore \quad \frac{Y_c}{Y_s} = \frac{L_c}{\Delta L_c} \times \frac{\Delta L_s}{L_s}$$

$$\text{or} \quad \frac{\Delta L_s}{\Delta L_c} = \frac{Y_c}{Y_s} \times \frac{L_s}{L_c} = \frac{1.11 \times 10^{11}}{2.0 \times 10^{11}} \times \frac{2.0}{2.2} = \frac{1}{2}$$

$$\therefore \quad \Delta L_c = 2\,\Delta L_s \qquad ...(iii)$$

$$\text{Also} \quad \Delta L_c + \Delta L_s = 1.2 \text{ mm} = 12 \times 10^{-4} \text{ m} \qquad ...(iv) \text{ (given)}$$

From eqs. (*iii*) and (*iv*), we have, $\Delta L_c = 8 \times 10^{-4}$ m.

Note that bulk moduli are *possessed by solids, liquids and gases. For example, the bulk modulus of steel is 16×10^{10} N/m^2 and that of water is 2.2×10^9 N/m^2. For air at normal pressure, $K = 10^5$ N/m^2. The bulk moduli of solids and liquids are very large indicating the fact that large forces are needed to produce even minute change in volume. Gases are more easily compressed and have correspondingly smaller bulk moduli. Thus greater the bulk modulus (K) of a material, the harder it is to change its volume.

Compressibility. *The compressibility of a material is a measure of how easily the material is compressed.* In other words, compressibility is just the reciprocal of bulk modulus *i.e.*,

$$\text{Compressibility, } k = \frac{1}{K}$$

The SI unit of compressibility is N^{-1} m^2 or Pa^{-1} (*i.e.*, the reciprocal of unit of K).

Solids and liquids are relatively incompressible *i.e.*, they have small values of compressibility (k) or large values of bulk modulus (K) and these values are almost independent of temperature and pressure. On the other hand, gases are easily compressed (small K and large k) and the values of K and k strongly depend on the pressure or temperature.

Example 18.7. A spherical ball contracts in volume by 0.01% when subjected to a normal uniform pressure of 10^8 N/m^2. Find the bulk modulus of the material.

Solution. Bulk modulus, $K = V\dfrac{\Delta p}{\Delta V}$

Here $\Delta p = 10^8$ N/m^2; $\Delta V/V = 0.01\% = 0.01/100 = 10^{-4}$

$$\therefore \quad K = \frac{V}{\Delta V} \times \Delta p = \left(\frac{1}{10^{-4}}\right) \times (10^8) = 10^{12} \text{ N/m}^2$$

Example 18.8. A cube is subjected to a pressure of 5×10^5 N/m^2. Each side of the cube is shortened by 1%. Find (*i*) the volumetric strain and (*ii*) bulk modulus of elasticity of cube material.

Solution. Let l be the initial length of each side of the cube.

$$\text{Initial volume, } V = l^3$$

$$\text{Final length of each side} = \left(1 - \frac{1}{100}\right) l = \frac{99}{100} l$$

$$\text{Final volume} = \left(\frac{99l}{100}\right)^3$$

$$\text{Change in volume, } \Delta V = \left(\frac{99l}{100}\right)^3 - l^3 = l^3\left[\left(\frac{99}{100}\right)^3 - 1\right]$$

(*i*) $$\text{Volumetric strain} = \frac{\Delta V}{V} = \frac{l^3\left[\left(\frac{99}{100}\right)^3 - 1\right]}{l^3} = \frac{-3}{100} = -0.03$$

(*ii*) $$\text{Bulk modulus, } K = \frac{\text{Normal stress}}{\text{Volumetric strain}} = \frac{5 \times 10^5}{0.03} = 1.67 \times 10^7 \text{ N/m}^2$$

Example 18.9. What will be the density of lead under a pressure of 2×10^8 N/m^2? Density of lead = 11.4×10^3 kg/m^3 and bulk modulus of lead $K = 8 \times 10^9$ N/m^2.

* Unlike Young's modulus and shear modulus which refers to solids only.

Solution. Bulk modulus, $K = -\dfrac{V\Delta p}{\Delta V}$

$\therefore$ Change in volume, $\Delta V = -\dfrac{\Delta p}{K}\times V = -\dfrac{2\times10^8}{8\times10^9}\times V = -\dfrac{V}{40}$

New volume of lead, $V' = V + \Delta V = V - \dfrac{V}{40} = \dfrac{39}{40}V$

If ρ' kg/m^3 is the new density of lead, then its mass

$$= V'\rho' = \frac{39}{40}V\rho' \text{ kg}$$

Since the mass is a constant quantity, we have,

$$\frac{39}{40}V\rho' = V\times11.4\times10^3$$

$\therefore$ $$\rho' = \frac{40}{39}\times11.4\times10^3 = \mathbf{11.69\times10^3\ kg/m^3}$$

PROBLEMS FOR PRACTICE

1. A sphere contracts in volume by 0.01% when taken to the bottom of sea 1 km deep. Find the bulk modulus of the material of the sphere. Density of sea water may be taken as 1000 kg m^{-3}. **[9.8 × 10^{10} N/m^2]**
2. A solid ball 300 cm in diameter is submerged in a lake of such depth that pressure exerted by water is 9.8 × 10^4 N/m^2. Find the change in volume of the ball at this depth. Bulk modulus of the material of the ball is 10^{12} N/m^2. **[1.386 c.c.]**

18.20. SHEAR MODULUS (OR ELASTICITY OF SHAPE)

This refers to situations in which the shape of a substance is changed by the application of tangential stress (or shear stress). *It is defined as the ratio of tangential stress to shear strain.* It is denoted by the symbol η.

$$\text{Shear modulus, } \eta = \frac{\text{Tangential stress}}{\text{Shear strain}}$$

Consider a rectangular block whose lower face *EFGH* is fixed [See Fig. 18.14]. Let a tangential force F be applied on the upper face *ABCD* of the block in the direction shown. The force provides a shear stress which displaces the upper face through a small distance $AA' = \Delta L$.

If $AF = L$ and area of the upper face is A, then,

$$\text{Tangential stress} = \frac{F}{A}$$

$$\text{Shear strain} = \theta \simeq \tan\theta = \frac{AA'}{AF} = \frac{\Delta L}{L}$$

$\therefore$ $$\eta = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta L}$$

or $$\eta = \frac{FL}{A\,\Delta L}$$

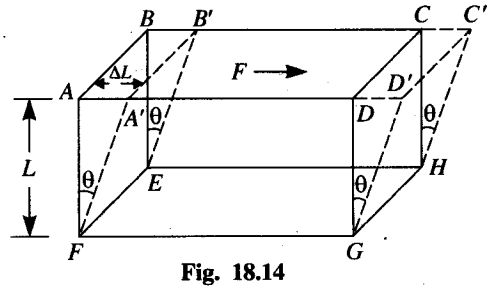


Fig. 18.14

The SI unit of shear modulus is N/m^2 or Pa.

The shear modulus has significance only for solid materials. A liquid or gas flows under the influence of a shear stress and cannot *permanently* support such a stress. The shear modulus is roughly one-third the value of Young's modulus for the same solid material. For example, Young's modulus for steel is 20 × 10^{10} N/m^2, whereas shear modulus for steel is 8 × 10^{10} N/m^2.

Example 18.10. An aluminium cube of each side 4 cm is subjected to a tangential force. The top face of the cube is sheared 0.012 cm *w.r.t.* the bottom. Find (*i*) shearing strain (*ii*) shear stress and (*iii*) shearing force. Given that modulus of rigidity is 2.08×10^{10} N/m^2.

Solution. $L = 4$ cm ; $\Delta L = 0.012$ cm ; $\eta = 2.08 \times 10^{10}$ N/ m^2

(*i*) Shearing strain, $\theta = \dfrac{\Delta L}{L} = \dfrac{0.012}{4} = 0.003$

(*ii*) $\eta = \dfrac{\text{Shear stress}}{\text{Shear strain}}$

$\therefore$ Shear stress $= \eta \times$ shear strain

$= (2.08 \times 10^{10}) \times (0.003) = \mathbf{6.24 \times 10^7\ N/m^2}$

(*iii*) Shearing force = Shear stress × area of cube face

$= (6.24 \times 10^7) \times (16 \times 10^{-4}) = \mathbf{9.98 \times 10^4\ N}$

Example 18.11. A rubber cube of side 20 cm has one side fixed while a tangential force equal to the weight of 400 kg is applied to the opposite face. Find (*i*) shearing strain and (*ii*) the distance through which the strained side moves. Given that modulus of rigidity for rubber is 8×10^6 N/m^2.

Solution. (*i*) Shear stress $= \dfrac{F}{A} = \dfrac{400 \times 9.8}{(20 \times 10^{-2})^2} = 9.8 \times 10^4$ N/m^2

Modulus of rigidity, $\eta = \dfrac{\text{Shear stress}}{\text{Shear strain}}$

$\therefore$ Shear strain, $\theta = \dfrac{\text{Shear stress}}{\eta} = \dfrac{9.8 \times 10^4}{8 \times 10^6} = \mathbf{0.0123}$

(*ii*) Now $\theta = \dfrac{\Delta L}{L}$

$\therefore$ $\Delta L = \theta \times L = 0.0123 \times 0.2 = 0.0025$ m $= \mathbf{0.25\ cm}$

Example 18.12. A square lead slab of side 50 cm and thickness 5.0 cm is subjected to a shearing force (on its narrow face) of magnitude 9.0×10^4 N. The lower edge is riveted to the floor. How much is the upper edge displaced if the shear modulus of the lead is 5.6×10^9 N/m^2?

Solution. The area of cross-section of the face where the force is applied is $A = 50 \times 5 = 250$ cm^2 $= 250 \times 10^{-4}$ m^2.

Now, $\eta = \dfrac{F/A}{\theta}$

$\therefore$ Shear strain, $\theta = \dfrac{F}{A\eta} = \dfrac{9.0 \times 10^4}{250 \times 10^{-4} \times 5.6 \times 10^9} = 6.4 \times 10^{-4}$

Now, $\theta = \dfrac{AA'}{AF}$

$\therefore$ Displacement of the upper edge is given by ;

$AA' = \theta \times AF = (6.4 \times 10^{-4}) \times 0.5 = \mathbf{3.2 \times 10^{-4}\ m}$

18.21. STRAIN ENERGY

When a body is strained, work has to be done to deform the body. This work done is stored in the body in the form of *elastic strain energy.* Consider a wire of length L and area of cross-section A. Suppose on applying a normal force F on the wire, the extension produced is l. If the

extension is increased by Δl, where Δl is so small that F can be considered constant, then work done ΔW is given by ;

$$\Delta W = F\,\Delta l$$

According to Hooke's law, $F = kl$

$\therefore$ The total work done (W) in increasing the extension from 0 to l is given by ;

$$W = \int_0^l kl\,dl = \frac{1}{2}kl^2 = \frac{1}{2}(kl)l = \frac{1}{2}Fl$$

This work done is stored in the wire as elastic potential energy U (the strain energy).

$\therefore$ Strain energy, $U = \frac{1}{2}Fl$

We can express this relation in another useful form.

$$U = \frac{1}{2}\times\left(\frac{F}{A}\right)\times\left(\frac{l}{L}\right)\times LA$$

$$= \frac{1}{2}\times \text{stress} \times \text{strain} \times \text{volume}$$

$\therefore$ Strain energy per unit volume, $u = \frac{1}{2}\times$stress $\times$ strain ...(*i*)

Therefore, the strain energy per unit volume in a deformed body is equal to one-half the product of the stress and strain. Although we have derived this expression for the case of longitudinal strain, it is applicable to any type of strain.

Example 18.13. Calculate the increase in energy of a brass bar of length 0.2 m and cross-sectional area 1 cm^2 when compressed with a force of 49 N along its length. Young's modulus of brass = 1.0 × 10^{11} N/m^2.

Solution. Initial length of bar, L = 0.2 m

Area of cross-section, A = 1 cm^2 = 10^{-4} m^2

Increase in energy of bar = W.D. in compressing

or $$U = \frac{1}{2}Fl$$

where F is the force applied and l is the decrease in length.

Now $$Y = \frac{FL}{Al} \text{ or } l = \frac{FL}{AY}$$

$\therefore$ $$U = \frac{1}{2}\times F \times \frac{FL}{AY} = \frac{F^2L}{2AY}$$

or $$U = \frac{(49)^2\times 0.2}{2\times(1.0\times10^{-4})\times(1.0\times10^{11})} = \mathbf{2.4 \times 10^{-5}\ J}$$

Example 18.14. A 40 kg boy whose leg bones are 4 cm^2 in area and 50 cm long falls through a height of 2 m without breaking his leg bones. If the bones can stand a stress of 0.9 × 10^8 N/m^2, calculate the Young's modulus for the material of the bone. Take g = 10 ms^{-2}.

Solution. Here m = 40 kg ; h = 2 m ; l = 0.5 m ; $A = 4 \times 10^{-4}$ m^2

Volume of leg = $Al = 4 \times 10^{-4} \times 0.5 = 2 \times 10^{-4}$ m^2

Loss in gravitational P.E. = Gain in elastic P.E. by both legs

or $$mgh = 2\left[\frac{1}{2}\times \text{stress} \times \text{strain} \times \text{volume of one leg}\right]$$

or $$40 \times 10 \times 2 = 2\left[\frac{1}{2} \times 0.9 \times 10^8 \times \text{strain} \times 2 \times 10^{-4}\right]$$

$$\therefore \quad \text{Strain} = \frac{40 \times 10 \times 2}{0.9 \times 2 \times 10^4} = \frac{2}{45}$$

$$\therefore \quad \text{Young's modulus, } Y = \frac{\text{Stress}}{\text{Strain}} = \frac{0.9 \times 10^8}{2/45} = 2.025 \times 10^9 \text{ Nm}^{-2}$$

18.22. POISSON'S RATIO

When a wire is suspended from one end and loaded at the other end, the length of the wire increases and its diameter decreases. Thus there occurs longitudinal strain as well as lateral strain.

The ratio of lateral strain to the longitudinal strain is called **Poisson's ratio (σ).**

Let L, D = original length and diameter respectively of the wire

ΔL, ΔD = slight increase in length and a corresponding slight decrease in the diameter of the wire

$$\text{Longitudinal strain} = \frac{\Delta L}{L}$$

$$\text{Lateral strain} = -\frac{\Delta D}{D}$$

The negative sign shows that if length increases, the diameter decreases.

$$\therefore \quad \text{Poisson's ratio, } \sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{-\Delta D / D}{\Delta L / L}$$

$$\therefore \quad \sigma = \frac{-\Delta D}{D} \times \frac{L}{\Delta L}$$

Poisson's ratio is never more than 0.5, its value is usually between 0.2 and 0.4.

18.23. RELATION AMONG ELASTIC CONSTANTS

We have studied various elastic constants. One might expect the existence of a relationship among these parameters describing the elastic properties of a material. By advanced treatment of mechanics, the following relationships exist among the various elastic constants :

(*i*) $Y = 3K(1-2\sigma)$ (*ii*) $Y = 2\eta(1+\sigma)$

(*iii*) $\sigma = \frac{3K-2\eta}{2\eta+6K}$ (*iv*) $\frac{9}{Y} = \frac{1}{K} + \frac{3}{\eta}$

Here the various elastic constants have their usual meanings.

18.24. APPLICATIONS OF ELASTICITY

The primary jobs of an engineer are the design, construction and maintenance of structures, machinery etc. One of his functions as a designer is to have a good knowledge of the elastic properties of the materials he proposes to use. This will enable him to predict the behaviour of the materials under the action of deforming forces. Although there are a very large number of practical applications of elasticity, we are briefing here a few by way of illustration.

(*i*) Most parts of structures and machinery are under some kind of stress. In their design, it has to be ensured that applied stresses do not exceed the elastic limits of the materials.

(*ii*) Due to particular elastic properties, some materials are used as vibration absorbers. For example, some types of rubber have particularly large hysteresis loops and so are useful as vibration absorbers. If a block of such a rubber is placed between a piece of vibrating

machinery and the floor, much of the energy of the mechanical vibrations is converted to heat energy in the rubber and so is not transmitted to the floor.

(*iii*) In a car, it is desirable that as little heat is generated as possible. For this reason, in the manufacture of car tyres that rubber is used which has small hysteresis loop.

(*iv*) Because of its strength and elastic properties, steel is used to make not only springs but in the construction of girders as well.

(*v*) The cross-section of steel girders has the form of the letter *I* as shown in Fig. 18.15. Most of the material in these *I*-beams is concentrated in the top and bottom flanges ; the piece joining the flanges, called the web, is of thinner cross-section. Thus when the beam is used horizontally in construction, the stress is predominantly in the top and bottom flanges — not in the central portion. One flange is sequeezed while the other is stretched. Between the top and bottom flanges is a stress-free region that acts principally to connect the top and bottom flanges together. This is the neutral layer where comparatively little material is needed. The flanges carry virtually all stresses in the beam. An *I*-beam is nearly as strong as a solid rectangular bar of the same overall dimensions, and its weight is considerably less. A large rectangular steel beam on a certain span might collapse under its own weight whereas an *I*-beam of the same depth would carry much added load.

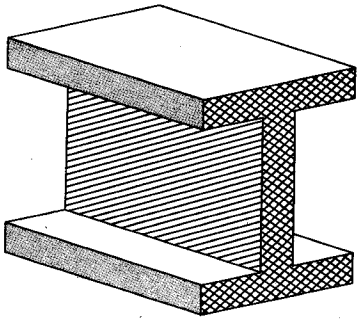


Fig. 18.15

(*vi*) A hollow shaft is stronger than a solid shaft made of same equal material. It is because the torque required to produce a given twist in a hollow shaft is greater than that required to twist a solid shaft of the same length and material through the same angle. For this reason, electric poles are made hollow.

CONCEPTUAL QUESTIONS

Q.1. Matter exists in the states solids, liquids and gaseous. Can there be any other state of matter?

Ans. In addition to solids, liquids and gases, there is a fourth state of matter, the least common state in our everyday environment—**plasma** (not to be confused with the clear, formless part of blood, also called plasma). A plasma looks and behaves like a high-temperature gas but with an important difference ; it conducts electricity. The atoms and molecules that make it up are ionised, stripped of one or more electrons as a result of the more violent collisions at high temperatures. In an ideal plasma, all atoms are completely stripped off electrons and are bare nuclei. Although the electrons and the ions are themselves electrically charged, the plasma as a whole is electrically neutral.

Q.2. Solids have fixed shapes and volumes. Why?

Ans. The molecules of a solid at temperatures above absolute zero oscillate about their equilibrium positions. Because their kinetic energy is low compared with their potential energy, the molecules of solids can merely vibrate about fixed positions. For this reason, a solid has both a fixed volume and a fixed shape.

Q.3. When the interatomic distance is r_0 = 0.9 Å, the potential energy is found to be minimum. Is the force attractive or repulsive at interatomic distance of (*i*) 0.6 Å (*ii*) 1.5 Å?

Ans. When interatomic distance is r_0 (= 0.9 Å), there is no net force between two atoms and their potential energy has its minimum value. At 0.6 Å which is less than r_0, the force is repulsive. However, at 1.5 Å, the force is attractive.

Q.4. Which determines the state of matter?

Ans. The interatomic and intermolecular forces arise from two main causes :

(*i*) The *potential energy* of the atoms/molecules which is due to the interaction with the surrounding atoms/molecules. The origin is principally electrical.

(*ii*) The *thermal energy* of the atoms/molecules (*i.e.*, kinetic energy of atoms/molecules) and it depends on the temperature of the substance concerned.

The particular state in which matter appears (*i.e.*, solid, liquid or gas) and the properties it then has, are determined by the relative magnitudes of these two energies.

Q.5. A liquid has a fixed volume but no fixed shape. Why?

Ans. The molecules of liquids, like those of solids, vibrate. However, the molecules of liquids have greater kinetic energies than those of solids. The increased kinetic energy results in large amplitude of vibration. Therefore, there is more likelyhood of a molecule being able to pass through the gaps between the molecules surrounding it. Hence there is a continual molecular migration superimposed on the vibrational motion. This accounts for the ability of a liquid to adopt the shape of its container. However, the molecules are close together and a change in volume would require that intermolecular forces were overcome. Therefore, liquids have fixed volumes.

Q.6. Which is more elastic iron or rubber?

Ans. Iron is more elastic than rubber. It is because for a given stress, the strain produced in iron is much smaller than that produced in the rubber.

Q.7. A 20 kg load is hung from the end of a spring. The spring then stretches a distance of 10 cm. If, instead, a 40 kg load is hung from the spring, how much will the spring stretch?

Ans. A 40 kg load has twice the weight of a 20 kg load. According to Hooke's law, $F \propto$ extension. Two times the applied force will result in two times the stretch. So spring will stretch 20 cm.

Q.8. Why do spring balances show wrong readings after they have been used for a long time?

Ans. If a material is repeatedly stressed and unstressed, it becomes weaker *i.e.*, the strain produced by a given amount of stress increases. For this reason, the spring balances which have been used for a long time give wrong readings.

Q.9. What is ultimate strength of a material?

Ans. The maximum stress that a material can experience before fracture is called ultimate strength of the material. For example, ultimate strength of steel is about 20×10^{10} N/m^2 (compression) and that of aluminium is 7×10^{10} N/m^2.

Q.10. Why are bridges declared unsafe after long use?

Ans. A bridge is subjected to forces of varying amounts due to the flow of traffic over it. In other words, the bridge is subjected to varying stresses. As a result, it becomes weaker *i.e.*, the strain produced by a given stress increases. If the elastic limit of the bridge is exceeded, it may collapse. For this reason, bridges are declared unsafe after long use.

Q.11. Of what particles may crystalline solids be composed? Give an example of a crystalline solid composed of each kind of particle.

Ans. Most of the non-living solid materials on the earth are in the form of crystals. A crystal is a solid made up of fundamental particles (atoms, molecules or ions) arranged in a definite, regular pattern. Examples are : Atoms – metals ; molecules – sugar ; ions – sodium chloride.

Q.12. Name five primary types of crystal formation.

Ans. The five primary types of crystal formation are :

(*i*) *Ionic crystal* ; composed of alternate positive and negative ions held together by electrostatic attraction.

(*ii*) *Polar crystal* ; composed of polar molecules and held together by the electrostatic attraction by the oppositely charged ends.

(*iii*) *Atomic crystal* ; composed of atoms (all of the same element or two different elements) held together by covalent bonds.

(*iv*) *Molecular crystal* ; composed of non-polar molecules or unionised atoms held together by the weak Vander Waal's forces.

(*v*) *Metallic crystal* ; composed of metal atoms. Each atom contributes about one electron (on the average) to the swarm of electrons that binds them together.

Q.13. Name the elements that compose most of the things we see everyday.

Ans. About a dozen elements compose most of the things we see everyday. Most common substances are formed out of the combinations of two or more of these most common elements : hydrogen (H), carbon (C), nitrogen (N), oxygen (O), sodium (Na), magnesium (Mg), aluminium (Al), silicon (Si), phosphorus (P), sulphur (S), chlorine (Cl), potassium (K), calcium (Ca) and iron (Fe).

Q.14. What is fatigue limit?

Ans. Mild steel and many other ferrous metals can safely undergo an infinite number of stress cycles provided that the maximum stress is kept below a particular value known as the *fatigue limit.* There is no such limit for non-ferrous materials.

Q.15. How does fatigue failure occur?

Ans. Fatigue fractures usually start on the surface at points of high stress *e.g.* at sharp corners and around rivet holes. It is believed that each time the material is stressed, a small amount of plastic strain is produced. Since it is plastic strain, the effects of repeated stressings are cumulative and eventually produce fracture.

Q.16. What is elastic after effect?

Ans. When deforming force is removed from an elastic body, the body tends to regain original configuration. It has been found that some bodies regain their original configuration immediately while others take appreciable time to do so. This delay in regaining the original configuration by a body after the removal of the deforming force is called elastic after effect. Elastic after effect is negligible for quartz and phosphor bronze. For this reason, we use the suspensions made from quartz and phosphor bronze in galvanometers.

Q.17. The elastic limit of earth's material is 3×10^8 N/m^2 and the density of rock material is 3×10^3 kg/m^3 . What is the maximum height a mountain on earth can possess?

Ans. The maximum height of a mountain on earth can be estimated from the elastic behaviour of earth. If h is the maximum height of the mountain, then,

Pressure exerted by mountain on earth = Elastic limit of earth material

$$\text{or} \quad h\rho g = 3\times10^8 \quad \therefore \quad h = \frac{3\times10^8}{\rho g} = \frac{3\times10^8}{3\times10^3\times9.8} \simeq 10^4 \text{ m}$$

It is interesting to note that the height of Mount Everest is nearly equal to this value.

Q.18. In Fig. 18.16, which material is more ductile?

Ans. The ductility of a material is the extent of plastic deformation. Clearly, it is greater for material *A*.

Q.19. In Fig. 18.16, which material is more brittle?

Ans. The plastic region for material *B* is small. Therefore, material *B* is more brittle.

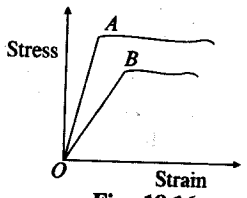


Fig. 18.16

VERY SHORT ANSWER QUESTIONS

Q.1. What is the order of (*i*) size of an atom (*ii*) size of a molecule?

Ans. (*i*) of the order of 10^{-10} m (*ii*) of the order of 3×10^{-10} m.

Q.2. Among the three states of matter which has its own shape and volume?

Ans. Solid state.

Q.3. Which forces are stronger ; interatomic or intermolecular?

Ans. Interatomic forces are stronger than intermolecular forces.

Q.4. Sand does not possess any definite shape and volume. Why is it still a solid?

Ans. Sand is divided rock. It has all the properties of solid even though its particles have indefinite volume like amorphous solids.

Q.5. What is the nature of interatomic/intermolecular forces?

Ans. These forces are electrical in nature.

Q.6. Which is the most important test for a crystalline solid?

Ans. A crystalline solid has a sharp melting point.

Q.7. What is a deforming force?

Ans. A force that produces a change in shape or size of a body is called deforming force.

Q.8. What is elasticity?

Ans. It is the property of a body to regain its shape and size when deforming force is removed.

Q.9. What is a perfectly elastic body?

Ans. If a body regains its exact shape and size on removal of the deforming force, it is called a perfectly elastic body.

Q.10. What is a perfectly plastic body?

Ans. If a body has no tendency at all to regain its original shape and size on the removal of deforming force, it is called a perfectly plastic body.

Q.11. Give two examples of nearly (*i*) perfectly elastic body (*ii*) perfectly plastic body.

Ans. (*i*) Quartz and Phosphor bronze (*ii*) Putty and Paraffin wax.

Q.12. Which is more elastic water or air and why?

Ans. Water. Bulk modulus, $K = 1/k$ where k is the compressibility of the material. Since water is less compressible, it has greater value of elasticity.

Q.13. Why are springs made of steel and not of copper?

Ans. It is because Young's modulus of steel is more than that of copper. Therefore, for a given deforming force, steel spring is stretched lesser than copper spring and regains its original state quickly on the removal of the deforming force.

Q.14. Which state of matter has volume elasticity?

Ans. All states of matter (solids, liquids and gases) have volume elasticity.

Q.15. Which state of matter has Young's modulus?

Ans. Only solids.

Q.16. What is the value of Young's modulus for a perfectly rigid body?

Ans. Young's modulus, $Y = \dfrac{\text{Normal stress}}{\text{Longitudinal strain}}$. The deforming force produces zero strain in a perfectly rigid body so that Y is infinite.

Q.17. Why is work done in stretching a wire?

Ans. When a wire is stretched, the interatomic forces oppose the increase in length of the wire. Therefore, work has to be done against these forces.

Q.18. What happens to work done during stretching of wire?

Ans. The work done in stretching the wire is stored in it in the form of elastic potential energy.

Q.19. What is the SI unit of elasticity?

Ans. Modulus of elasticity = stress/strain. Since strain has no units, modulus of elasticity has the same units as that of stress *i.e.*, N/m^2.

Q.20. The length of a wire increases by 1% on suspending a mass of 2 kg from it. What is the linear strain in the wire?

Ans. Linear strain $= \frac{\Delta L}{L} = 1\% = \frac{1}{100} = 0.01$.

Q.21. What is (*i*) tensile stress (*ii*) compressive stress?

Ans. (*i*) When the deforming force increases the length of the body, the stress produced is called tensile stress.

(*ii*) When the deforming force decreases the length of the body, the stress produced is called compressive stress.

Q.22. What is breaking stress?

Ans. The maximum stress that can be applied to a material without fracture (or breaking) is called breaking stress. It is also called *ultimate tensile strength.*

Q.23. Which state of matter has only bulk modulus?

Ans. Liquids and gases have only bulk modulus.

Q.24. Young's modulus of elasticity and modulus of rigidity do not exist for liquids and gases. Why ?

Ans. It is because liquids and gases cannot be deformed along one dimension only and also cannot undergo shear strain.

Q.25. Which type of modulus of elasticity is involved in (*i*) stretching coiled spring (*ii*) water lift pump (*iii*) power transmission by an automobile shaft?

Ans. (*i*) modulus of rigidity (*ii*) bulk modulus (*iii*) modulus of rigidity.

Q.26. What is the modulus of rigidity of a fluid (liquid or gas)?

Ans. Zero, because a fluid does not possess a shape of its own.

Q. 27. What is the effect of temperature on the three types of moduli?

Ans. With the rise in temperature, the values of Y, K and η decrease.

Q.28. What is the SI unit of Poisson's ratio?

Ans. It is unitless because it is ratio of two strains.

Q.29. A steel wire is stretched by a weight of 1 kg. If the radius of the wire is doubled, how will Young's modulus Y of wire be affected?

Ans. The value of Y remains unaffected because it depends upon the nature of the material.

Q.30. Write copper, steel, rubber and glass in the increasing order of their elasticity.

Ans. Rubber, glass, copper and steel.

SHORT ANSWER QUESTIONS

Q.1. What is elastic fatigue?

Ans. The loss of strength of a material because of repeated stresses is known as elastic fatigue. Due to the repeated stresses, there is a slight permanent change in the structure of the material. As this process continues, some portions of the material are weakened and material may fracture.

Q.2. How do you explain the elastic behaviour of a body?

Ans. The deforming force changes the distance between the molecules of the body, either pulling them further apart or pushing them close together. When the deforming force is removed, the molecular forces of the body restore the molecules to their normal spacing. Consequently, the body regains its original state.

Q.3. Explain the terms elastic limit, yield point and breaking stress for an elastic material.

Ans. Refer to Art. 18.14.

Q.4. Explain the three types of strain that a deforming force can produce.

Ans. Refer to Art. 18.9.

Q.5. What is elastic limit? What happens if it is exceeded?

Ans. Elastic limit is the maximum stress that a body can sustain and still regain its original size and shape on the removal of the deforming force. If the elastic limit is exceeded, there will be some permanent change in size/shape of the body.

Q.6. If in a wire of Young's modulus Y, longitudinal strain X is produced, what will be potential energy stored per unit volume?

Ans. Elastic P.E. stored per unit volume in a strained body

$$= \frac{1}{2}\times\text{stress}\times\text{strain} = \frac{1}{2}\times(Y\times\text{strain})\times\text{strain}$$

$$= \frac{1}{2}\times Y\times(\text{strain})^2 = \frac{1}{2}YX^2$$

Q.7. A force of 1000 N stretches the length of a hanging wire by 1 mm. What force is required to stretch a wire of the same material and length but having four times the diameter by 1 mm?

Ans. $Y = \dfrac{F/A}{\Delta L/L} = \dfrac{FL}{A\,\Delta L}$. Since L, ΔL and Y are constant, $\dfrac{F}{A}$ = Constant.

$$\therefore \frac{F_2}{A_2} = \frac{F_1}{A_1} \text{ or } \frac{F_2}{D_2^2} = \frac{F_1}{D_1^2} \text{ or } \frac{F_2}{(4)^2} = \frac{1000}{(1)^2} \quad \therefore F_2 = 16000 \text{ N}$$

Q.8. The breaking force for a wire is F. What will be the breaking force for (*i*) two parallel wires of the same size (*ii*) a single wire of double thickness?

Ans. (*i*) When the two wires of the same size are suspended in parallel, the breaking force for the parallel combination $= F + F = 2F$.

(*ii*) $Y = \dfrac{FL}{A\,\Delta L}$ or $F = \dfrac{YA\,\Delta L}{L}$ $\therefore F \propto D^2$

If the thickness (diameter) of the wire is doubled, the breaking force will be $4F$.

Q.9. Distinguish between interatomic and intermolecular forces.

Ans. Refer to Art. 18.1.

Q.10. The length of a metallic wire is L_1 when the tension in the wire is T_1 and is L_2 when the tension is T_2. What is the original length of the wire?

Ans. Let L and A be the length and area of cross-section of the metallic wire. Now, $Y = FL/A\Delta L$.

For the first case : $F = T_1$ and $\Delta L = L_1 - L$ $\therefore$ $Y = \dfrac{T_1 L}{A(L_1 - L)}$ ...(*i*)

For the second case : $F = T_2$ and $\Delta L = L_2 - L$ $\therefore$ $Y = \dfrac{T_2 L}{A(L_2 - L)}$ ...(*ii*)

From eqs. (*i*) and (*ii*), $\dfrac{T_1 L}{A(L_1 - L)} = \dfrac{T_2 L}{A(L_2 - L)}$

or $L(T_2 - T_1) = T_2 L_1 - T_1 L_2$ $\therefore$ $L = \dfrac{T_2 L_1 - T_1 L_2}{T_2 - T_1}$

Q.11. Why is the stretching of a coil spring determined by its shear modulus?

Ans. It is because when a coil spring is stretched, there is neither a change in the length of the coil nor a change in its volume. Only change that takes place is the change in shape of the coil spring. Therefore, stretching of a coil spring is determined by its shear modulus.

Q.12. The length of a wire is cut to half. Why this change has no effect on the maximum load the wire can support?

Ans. Breaking force = Breaking stress × Area of cross-section. When the wire is cut to half, there is no change in the area of cross-section of the wire. Therefore, there is no change in the maximum load (*i.e.,* breaking force) the wire can support.

Q.13. A cable is cut to half of its original length. What will be the effect on the increase in its length under a given load?

Ans. $Y = \frac{FL}{A\,\Delta L}$. Now F, A and Y are fixed so that $\Delta L \propto L$. Therefore, change in length (ΔL) is directly proportional to the original length (L). Since the original length is halved, the increase in length will be reduced to half.

Q.14. A wire is replaced by another wire of the same length and material but of twice diameter. How does this affect elongation under a given load?

Ans. $Y = \frac{FL}{A\,\Delta L}$. Now F, L and Y are fixed so that $\Delta L \propto 1/A$. Therefore, elongation (ΔL) is inversely proportional to the area of X-section (A) of the wire. Since the diameter of the wire is doubled, the area of cross-section becomes four times so that extension will be reduced to one-fourth on replacing the wire.

Q.15. A wire of length L and area of X-section A is made of a material of Young's modulus Y. It is stretched by an amount x. What is the work done?

Ans. $Y = \frac{FL}{A\,\Delta L} \quad \therefore F = \frac{YA\,\Delta L}{L}$. Average extension $= \frac{0+x}{2} = \frac{x}{2}$

$\therefore$ Work done = Force × Average extension $= \frac{YA\,\Delta L}{L} \times \frac{x}{2} = \frac{YA\,x}{L} \times \frac{x}{2} = \frac{YA\,x^2}{2L}$.

Q.16. Why do spring balances show wrong readings after they have been used for a long time?

Ans. After repeated use of a spring balance, its spring gets fatigued and there is a loss of elastic strength of the spring. Therefore, for a given load, the extension in the spring is much more than the identical new spring. As a result, the spring balance gives wrong reading.

Q.17. What will happen to the potential energy of the atoms of a solid when it is (*i*) compressed (*ii*) stretched?

Ans. (*i*) The potential energy of the atoms will increase. It is because the distances between the atoms will decrease compared to the normal spacing (minimum potential energy) of the atoms.

(*ii*) Again, the potential energy will increase because now the spacing between the atoms will increase compared to the normal spacing.

Q.18. A wire stretches by a certain amount under a load. If the load and radius are both increased to four times, find the stretch caused in the wire.

Ans. For the first case, $Y = \frac{F/A}{\Delta L/L} = \frac{F}{A} \times \frac{L}{\Delta L}$

For the second case, $Y = \frac{4F}{16\,A} \times \frac{L}{\Delta L'}$

$\therefore \quad \frac{F}{A} \times \frac{L}{\Delta L} = \frac{4F}{16A} \times \frac{L}{\Delta L'} \quad \therefore \Delta L' = \frac{\Delta L}{4}$

Q.19. A metallic wire is suspended by attaching some weight to it. If α is the longitudinal strain and Y is the Young's modulus, find the ratio between elastic potential energy and the energy density.

Ans. Elastic potential energy $= \frac{1}{2}$ × stress × strain × volume

Energy density = Elastic potential energy/volume

$\therefore \quad \frac{\text{Elastic potential energy}}{\text{Energy density}}$ = Volume of wire

Q.20. A heavy wire is suspended from a roof but no weight is attached to its lower end. Is it under stress?

Ans. Yes. It is because the weight of wire is acting as a deforming force.

LONG ANSWER QUESTIONS

1. What do you understand by intermolecular forces? Discuss the characteristics of intermolecular forces. **[Refer to Art. 18.1]**
2. State and explain Hooke's law. Discuss two applications of elasticity. **[Refer to Arts. 18.11 and 18.24]**
3. Explain the terms elastic limit, yield point, breaking point and permanent set from stress-strain curve. **[Refer to Art. 18.14]**
4. Define (*i*) Young's modulus (*ii*) Bulk modulus and (*iii*) Modulus of rigidity. Give one example of each. **[Refer to Arts. 18.13, 18.19 and 18.20]**
5. Derive an expression for the elastic potential energy of a wire when stretched. **[Refer to Art. 18.21]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The forces acting between atoms due to electrostatic interaction between the charges of the atoms are called interatomic forces. The interatomic forces are active if the distance between the two atoms is of the order of atomic size ($\simeq 10^{-10}$m).
2. The forces acting between the molecules due to electrostatic interaction between the charges of the molecules are called intermolecular forces. The intermolecular forces are active if the separation between two molecules is of the order of molecular size ($\simeq 10^{-9}$m).
3. The interatomic and intermolecular forces are electrical in nature. Both forces are attractive upto a certain distance between atoms/molecules and become repulsive when the distance between them becomes less than this value.
4. Interatomic forces are about 50 to 100 times stronger than intermolecular forces.
5. There are four states of matter *viz.* (*i*) solid state (*ii*) liquid state (*iii*) gaseous state and (*iv*) plasma state. In plasma state, the medium is in the form of positive and negative ions.
6. The intermolecular forces are stronger for solids as compared to liquids and gases.
7. The intermolecular forces in liquids are less stronger than solids but more stronger than gases.
8. When the deforming force increases the length of the body, the stress is called **tensile stress.** When the deforming force decreases the length of the body, the stress is called **compressive stress.**
9. A material is said to be elastic if it returns to its original size and shape after the removal of the deforming force.
10. The **elastic limit** is the maximum load which can be applied to a body and it still regains its original size and shape once the load has been removed.
11. Upto elastic limit, stress $\propto$ strain. This is **Hooke's Law.** Therefore, upto elastic limit, the curve between stress and strain is a straight line passing through the origin.
12. If the stress is increased beyond the elastic limit, a point is reached at which there is a marked increase in extension. This is called **yield point** and the material is said to be showing **plastic** behaviour.
13. If between elastic limit and yield point, the load is removed, all the extension is not recoverable.
14. The maximum stress that can be applied to a material without fracture (or breaking) is called **breaking stress**. It is also called **ultimate tensile strength**.

$$\text{Breaking stress} = \frac{\text{Breaking force}}{\text{Area of X-section}}$$

The breaking stress is independent of length of the body.

15. $$\text{Youngs' modulus, } Y = \frac{\text{Normal stress}}{\text{Longitudinal strain}} = \frac{F/A}{\Delta L/L}$$

$$\text{Bulk modulus, } K = \frac{\text{Normal stress}}{\text{Volumetric strain}} = -\frac{F/A}{\Delta V/V}$$

$$= -\frac{FV}{A\Delta V} = \frac{-V\Delta p}{\Delta V} \qquad \left(\because \Delta p = \frac{F}{A}\right)$$

$$\text{Shear modulus, } \cdot = \frac{\text{Tangential stress}}{\text{Tangential strain}}$$

$$= \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} = \frac{F}{A\theta} \qquad \left(\because \frac{\Delta L}{L} = \theta\right)$$

16. Only solids have all three types of moduli. Liquids and gases have only bulk modulus.
17. The bending of a beam involves both tensile and compressive stress — the outer surface is stretched and the inner surface is compressed.
18. Compressibility, $k = \frac{1}{K} = -\frac{\Delta V}{\Delta p V}$

    A material that is easily compressed has a small bulk modulus (K). Obviously, gases have much smaller bulk modulus.
19. A **perfectly plastic material** is one which shows no tendency to return to its original size and shape when the deforming force is removed. In this sense, a perfectly plastic material is the opposite of a perfectly elastic material.
20. Although both pressure and stress are defined as force/area, the following differences are worth noting :
    (*i*) Pressure is always normal to the area whereas stress can be either normal or tangential.
    (*ii*) Pressure on a body is always compressive whereas stress can be either tensile or compressive.
21. The units of pressure, stress and modulus of elasticity are the same *i.e.* N/m^2.
22. The three elastic moduli have the same order of magnitude for any solid material and are applicable to the region where stress $\propto$ strain.
23. Rubber does not obey Hooke's law.
24. The stress-strain curve of an elastic material is slightly different from the corresponding load-extension curve. It is because as the length of the wire increases, its area of cross-section decreases.
25. Solids and liquids are relatively incompressible *i.e* they have small values of compressibility or large values of bulk modulus (K).
26. For most materials, the value of Young's modulus decreases with increasing temperature. In case of rubber, however, Young's modulus increases with increasing temperature.
27. Poisson's ratio, $\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{-\Delta D/D}{\Delta L/L} = -\frac{\Delta D \cdot L}{D\,\Delta L}$

    Since it is a ratio of two similar quantities, it has no units. It is a pure number and hence dimensionless.
28. The work done in stretching a wire is stored as elastic potential energy in the wire. On a molecular level, this corresponds to the increased potential energy of the molecules which results from their increased separation.
29. Elastic potential energy stored per unit volume in a strained body
$$= \frac{1}{2} \times \text{stress} \times \text{strain} = \frac{(\text{stress})^2}{2Y} = \frac{Y \times (\text{strain})^2}{2}$$
30. Elastic potential energy per unit volume is equal to the area under stress-strain curve.
31. Work done in stretching a wire is
$$W = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$
$$= \frac{1}{2} \times \frac{F}{A} \times \frac{l}{L} \times AL = \frac{1}{2} F l = \frac{1}{2} \times \text{Load} \times \text{Extension}$$
32. If there is no change in the volume of a wire on loading, then Poisson's ratio is 0.5 (maximum possible value).

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. If S is the stress and Y is Young's modulus of the material of a wire, then energy stored in the wire per unit volume is **[AIIMS 1997]**

   (a) $\frac{S^2}{2Y}$ (b) $\frac{2Y}{S^2}$

   (c) $\frac{S}{2Y}$ (d) $2S^2Y$

2. The following four wires are made of same material and same tension is applied on them. Which will have maximum increase in length? **[MP-PMT 1998]**

   (a) length = 100 cm ; diameter = 1 mm

   (b) length = 50 cm ; diameter = 0.5 mm

   (c) length = 200 cm ; diameter = 2 mm

   (d) length = 300 cm ; diameter = 3 mm

3. A rod elongates by l when a body of mass M is suspended from it. The work done is **[CPMT 1999]**

   (a) $\frac{1}{2}Mgl$ (b) Mgl

   (c) $2\ Mgl$ (d) zero

4. If in a wire of Young's modulus Y, longitudinal strain X is produced, then the value of potential energy stored in its unit volume will be **[Pb-PMT 1999]**

   (a) $Y X^2$ (b) $2Y X^2$

   (c) $0.5Y^2X$ (d) $0.5Y X^2$

5. A stretched rubber has **[AIIMS 1999]**

   (a) increased kinetic energy

   (b) increased potential energy

   (c) decreased kinetic energy

   (d) decreased potential energy

6. What percent length of a wire will increase by applying a stress of 9.8 N/mm^2 on it? $Y = 1\times10^{11}$ N/m^2. **[MNR 1999]**

   (a) 0.0078% (b) 0.0088%

   (c) 0.0098% (d) 0.0067%

7. An iron rod of length 2 m and cross-sectional area of 50 mm^2 stretches by 0.5 mm when a mass of 250 kg is hung from its lower end. Young's modulus of iron rod is **[Pb PMT 2001]**

   (a) 19.6×10^{10} N/m^2 (b) 19.6×10^{15} N/m^2

   (c) 19.6×10^{18} N/m^2 (d) 19.6×10^{20} N/m^2

8. The breaking stress of a material depends upon **[Pb PMT 1997]**

   (a) material of wire (b) length of wire

   (c) radius of wire

   (d) shape of cross-section

9. Let Y_g and Y_r represent Young's modulus for glass and rubber respectively. It is said that glass is more elastic than rubber. Therefore, it follows that **[AMU PMT 2000]**

   (a) $Y_g = Y_r$ (b) $Y_g < Y_r$

   (c) $Y_g > Y_r$ (d) $Y_g / Y_r = 0$

10. According to Hooke's law of elasticity, if stress is increased, the ratio of stress to strain **[AIIMS 2001]**

    (a) increases (b) decreases

    (c) becomes zero (d) remains constant

11. An elastic material of Young's modulus Y is subjected to stress S. The elastic energy per unit volume is **[MP PMT 1996]**

    (a) $2\ S/Y$ (b) $S/2Y$

    (c) $S^2/2Y$ (d) $SY/2$

12. A wire of length L and radius r is rigidly fixed at one end. On stretching the other end of the wire with a force F, the increase in length is l. If another wire of the same material but of length $2L$ and radius $2r$ is stretched with a force $2F$, the increase in length is **[MP PMT 1996]**

    (a) $l/4$ (b) $l/2$

    (c) l (d) $2l$

13. If stress-strain relation for volumetric change is in the form $\frac{\Delta V}{V_0} = KP$ where P is applied uniform pressure, then K stands for **[AMU PMT 2000]**

    (a) Bulk modulus (b) Young's modulus

    (c) Compressibility (d) Shear modulus

14. The SI unit of Poisson's ratio is **[Pb PMT 1998]**

    (a) Jm^{-1} (b) Nm^{-2}

    (c) Nm^2 (d) unitless

15. The reciprocal of bulk modulus of elasticity is called **[MP PMT 1995]**

    (a) compressibility (b) rigidity

    (c) viscosity

    (d) modulus of elasticity

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*a*)	**2.** (*b*)	**3.** (*a*)	**4.** (*d*)	**5.** (*b*)
6. (*c*)	**7.** (*a*)	**8.** (*a*)	**9.** (*c*)	**10.** (*d*)
11. (*c*)	**12.** (*c*)	**13.** (*c*)	**14.** (*d*)	**15.** (*a*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Energy stored per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain} = \frac{1}{2} \times \text{stress} \times \frac{\text{stress}}{Y} = \frac{1}{2}\frac{S^2}{Y}$$

2. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} = \frac{FL}{\frac{\pi}{4}D^2 \times \Delta L} \quad \therefore \text{Extension, } \Delta L = \frac{4F}{\pi D^2} \times \frac{L}{Y}$$

Since F and Y are constant, $\Delta L \propto L/D^2$. The value of L/D^2 is maximum for option (*b*).

3. Work done $= \frac{1}{2} \times \text{load} \times \text{extension} = \frac{1}{2} \times Mg \times l = \frac{1}{2} Mgl$

4. P.E. stored per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain} = \frac{1}{2} \times (Y \times \text{strain}) \times \text{strain} = \frac{1}{2} \times Y\,(\text{strain})^2 = \frac{1}{2} YX^2$$

5. When rubber is stretched, the work done to do so is stored as elastic potential energy in the rubber. Therefore, potential energy increases.

6. $\text{Strain} = \frac{\Delta L}{L} = \frac{\text{Stress}}{Y} = \frac{9.8 \times 10^6}{1 \times 10^{11}} = 9.8 \times 10^{-5}$

$$\therefore \%\text{age increase in length} = \frac{\Delta L}{L} \times 100 = 9.8 \times 10^{-5} \times 100 = 0.0098\%$$

7. Young's modulus, $Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta L}$

Here $F = mg = 250 \times 9.8$ N ; $L = 2$ m

$A = 50\ \text{mm}^2 = 50 \times 10^{-6}\ \text{m}^2$; $\Delta L = 0.5\ \text{mm} = 0.5 \times 10^{-3}$ m

$$\therefore \quad Y = \frac{(250 \times 9.8) \times 2}{(50 \times 10^{-6}) \times 0.5 \times 10^{-3}} = 19.6 \times 10^{10}\ \text{N/m}^2$$

8. The maximum stress that can be applied to a material without rupturing it is called breaking stress. Its value depends upon the nature of the material of the wire.

9. $\text{Strain} = \frac{\text{Stress}}{Y}$. For a given stress, the greater the Y, the lesser is the strain. Lesser strain means the body will regain its original size more quickly. Hence $Y_g > Y_r$.

10. According to Hooke's law, within elastic limit, the stress set up in a body is directly proportional to strain. Therefore, stress/strain is constant.

11. Energy stored per unit volume $= \frac{1}{2} \times \text{stress} \times \text{strain} = \frac{1}{2} \times S \times \frac{S}{Y} = \frac{1}{2}\frac{S^2}{Y}$

12. $Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$. Since Y is constant, $\frac{FL}{A\,\Delta L} = \text{constant} \quad \therefore \frac{F_1 L_1}{A_1 \Delta L_1} = \frac{F_2 L_2}{A_2 \Delta L_2}$

or $$\Delta L_2 = \frac{F_2 L_2}{A_2} \times \frac{A_1 \Delta L_1}{F_1 L_1} = \left(\frac{F_2}{F_1}\right) \times \left(\frac{L_2}{L_1}\right) \times \left(\frac{A_1}{A_2}\right) \times \Delta L_1 = (2) \times (2) \times \left(\frac{1}{2}\right)^2 \times \Delta L_1$$

$\therefore$ $\Delta L_2 = \Delta L_1 = l$

13. **Compressibility** $= \dfrac{1}{\text{Bulk modulus}} = \dfrac{\Delta V / V_0}{P} = K$

14. Poisson's ratio $= \dfrac{\text{Lateral strain}}{\text{Longitudinal strain}}$

Since it is a ratio of two similar quantities, it has **no unit**. It is a number.

15. **Compressibility** $= \dfrac{1}{\text{Bulk modulus}}$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A force of 1000 N stretches the length of a hanging wire by 1 mm. The force required to stretch a wire of the same material and length but having four times the diameter by 1 mm is **[MP PET 1995]**

(*a*) $\frac{1}{16} \times 10^3$ N (*b*) $\frac{1}{4} \times 10^5$ N

(*c*) 4×10^3 N (*d*) 16×10^3 N

2. Within elastic limit, stress is **[CET Karnataka 1999]**

(*a*) zero

(*b*) independent of strain

(*c*) directly proportional to strain

(*d*) inversely proportional to strain

3. The length of a rod is 20 cm and area of cross-section is 2 cm^2. The Young's modulus of the material of the rod is 1.4×10^{11} N/m^2. If the rod is compressed by 5 kgwt, then increase in the energy of the rod is **[CET Haryana 1999]**

(*a*) 22.5×10^{-4} J (*b*) 9.8×10^{-5} J

(*c*) 45×10^{-5} J (*d*) 8.57×10^{-6} J

4. A cube is subjected to a uniform volume compression. If the side of the cube decreases by 1%, the bulk strain is **[EAMCET 1995]**

(*a*) 0.01 (*b*) 0.02

(*c*) 0.03 (*d*) 0.06

5. Which one of the following quantities does not have the unit of force per unit area? **[CET Haryana 1997]**

(*a*) stress (*b*) strain

(*c*) Young's modulus (*d*) pressure

6. The modulus of elasticity is dimensionally equivalent to **[CET Karnataka 1998]**

(*a*) stress (*b*) strain

(*c*) surface tension (*d*) acceleration

7. A wire is stretched by 0.01 m when it is stretched by a certain force. Another wire of the same material but double the length and double the diameter is stretched by the same force. The elongation is **[EAMCET 1995]**

(*a*) 0.005 m (*b*) 0.01 m

(*c*) 0.02 m (*d*) 0.04 m

8. The spring balance does not read properly after its long use because **[CET Haryana 1999]**

(*a*) its plastic power increases

(*b*) its plastic power decreases

(*c*) the elasticity decreases

(*d*) the elasticity increases

9. A wire of length L and area of cross-section A is made of material of Young's modulus Y. It is stretched by an amount x. The work done is **[MP PET 1996]**

(*a*) $\frac{Y x A}{2L}$ (*b*) $\frac{Y x^2 A}{A}$

(*c*) $\frac{Y x^2 A}{2L}$ (*d*) $\frac{2Y x^2 A}{L}$

10. Two wires A and B of the same material have radii in the ratio 2 : 1 and lengths in the ratio 4 : 1. The ratio of normal force required to produce the same change in length of these two wires is **[CET Haryana 1996]**

(*a*) 1 : 1 (*b*) 2 : 1

(*c*) 1 : 4 (*d*) 1 : 2

11. A rubber ball is taken to a 100 m deep lake and its volume changes by 0.1%. The bulk modulus of rubber is nearly **[CET Karnataka 1998]**

(a) 1×10^6 N/m^2 (b) 1×10^7 N/m^2
(c) 1×10^8 N/m^2 (d) 1×10^9 N/m^2

12. The work done in increasing the length of a 1 m long wire of cross-sectional area 1 mm^2 through 1 mm will be ($Y = 2 \times 10^{11}$ N/m^2) **[CET Rajasthan 1999]**

(a) 0.1 J (b) 5 J
(c) 10 J (d) 250 J

13. Two wires A and B of equal lengths are made of the same material. Wire A has a diameter that is twice that of B. If identical weights are suspended from the ends of these wires, the increase in length is **[MP PET 1995]**

(a) four times for wire A as for wire B
(b) twice for wire A as for wire B
(c) half for wire A as for wire B
(d) one-fourth for wire A as for wire B

14. The bulk modulus of a metal is 8×10^9 N/m^2 and its density is 11 g/cm^3. The density of this metal under a pressure of 20,000 N/cm^2 will be **[MP PET 1990]**

(a) $\frac{440}{39}$ g/cm^3 (b) $\frac{239}{39}$ g/cm^3
(c) $\frac{474}{39}$ g/cm^3 (d) $\frac{512}{39}$ g/cm^3

15. The length of a wire increases by 1% on suspending 2 kg wt from it. The linear strain in the wire is **[CET Haryana 1997]**

(a) 0.01 (b) 0.001
(c) 0.1 (d) 0.0001

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (d)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (b)
6. (a)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (a)
11. (d)	**12.** (a)	**13.** (d)	**14.** (a)	**15.** (a)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. $Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$. Since L, ΔL and Y are constant, $\therefore \frac{F}{A} = \text{constant}$

$\therefore \frac{F_2}{A_2} = \frac{F_1}{A_1}$ or $\frac{F_2}{D_2^2} = \frac{F_1}{D_1^2}$

$\therefore F_2 = F_1 \times \frac{D_2^2}{D_1^2} = 10^3 \times (4)^2 = \mathbf{16 \times 10^3\ N}$

2. According to Hooke's law, within elastic limit, stress $\propto$ strain or stress/strain is constant.

3. $Y = \frac{F/A}{l/L} = \frac{FL}{Al}$ $\therefore$ Change in length, $l = \frac{FL}{AY}$

$$\text{Increase in energy} = \frac{1}{2} \times \text{force} \times l = \frac{1}{2} \times (5 \times 9.8) \times \frac{FL}{AY}$$

$$= \frac{1}{2} \times (5 \times 9.8) \times \frac{(5 \times 9.8) \times 20 \times 10^{-2}}{2 \times 10^{-4} \times 1.4 \times 10^{11}} = \mathbf{8.57 \times 10^{-6}\ J}$$

4. Suppose l is the initial length of each side of the cube.

$$\text{Final length, } l' = l\left(1 - \frac{1}{100}\right) = \frac{99l}{100}\text{; Initial volume, } V = l^3\text{;}$$

$$\text{Final volume} = \left(\frac{99l}{100}\right)^3$$

$$\text{Change in volume, } \Delta V = l^3 - \left(\frac{99}{100}\right)^3 l^3 = l^3\left[1 - \left(\frac{99}{100}\right)^3\right] = \frac{3l^3}{100}$$

$$\therefore \quad \text{Bulk strain} = \frac{\Delta V}{V} = \frac{3l^3}{100} \times \frac{1}{l^3} = \mathbf{0.03}$$

5. Strain is a dimensionless and unitless quantity.

6. Modulus of elasticity $= \dfrac{\text{Stress}}{\text{Strain}}$

Since strain is a dimensionless quanity, modulus of elasticity is dimensionally equivalent to stress.

7. $Y = \dfrac{F/A}{\Delta L/L} = \dfrac{FL}{A\,\Delta L}$. Since Y and F are constant, $\dfrac{L}{A\,\Delta L} = \text{constant}$.

$$\therefore \quad \frac{L_1}{A_1\,\Delta L_1} = \frac{L_2}{A_2\,\Delta L_2} \quad \text{or} \quad \Delta L_2 = \frac{L_2}{A_2} \times \frac{A_1 \Delta L_1}{L_1} \quad \text{or} \quad \Delta L_2 = \Delta L_1 \times \left(\frac{L_2}{L_1}\right) \times \left(\frac{A_1}{A_2}\right)$$

$$\text{or} \quad \Delta L_2 = \Delta L_1 \times \left(\frac{L_2}{L_1}\right) \times \left(\frac{D_1}{D_2}\right)^2 = 0.01 \times (2) \times \left(\frac{1}{2}\right)^2 = \frac{0.01}{2} = \mathbf{0.005\ m}$$

8. Due to continuous use, the original length of the spring is increased and, therefore, the elasticity is decreased.

9. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta L} \quad \therefore \ F = \frac{YA\,\Delta L}{L}$$

$$\text{Average extension} = \frac{0 + x}{2} = \frac{x}{2}$$

$$\text{W.D.} = \text{Force} \times \text{Average extension} = \frac{YA\,\Delta L}{L} \times \frac{x}{2} = \frac{YAx}{L} \times \frac{x}{2} = \frac{YAx^2}{2L}$$

10. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{\Delta L}$$

Since Y and ΔL are constant, $\dfrac{FL}{A}$ = constant.

$$\therefore \quad \frac{F_A L_A}{A_A} = \frac{F_B L_B}{A_B} \quad \text{or} \quad \frac{F_A}{F_B} = \left(\frac{L_B}{L_A}\right) \times \left(\frac{A_A}{A_B}\right) = \left(\frac{1}{4}\right) \times (2)^2 = 1$$

$$\therefore \quad F_A : F_B = 1 : 1$$

11. $\Delta p = h\rho g = 100 \times 10^3 \times 9.8\ \text{N/m}^2$; $\Delta V/V = 0.1/100$

$$\therefore \quad K = \frac{\Delta p}{\Delta V/V} = \frac{100 \times 10^3 \times 9.8 \times 100}{0.1} = 9.8 \times 10^8\ \text{N/m}^2 \simeq 10^9\ \text{N/m}^2$$

12. $$\text{Work done} = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume} = \frac{1}{2} \times Y \times (\text{strain})^2 \times \text{volume}$$

$$= \frac{1}{2} Y \times \left(\frac{l}{L}\right)^2 \times AL = \frac{1}{2} \times Y \times \frac{l^2}{L} \times A$$

$$= \frac{1}{2} \times \left(2 \times 10^{11}\right) \times \frac{(1 \times 10^{-3})^2}{1} \times 1 \times 10^{-6} = \mathbf{0.1\ J}$$

13. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta L}$$

Since Y, F and L are constant $= \dfrac{1}{A\,\Delta L}$ = constant.

$$\therefore \quad \frac{1}{A_A\,\Delta L_A} = \frac{1}{A_B\,\Delta L_B} \quad \text{or} \quad \frac{\Delta L_A}{\Delta L_B} = \frac{A_B}{A_A} = \left(\frac{D_B}{D_A}\right)^2$$

or $\frac{\Delta L_A}{\Delta L_B} = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad \therefore \ \Delta L_A : \Delta L_B = 1 : 4$

14. Bulk modulus, $K = \frac{\Delta p V}{\Delta V}$ $\therefore$ Change in volume, $\Delta V = \frac{\Delta p V}{K} = \frac{2\times10^8 \times V}{8\times10^9} = \frac{V}{40}$

New volume of metal, $V' = V - \Delta V = V - \frac{V}{40} = \frac{39V}{40}$

Mass of metal $= V' \times \rho' = \frac{39V}{40}\rho'$

or $11 \times V = \frac{39V}{40}\rho' \ \therefore \ \rho' = \frac{40\times11}{39} = \frac{440}{39}$ g/cm^3

15. Strain $= \frac{\Delta L}{L} = 1\% = \frac{1}{100} = 0.01$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **By how much a rubber string of length 10 m increases in length under its own weight when suspended vertically? Density of rubber = 1.5×10^3 kg/m^3; $Y = 5 \times 10^8$ N/m^2; $g = 10$ ms^{-2}.**

Hint. Let A be the area of cross-section of the string in SI units.

Mass of string, m = Volume × Density = $(A \times 10) \times 1.5 \times 10^3 = 1.5 \times 10^4 A$ kg

Stress in the string $= \frac{mg}{A} = \frac{1.5\times10^4 \times A \times 10}{A} = 1.5\times10^5$ N/m^2

The weight of the string acts at its centre of gravity. Therefore, it will produce extension only in 5 m length of the wire.

$$\text{Strain} = \frac{\Delta L}{L} = \frac{\Delta L}{5}$$

$$Y = \frac{\text{Stress}}{\text{Strain}} \quad \text{or} \quad 5\times10^8 = \frac{1.5\times10^5}{\Delta L/5}$$

$$\therefore \quad \Delta L = \frac{1.5\times10^5\times5}{5\times10^8} = 1.5\times10^{-3} \text{ m} = 1.5 \text{ mm}$$

2. **In a large lecture hall, a pendulum is to be made by suspending a 40 kg ball from the end of a steel wire 15 m long.**
   (*i*) What cross-sectional area should the wire have if the applied stress in it is to be only 10% of its breaking stress?
   (*ii*) How far will the ball stretch the wire?
   Tensile strength of steel is 0.48×10^9 N/m^2 and $Y = 200 \times 10^9$ N/m^2.

Hint. (*i*) The breaking stress of a material is its tensile strength.

Stress to be applied = 10% of $0.48 \times 10^9 = 0.48 \times 10^8$ N/m^2

$$\text{Stress} = \frac{F}{A} = \frac{mg}{A} = \frac{40\times9.8}{A} = \frac{392}{A}$$

$\therefore$ Area of cross-section, $A = \frac{392}{\text{stress}} = \frac{392}{0.48\times10^8} = 8.1 \times 10^{-6}$ m^2

(*ii*) $$Y = \frac{F/A}{\Delta L/L}$$

$$\therefore \quad \frac{\Delta L}{L} = \frac{F/A}{Y} = \frac{0.48\times10^{8}}{200\times10^{9}} = 2.4\times10^{-4}$$

or $$\Delta L = (2.4 \times 10^{-4}) \times 15 = 3.6 \times 10^{-3}\ \text{m} = \mathbf{3.6\ mm}$$

3. A steel wire 8 m long and 4 mm in diameter is fixed to two rigid supports. Calculate the increase in tension when the temperature falls by 10°C. Given $\alpha = 12 \times 10^{-6}$/°C ; $Y = 2 \times 10^{11}$ N/m^2.

Hint. Increase in length, $\Delta L = L\alpha t$. Now, strain $= \frac{\Delta L}{L} = \frac{L\alpha t}{L} = \alpha t$

Stress $= Y \times \text{strain} = Y\alpha t$

$\therefore$ Increase in tension = Stress × Area of cross-section

$$= (Y\alpha t)\times\pi r^2$$

$$= (2\times10^{11}\times12\times10^{-6}\times10)\times\pi\times(2\times10^{-3})^2 = \mathbf{301.7\ N}$$

4. A uniform rod of 2 sq mm cross-section is heated from 0°C to 25°C. Find the force which must be exerted to prevent it from expanding. Given $\alpha = 2 \times 10^{-6}$/ °C ; $Y = 2 \times 10^{11}$ N/m^2.

Hint. Stress $= Y\alpha t = 2 \times 10^{11} \times 2 \times 10^{-6} \times 25 = 10^7$ N/m^2

Force required = Stress × Area of cross-section = $10^7 \times 2 \times 10^{-6} = \mathbf{20\ N}$

5. Caculate the pressure required to stop the increase in volume of copper block when it is heated from 50°C to 70°C. Coefficient of linear expansion of copper = 8×10^{-6} per °C and bulk modulus of elasticity = 3.6×10^{11} N/m^2.

Hint. If V is the original volume of the block and ΔV is the increase in volume when heated from temperature t_1 to t_2, then,

$$\Delta V = V\times\gamma\times(t_2 - t_1)$$

Here γ = coefficient of volume expansion = 3α

$\therefore$ Volume strain $= \frac{\Delta V}{V} = \gamma(t_2 - t_1)$

Bulk modulus, $K = \frac{\text{Change in pressure}}{\text{Volume strain}} = \frac{p}{\gamma(t_2 - t_1)}$

or $$p = K\gamma(t_2 - t_1) = (3.6\times10^{11})\times(3\times8\times10^{-6})\times20 = \mathbf{1.728\times10^{8}\ N/m^2}$$

6. A spherical ball contracts in volume by 0.1% when subjected to a uniform normal pressure of 100 atmospheres. Calculate the bulk modulus of the material of the ball. Take 1 atmosphere =10^5 N/ m^2.

Hint. Volume strain $= \Delta V/V = 0.1\% = 0.1/100 = 10^{-3}$

Normal stress = 100 atmospheres = $100 \times 10^5 = 10^7$ N/m^2

$\therefore$ Bulk modulus of the material of the ball is

$$K = \frac{\text{Normal stress}}{\text{Volume strain}} = \frac{10^7}{10^{-3}} = \mathbf{10^{10}\ N/m^2}$$

7. A rod *AD* consisting of three segments *AB*, *BC* and *CD* is hanging vertically from a fixed support *A*. The lengths of the segments are respectively 0.1 m, 0.2 m and 0.15 m. The cross-section of the rod is uniformly 10^{-4} m^2. A weight of 10 kg is hung from *D*. Calculate the displacements of points *B*, *C* and *D*. Neglect the weight of the rod. Given $Y_{AB} = 2.5 \times 10^{10}$ N/m^2 ; $Y_{BC} = 4 \times 10^{10}$ N/m^2 ; $Y_{CD} = 10^{10}$ N/m^2.

[Roorkee Engg.]

Hint. Figure 18.17 shows the conditions of the problem. Here *ABCD* is the initial position of the rod. When a weight 10 kg is applied at the end *D*, the rod takes the position *AB'C'D'*.

$$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta L} = \frac{mgL}{\pi r^2\,\Delta L}$$

$\therefore$ Increase in length, $\Delta L = \dfrac{mgL}{\pi r^2 Y}$

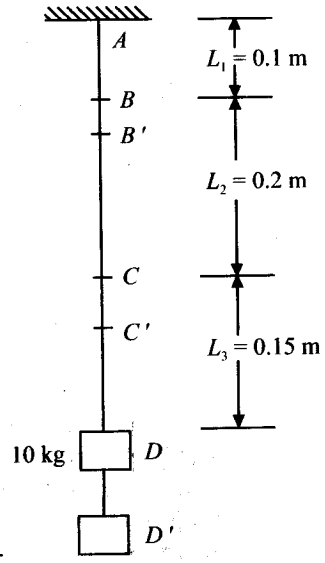


Fig. 18.17

For rod *AB*

Increase in length, $\Delta L_1 = \dfrac{mgL_{AB}}{\pi r^2 Y_{AB}}$

$$= \frac{10\times 9.8\times 0.1}{10^{-4}\times 2.5\times 10^{10}} = 3.92\times 10^{-6}\text{ m}$$

Therefore, displacement of point *B* is **3.92×10^{-6} m**

For rod *BC*

Increase in length, $\Delta L_2 = \dfrac{mgL_{BC}}{\pi r^2 Y_{BC}}$

$$= \frac{10\times 9.8\times 0.2}{10^{-4}\times 4\times 10^{10}} = 4.9\times 10^{-6}\text{ m}$$

Therefore, displacement of point $C = \Delta L_1 + \Delta L_2 = (3.92 + 4.9)\ 10^{-6}$ m = **8.82×10^{-6} m**

For rod *CD*

Increase in length, $\Delta L_3 = \dfrac{mgL_{CD}}{\pi r^2 Y_{CD}} = \dfrac{10\times 9.8\times 0.15}{10^{-4}\times 10^{10}} = 14.7\times 10^{-6}$ m

Therefore, displacement of point $D = \Delta L_1 + \Delta L_2 + \Delta L_3$ = **23.52×10^{-6} m**

8. **A copper wire of negligible mass, 1 m long and cross-sectional area 10^{-6} m^2 is kept on a smooth horizontal table with one end fixed. A ball of mass 1 kg is attached to the other end. The wire and the ball are rotating with an angular velocity of 20 rad/s. If the elongation of the wire is 10^{-3} m, find Young's modulus of wire. If on increasing the angular velocity to 100 rad/s, the wire breaks down, find the breaking stress.** **[Roorkee Engg.]**

Hint. Stretching force on the wire is given by ;

$$F = \frac{mv^2}{r} = mr\omega^2 = 1\times 1\times (20)^2 = 400\text{ N}$$

$$\text{Stress in wire} = \frac{F}{A} = \frac{400}{10^{-6}} = 4\times 10^8\text{ N/m}^2$$

$$\text{Strain in wire} = \frac{\Delta L}{L} = \frac{10^{-3}}{1} = 10^{-3}$$

$\therefore$ $$\text{Young's Modulus} = \frac{\text{Stress}}{\text{Strain}} = \frac{4\times 10^8}{10^{-3}} = \mathbf{4\times 10^{11}\ N/m^2}$$

$$\text{Breaking stress} = \frac{mr\omega_B^{\,2}}{A} = \frac{1\times 1\times (100)^2}{10^{-6}} = \mathbf{10^{10}\ N/m^2}$$

Note that here ω_B (= 100 rad/s) is the breaking angular velocity.

9. **A stone of 0.5 kg mass is attached to one end of a 0.8 m long aluminium wire of 0.7 mm diameter and suspended vertically. The stone is now rotated in a horizontal plane at a rate such that the wire makes an angle of 85° with the vertical. Find the increase in length of the wire. The Young's modulus of aluminium = 7×10^{10} N/m^2.** **[Roorkee Engg.]**

Hint. Figure 18.18 shows the conditions of the problem. Let the stone rotate with a speed v in a horizontal circular path of radius r. Then,

$$r = l \sin\theta \quad \text{where } l = \text{length of wire}$$

The various forces acting on the stone are shown. It is clear that :

$$T\cos\theta = mg \quad \text{and} \quad T\sin\theta = \frac{mv^2}{r}$$

$$\therefore \quad T = \frac{mg}{\cos\theta} = \frac{0.5\times 9.8}{\cos 85^\circ} = 56.19 \text{ N}$$

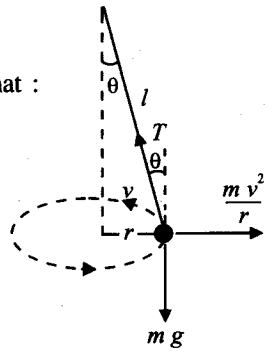


Fig. 18.18

Now $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} = \frac{TL}{A\Delta L}$$

$\therefore$ Extension in wire is given by ;

$$\Delta L = \frac{TL}{YA} = \frac{56.19\times 0.8}{(7\times 10^{10})\times\pi(0.35\times 10^{-3})^2} = 1.67\times 10^{-3} \text{ m} = 1.67 \text{ mm}$$

10. The rubber cord of catapult has a cross-sectional area of 1 mm^2 and total unstretched length 10 cm. It is stretched to 12 cm and then released to project a missile of mass 5 g. If Young's modulus Y for rubber is 5 $\times 10^8$ N/m^2, calculate the velocity of projection. Assume that total elastic energy of catapult is converted into kinetic energy without any heat loss.

Hint. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} \quad \therefore \quad F = \frac{YA\Delta L}{L}$$

Here $\Delta L = 12 - 10 = 2$ cm $= 2\times 10^{-2}$ m ; $L = 10\times 10^{-2}$ m ; $A = 1\times 10^{-6}$ m^2

$$\therefore \quad \text{Load, } F = \frac{(5\times 10^8)\times(1\times 10^{-6})\times(2\times 10^{-2})}{10\times 10^{-2}} = 100 \text{ N}$$

If v is the velocity of projection in SI units, we have,

Elastic energy of catapult = K.E. of missile

$$\frac{1}{2}\times \text{load} \times \text{extension} = \frac{1}{2}mv^2 \quad \text{or} \quad \frac{1}{2}\times 100\times 2\times 10^{-2} = \frac{1}{2}\times(5\times 10^{-3})\times v^2$$

$$\therefore \quad v^2 = \frac{100\times 2\times 10^{-2}}{5\times 10^{-3}} = 400 \quad \text{or} \quad v = \sqrt{400} = 20 \text{ ms}^{-1}$$

11. A thin rod of negligible mass and of area of cross-section 4 $\times 10^{-6}$ m^2, suspended vertically from one end, has a length of 0.5 m at 100°C. The rod is cooled to 0°C but prevented from contracting by attaching a mass at the lower end. Find (*i*) the mass (*ii*) the energy stored in the rod. Given Young's modulus = 10^{11} N/m^2, coefficient of expansion = 10^{-5} K^{-1}; g = 10 N/ kg. [I.I.T.]

Hint. Here $l_{100} = 0.5$ m ; $\alpha = 10^{-5}$ K^{-1} ; t = fall in temperature = 100 K

(*i*) Contraction of rod due to cooling is

$$l = l_{100} \times \alpha \times t$$

$$\therefore \text{ Strain developed, } \frac{l}{l_{100}} = \alpha\times t = 10^{-5}\times 100 = 10^{-3}$$

$$\text{Stress developed} = Y \times \text{strain} = 10^{11} \times 10^{-3} = 10^8 \text{ N/m}^2$$

$$\text{Tension in wire, } T = \text{Stress} \times \text{cross-sectional area} = 10^8 \times (4\times 10^{-6}) = 400 \text{ N}$$

$$\therefore \quad \text{Mass to be attached} = \frac{T}{g} = \frac{400}{10} = 40 \text{ kg}$$

(*ii*) Energy stored in the rod is $U = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$

$$= \frac{1}{2} \times (10^8) \times (10^{-3}) \times (4 \times 10^{-6} \times 0.5) = 0.1 \text{ J}$$

12. Two rods of different materials but of equal cross-sections and lengths (1.0 m each) are joined to make a rod of length 2.0 m. The metal of one rod has coefficient of linear thermal expansion 10^{-5}/°C and Young's modulus 3×10^{10} N/m^2. The other metal has the values 2×10^{-5} /°C and 10^{10} N/m^2 respectively. How much pressure must be applied to the ends of the composite rod to prevent its expansion when the temperature is raised by 100°C? What will be the separate lengths of the two rods at the new temperature? [Roorkee Engg.]

Hint. Figure 18.19 shows the conditions of the problem. The length of each rod is $L = 1.0$ m. Due to temperature increase t (= 100°C) the rods will expand. Let ΔL_1 and ΔL_2 be the increase in the lengths of rods 1 and 2 due to rise in temperature. Then,

$$\Delta L_1 = \alpha_1 Lt \quad \text{and} \quad \Delta L_2 = \alpha_2 Lt$$

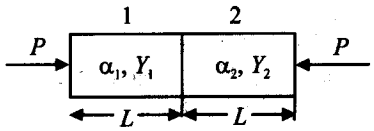


Fig. 18.19

Total increase in the length of the composite rod

$$= \Delta L_1 + \Delta L_2 = (\alpha_1 + \alpha_2)Lt \quad \ldots (i)$$

Suppose P is the pressure applied at the ends of the rod to prevent this increase in length. Let $\Delta L_1'$ and $\Delta L_2'$ be the contractions in lengths of rods 1 and 2 due to pressure P. Then,

$$Y = \frac{\text{Stress (pressure)}}{\text{Strain}} = \frac{P}{\Delta L/L} = \frac{PL}{\Delta L}$$

$$\therefore \quad \Delta L_1' = \frac{PL}{Y_1} \quad \text{and} \quad \Delta L_2' = \frac{PL}{Y_2}$$

$$\text{Total contraction} = \Delta L_1' + \Delta L_2' = \frac{PL}{Y_1} + \frac{PL}{Y_2} = PL\left(\frac{1}{Y_1} + \frac{1}{Y_2}\right)$$

$$= PL\left(\frac{Y_1 + Y_2}{Y_1 Y_2}\right) \quad \ldots (ii)$$

As per conditions of the problem, eq. (*i*) = eq. (*ii*). Therefore,

$$(\alpha_1 + \alpha_2)Lt = PL\left(\frac{Y_1 + Y_2}{Y_1 Y_2}\right)$$

or

$$P = \frac{Y_1 Y_2 (\alpha_1 + \alpha_2)t}{Y_1 + Y_2}$$

$$= \frac{3 \times 10^{10} \times 10^{10} \times (1+2) \times 10^{-5} \times 100}{3 \times 10^{10} + 10^{10}} = 2.25 \times 10^7 \text{ N/m}^2$$

Let the separate lengths of rods 1 and 2 be L_1 and L_2 at the new temperature respectively. Then,

$$L_1 = L + \Delta L_1 - \Delta L_1' = L + \alpha_1 Lt - PL/Y_1$$

$$= L[1 + \alpha_1 t - P/Y_1] = 1.0\left[1 + 10^{-5} \times 100 - \frac{2.25 \times 10^7}{3 \times 10^{10}}\right] = 1.00025 \text{ m}$$

Similarly, $L_2 = L + \Delta L_2 - \Delta L_2' = 0.99975$ m

19

Hydrostatics (Fluids at Rest)

INTRODUCTION

The three states of matter are solid, liquid and gas. A solid maintains a fixed shape and a fixed volume even if a large external force is applied to it. The liquids and gases do not have their own shape and take the shape of the containing vessel. Due to this reason, the liquids and gases have the ability to flow under the action of external force. Hence liquids and gases are called fluids. *The branch of physics which deals with the study of the properties of fluids at rest is called* **hydrostatics**. In this chapter, we shall study the various aspects of hydrostatics including **pressure** and **surface tension**.

19.1. IDEAL FLUID

A fluid which is incompressible is called an **ideal fluid**.

This means that if a force is applied on an ideal fluid, there will be no change in its volume. Liquids (*e.g.,* water) are nearly ideal fluids. However, gases (*e.g.,* air) are not ideal fluids because they can be easily compressed. *In this chapter, we shall deal with ideal fluids (i.e., liquids) unless stated otherwise.*

19.2. LIQUID AT REST

In a liquid at rest, the forces exerted by the liquid are always perpendicular to the surfaces in contact with the liquid. The surface in contact with the liquid may be a solid surface (*e.g.,* bottom or walls of the containing vessel, a body immersed in the liquid) or a liquid surface.

(*i*) **Liquid in contact with solid surface.** Consider a liquid at rest in a vessel as shown in Fig. 19.1. Then force exerted by the liquid at each point on the bottom or walls is perpendicular to the surface at that point. To prove this fact, consider a point O at the bottom of the vessel. Suppose for a moment that force F exerted by the liquid at point O is not perpendicular to the

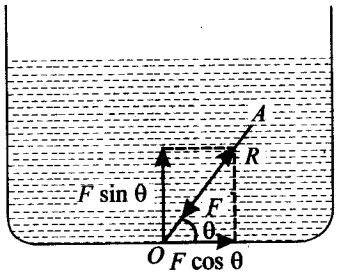


Fig. 19.1

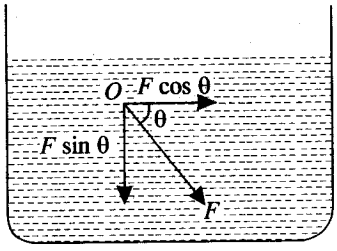


Fig. 19.2

bottom but makes an angle θ with the horizontal as shown in Fig. 19.1. According to Newton's third law, the surface exerts an equal and opposite force $R = F$. The reaction R (= F) has horizontal

and vertical components $F\cos\theta$ and $F\sin\theta$ respectively. The horizontal component $F\cos\theta$ would cause the liquid to flow which contradicts our original assumption that the liquid is at rest. Therefore, $F\cos\theta$ must be zero *i.e.*, $\theta = 90°$. Similarly, the force exerted by the liquid is perpendicular to any point of the surface of the container in contact with it.

Therefore, a liquid at rest always exerts a normal force on the solid surface in contact with it.

(*ii*) **Liquid in contact with liquid surface.** Consider a liquid at rest in a vessel as shown in Fig. 19.2. Consider a point O inside the liquid. The force at point O must be perpendicular to the surface in contact with O. Suppose this were not true and force F at point O acts making an angle θ with the horizontal. The horizontal and vertical components of the force are $F\cos\theta$ and $F\sin\theta$ respectively. The horizontal component $F\cos\theta$ would cause the liquid to flow which contradicts our original assumption that the liquid is at rest. Therefore, $F\cos\theta$ must be zero *i.e.*, $\theta = 90°$.

Thus the forces acting on a liquid at rest are always perpendicular to the surface of the liquid.

19.3. THRUST OF THE LIQUID

We have seen that a liquid at rest always exerts a force perpendicular to the surface of contact. The surface may be the bottom or walls of the containing vessel or a body immersed in the liquid.

The total perpendicular force exerted by a liquid at rest on a surface in contact with it is called **thrust** *of the liquid on that surface.*

Since thrust is a force, its SI unit is newton (N). The dimensional formula of thrust is $[MLT^{-2}]$.

19.4. PRESSURE OF THE LIQUID

While dealing with fluids, pressure exerted by the fluid is an important physical quantity. It is defined as under.

The normal force (thrust) exerted by liquid at rest on a unit area of the surface in contact with it is called **pressure of the liquid** *on the surface.*

If F is the normal force exerted by a liquid on a surface of area A, then pressure of the liquid on the surface is

$$\text{Pressure, } P = \frac{F}{A}$$

The SI unit of force is N and that of area is m^2. Therefore, SI unit of pressure is N/m^2. Recall that SI unit of stress is also N/m^2. When we describe pressure, we sometimes give special name to N/m^2—the **pascal** (Pa) *i.e.*, 1 N/m^2 = 1 Pa. The dimensional formula for pressure is $[ML^{-1}T^{-2}]$.

Discussion. The following points may be noted :

(*i*) Although force is a vector quantity, **pressure is a scalar quantity**. In a liquid at rest, pressure at a given point is exerted equally in all directions. Therefore, it makes no sense to associate direction with pressure.

(*ii*) The concept of pressure is particularly useful in dealing with fluids *i.e.*, liquids and gases. It is because fluids often lack definite shape and volume and it is convenient to use the quantities pressure and density rather than force and mass when studying hydrostatics and hydrodynamics.

19.5. PRACTICAL APPLICATIONS OF PRESSURE

We know that pressure P (= F/A) is the force acting normally on a unit area of the surface. The concept of pressure finds wide practical applications.

(*i*) **Nails and pins have pointed ends.** Because of pointed ends, nails and pins have very small area. A small force on the head of a pin or nail will exert considerable pressure ($P = F/A$) to drive it into the surface easily.

(*ii*) **Wide wooden or concrete sleepers are placed below the railway tracks.** Because of this arrangement, the area over which the weight of the train acts is increased. As a result, the pressure on the ground is reduced, providing safety to railway tracks.

(*iii*) **Buildings have wide foundations.** Because of wide foundation, the area over which the weight of the building acts is increased. This reduces the pressure exerted by the building on the ground.

(*iv*) **It is painful to walk barefooted on road covered with pebbles having sharp edges.** The sharp-edged pebbles have very small area. Therefore, the weight of the body exerts considerable pressure on the pebbles through the feet. The pebbles exert equal reaction on the feet. For this reason, it is painful to walk barefooted on pebbles.

19.6. PRESSURE EXERTED BY A LIQUID COLUMN

Consider a cylinder of cross-sectional area A filled with a liquid of uniform density ρ to a height h (See Fig. 19.3). A normal force acts on the bottom of the cylinder due to the weight of the liquid inside. Therefore, pressure P exerted by the liquid on the bottom of the cylinder is

$$P = \frac{\text{Force}}{\text{Area of bottom}} = \frac{F}{A}$$

If m is the mass of the liquid in the cylinder, then, $F = mg$.

$$\therefore \quad P = \frac{mg}{A}$$

Now m = Volume × density = $(Ah) \times \rho = A\,h\rho$

$$\therefore \quad P = \frac{(Ah\rho)g}{A} = h\rho g$$

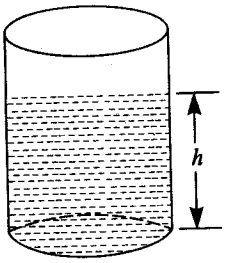


Fig. 19.3

Thus pressure due to a liquid (*at rest*) of density ρ at a depth h below the free surface is $h\,\rho\,g$.

Discussion. The following points may be noted :

(*i*) The pressure exerted by a liquid at rest is independent of the size and shape of the container ($\because P = h\rho g$).

(*ii*) All the points at the same depth have the same pressure ($\because P \propto h$; ρ and g are constant). As depth increases, pressure increases and *vice-versa.*

Explanation of fluid pressure. The pressure of a liquid (or fluid) contained in a container can be explained on the basis of molecular theory of matter. A liquid is composed of molecules which are in a state of continuous random motion. They suffer collisions with each other and with the walls of the container. If we consider any surface placed within the liquid, quite a large number of molecules strike the surface and thereby impart momentum to the surface. The momentum transferred per second to the surface gives the average force or thrust on the surface. The liquid pressure is the thrust per unit area of the surface.

19.7. PASCAL'S LAW

Consider a liquid contained in a vessel as shown in Fig. 19.4. The pressure difference between points A and B is

$$P_A - P_B = h\,\rho\,g$$

Suppose we increase the pressure on the surface of the liquid (*i.e.,* at point B) by any means. Now $P_A - P_B$ must remain the same because h, ρ and g are unchanged. Thus if pressure at point B increases by ΔP, then pressure at point A must increase by ΔP so that $P_A - P_B$ remains constant. This is true for all points in the liquid because ΔP is independent of h. This fact was recognised by Pascal and is contained in the statement known as Pascal's law. It is stated below:

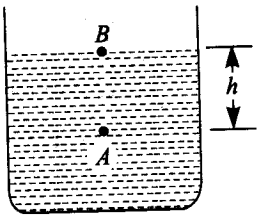


Fig. 19.4

An increase in pressure at one point of the enclosed liquid at rest is transmitted equally to every part of the liquid and to the walls of the container, provided the effect of gravity is neglected.

Thus if the pressure at the pumping station of a city water system is increased by 10^4 N/m^2, then the pressure at all the homes connected by water lines to the pumping station also increases by 10^4 N/m^2. The pressure varies in different homes, depending on their elevation relative to the pumping station. But an increase in pressure at the station causes an equal increase throughout the water system.

Experimental proof. In order to prove Pascal's law experimentally, consider a spherical vessel having four cylindrical tubes A, B, C and D fitted with water-tight frictionless pistons as shown in Fig. 19.5. Suppose the area of cross-section of A, B, C and D are a, $a/2$, $2a$ and $3a$ respectively. Let the vessel be completely filled with water and pistons be kept in position. If a force F is applied on piston A, the pressure developed on the liquid is $P = F/a$. It will be found that all other pistons (B, C and D) are pushed outwards. In order to keep the piston B, C and D in their original positions, forces $F/2$, $2F$ and $3F$ respectively have to be applied on the pistons. Therefore, pressures on the pistons are :

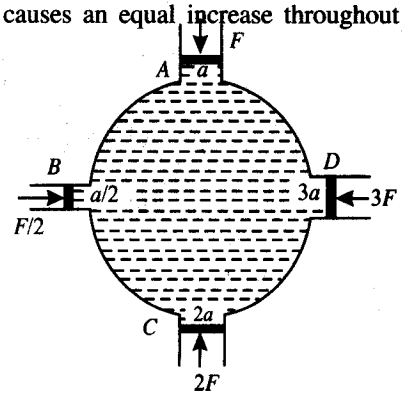


Fig. 19.5

$$\text{On } B = \frac{F/2}{a/2} = \frac{F}{a} = P \; ; \quad \text{On } C = \frac{2F}{2a} = \frac{F}{a} = P \; ; \quad \text{On } D = \frac{3F}{3a} = \frac{F}{a} = P$$

This indicates that pressure P (= F/a) applied on piston A is transmitted equally to all parts of the liquid. This experiment proves Pascal's law.

19.8. HYDRAULIC LIFT

A practical application of Pascal's law is the hydraulic lift which is used to lift a heavy load with a small force. Fig. 19.6 shows the principle of a hydraulic lift. It consists of two vertical cylinders A and B of different cross-sectional areas A_1 and A_2 ($A_2 > A_1$) connected by a horizontal tube C. Pistons are fitted in both the cylinders. The system is filled with water or oil. The load to be lifted is placed on the cylinder of larger cross-sectional area (A_2) and lifting force is applied at the smaller cylinder of smaller area of cross-section (A_1).

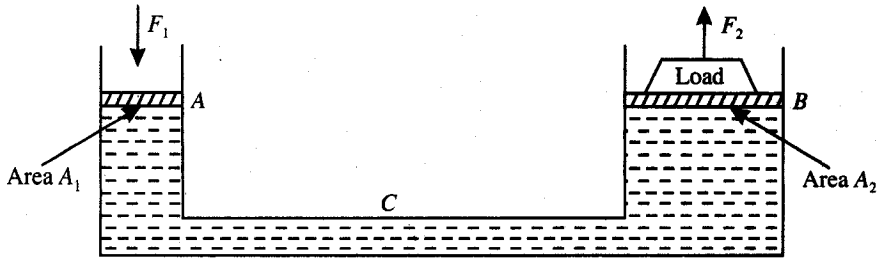


Fig. 19.6

Working. When downward force F_1 is applied to the smaller piston A, the pressure of liquid under this piston increases to P (= F_1/A_1). According to Pascal's law, this increased pressure P is transmitted to every part of the liquid including piston B. In other words, pressure P (= F_1/A_1) is exerted on piston B. Therefore, upward force F_2 on piston B is

$$\text{Upward force on piston } B,\ F_2 = P \times A_2 = \frac{F_1}{A_1} \times A_2$$

or $$F_2 = F_1 \times \frac{A_2}{A_1}$$

Since $A_2 > A_1$, $F_2 > F_1$. Therefore, with a small force (F_1), we can lift a heavy load (F_2).

19.9. HYDRAULIC BRAKING SYSTEM

Figure 19.7 shows a simplified diagram of an automobile braking system. It is based on the same principle as that of hydraulic lift. The system includes a master cylinder containing a piston

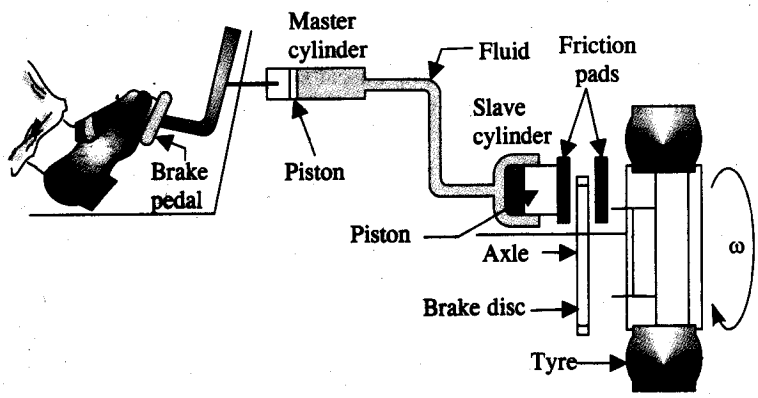


Fig. 19.7

actuated by the brake pedal and 'slave' cylinders at each wheel. The cylinders are connected by metal and rubber houses and the whole system is filled with oil. When you step on the brake pedal, you force fluid out of the master cylinder. As a result, the pistons in the slave cylinders move outward to keep the volume of the oil constant. These slave pistons push friction pads against the rotating discs, thus dissipating the kinetic energy of the car. The master cylinder has a much smaller area of cross-section (A_1) than that of slave cylinders (A_2). Applying a force F_1 to the master cylinder causes an increase of pressure $P = F_1/A_1$. This pressure increase is transmitted undiminished (Pascal's principle) throughout the system. The force F_2 on each slave cylinder is then,

$$F_2 = PA_2 = \frac{F_1}{A_1} \times A_2 = \left(\frac{A_2}{A_1}\right) \times F_1$$

Since A_2 is greater than A_1, F_2 is also greater than F_1. Thus the hydraulic system amplifies the relatively weak force of your foot into the large force that ultimately stops the rotating discs and the wheels attached to it.

19.10. HYDROSTATIC PARADOX

Fig. 19.8 shows a vessel consisting of four sections (*A*, *B*, *C* and *D*) of different shapes but with its base horizontal. If a liquid is poured into the vessel, it will be seen that liquid rises to the same height in all the sections. It is because the pressure within a liquid is directly proportional ($P = h\rho g$) to its depth below the free surface. Since the base of the vessel is horizontal, the pressure of the liquid at all points on the base must be the same. Therefore, the liquid will rise to the same level regardless of the shape of the sections.

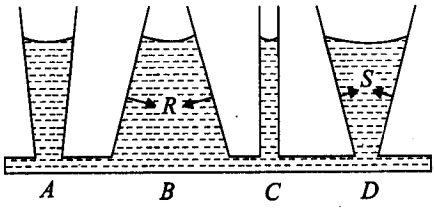


Fig. 19.8

At first glance, there appears to be a **paradox** here. The weight of the liquid is obviously different in each section and since liquid pressure depends upon the weight of the liquid above a reference point, it would seem that pressure should be different at the base of sections. No doubt the cross-

sectional area of B at the base is greater than that at D but the force at the base of B is the sum of the weight of water above the base together with the downward component of reaction R of the sides of the section B. However, the force on the base of D is the weight of water above the base *minus* the upward component of the reactions S of the walls of section D. Therefore, the value of pressure at the base of section B is the same as that on the base of section D. This is also true for other sections.

19.11. EFFECT OF GRAVITY ON LIQUID PRESSURE

Consider a liquid of density ρ contained in a vessel as shown in Fig. 19.9 (*i*). Let X and Y be two points in the liquid separated by a vertical distance h. It is desired to find the pressure difference between points X and Y. For this purpose, consider an imagninary cylinder of liquid of arbitrary cross-sectional area A such that points X and Y lie at the upper and lower faces of the cylinder. Suppose P_1 and P_2 are the values of liquid pressure at points X and Y respectively. The free body diagram of the cylinder of liquid is shown in Fig. 19.9 (*ii*).

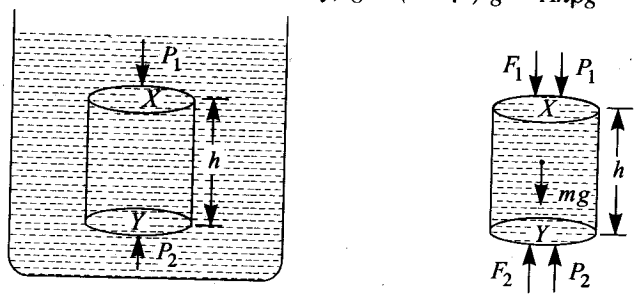
Referring to Fig. 19.9 (*ii*), the various vertical forces acting are :

(*i*) The downward force $F_1 = P_1 A$ acting on the upper face.

(*ii*) The upward force $F_2 = P_2 A$ acting on the lower face.

(*iii*) The weight mg of the liquid in the considered cylinder of height h. This force acts downward. Now volume of this cylinder is $A\ h$.

$$\therefore \qquad mg = (\text{Volume} \times \text{density})\ g = (A\ h\rho)\ g = Ah\rho g$$



(*i*) (*ii*)

Fig. 19.9

Since the liquid is not moving and, therefore, is in equilibrium. Consequently, these three vertically directed forces must add to zero *i.e.*,

$$(F_1 + m\ g) - F_2 = 0$$

or $$P_1 A + m\ g = P_2 A$$

or $$P_1 A + A\ h\rho\ g = P_2 A$$

or $$P_1 + h\rho\ g = P_2$$

$$\therefore \qquad P_2 - P_1 = h\rho\ g \qquad ...(i)$$

When gravity is present. Eq. (*i*) gives pressure difference between points X and Y when gravity is present. Note that pressure difference exists between two points located vertically one above the other because of gravity.

When gravity is absent. If gravity is absent or neglected, then eq. (*i*) becomes :

$$P_2 - P_1 = 0 \quad \text{or} \quad P_2 = P_1$$

Therefore, in the absence of gravity, no pressure difference would exist between different points in the liquid.

Example 19.1. **A large can crusher is shown in Fig. 19.10. The large piston has an area of 8 m^2 and exerts a force F_2 of magnitude 2×10^6 N on the cans. Calculate the magnitude of the force F_1 exerted by the small piston (area 10 cm^2) on the fluid. Do not ignore the fact that the large piston is 1 m higher than the small piston. The fluid used is water.**

Solution. The pressure P_2 in the fluid at the top of the crusher is

$$P_2 = \frac{F_2}{A_2} = \frac{2\times10^6}{8} = 2.5\times10^5 \text{ N/m}^2$$

The pressure P_1 on the fluid at the small piston is

$$P_1 = P_2 + h\rho g \text{ (Here } h = 1 \text{ m)}$$

$$= 2.5\times10^5 + 1\times10^3\times9.8 = 2.6\times10^5 \text{ N/m}^2$$

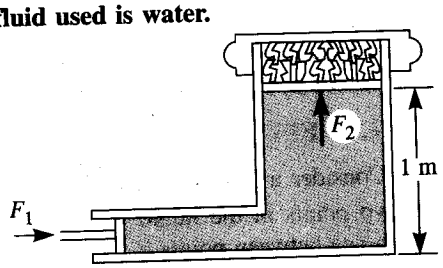


Fig. 19.10

Since the small piston has area $A_1 = 10 \text{ cm}^2 = 10^{-3} \text{ m}^2$, the magnitude of force F_1 on this piston is

$$F_1 = P_1 A_1 = (2.6 \times 10^5) \times 10^{-3} = 260 \text{ N}$$

Example 19.2. **Two pistons of a hydraulic lift have diameters of 30 cm and 2.5 cm. What is the force exerted by the larger piston when 50 kg wt is placed on the smaller piston? If the stroke of the smaller piston is 4 cm, through what distance will the larger piston move after 10 strokes?**

Solution. Area of smaller piston, $A_1 = \pi\left(\frac{2.5}{2}\right)^2$; Area of larger piston, $A_2 = \pi\left(\frac{30}{2}\right)^2$

Now $$\frac{F_1}{A_1} = \frac{F_2}{A_2}$$

$\therefore$ $$F_2 = F_1\times\frac{A_2}{A_1} = 50\times\frac{\pi(30/2)^2}{\pi(2.5/2)^2} = 7200 \text{ kg wt}$$

In one stroke, input work done = output work done

or $$F_1\times l_1 = F_2\times l_2$$

$\therefore$ $$l_2 = \frac{F_1 l_1}{F_2} = \frac{50\times4}{7200} = 0.028\text{cm}$$

Distance through which larger piston moves in 10 strokes

$$= 10 l_2 = 10 \times 0.028 = 0.28 \text{ cm}$$

Example 19.3. **A rectangular tank is 10 m long, 5 m broad and 3 m high. It is filled to brim with water of density 10^3 kgm^{-3}. Calculate the thrust at the bottom and walls of the tank due to hydrostatic pressure. Take $g = 9.8$ ms^{-2}.**

Solution. Pressure at the bottom of the tank is

$$P_{bottom} = h\rho g = 3 \times 10^3 \times 9.8 = 29.4 \times 10^3 \text{ N/m}^2$$

Area of bottom, A_{bottom} = Length × Breadth = 10 × 5 = 50 m^2

$\therefore$ Thrust on the bottom = $P_{bottom} \times A_{bottom} = 29.4 \times 10^3 \times 50 = 1.47 \times 10^6$ N

Average pressure on the walls is

$$P_{walls} = \frac{0 + h\rho g}{2} = \frac{0 + 29.4\times10^3}{2} = 14.7\times10^3 \text{ N/m}^2$$

Area of broad walls = 2 × length × height = 2 × 10 × 3 = 60 m^2

Area of narrow walls = 2 × breadth × height = 2 × 5 × 3 = 30 m^2

Total area of walls, $A_{walls} = 60 + 30 = 90 \text{ m}^2$

Thrust on walls = $P_{walls} \times A_{walls} = 14.7 \times 10^3 \times 90 = 1.323 \times 10^6$ N

PROBLEMS FOR PRACTICE

1. What will be the length of mercury column in a barometer tube, when the atmospheric pressure is 75 cm of mercury and the tube is inclined at an angle of 60° with the horizontal direction ? **[86.6 cm]**
2. To lift an automobile of 2000 kg, a hydraulic lift with a larger piston 900 cm^2 in area is used. Calculate the force that must be applied to the smaller piston of area 10 cm^2 to accomplish this task. **[217.8 N]**

19.12. IMPORTANT PROPERTIES OF LIQUIDS AT REST

The following are the important properties of liquids (fluids) at rest :

(*i*) *A liquid at rest always exerts a normal force (i.e., perpendicular force) on the surface in contact with it.* This is explained in Art. 19.2.

(*ii*) *In a liquid at rest, the net force on any volume element of the liquid must be zero.* This follows from Newton's second law. If there were any force on any part of the liquid, it would cause that part to flow. This contradicts our assumption that liquid is at rest. In other words, *at any point in a liquid, pressure is the same in all directions.*

(*iii*) *At any point at a depth h below the surface of a liquid of density* ρ, *the pressure due to the weight of the liquid is* $P = \rho gh$. This is proved in Art. 19.6.

(*iv*) *If an external force causes an increase in pressure at any point in an enclosed liquid, every point in the liquid experiences the same pressure change.* This is known as Pascal's law and is discussed in Art. 19.7.

19.13. ABSOLUTE AND GAUGE PRESSURE

Consider a container of liquid open at the top as shown in Fig. 19.11. The free surface at the top experiences the pressure P_a of the atmosphere pressing down at the surface. Pascal's principle states that every point in the container experiences this added pressure. The total pressure at a point A at a depth h below the free surface is

$$P = P_a + h\rho g$$

Thus the absolute or actual pressure at point A is $P_a + h\rho g$.

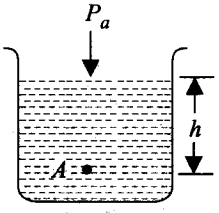


Fig. 19.11

Generally, when we use a gauge to measure pressure in a container, we are doing so with P_a surrounding us and the gauge. What the gauge measures is the *difference* between the pressure P in the container and the atmospheric pressure P_a. *The difference between the absolute pressure and the atmospheric pressure (i.e., $P - P_a$) is called the* **gauge pressure P_G.**

$$\therefore \quad P_G = P - P_a = h\rho g$$

$$\therefore \quad \text{Absolute pressure, } P = P_G + P_a \text{ where } P_G = h\rho g$$

Example 19.4. A household hot water heating system has an expansion tank 12 m above the boiler. If the tank is open to the atmosphere, calculate the absolute and gauge pressure in the boiler. Given that $P_a = 1.01 \times 10^5$ Pa.

Solution. Absolute pressure P in the boiler is

$$P = P_a + h\rho g$$

Here $P_a = 1.01 \times 10^5$ Pa ; $h = 12$ m ; $\rho = 1000$ kg m^{-3} ; $g = 9.8$ ms^{-2}

$$\therefore \quad P = 1.01 \times 10^5 + 12 \times 1000 \times 9.8 = \mathbf{2.19 \times 10^5 \text{ Pa}}$$

$$\text{Gauge pressure, } P_G = P - P_a = 2.19 \times 10^5 - 1.01 \times 10^5 = \mathbf{1.18 \times 10^5 \text{ Pa}}$$

If there is a pressure gauge on the furnace, it is always calibrated to read gauge rather than absolute pressure.

Example 19.5. The interior of a submarine located at a depth of 50 m in sea water is maintained at sea level atmospheric pressure. Find the force acting on a window 20 cm square. The density of sea water is 1.03×10^3 kg m^{-3}.

Solution. Pressure outside the submarine is

$$P = P_a + h\rho g$$

Pressure inside the submarine $= P_a$

$\therefore$ Net pressure on window, $P' = P - P_a = h\rho g$

or $P' = 50 \times 1.03 \times 10^3 \times 9.8 = 5.05 \times 10^5$ Pa

Area of window, $A = 20 \times 20 = 400$ cm^2 $= 0.04$ m^2

$\therefore$ Force on window, $F = P'A = 5.05 \times 10^5 \times 0.04 =$ **2.02×10^4 N**

Example 19.6. A liquid stands at the same level in the *U*-tube when at rest. If *A* is the area of cross-section of the tube and *g* is the acceleration due to gravity, what will be the difference in height of the liquid in the two limbs when the system is given an acceleration *'a'* towards the right as shown in Fig. 19.12.

Solution. Fig. 19.12 shows the conditions of the problem. Let l be the length of the horizontal portion CD of the tube.

Mass of liquid in the portion CD

= Volume × density

$= (Al) \times \rho = A\,l\rho$

Force on the above mass towards left

$= A\,l\rho \times a$

Due to difference h in height of the liquid, the downward force exerted on the liquid in the horizontal portion $CD = h\rho g \times A$.

$\therefore$ $h\rho g \times A = A\,l\rho \times a$

or $h = \dfrac{al}{g}$

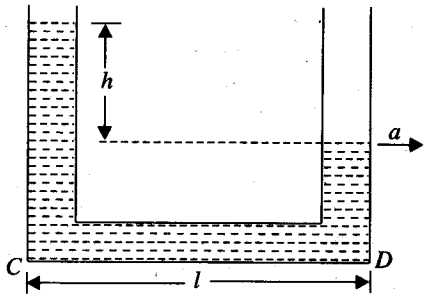


Fig. 19.12

Example 19.7. The liquids shown in Fig. 19.13 in the two arms are mercury (specific gravity = 13.6) and water. If the difference of heights of the mercury columns is 2 cm, find the height *h* of the water column.

Solution. Figure 19.13 shows the conditions of the problem. Let the atmospheric pressure be P_a in SI units.

Pressure at A, $P_A = P_a + h\rho g = P_a + h \times 1000 \times g$

Pressure at B, $P_B = P_a + h'\rho' g = P_a + 0.02 \times 13600 \times g$

Since A and B are at the same level, the pressures are equal at A and B.

$\therefore$ $P_a + h \times 1000 \times g = P_a + 0.02 \times 13600 \times g$

or $h = \dfrac{0.02 \times 13600}{1000} = 0.27$ m $=$ **27 cm**

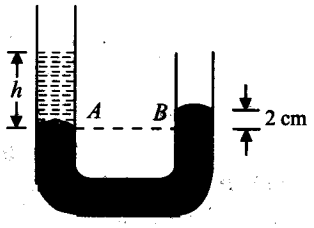


Fig. 19.13

PROBLEMS FOR PRACTICE

1. A diver descends to a depth of 50 m in sea water which has a density of 1025 kg m^{-3}. What is the pressure (gauge) on his body? **[5.02 × 10^5 N/m^2]**
2. A rectangular tank is 3 m long, 2 m wide and 1.5 m in height. It contains water to a depth of 1 m. The density of water is 1000 kg/m^3. Find (*i*) the pressure at the bottom of the tank (*ii*) force on the longer vertical wall and (*iii*) force on the shorter vertical wall. **[(*i*) 9800 Pa (*ii*) 14700 N (*iii*) 9800 N]**
3. Convert pressure head of 6 m of water to pressure head of an oil of density 750 kg/m^3. The density of water is 1000 kg/m^3. **[8 m]**
4. What is the pressure due to the water at the bottom of a 12 m deep lake? **[1.18 × 10^5 Pa]**
5. A glass tube is bent into a *U* shape. Water is poured into the tube until it stands 12 cm high on each side. Kerosene (ρ = 870 kg/m^3) is added slowly to one side until water on the other side rises 5 cm. What is the length of kerosene column? **[11.5 cm]**

19.14. ATMOSPHERE AND ATMOSPHERIC PRESSURE

(*i*) **Atmosphere.** *The gaseous envelope around the earth is called* **earth's atmosphere.**

The earth's atmosphere extends to a height of about 400 km above the surface of earth. The earth's atmosphere is mainly composed of nitrogen (78%) and oxygen (21%). It also contains minute quantities of carbon dioxide, neon, water vapours, dust particles etc.

(*ii*) **Atmospheric pressure.** *The pressure exerted by atmosphere is called* **atmospheric pressure.**

Obviously, the atmospheric pressure is the weight of air over 1 m^2 surface ($P_a = F/A$). The density of air decreases as we go up from the surface of earth. Therefore, atmospheric pressure P_a (= $h\ \rho\ g$) is maximum on earth's surface and goes on decreasing as we go up from the earth. Since the density of air is not constant, we cannot use the formula $P = h\rho g$ to calculate atmospheric pressure accurately. However, we can measure it with a device called *barometer.* The measured value of atmospheric pressure on earth is

$$P_a = 1.013 \times 10^5 \text{ N/m}^2 \text{ or Pa}$$

This means that weight of air on 1 m^2 surface of earth is about 10^5 N. The atmospheric pressure varies with altitude, temperature and weather conditions.

19.15. MERCURY BAROMETER

A mercury barometer is an instrument used to measure the atmospheric pressure. A long tube (about 1 m) closed at one end is filled with mercury and then inverted into a trough of mercury (See Fig. 19.14). It is seen that the mercury level in the tube falls till the height h becomes nearly 76 cm above the mercury level in the trough. The upper end of the closed tube is nearly a vacuum and so its pressure can be taken as zero. We say that *atmospheric pressure is 76 cm of mercury.*

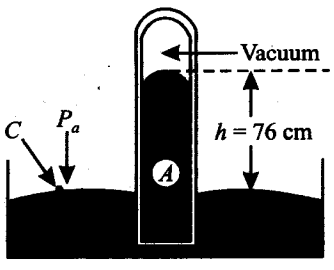


Fig. 19.14

Theory. To understand how simple barometer measures atmospheric pressure, refer to Fig. 19.14. At point *C* on the surface of mercury in the trough, the pressure is the atmospheric pressure P_a. According to Pascal's principle, this pressure is transmitted uniformly through the mercury in the trough. Therefore, at point *A* inside the tube, at the level of point *C*, the pressure is the same as at the point *C* (*i.e.,* atmospheric pressure). Now the pressure at point *A* inside the tube is due only to the height h of mercury in the tube and is $h\rho g$ where ρ is the density of mercury.

$\therefore$ Atmospheric pressure, $P_a = h\rho g$

Now at normal temperature and pressure at sea level, h = 76 cm = 0.76 m. Also density of mercury is $\rho = 13.6 \times 10^3$ kg/m^3.

$\therefore$ Standard atmospheric pressure is

$$P_a = (0.76) \times (13.6 \times 10^3) \times 9.8 = 1.01 \times 10^5 \text{ N/m}^2 \text{ or Pa}$$

Discussion. The following points may be noted :

(*i*) The atmospheric pressure is directly proportional (ρ and g being constant) to the height h of the mercury column. Therefore, atmospheric pressure can be conveniently expressed in terms of the height of mercury column. We say that standard atmospheric pressure is 76 cm of mercury. It means that the weight of 76 cm column of mercury is equal to the weight of a column of the atmosphere of the same cross-section, extending to the top of earth's atmosphere.

(*ii*) The vertical height of the mercury column remains constant even when the tube is tilted, unless the top of the tube is less than 76 cm above the level in the trough — in which case the mercury completely fills the tube.

19.16. HEIGHT OF ATMOSPHERE

When molecules of air or of any gas collide with a solid or liquid surface, they exert force against that surface. The force is of impulsive nature, much like the force of a tennis ball hitting the practice board. At sea level, collisions of air molecules with a solid surface cause an average force of 1.01×10^5 N to be exerted on each 1 m^2 of the surface. Thus atmospheric pressure at sea level is 1.01×10^5 N/m^2. Let us now calculate the height of air column that would cause a pressure of 1.01×10^5 N/m^2 on the surface of earth. Suppose the required height is h metres above earth's surface. Although the density of air goes on decreasing as we go up, we can assume it to be constant at 1.293 kg/m^3 throughout. Then our estimate of h is

$$h\rho g = 1.01 \times 10^5$$

or $$h \times 1.293 \times 9.8 = 1.01 \times 10^5$$

$\therefore$ $$h = \frac{1.01 \times 10^5}{1.293 \times 9.8} \simeq 8000 \text{ m} = 8 \text{ km}$$

Units. There are two common units of pressure worth mentioning at this point. One, called the **torr** is named after the inventor of the barometer, the Italian physicist Torricelli (1608 – 1647). The other, the **bar**, is commonly used in the science of meterology. These units have the values :

$$1 \text{ torr} = 1 \text{ mm of Hg} \; ; \; 1 \text{ bar} = 10^5 \text{ Pa (exactly)}$$

19.17. MANOMETER

A manometer is a device that is used to measure the pressure of a gas. Although it has many variations, a manometer is basically a *U*-shaped tube partly filled with a liquid, often mercury. Fig. 19.15 shows the simple or open type manometer. One limb of the manometer is open to the atmosphere and the other is connected to the chamber C of which the gas pressure is to be measured.

(*i*) When the tube is open on both ends, air pushes down on each column with equal pressure and the liquid stands at equal levels in the two arms of the tube.

(*ii*) When one end of the tube is connected to the chamber C whose gas pressure P is to be measured, the levels of the liquid will be different in the two arms of the tube. If P is greater than the atmospheric pressure P_a, then level of liquid in the open arm will be at a height h above the level in the other arm as shown in Fig. 19.15. The pressure at point D is atmospheric pressure P_a since this arm is open above D. The pressure in the

liquid of manometer at B is $P_a + h\ \rho g$ where ρ is the density of the liquid in the manometer. The pressure at point A is the same as the pressure at point B since they are at the same level. According to Pascal's principle, pressure at A is the same as at C. Therefore, pressure P in the chamber is

$$P = P_a + h\ \rho\ g$$

The total pressure P is called the absolute pressure while $P - P_a\ (= h\ \rho\ g)$ is called the gauge pressure. Note that this manometer measures the gauge pressure of the gas in the chamber C.

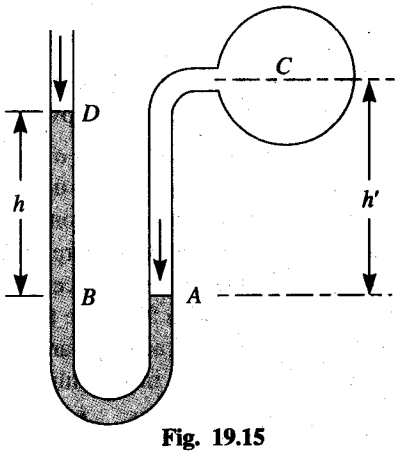


Fig. 19.15

Note. For small pressure differences, accuracy can be obtained by using a liquid less dense than mercury. The difference in heights is increased by a factor of 13600/ρ where ρ is the SI density of the liquid used instead of mercury. Notice if water were used in a barometer to measure P_a, the water column would be about 76 × 13600/1000 = 1034 cm, nearly the height of a three-storey building.

Example 19.8. Calculate (*i*) absolute pressure (*ii*) gauge pressure of the gas in the bulb shown in Fig. 19.16. Atmospheric pressure = 1.01×10^5 Pa, density of mercury = 13.6×10^3 kg m^{-3}, g = 9.8 m s^{-2}.

Solution. Pressure at A = Atmospheric pressure, $P_a = 1.01 \times 10^5$ Pa

Pressure at $B = P_a + h\rho g$

$= 1.01 \times 10^5 + 0.120 \times 13.6 \times 10^3 \times 9.8$

$= 1.01 \times 10^5 + 0.16 \times 10^5 = 1.17 \times 10^5$ Pa

(*i*) Since C is at the same level as B, the absolute pressure P of the gas in the bulb is

$$P = 1.17 \times 10^5 \text{ Pa}$$

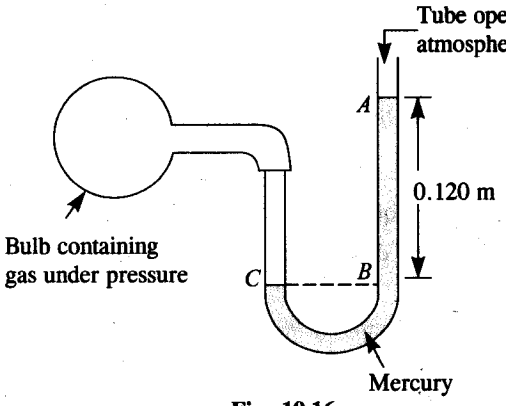


Fig. 19.16

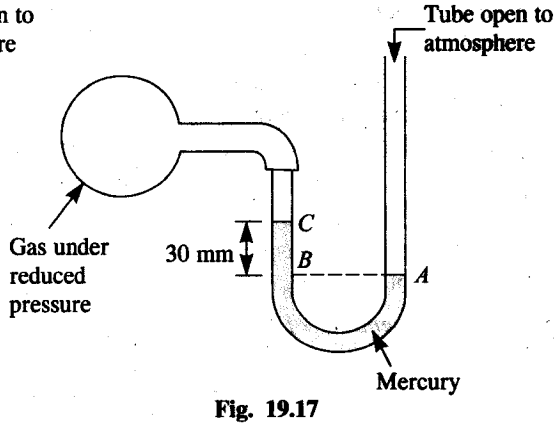


Fig. 19.17

(*ii*) Gauge pressure of the gas in the bulb is

$$P_G = P - P_a = h\ \rho\ g = 0.16 \times 10^5 \text{ Pa}$$

Example 19.9. Calculate the pressure of the gas in the bulb shown in Fig. 19.17. Atmospheric pressure is 760 mm of Hg.

Solution. Pressure at A = 760 mm

$\therefore$ Pressure at B = 760 mm

$\therefore$ Pressure at C = 760 − 30 = 730 mm of Hg

i.e., Pressure of the gas = **730 mm of Hg**

19.18. BUOYANT FORCE

When a body is partially or wholly immersed in a fluid, it displaces a certain amount of fluid. The displaced fluid exerts an upward force on the body, called buoyant force (F_B).

The upward force acting on a body partially or wholly immersed in a fluid is called **buoyant force** *(F_B).* It is also known as **upward thrust** or **buoyancy.**

The buoyant force arises from the fact that pressure in a fluid increases with the increase in depth. Therefore, upward pressure on the bottom surface of the immersed body is greater than the downward pressure on its top face. The resultant force is then directed up. Thus we find that when a body is immersed in a fluid, the force of gravity ($W = mg$) acts downward while the buoyant force F_B exerted by the fluid acts upward. The relative magnitudes of W and F_B will decide whether the body will sink or float in the fluid.

19.19. ARCHIMEDES' PRINCIPLE

Archimedes' principle may be stated as under :

When a body is partially or wholly immersed in a liquid, the buoyant force acting on the body is equal to the weight of the liquid displaced by the body i.e.,

$$\text{Buoyant force, } F_B = \text{Weight of liquid displaced by the body}$$
$$= \text{Mass of liquid displaced by the body} \times g$$
$$= (\rho_l V_l) \times g$$
$$\therefore \quad F_B = \rho_l g V_l$$

where ρ_l = Density of the liquid
g = Acceleration due to gravity
V_l = Volume of liquid displaced by the body

If the body is completely immersed in the liquid, then V_l = volume V of the body. If the body is partially immersed in the liquid, then V_l is equal the volume of the body below the surface of the liquid.

Discussion. The following points may be noted :

(*i*) The apparent weight of an immersed body is equal to the difference between the weight W of the body in air and the buoyant force F_B *i.e.,*

Apparent weight of body = $W - F_B$

Therefore, apparent weight of a body is less in the liquid than it would be in air.

(*ii*) Note that *volume of the immersed part of the body* determines the magnitude of the buoyant force F_B.

(*iii*) *The magnitude of buoyant force (F_B) depends on the density of the liquid (ρ_l) and volume (V_l) of immersed part of the body and not on the composition or shape of the body.*

19.20. PROOF OF ARCHIMEDES' PRINCIPLE

Consider a metallic cylinder *completely* immersed in a liquid as shown in Fig. 19.18.

Let h = Height of the cylinder
A = Area of top and bottom ends of cylinder
ρ_l = Density of the liquid.
h_1 = Depth of top end of cylinder below the liquid surface
h_2 = Depth of bottom end of cylinder from liquid surface
V = Volume of cylinder = $Ah = V_l$

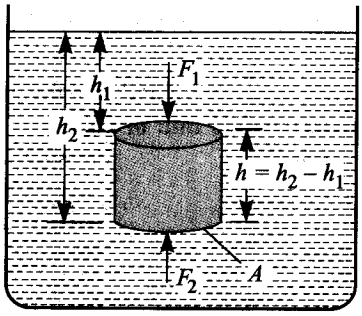


Fig. 19.18

The liquid exerts a pressure $P_1 = \rho_l g h_1$ at the top surface of the cylinder and downward force due to this pressure on the top of the cylinder is

$$F_1 = P_1 A = h_1 \rho_l g A \quad \text{...vertically downward}$$

Similarly, the liquid exerts an upward force F_2 on the bottom of the cylinder. The magnitude of this force is given by ;

$$F_2 = P_2 A = (h_2 \rho_l g) A = h_2 \rho_l g A \quad \text{...vertically upward}$$

The net force on the cylinder due to the liquid pressure acts upward. This is the buoyant force F_B and its magnitude is given by ;

$$\text{Buoyant force, } F_B = F_2 - F_1 = h_2 \rho_l g A - h_1 \rho_l g A$$

$$= \rho_l g A (h_2 - h_1) = \rho_l g A h \quad (\because h = h_2 - h_1)$$

or $$F_B = \rho_l g A h = \rho_l g V_l \quad (\because Ah = V = V_l)$$

Now $\rho_l V_l = m$, the mass of the liquid displaced.

$\therefore$ $$F_B = mg = \text{Weight of liquid displaced by immersed body}$$

Thus, Archimedes' principle stands proved. Note that the result clearly does not depend upon the fact that we have considered a cylinder. It is valid no matter what the shape an object has. Archimedes' principle can be verified by experiment.

19.21. PRINCIPLE OF FLOATATION

A body wholly or partially submerged in a liquid experiences two vertical forces *viz.*, (*i*) the actual weight W of the body acting vertically downward and (*ii*) vertically upward buoyant force F_B exerted by the liquid on the body. The relative magnitudes of W and F_B will decide whether the body will sink or float in the liquid. Consider a body *wholly* immersed in a liquid.

Let M = Mass of the immersed body

ρ = Density of the body

ρ_l = Density of the liquid

$\therefore$ Volume of body, $V = \dfrac{M}{\rho}$

Actual weight of body, $W = Mg$

Buoyant force, $F_B = \rho_l g V$ ...body fully immersed

$$= \rho_l g \left(\frac{M}{\rho}\right) = Mg \frac{\rho_l}{\rho}$$

$\therefore$ $$F_B = W \frac{\rho_l}{\rho} \quad (\because W = Mg)$$

Apparent wt. in liquid, $W_a = W - F_B = W - W \dfrac{\rho_l}{\rho} = W\left(1 - \dfrac{\rho_l}{\rho}\right)$

$\therefore$ $$W_a = W - F_B = W\left(1 - \frac{\rho_l}{\rho}\right) \quad \text{...(i)}$$

Let us discuss the three possibilities.

(*i*) When $W > F_B$. Clearly, the downward force (W) is greater than the upward force (F_B). Therefore, the **body will sink** to the bottom of the liquid. Let us see when it happens.

Now $W > F_B$ or $W > W \dfrac{\rho_l}{\rho}$

$\therefore$ $\rho > \rho_l$

Therefore, if the density of the body is more than that of the liquid in which it is immersed, the body will sink to the bottom of the liquid.

(*ii*) When $F_B > W$. Clearly, the upward force (F_B) is greater than the downward force (W). Therefore, the **body will float**. Let us see when it happens.

Now $F_B > W$ or $W\frac{\rho_l}{\rho} > W$

$\therefore$ $\rho < \rho_l$

Therefore, if the density of the body is less than that of the liquid in which it is immersed, the body will float.

In this case, the body will rise above the surface of the liquid to such an extent that the weight of the liquid displaced by the immersed part of the body (*i.e.*, F_B) becomes equal to the weight of the body (*i.e.*, W).

(*iii*) When $W = F_B$. In this case, the resultant force on the body is zero. Therefore, the **body floats** within the liquid at any depth below the free surface of the liquid. Let us see when it happens

Now $W = F_B$ or $W = \frac{\rho_l}{\rho}W$

$\therefore$ $\rho = \rho_l$

Therefore, if the density of the body is equal to that of the liquid in which it is immersed, the body will float within the liquid.

Law of Floatation. We have seen above that a body will float in a liquid if $F_B = W$ or $F_B > W$. Therefore, law of floatation may be stated as under :

A body will float in a liquid if the buoyant force (F_B) acting on the body is equal to or greater than the weight (W) of the body i.e., $F_B \geq W$.

or

A body will float in a liquid if the density of the body (ρ) is less than or equal to the density of the liquid (ρ_l) i.e., $\rho \leq \rho_l$.

19.22. FRACTION OF VOLUME SUBMERGED OF A FLOATING BODY

Consider a body floating in a liquid.

Let V = Total volume of the body

V' = Volume of body submerged in liquid

ρ = Density of the body

ρ_l = Density of liquid

Mass of liquid displaced by body = $V'\rho_l$

Weight of liquid displaced by body = $V'\rho_l g$

$\therefore$ Buoyant force, $F_B = V'\rho_l g$

Mass of body = Volume × density = $V\rho$

Weight of body = $V\rho g$

According to principle of floatation, F_B = Weight of body

or $V'\rho_l g = V\rho g$

$\therefore$ $\frac{V'}{V} = \frac{\rho}{\rho_l}$ ...(*i*)

Volume of body outside the liquid $= V - V' = V - \frac{\rho}{\rho_l}V$

$= V\left(1 - \frac{\rho}{\rho_l}\right)$ ...(*ii*)

Note. From eq. (*i*), $\frac{V'}{V} = \frac{\rho}{\rho_l}$. If a regular body is floating in water with 3/4th part immersed in water, then density of the body is 0.75 g/cm^3.

Example 19.10. A piece of metal weighs 45 N in air and 28.3 N when fully submerged in water. Determine the specific gravity of the metal.

Solution. The apparent weight W_a of the submerged object is equal to the actual weight W minus the buoyant force F_B *i.e.*,

$$W_a = W - F_B = \rho g V - \rho_w g V$$

where V is the volume of the object, ρ its density and ρ_w density of water.

Now $$W - W_a = F_B = \rho_w g V$$

$$\therefore \quad \frac{W}{W - W_a} = \frac{\rho g V}{\rho_w g V} = \frac{\rho}{\rho_w}$$

But ρ/ρ_w is the specific gravity of the metal.

$$\therefore \quad \text{Specific gravity of metal} = \frac{W}{W - W_a} = \frac{45}{45 - 28.3} = 2.69$$

Example 19.11. An object floats on water with 20% of its volume above the water line. What is the density of the object? Density of water = 1000 kg m^{-3}.

Solution. Suppose the volume of the entire object is V. Then the volume of the object under water is 0.8 V.

$$\text{Buoyant force, } F_B = 0.8\ V \times \rho_w \times g$$

If ρ is the density of the object, then,

$$\text{Weight of object, } W = \rho\ V\ g$$

According to principle of floatation, $F_B = W$

or $$0.8\ V \times \rho_w \times g = \rho\ V\ g$$

$$\therefore \quad \rho = 0.8 \times \rho_w = 0.8 \times 1000 = \mathbf{800\ kg/m^3}$$

Example 19.12. A glass tumbler is full to the brim with water. An ice cube floats on the top surface. (*i*) What fraction of the ice cube is immersed? (*ii*) What happens to water level after the ice is melted? Density of ice is 910 kg m^{-3} and that of water is 1000 kg m^{-3}.

Solution. (*i*) Suppose the volume of the entire cube is V_i and its volume under water is V_w. The density of ice is ρ_i and that of water is ρ_w.

$$\text{Weight of ice cube, } W_i = \rho_i V_i\ g$$

$$\text{Buoyant force, } F_B = \rho_w V_w\ g$$

According to principle of floatation, $F_B = W_i$

or $$\rho_w V_w g = \rho_i V_i g$$

or $$\rho_w V_w = \rho_i V_i$$

$$\therefore \quad \frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} = \frac{910}{1000} = 0.91$$

(*ii*) As the cube melts, its volume shrinks from V_i to V_w as its density goes up from ρ_i to ρ_w. The final water level is just the same as before.

Example 19.13. A copper cube weighs 0.50 kg in air and 0.40 kg in water. Is the cube hollow or solid? Density of copper = 8.96×10^3 kg/m^3 ; density of water = 10^3 kg/m^3.

Solution. Suppose V is the volume of the copper cube.

Loss of weight in water = $0.50 \times g - 0.40 \times g = 0.1 \times g$

Weight of water displaced = Volume × density × $g = V \times 10^3 \times g$

According to Archimedes' principle,

$$V \times 10^3 \times g = 0.1 \times g$$

$$\therefore \quad V = \frac{0.1}{10^3} = 10^{-4}\text{m}^3$$

$$\text{Density of copper cube} = \frac{\text{Mass}}{\text{Volume}} = \frac{0.5}{10^{-4}} = 0.5 \times 10^4 \text{ kg/m}^3$$

Since it is less than the density of copper, the cube is hollow.

Example 19.14. An object that weighs 40 N in air, "weighs" 20 N when submerged in water and 30 N when submerged in a liquid of unknown density. What is the density of the liquid?

Solution. Suppose V is the volume of the object.

In water. $W = 40$ N ; $W_a = 20$ N ; $F_B = V\rho_w g$

Now $$F_B = W - W_a$$

or $$V\rho_w g = W - W_a \quad \therefore V = \frac{W - W_a}{\rho_w g} \qquad ...(i)$$

In liquid. $W = 40$ N ; $W_a' = 30$ N; $F_B' = V\rho_l g$

Now $$F_B' = W - W_a'$$

or $$V\rho_l g = W - W_a'$$

$$\therefore \quad \rho_l = \frac{W - W_a'}{V g} = \frac{W - W_a'}{W - W_a} \times \rho_w \qquad \left(\because V = \frac{W - W_a}{\rho_w g}\right)$$

$$= \left(\frac{40 - 30}{40 - 20}\right) \times 1000 = 500 \text{ kg/m}^3$$

Examples 19.15. A piece of pure gold ($\rho = 19.3$ g/cm^3) is suspected to be hollow from inside. It weighs 38.250 g in air and 33.865 g in water. Calculate the volume of the hollow portion of the gold, if any.

Solution. Volume of pure gold piece in air is

$$V = \frac{\text{Mass of pure gold piece in air}}{\text{Density of gold}} = \frac{38.250}{19.3} = 1.982 \text{ cm}^3$$

Loss of weight of given gold piece in water = $38.250 \times g - 33.865 \times g = 4.385 \times g$

Let V' be the volume of given gold piece.

$\therefore$ Weight of water displaced = Volume × density × $g = V' \times 1 \times g$

According to Archimedes' principle, we have,

$$V' \times 1 \times g = 4.385 \times g \quad \text{or} \quad V' = 4.385 \text{ cm}^3$$

$\therefore$ *Volume of hollow portion of gold piece

$$= V' - V = 4.385 - 1.982 = \mathbf{2.403\ cm^3}$$

Example 19.16. A sample of milk diluted with water has a density of 1032 kg m^{-3}. If pure milk has a density of 1080 kg m^{-3}, find the percentage of water by volume in the milk.

Solution. Let V be the volume of diluted milk and v be the volume of water in it. Then,

$$\text{Mass of pure milk} = (V - v) \times \text{density} = (V - v) \times 1080$$

$$\text{Mass of water} = v \times \text{density} = v \times 1000$$

$$\text{Density of diluted milk} = \frac{\text{Mass of pure milk + Mass of water}}{V}$$

or $$1032 = \frac{(V - v) \times 1080 + v \times 1000}{V}$$

or $$1032\ V = 1080\ V - 1080\ v + 1000\ v$$

or $$\frac{v}{V} = \frac{48}{80} = 0.6$$

$\therefore$ %age of water by volume in milk = $0.6 \times 100 = \mathbf{60\%}$

Example 19.17. A piece of wood of relative density 0.25 floats in a pail containing oil of relative density 0.81. What is the percentage volume of the wood above the surface of the oil?

Solution. Density of wood, ρ_w = Relative density × Density of water

$$= 0.25 \times 1 = 0.25\ \text{g/cm}^3$$

$$\text{Density of oil, } \rho_O = 0.81 \times 1 = 0.81\ \text{g/cm}^3$$

Let V cm^3 be the volume of the piece of wood.

$\therefore$ Weight of wood = $V\rho_w g$

Let V' be the volume of the immersed portion of the wood.

$\therefore$ Weight of oil displaced = $V' \rho_O\ g$

According to principle of floatation, we have,

Weight of oil displaced = Weight of wood

or $$V'\ \rho_O\ g = V\ \rho_w\ g$$

or $$V' = V \times \frac{\rho_w}{\rho_O} = V \times \frac{0.25}{0.81} = 0.309\ V$$

$\therefore$ %age of volume of wood above oil surface

$$= \frac{V - V'}{V} \times 100 = \frac{V - 0.309V}{V} \times 100 = \mathbf{69.1\%}$$

Example 19.18. A cube of wood floating in water supports a 200 g mass at the centre of its top face. When the mass is removed, the cube rises by 2 cm. Determine the volume of the cube.

Solution. Let l cm be each side of the cube.

Volume occupied by 2 cm depth of cube = $2 \times l \times l = 2l^2$ cm^3

According to principle of floatation, we have,

$$2l^2 \times \text{density of water} \times g = 200 \times g$$

or $$2\ l^2 \times 1 \times g = 200\ g$$

or $$l = \sqrt{100} = 10\ \text{cm}$$

$\therefore$ Volume of cube = $l^3 = (10)^3 = \mathbf{1000\ cm^3}$

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* Had the gold been pure, its volume should have been 1.982 cm^3. Since the volume of gold is more (4.385 cm^3), it means the average density is less than the density of pure gold. For this reason, the volume is more.

PROBLEMS FOR PRACTICE

1. A box of treasure with a mass of 92 kg and a volume of 0.031 m^3 lies at the bottom of the ocean. How much force is needed to lift it? **[590 N]**
2. A cork has a density of $\rho = 200$ kg/m^3. What fraction of the volume of the cork is submerged when the cork floats in water? Density of water = 1000 kg/m^3. **[1/5]**
3. A string supports a solid iron object of mass 180 g totally immersed in a liquid of density 800 kg m^{-3}. Calculate the tension in the string if the density of iron is 8000 kg m^{-3}. **[1.62 N]**
4. When a 60 N stone is attached to a scale and submerged in water, the scale reads 40 N. Find (*i*) the specific gravity of the stone and (*ii*) volume of the stone. **[(*i*) 3 (*ii*) 2.04 litres]**
5. The specific gravity of sea water is 1.03 and that of ice is 0.92. What fraction of an iceberg is above the surface of water? **[0.107]**

19.23. TERMS CONCERNING SURFACE TENSION

According to the molecular theory, matter is made up of minute particles called molecules which attract each other. The intermolecular forces may be classified as under :

(*i*) Force of cohesion (*ii*) Force of adhesion

(*i*) **Force of Cohesion.** *The force of attraction between the molecules of the same substance is called force of cohesion or cohesive force.* It is maximum in solids, lesser in liquids and least in gases. The effect of cohesive forces are observed in daily life. For example, definite shape of the solids is due to strong forces of cohesion among their molecules. Further, it is due to cohesive force that two drops of a liquid when brought in contact coalesce into one.

(*ii*) **Force of Adhesion.** *The force of attraction between the molecules of the different substances is called the force of adhesion or adhesive force.* Force of adhesion is different for different substances. For example, gum has greater adhesive force for a solid surface than water. When we pour water on a glass plate, the plate becomes wet because of the adhesive force between the molecules of glass and water.

Practical Examples of Cohesive and Adhesive Forces. A few practical examples of these forces are given below :

(*a*) The ink sticks on paper. It is because the adhesive force between ink and paper is greater than the cohesive force of ink molecules.

(*b*) Water wets the glass. It is because the adhesive force between water molecules and glass molecules is greater than the force of cohesion between water molecules.

(*c*) Mercury does not wet the glass because the adhesive force between mercury molecules and glass molecules is less than the cohesive force between mercury molecules.

(*d*) We are able to write on the blackboard with a piece of chalk because the adhesive force between chalk molecules and wood molecules is much greater than the cohesive force between chalk molecules.

Some facts about molecular forces

(*i*) On the average, the molecules in solids and liquids are separated by a distance of the order of 10^{-9} m.

(*ii*) The molecules of a liquid or solid attract each other if distance between them is 10^{-9} m or less.

(*iii*) If the distance between the molecules becomes greater than 10^{-9} m, the attraction between the molecules becomes negligible.

(*iv*) The maximum distance (10^{-9} m) upto which two molecules attract each other is called *molecular range.*

(*v*) The force of attraction between molecules is due to electrical interaction between the charges.

(*vi*) Molecular forces do not obey inverse square law.

19.24. SURFACE TENSION

The molecules of a liquid attract each other ; otherwise the liquid would not have a definite volume. Consider a water molecule *A* well within the volume of the liquid as shown in Fig. 19.19. The molecule *A* is attracted by water molecules from all sides and hence the resultant molecular force on it is zero. Now consider a water molecule *B* at the surface of the liquid. The molecule *B* is surrounded by water molecules on one side of the surface, while on the other side, it is surrounded by air molecules. The air molecules exert a small force on molecule *B* compared to that exerted by water molecules. Therefore, there is net downward molecular force on molecule *B*. A similar situation exists for other molecules at the liquid surface. The result is that the surface of a liquid behaves as a stretched membrane. In other words, the attractive intermolecular forces on the surface of the liquid tend to compress the liquid surface *i.e.,* liquid tries to minimise its surface area. This effect is called *surface tension.*

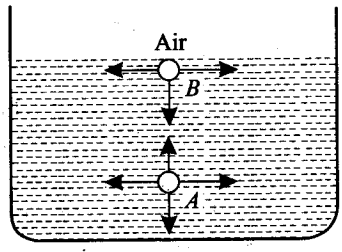


Fig. 19.19

The property of a liquid at rest by virtue of which its free surface behaves like a stretched membrane under tension and tries to occupy as small area as possible is called **surface tension.**

Discussion. The following points may be noted :

(*i*) The molecules on the surface of a liquid have more potential energy than those within the liquid. It is because to take the molecule from the interior of the liquid to its surface, work must be done against the downward molecular forces. This work done is stored as potential energy in the molecules at the surface of the liquid.

(*ii*) Every system tends to arrange itself in such a way that it has minimum potential energy. Therefore, the free surface of a liquid also tends to have minimum potential energy. This is possible only if the area of the liquid surface is minimum. *In order to acquire minimum area, the surface of a liquid tends to contract and hence behaves as a stretched membrane.*

(*iii*) For a given volume, a sphere has the minimum surface area. Therefore, every surface of a liquid tries to become spherical. But due to the presence of gravitational force it fails to do so unless the gravitational force is negligible.

19.25. MEASUREMENT OF SURFACE TENSION

Surface tension is due to the intermolecular attractions on the liquid surface and these forces produce a skin effect on the surface. Imagine a line *AB* drawn tangentially on the surface of the liquid as shown in Fig. 19.20. The line divides the surface into two parts ; one surface to the left of the line and the other surface to the right of the line. It is found that the two parts of the surface pull each other with a force directly proportional to the length of the line. *These forces of pull are perpendicular to the line separating the two parts and tangential to the surface.*

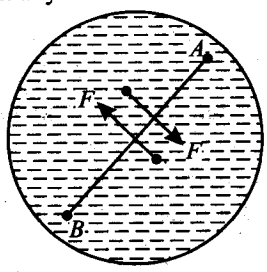


Fig. 19.20

Let *F* be the magnitude of the forces exerted on each other by the two parts of the surface across the line *AB* of length *l*. Then force acting per unit length of the line is called surface tension *T i.e.,*

Surface tension, $T = \dfrac{\text{Force on either side of line } AB}{\text{Length of } AB} = \dfrac{F}{l}$

If $l = 1$, then $T = F$.

Thus **surface tension of a liquid** *can be measured as the force per unit length in the plane of the liquid surface acting at right angles on either side of an imaginary line drawn on the liquid surface.*

Since surface tension T is a force per unit length, its SI unit is Nm^{-1}.

$$\text{Dimensions of surface tension} = \frac{\text{Dimensions of force}}{\text{Dimensions of length}} = \frac{[M\,LT^{-2}]}{[L]} = [M\,T^{-2}]$$

The surface tension of water is about 0.075 N/m.

Discussion. The following points are worth nothing :

(*i*) Surface tension is a **scalar quantity** because it has no specific direction for a given liquid.

(*ii*) Surface tension is not simply a force but is a force divided by a length ($T = F/l$).

(*iii*) *Surface tension is the force per unit length needed to hold the liquid surface together. Surface tension is always tangent to the liquid surface.*

(*iv*) The surface tension of a liquid decreases with the increase in temperature and *vice-versa.*

Example 19.19. Water is kept in a beaker of radius 5.0 cm. Consider a diameter of the beaker on the surface of the water. Find the force by which the surface on one side of the diameter pulls the surface on the other side. Surface tension of water is 0.075 N/m.

Solution. Length of diameter, $l = 2 \times 5 = 10$ cm $= 0.1$ m

$$\text{Now surface tension, } T = \frac{F}{l} \text{ or } F = T \times l = 0.1 \times 0.075 = 7.5 \times 10^{-3} \text{ N}$$

Example 19.20. A wire ring of 3 cm radius resting flat on the surface of a liquid is raised. The pull required is 3.03 gf more before the film breaks than it is afterwards. Find the surface tension of the liquid.

Solution. When the wire ring is resting flat on the surface of the liquid, a thin film of liquid is in contact with the ring. The surface tension force acts along the circumference of the ring (downward). Therefore, in order to raise the ring, the force needed is equal to the sum of the weight of the ring and the force due to surface tension. However, when the film breaks, the force required to raise the ring is equal to the weight of the ring because no force is to be applied against surface tension. Clearly, the difference between the forces applied before and after breaking of the film is equal to the force due to surface tension.

$\therefore$ Surface tension force $= 3.03$ gf $= 3.03 \times 980$ dynes

The film has two surfaces, one inside and one outside the ring. Therefore, the total length over which the surface tension acts is $2(2\pi r) = 4\pi r$. If T is the surface tension of the liquid,

$$\text{Surface tension force} = 4\pi r \times T$$

$$\therefore \quad 4\pi r \times T = 3.03 \times 980$$

$$\text{or} \quad T = \frac{3.03 \times 980}{4\pi \times 3} = 78.80 \text{ dynes/cm}$$

Example 19.21. The maximum force, in addition to the weight required to pull a wire 5 cm long from the surface of water at 20°C is 728 dynes. Calculate the surface tension of water.

Solution. There is a film on either side of the wire. Therefore, the length over which the surface tension acts is $l = 2 \times 5 = 10$ cm.

$$\therefore \quad \text{Surface tension of water, } T = \frac{F}{l} = \frac{728}{10} = 72.8 \text{ dynes/cm}$$

Example 19.22. A glass plate of length 10 cm, breadth 1.54 cm and thickness 0.20 cm has a mass of 8.2 g in air. It is held vertically with long side horizontal and the lower half under water. Find the apparent weight of the plate. Surface tension of water = 7.3×10^{-2} N/m and $g = 9.8$ ms^{-2}.

Solution. Weight of glass plate, $W = m\,g = (8.2 \times 10^{-3}) \times 9.8 = 80.36 \times 10^{-3}$ N

Total length of the plate in contact with water is

$$L = 2\,(10 + 0.2)\text{ cm} = 20.4\text{ cm} = 20.4 \times 10^{-2}\text{ m}$$

Downward pull on the plate due to surface tension is

$$F = T \times L = (7.3 \times 10^{-2}) \times 20.4 \times 10^{-2} = 14.89 \times 10^{-3}\text{ N}$$

Total downward force on the plate is

$$R = W + F = (80.36 + 14.89)\ 10^{-3}\text{ N} = 95.25 \times 10^{-3}\text{ N}$$

Volume of plate immersed in water is

$$V = \frac{1}{2}\,[10 \times 1.54 \times 0.2] = 1.54\text{ cm}^3 = 1.54 \times 10^{-6}\text{ m}^3$$

$$\text{Upthrust, } F_B = V\rho g = (1.54 \times 10^{-6}) \times (1000) \times 9.8 = 15.09 \times 10^{-3}\text{ N}$$

Apparent weight of plate $= R - F_B = (95.25 - 15.09) \times 10^{-3}\text{ N} = \mathbf{80.16 \times 10^{-3}\text{ N}}$

PROBLEMS FOR PRACTICE

1. A *U*-shaped wire is dipped in a soap solution and removed. The thin soap film formed between the wire and a light slider supports a weight of 4.2×10^{-3} N (which includes the small weight of the slider). The length of the slider is 0.05 m. What is the surface tension of the liquid? **[4.2×10^{-2} N/m]**
2. A thin circular wire of 1.12 cm radius is supported horizontally in a liquid and withdrawn slowly by an upward force of 5.53×10^{-3} N. If the surface tension of the liquid is 0.032 N/m, what is the mass of the wire? **[1.05×10^{-4} kg]**
3. The tension in the skin of the left ventricle of the heart is 4.90 N/m. A 4 cm cut is made on the left ventricle during open-heart surgery. How many stitches must be used to repair the cut? Each stitch will safely support a tension of 0.4 N. **[29 stitches]**

19.26. SURFACE ENERGY

We have seen that surface tension is the force per unit length needed to hold the liquid surface together. Therefore, in order to increase the surface area of a liquid, work will have to be done against the force of surface tension. This work done is stored in the surface layer of the liquid in the form of potential energy. The liquid surface has now more surface energy due to the increased surface area. Work done against the force of surface tension to increase unit surface area is called surface energy.

The work done (against force of surface tension) per unit area in increasing the surface area of a liquid at constant temperature is called **surface energy** *of the liquid i.e.,*

$$\text{Surface energy, } \sigma = \frac{\text{Work done in increasing surface area}}{\text{Increase in surface area}}$$

This is also called the *free surface energy* because the mechanical work done can be released when the surface contracts.

19.27. RELATION BETWEEN SURFACE TENSION AND SURFACE ENERGY

We now find the relation between surface energy of a liquid and its surface tension. Consider a rectangular frame of wire *ABCD* in which the wire *BC* is movable. If we dip the frame in a soap solution, a thin film is formed which pulls the wire *BC* toward left due to surface tension. If T is the surface tension of the film and l is the length of the wire *BC*, then force F on *BC* due to surface tension is given by ;

$$F = T \times 2l$$

Since the film has both an upper and lower surface, the total length over which the surface tension acts on BC is $2l$.

Suppose the wire is now moved a distance x from BC to $B'C'$ (Fig. 19.21) against the surface tension force F so that the surface area of the film increases. In order to increase the film area, work has to be done against F.

Work done in enlarging surface area is

Work done = Force × Distance

$$= (T \times 2\,l) \times x = T \times 2\,l\,x$$

The increase in surface area is $2l\,x$ (upper and lower surfaces).

$$\therefore \quad \text{Surface energy, } \sigma = \frac{\text{W.D. in increasing surface area}}{\text{Increase in surface area}}$$

$$= \frac{T \times 2l\,x}{2l\,x} = T$$

$$\therefore \quad \sigma = T$$

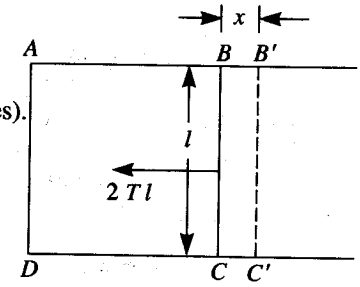


Fig. 19.21

Thus surface tension T is numerically equal to the surface energy σ. This provides second definition of T.

Hence **surface tension** *is the work done (against force of surface tension) at constant temperature to increase the surface area of the liquid by unit area.*

Units. The SI unit of work done is Nm (or joule) and that of area is m^2. Therefore, SI unit of surface energy is N/m (or J/m^2). Note that surface tension and surface energy have the same units and dimensions.

Discussion. The following points may be noted :

(*i*) The surface tension (T) is numerically equal to the surface energy (σ).

(*ii*) The surface tension may be defined as surface force per unit length or surface energy per unit area.

Example 19.23. **Calculate the work done against surface tension forces in blowing a soap bubble of diameter 1 cm. The surface tension of soap solution is 2.5×10^{-2} N/m.**

Solution. Here $T = 2.5 \times 10^{-2}$ N/m ; $r = \frac{1}{2}$ cm = 0.5 cm

Original surface area of bubble = 0

Final surface area of bubble, $A = 2 \times (4\pi r^2) = 8\pi r^2$

The factor 2 is necessary because the soap bubble has two free surfaces.

$$\therefore \quad \text{Increase in surface area} = A - 0 = A = 8\pi r^2 = 8\pi \times (0.5)^2 \times 10^{-4}\,m^2$$

$$\therefore \quad \text{Work done} = T \times \text{Increase in surface area}$$

$$= (2.5 \times 10^{-2}) \times [8\pi \times (0.5)^2 \times 10^{-4}] = 1.57 \times 10^{-5}\ J$$

Example 19.24. **Find the work required to break up a drop of water of radius 0.5 cm into drops of water each of radii 1 mm. The surface tension of water is 7×10^{-2} N/m.**

Solution. The original radius of drop is R (= 0.5 cm) and the final radius of each drop is r (= 1 mm = 0.1 cm).

Since the volume of a drop is $(4/3)\pi$ (radius)^3

$$\therefore \quad \text{Number of drops formed} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3} = \frac{R^3}{r^3} = \frac{(0.5)^3}{(0.1)^3} = 125$$

Final surface area of drops $= 125\times\left(4\pi r^2\right)$

$= 125 \times [4\pi \times (0.1)^2 \times 10^{-4}]\ m^2 = 5\pi\times10^{-4} m^2$

Original surface area of drop $= 4\pi R^2 = 4\pi\times\left[(0.5)^2\times10^{-4}\right] m^2 = \pi \times 10^{-4}\ m^2$

$\therefore$ Work done = T × Increase in surface area

$= (7 \times 10^{-2}) \times [5\pi \times10^{-4} - \pi\times10^{-4}] = 8.8 \times 10^{-5}$ J

Example 19.25. **What amount of energy will be liberated if 1000 droplets of water each 10^{-8}m diameter coalesce to form a large spherical drop? Surface tension of water is 7.2×10^{-2} N/m.**

Solution. The radius of each smaller drop is r (= 0.5×10^{-8} m) and that of the large drop is R. Since mass is conserved,

$\therefore \quad \left(\frac{4}{3}\pi R^3\right)\rho = 1000\left(\frac{4}{3}\pi r^3\right)\rho$

or $R^3 = 1000\ r^3$

or $R = 10\ r = 10 \times (0.5 \times 10^{-8}) = 5 \times 10^{-8}$ m

Decrease in surface area $= 1000\times4\pi r^2 - 4\pi R^2$

$= 4\pi\ (1000 \times 25 \times 10^{-18} - 25 \times 10^{-16})\ m^2$

$= 4\pi \times 225 \times 10^{-16}\ m^2$

$\therefore$ Energy liberated = T × Decrease in surface area

$= (7.2 \times10^{-2}) \times (4\pi\times 225 \times 10^{-16})$

$= 2.035 \times 10^{-14}$ J

Example 19.26. **A film of water is formed between two straight parallel wires each 10 cm long at a separation of 0.5 cm. Calculate the work required to increase 1 mm distance between the wires. Surface tension of water = 72×10^{-3} N/m.**

Solution. Initial surface area of film = Length × Separation = $10 \times 0.5 = 5\ cm^2 = 5 \times 10^{-4}\ m^2$

Final surface area of film = $10 \times (0.5 + 0.1) = 6\ cm^2 = 6 \times 10^{-4}\ m^2$

Increase in surface area = $(6 - 5) \times 10^{-4} = 1.0 \times 10^{-4}\ m^2$

The film has two surfaces. Therefore, net increase in the surface area of the film is

$\Delta A = 2 \times (1.0 \times 10^{-4}) = 2 \times 10^{-4}\ m^2$

$\therefore$ Work done = Surface tension × Increase in area

$= (72 \times 10^{-3}) \times (2 \times 10^{-4}) = 144 \times 10^{-7}$ J

Example 19.27. **Suppose that 64 raindrops combine into a single drop. Calculate the ratio of the total energy of the 64 drops to that of a single drop.**

Solution. Let R be the radius of single drop and r the radius of each smaller drop.

Volume of single drop = Volume of 64 drops

or $\frac{4}{3}\pi R^3 = 64\times\left(\frac{4}{3}\pi r^3\right)$

$\therefore \quad R = 4r$

Surface energy of 64 drops, $S_1 = 64 \times 4\pi r^2 \times T$

Surface energy of bigger drop, $S_2 = 4\ \pi\ R^2 \times T$

$\therefore \quad \frac{S_1}{S_2} = \frac{64\times4\pi r^2\times T}{4\pi R^2\times T} = \left(\frac{r}{R}\right)^2\times64 = \frac{64}{16} = 4$

Example 19.28. If a number of little droplets of water, all of the same radius r, coalesce to form a single drop of radius R, show that the rise in temperature is given by ;

$$\frac{3T}{\rho J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

where T is the surface tension of water and J is the mechanical equivalent of heat.

Solution. Let n be the number of little droplets which coalesce to form a single drop.

Surface area of n droplets $= n \times 4\pi r^2$

Surface area of single drop $= 4\pi R^2$

$\therefore$ Decrease in surface area $= n \times 4\pi r^2 - 4\pi R^2$

$\therefore$ Decrease in surface tension energy $= [n \times 4\pi r^2 - 4\pi R^2] \times T$

Heat produced in the process $= [n \times 4\pi r^2 - 4\pi R^2] \times \frac{T}{J}$

If $\Delta\theta$ is the increase in temperature in the process, then,

$$mc\Delta\theta = [n \times 4\pi r^2 - 4\pi R^2] \times \frac{T}{J}$$

or $$\left(\frac{4}{3}\pi R^3\right)\rho \times \Delta\theta = 4\pi(nr^2 - R^2)\frac{T}{J}$$

or $$\Delta\theta = \frac{3T}{\rho J}\left(\frac{nr^2}{R^3}-\frac{1}{R}\right) \quad ...(i)$$

Since mass is a conserved quantity, the total mass of n droplets is equal to the mass of single drop *i.e.*,

$$n \times \frac{4}{3}\pi r^3 \times \rho = \frac{4}{3}\pi R^3 \times \rho$$

$\therefore$ $$n = \left(\frac{R}{r}\right)^3$$

Putting the value of n $[= (R/r)^3]$ in eq. (i), we have,

$$\Delta\theta = \frac{3T}{\rho J}\left[\frac{1}{r}-\frac{1}{R}\right]$$

PROBLEMS FOR PRACTICE

1. The surface tension of soap solution is 0.03 N/m. What amount of work is required to produce a bubble of radius 0.5 m? **[1.884×10^{-3} J]**
2. A spherical drop of mercury of radius 2 mm falls to the ground and breaks into 10 smaller drops of equal size. Calculate the amount of work that has to be done in the process. Surface tension of mercury is 4.72×10^{-1} Nm^{-1}. **[2.74×10^{-5} J]**
3. Calculate the change in surface energy of a soap bubble when its radius decreases from 5 cm to 1 cm. Surface tension of soap solution = 2.0×10^{-2} Nm^{-1}. **[1.2×10^{-3} J]**

19.28. SOME SURFACE TENSION PHENOMENA

There are a large number of phenomena that result from liquid surface tension. However, we shall deal with a few of them by way of illustration.

(*i*) **Thread on a soap film.** The tendency of a soap film to contract due to surface tension can be demonstrated by placing a thread B carefully on soap film formed in a metal ring A as shown in Fig. 19.22 (*i*). The surface tension forces on the both sides of the thread counterbalance as shown in Fig. 19.22 (*i*). However, if the film enclosed by the thread is pierced, the thread is pulled out into a circle by the surface tension forces F which now act on one side only of the thread [See Fig. 19.22 (*ii*)]. Note that the film has contracted to minimum area. It is because the thread

is pulled into a circle (the shape with maximum area for a given perimeter) and therefore the liquid film has the minimum possible area.

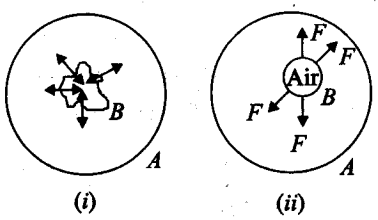


Fig. 19.22

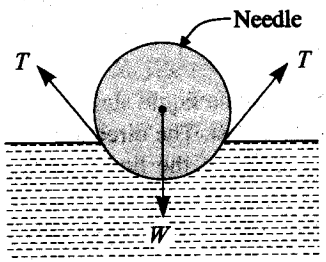


Fig. 19.23

(*ii*) **Floating needle.** If we place a thin needle on a piece of blotting paper and place the paper lightly on the surface of water, the blotting paper will soon sink while the needle will remain floating on the surface [See Fig. 19.23]. The needle creates a depression in the liquid surface so that the surface tension forces T now have an upward directed component which is capable of supporting the weight of the needle. However, if we add oil in water, the needle sinks. It is because oil lowers the surface tension of water and the weight of needle exceeds the upward directed component of surface tension force.

(*iii*) **Small drops of mercury are spherical while large ones are flat.** A small drop of mercury is spherical because the gravitational force on it is very small and the liquid shape is then mainly due to surface tension forces. Although the density of mercury is high, small drops of mercury are spherical. The ratio of surface area ($4\pi r^2$) to weight (or volume $4\pi r^3/3$) of a sphere is proportional to the ratio r^2/r^3 or $1/r$. Thus the smaller the radius, the greater is the influence of surface tension forces compared with the weight. However, large mercury drops are flattened on top. This time the gravitational force is relatively greater.

(*iv*) **Formation of lead shots.** In order to manufacture lead shots, melted lead is allowed to fall in water by spraying it from a sufficient height. During its fall, the melted lead forms small spherical drops due to surface tension forces and on entering water, they become solid.

(*v*) **When a dry brush is dipped in water, its hair spread out.** When a camel-hair brush is dipped in water, the hair spread out but when the brush is removed from water, its hair cling together (See Fig. 19.24). The reason is simple. When the brush is in water, there is water all around its hair. Consequently, there is no free surface and the effect of surface tension is absent. However, when the brush is taken out of water, the forces of surface tension contract the hair.

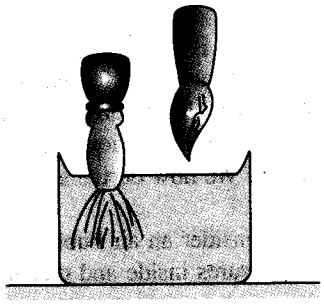


Fig. 19.24

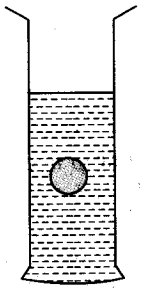


Fig. 19.25

(*vi*) **Eliminating the effects of gravity.** Soap bubbles are almost perfect spheres because they have large surface areas and negligible masses *i.e.*, effect of gravity is almost nil. The effect of gravity can be eliminated as shown in Fig. 19.25. A drop of olive oil is left gently inside a mixture of spirit and water (whose density is exactly equal to density of oil) with the help of pipettee. It

is found that olive oil drop is perfectly spherical in shape. It is because the density of mixture is equal to the density of olive oil and therefore the upthrust on the oil drop is exactly equal to its weight. Since the effect of gravity is eliminated, the shape of oil drop is perfectly spherical.

19.29. EXCESS PRESSURE ON CURVED LIQUID SURFACE

Every molecule on the liquid surface experiences force of surface tension that acts tangentially to the liquid surface at rest. The direction of the resultant force on a molecule on the liquid surface depends upon the shape of the liquid surface.

(*i*) When liquid surface is plane. If the liquid surface is plane [See Fig. 19.26 (*i*)], forces due to surface tension (*T*, *T*) act tangentially to the liquid surface in the opposite directions. Hence the resultant force on the molecule *M* is zero. Therefore, pressure on the liquid side is equal to pressure on the vapour side.

Therefore, in case of plane liquid surface, the pressure on the liquid side is equal to the pressure on the vapour side.

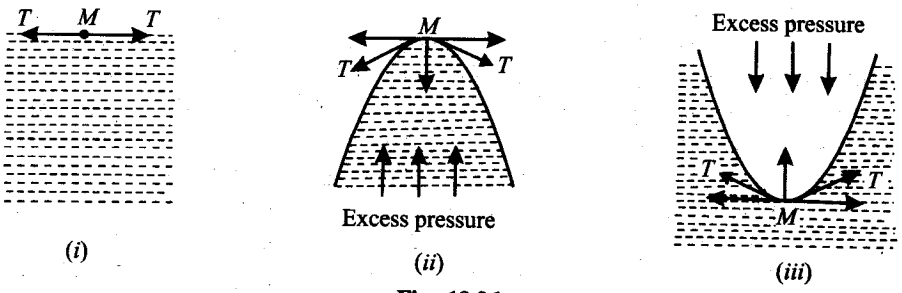


Fig. 19.26

(*ii*) When liquid surface is curved. If the liquid surface is curved, every molecule on the liquid surface experiences forces (*T*, *T*) due to surface tension along the tangent to the surface. Resolving these forces into rectangular components, we find that horizontal components cancel each other while vertical components add up. Therefore, resultant force normal to the surface acts on the curved surface of a liquid. For a **convex surface** [See Fig. 19.26 (*ii*)], the resultant force is directed inward towards the centre of curvature. On the other hand, for a **concave surface** [See Fig. 19.26 (*iii*)], the resultant force is directed outward from the centre of curvature. Clearly, for the equilibrium of the curved liquid surface, there must be an excess of pressure on its concave side so that excess pressure force balances the resultant force due to surface tension.

Therefore, for a curved liquid surface in equilibrium, the pressure on its concave side is greater than the pressure on its convex side.

19.30. EXCESS PRESSURE INSIDE A BUBBLE AND LIQUID DROP

The small bubbles and liquid drops are spherical because the forces of surface tension predominate the gravitational forces. The fact that a bubble or liquid drop does not collapse due to the combined effect of external pressure and force of surface tension indicates that pressure inside a bubble or liquid drop is greater than that outside it. We now derive expressions for excess pressure inside a bubble and a liquid drop.

1. Excess pressure inside Air bubble in a liquid. Consider an air bubble of radius *R* inside a liquid having surface tension *T*. Let p_2 and p_1 be the pressures inside and outside the air bubble. Then excess pressure within the air bubble is $p_2 - p_1$.

In order to find the excess pressure inside the air bubble, we consider the forces acting on one half *B* of the air bubble as shown in Fig. 19.27. The three forces acting on hemisphere *B* are :

(*i*) The half bubble not shown exerts a force due to surface tension around the rim of the half bubble we are considering. This force is directed to the right. Since the length of the rim is $2\pi R$, the magnitude of this force is $2\pi\ RT$ (See Fig. 19.27).

$\therefore$ Force due to surface tension, $F_T = 2\pi\ RT$ ...towards right

(*ii*) The pressure p_1 outside also exerts a force to the right acting perpendicular to an area πR^2 (area of the flat face of hemisphere).

$\therefore$ Force due to p_1, $F_{p_1} = p_1 \pi R^2$ ...towards right

(*iii*) The force due to pressure inside the bubble is to the left and its magnitude is $p_2 \pi R^2$ *i.e.*,

Force due to p_2, $F_{p_2} = p_2 \pi R^2$ ...towards left

As the hemisphere B is in equilibrium under the action of these three forces,

$$\therefore \quad F_{p_2} = F_T + F_{p_1}$$

$$\text{or} \quad p_2 \pi R^2 = 2\pi RT + p_1 \pi R^2$$

$$\text{or} \quad \pi R^2 (p_2 - p_1) = 2\pi RT$$

$$\therefore \quad \text{Excess pressure, } p_2 - p_1 = \frac{2T}{R} \qquad ...(i)$$

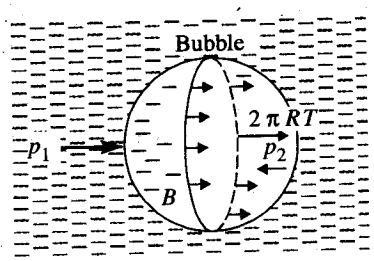


Fig. 19.27

Thus pressure inside the air bubble is greater than the pressure outside by an amount 2T/R. The smaller the air bubble (*i.e.*, smaller R), the greater is the excess pressure.

2. Excess pressure inside a soap (liquid) bubble. Consider a soap bubble of radius R and suppose the surface tension of the soap solution is T. Let p_2 and p_1 be the pressures inside and outside the soap bubble. In order to find excess pressure inside the soap bubble, we consider the forces acting on one half B of the bubble as shown in Fig. 19.28. *A soap bubble has two liquid surfaces in contact with air ; one inside the bubble and the other outside the bubble.* Therefore, force on one half B of the soap bubble due to surface tension is $2 \times (2\pi RT)$ or $4\pi RT$ (See Fig. 19.28). The various forces acting on the hemisphere are :

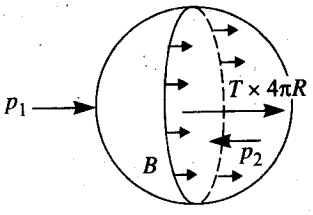


Fig. 19.28

(*i*) Force due to surface tension = $4\pi RT$ ... towards right

(*ii*) Force due to outside pressure $p_1 = p_1 \pi R^2$ ...towards right

(*iii*) Force due to inside pressure $p_2 = p_2 \pi R^2$ ... towards left

As the hemisphere B is in equilibrium under the action of these three forces,

$$\therefore \quad p_2 \pi R^2 = 4\pi RT + p_1 \pi R^2$$

$$\text{or} \quad \pi R^2 (p_2 - p_1) = 4\pi RT$$

$$\therefore \quad \text{Excess pressure, } p_2 - p_1 = \frac{4T}{R}$$

Thus the pressure inside the soap bubble is greater than the pressure outside by an amount $4T/R$. Again, the smaller the soap bubble, the greater is the excess pressure.

3. Excess pressure inside the liquid drop. Consider a liquid drop of radius R and suppose the surface tension of the liquid is T. Let p_2 and p_1 be the pressures inside and outside the liquid drop. Like air bubble in water, a liquid drop has only one liquid surface in contact with air. Therefore, force of surface tension on one half of liquid drop (Refer back to Fig. 19.27) is $2\pi RT$. For equilibrium of the one half of liquid drop, we have,

$$p_2 \pi R^2 = 2\pi RT + p_1 \pi R^2$$

$$\text{or} \quad \pi R^2 (p_2 - p_1) = 2\pi RT$$

$$\therefore \quad \text{Excess pressure, } p_2 - p_1 = \frac{2T}{R}$$

Discussion. The following points are worth noting :

(*i*) An air bubble in water or a liquid drop has only one liquid surface in contact with air. However, a soap bubble (or a liquid bubble) has two liquid surfaces in contact with air. Therefore, excess pressure is double in the latter case compared to the former case.

(*ii*) In case of bubbles or liquid drops, the pressure on the concave side of liquid-air interface is greater than that of the convex side.

Example 19.29. What should be the pressure inside a small air bubble of 0.1 mm radius situated just below the water surface? Surface tension of water = 7.2×10^{-2} Nm^{-1} and atmospheric pressure = 1.013×10^5 N/m^2.

Solution. Radius of air bubble, $R = 0.1$ mm $= 0.1 \times 10^{-3}$ m

Surface tension of water, $T = 7.2 \times 10^{-2}$ Nm^{-1}

The excess pressure inside an air bubble is given by ;

$$p_2 - p_1 = \frac{2T}{R} = \frac{2\times(7.2\times10^{-2})}{0.1\times10^{-3}} = 1.44\times10^3 \text{ N/m}^2$$

$\therefore$ Pressure inside air bubble, $p_2 = p_1 + 1.44 \times 10^3$

$= 1.013 \times 10^5 + 1.44 \times 10^3 = \mathbf{1.027 \times 10^5}$ $\mathbf{N/m^2}$

Example 19.30. If excess pressure is balanced by that due to a column of oil (specific gravity 0.8) 2 mm high where R = 1.0 cm, find surface tension of the soap bubble.

Solution. The excess pressure inside a soap bubble is given by ;

$$p_2 - p_1 = \frac{4T}{R}$$

It is given that $p_2 - p_1 = h\rho g$.

$\therefore$ $$h\rho g = \frac{4T}{R}$$

or $$T = \frac{h\rho gR}{4}$$

Here $h = 2$ mm $= 2 \times 10^{-3}$ m ; $\rho = 800$ kg m^{-3}; $R = 1.0$ cm $= 1.0 \times 10^{-2}$ m

$\therefore$ Surface tension, $T = \dfrac{(2\times10^{-3})\times(800)\times(9.8)\times(1.0\times10^{-2})}{4}$

$= \mathbf{3.92 \times 10^{-2}}$ $\mathbf{Nm^{-1}}$

Example 19.31. The lower end of a capillary tube of diameter 2.0 mm is dipped 8.0 cm below the surface of water in a beaker. What is the pressure required in the tube in order to blow a hemispherical bubble at its end in water? The surface tension of water at the temperature of the experiment is 7.3×10^{-2} Nm^{-1}; 1 atmosphere = 1.01×10^5 Pa ; density of water = 1000 kg m^{-3} and g = 9.8 ms^{-2}.

Solution. The pressure required in the tube is equal to the pressure inside the air bubble.

Pressure inside the bubble

= Pressure outside the bubble + excess pressure

$$= (P_a + h\rho g) + \frac{2T}{R}$$

$$= (1.01 \times 10^5 + 8 \times 10^{-2} \times 1000 \times 9.8) + \frac{2\times7.3\times10^{-2}}{1.0\times10^{-3}}$$

$= (1.02 \times 10^5) + 146 = \mathbf{1.02 \times 10^5}$ **Pa**

Example 19.32. A small hollow sphere having a small hole in it is immersed into water to a depth of 20 cm before any water penetrates into it. If the surface tension of water is 73 dynes cm^{-1}, find the radius of the hole.

Solution. When the sphere is immersed in water, an air bubble is formed at the hole. The pressure due to water will be opposite to the excess pressure inside the air bubble. At a depth of 20 cm, the two become equal and water just enters the hole.

$\therefore$ At equilibrium position, $\frac{2T}{R} = h\rho g$ ...R is radius of hole

or $$R = \frac{2T}{h\rho g} = \frac{2\times73}{20\times1\times980} = \mathbf{0.0075\ cm}$$

PROBLEMS FOR PRACTICE

1. Find the gauge pressure inside a soap bubble of diameter 5 cm. The surface tension of soap solution is 25×10^{-3} N m^{-1}. **[4 Pa]**
2. A glass tube of 1 mm bore is dipped vertically into a container of mercury with its lower end 2 cm below the mercury surface. What must be the gauge pressure of air in the tube in order to blow a hemispherical bubble at its lower end? Given density of mercury = 13600 kg m^{-3} and surface tension of mercury = 35×10^{-3} N m^{-1}. **[2805.6 Pa]**
3. Calculate the depth of water at which an air bubble of radius 0.4 mm may remain in equilibrium. Surface tension of water = 7×10^{-2} N m^{-1} and g = 9.8 ms^{-2}. **[3.57 cm]**
4. A small hollow sphere which has a small hole in it is immersed in water to a depth 40 cm before any water penetrates into it. If the surface tension of water is 7.5×10^{-2} N m^{-1}, find the radius of the hole. **[0.003827 cm]**
5. A soap bubble has a diameter of 4 mm. Calculate the pressure inside it if the atmospheric pressure is 10^5 Pa. Surface tension of soap solution = 2.8×10^{-2} Nm^{-1}. **[1.00056 × 10^5 Pa]**

19.31. ANGLE OF CONTACT

When a liquid surface touches a solid surface, the shape of the liquid surface near the contact is generally curved (See Fig. 19.29). This effect is due to the presence of cohesive forces (*i.e.,* forces of attraction between the molecules of the liquid) and adhesive forces (*i.e.*, forces of attraction between the molecules of solid and liquid).

The **angle of contact** *between a liquid and a solid is defined as the angle* (θ) *which the tangent to the liquid surface at the point of contact* (*i.e., point A in Fig. 19.29*) *makes with the solid surface inside the liquid.*

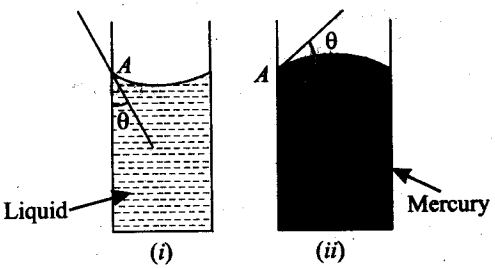


Fig. 19.29

Note that the meniscus of the liquid is concave when angle of contact (θ) is less than 90° and it is convex when θ is greater than 90°. The value of angle of contact depends on the following factors :

(*i*) The nature of the liquid and the solid in contact.
(*ii*) The medium that exists above the free surface of the liquid.
(*iii*) Impurities present in the liquid ; angle of contact θ decreases on adding impurities to the liquid.
(*iv*) Temperature ; contact angle increases with the increase in temperature.

It may be noted that for a given liquid-solid pair, the angle of contact is the same whether the liquid is contained in a vessel made of that solid material or a rod of that solid material is dipped in the liquid.

1. When angle of contact is acute. Fig. 19.29 (*i*) shows that angle of contact between the liquid (*not water*) and the solid surface (glass wall) is acute (*i.e.*, $\theta < 90°$). In this case, the resultant adhesive force is greater than the resultant cohesive force. Therefore, the glass wall pulls up liquid and liquid tends to stick (*i.e.*, wet) to the glass. As a result, the meniscus of the liquid becomes concave. To sum up, when angle of contact is acute, the :

(*a*) liquid will wet the solid.

(*b*) meniscus of the liquid will be concave.

(*c*) liquid will rise in the capillary tube made of such a solid.

2. When the angle of contact is obtuse. Fig. 19.29 (*ii*) shows that angle of contact between mercury and the solid is obtuse (*i.e.*, $\theta > 90°$). In this case, the resultant cohesive force is greater than the resultant adhesive force. Therefore, the mercury gets depressed near the glass wall and mercury surface pulls from the glass wall (*i.e.*, mercury does not wet the glass). As a result, the meniscus of mercury becomes convex. To sum up, when angle of contact is obtuse, the :

(*a*) liquid will not wet the solid.

(*b*) meniscus of the liquid will be convex.

(*c*) liquid will get depressed in the capillary tube made of such a solid.

3. When angle of contact is zero. The angle of contact for pure water and clean glass is zero. In this case, the resultant adhesive force is equal to the resultant cohesive force. Therefore, in the glass tube, the meniscus of water will be exactly hemispherical.

Note. A surface will be wetted by water if the angle of contact is less than 90°. Addition of soap to water reduces the angle of contact and is, therefore, useful in washing.

19.32. CAPILLARITY

When a glass *capillary tube open at both ends is dipped vertically in water, the water in the tube will rise above the level of water in the vessel as shown in Fig. 19.30 (*i*). In case of mercury, the liquid is depressed in the tube below the level of mercury in the vessel as shown in Fig. 19.30 (*ii*). A liquid whose angle of contact is less than 90° suffers capillary rise. On the other hand, a liquid whose angle of contact is greater than 90° suffers capillary depression. If $\theta = 90°$, the liquid will neither rise nor fall. The narrower the tube, the greater is the rise or fall of the liquid and *vice-versa.*

The rise or fall of a liquid in a narrow tube is called **capillarity** *or* **capillary action.**

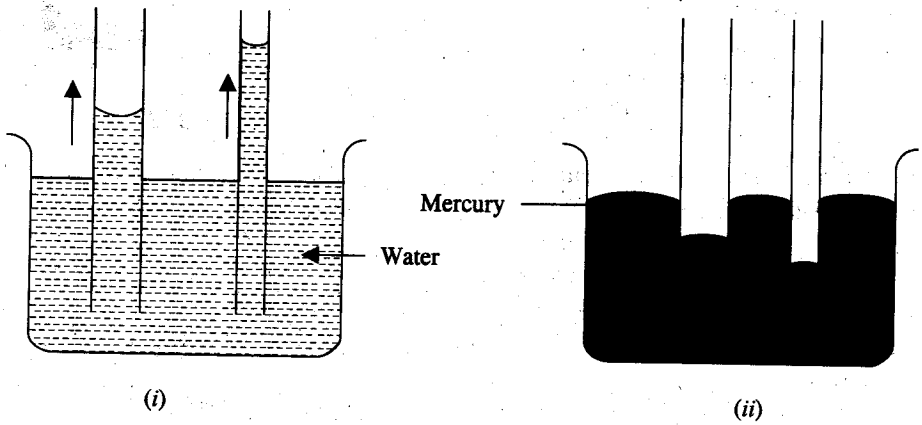


Fig. 19.30

* A tube of very small diameter is called a capillary tube. Capillary means hair-like.

When the capillary tube is dipped into water, the adhesion between the glass and the water causes a thin film of water to be drawn up over the surface of the tube. Surface tension causes this film to contract. The film on the inner surface continues to contract, raising water with it until the adhesive force is balanced by the weight of water lifted. In a narrow tube, the weight of water in the tube is small and water is lifted higher than if the tube bore were large. Similarly, the capillary depression of mercury can be explained considering the fact that the cohesive force between mercury molecules is much larger than the adhesive force between mercury and glass.

19.33. CAPILLARY RISE (ASCENT FORMULA)

Capillary action is a result of both adhesion and surface tension. Suppose a capillary tube is held vertically in a liquid (*e.g.*, water) which has a concave meniscus. We know that there always exists an excess pressure equal to $2T/R$ on the concave side of liquid-air interface, where T is the surface tension of the liquid and R is the radius of curvature of the meniscus. Therefore, the pressure just below the meniscus is less than the pressure just above it by $2T/R$. Now the capillary tube is open to atmosphere so that the pressure on the upper side of the meniscus is atmospheric pressure P. Hence pressure just below the meniscus is $P - (2T/R)$ as shown in Fig. 19.31 (*i*).

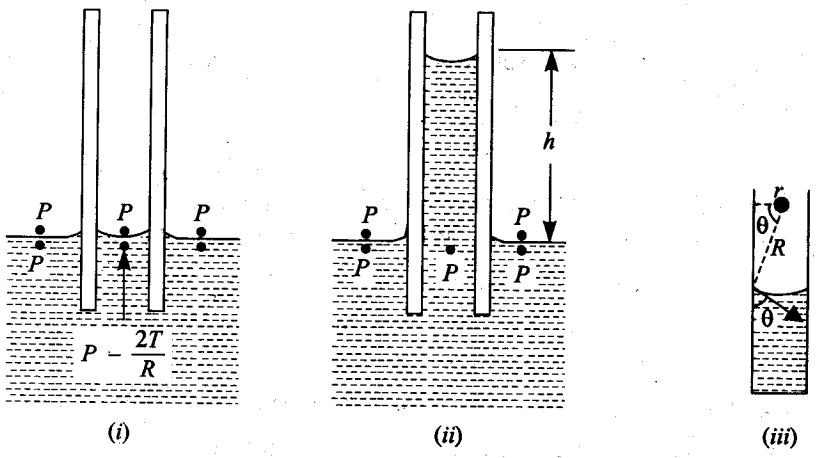


Fig. 19.31

The pressure outside the tube at the level of the meniscus is the atmospheric pressure P. Since pressure at all points in the same level of liquid must be the same, the liquid is pushed up in the tube. The rising of liquid in the capillary stops at a certain height h as shown in Fig. 19.31 (*ii*). In this position, the pressure of the liquid-column of height h becomes equal to $2T/R$ *i.e.*,

$$h\rho g = \frac{2T}{R} \qquad ...(i)$$

Here ρ is the density of the liquid and g is acceleration due to gravity.

Suppose r is the radius of the capillary tube and θ is the angle of contact of the liquid as shown in Fig. 19.31 (*iii*). Then radius of curvature R of the meniscus is given by ;

$$R = \frac{r}{\cos\theta}$$

Putting $R = r/\cos\theta$ in eq. (*i*), we have,

$$h\rho g = \frac{2T}{r/\cos\theta}$$

or

$$h = \frac{2T\cos\theta}{r\rho g} \qquad ...(ii)$$

Eq. (ii) gives the expression for capillary rise for a liquid which has a concave meniscus i.e., angle of contact is acute. Note that narrower the tube, the greater is the height to which the liquid rises.

If the capillary tube is held vertically in a liquid that has a convex meniscus, then angle of contact θ is obtuse. Therefore, $\cos\theta$ is negative and so is h. This means that the liquid will suffer capillary depression.

Rise of liquid in a tube of insufficient length. Suppose that a capillary tube is held vertically in a liquid which has a concave meniscus. If ρ and T are the density and surface tension of the liquid respectively, then capillary rise h is given by ;

$$h\rho g = \frac{2T}{R}$$

where R is the radius of curvature of the liquid meniscus in the tube.

or $$hR = \frac{2T}{\rho g} = \text{Constant (for a given liquid)}$$

When the length of the tube is greater than h, the liquid rises in the tube so as to satisfy the above relation. But if the length of the tube is insufficient (*i.e.*, less than h), say h', the liquid rises up to the top of the tube and spreads itself on the material of the tube. The radius of curvature increases to R' in such a way that :

$$h'R' = hR = \frac{2T}{\rho g}$$

It is clear that the liquid cannot emerge in the form of a fountain from the upper end of a short capillary tube. It is because it will violate the law of conservation of energy.

19.34. PRACTICAL APPLICATIONS OF CAPILLARITY

We encounter several examples of capillarity in our everyday life. However, a few practical applications of capillarity are given below by way of illustration.

(*i*) Oil rises in the cotton wicks of lamps through the small capillaries between the threads.

(*ii*) Due to capillary action, water rises to the surface of the fields and gets evaporated. If the soil is made loose by ploughing, the effective diameters of the capillaries in the soil are increased. This prevents the rise of water from the lower layers to the surface.

(*iii*) Water comes out of an earthen pot through small pores by capillary action.

(*iv*) Water rises in the plants up the leaves through fine pores of the fibres of wood.

(*v*) To avoid damping of bricks on account of rise of moisture, damp proof materials are used in the foundations of buildings.

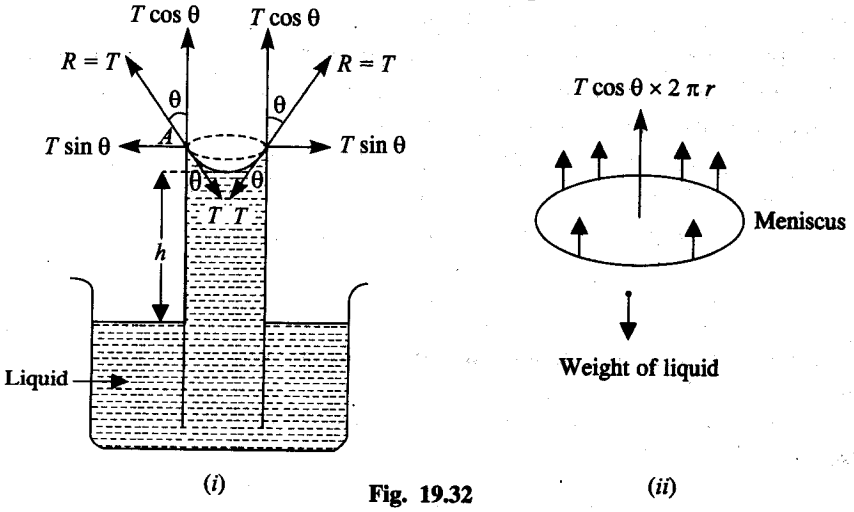
(*vi*) Writing nib is split in the middle so that a fine capillary is formed in it. When it is dipped in ink, the ink rises in the capillary.

(*vii*) Coffee powder is easily soluble in water because water immediately wets the fine granules of coffee by capillary action.

19.35. ALTERNATIVE PROOF FOR FORMULA OF CAPILLARY RISE

We have already derived the formula (See Art. 19.33) for the capillary rise of a liquid by using excess pressure formula ($p_2 - p_1 = 2T/R$). We now consider an alternative proof. Suppose that a capillary tube is held vertically in a liquid (*e.g.*, water) which has a concave meniscus as shown in Fig. 19.32 (*i*).

Let θ = Angle of contact
h = Height to which the liquid rises (See Fig. 19.32)
r = Radius (internal) of the tube
ρ = Density of the liquid
T = Surface tension of the liquid.



(*i*) **Fig. 19.32** (*ii*)

The surface tension forces cause the liquid to exert a downward directed force T on the walls of the tube. This force T acts along the tangent at the point of contact 'A' . In accordance with Newton's third law, the tube exerts an equal and opposite reaction R (= T). The reaction R (= T) can be resolved into two rectangular components *viz,* $T \sin\theta$ and $T \cos\theta$. The horizontal component $T \sin\theta$ plays *no part and the vertical component $T \cos\theta$ pulls the liquid upward. The component $T \cos\theta$ acts along the whole circumference of the meniscus [See Fig. 19.32 (*ii*)].

$\therefore$ Total upward force = $T\cos\theta \times 2\pi r$

Volume of liquid in the tube above the free surface of liquid is

V = Volume of cylinder of height h and radius r + Volume of cylinder of height r and radius r – Volume of hemisphere of radius r

$$= \pi r^2 h + \pi r^2 \times r - \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$= \pi r^2 h + \pi r^3 - \frac{2}{3}\pi r^3 = \pi r^2\left(h + \frac{r}{3}\right)$$

Weight of liquid column = $V\rho g = \pi r^2\left(h + \frac{r}{3}\right)\rho g$

The upward force (= $T\cos\theta \times 2\pi r$) supports the weight of liquid column above the free surface of the liquid.

$\therefore$ $$T\cos\theta \times 2\pi r = \pi r^2\left(h + \frac{r}{3}\right)\rho g$$

or $$h + \frac{r}{3} = \frac{2T\cos\theta}{r\rho g}$$

or $$h = \frac{2T\cos\theta}{r\rho g} - \frac{r}{3}$$

* The horizontal components all round the tube cancel out and hence they play no part.

If the tube is narrow, $r/3$ can be neglected as compared to h.

$$\therefore \quad h = \frac{2T\cos\theta}{r\rho g} \qquad ...(i)$$

This is the same result as derived in Art. 19.33. This formula is also known as **ascent formula**.

Discussion. This result has several notable features :

(*i*) The rise of liquid in the tube is inversely proportional to the radius (or diameter) of the tube. The narrower the tube, the greater is the height to which the liquid rises.

(*ii*) If θ is less than 90°, $\cos\theta$ is positive and hence h is positive. This means a liquid whose angle of contact is acute will rise in the tube.

(*iii*) If θ is greater than 90°, $\cos\theta$ is negative and hence h is negative. This means a liquid whose angle of contact is obtuse will suffer capillary depression.

(*iv*) If $\theta = 90°$, $h = 0$ and the liquid neither rises nor is depressed.

(*v*) From eq. (*i*) above, we have,

$$T = \frac{hr\rho g}{2\cos\theta}$$

Since the values of h, r, ρ, g and θ are known, the surface tension T of the liquid can be determined.

Example 19.33. **Water rises in a capillary tube to a height of 2.0 cm. In another capillary tube whose radius is one-third of it, how much the water will rise?**

Solution. $$h = \frac{2T\cos\theta}{r\rho g}$$

or $$h\,r = \frac{2T\cos\theta}{\rho g}$$

or $$h\,r = \text{Constant} \qquad [\because T, \theta, \rho, g \text{ are same}]$$

Suppose in a capillary tube of radius one-third, the water rises to a height h'. Then,

$$h'\,(r/3) = h\,r$$

or $$h' = \frac{3hr}{r} = 3h = 3 \times 2.0 = \mathbf{6.0\ cm}$$

Example 19.34. **A glass *U*-tube is inverted with open ends of the straight limbs, of diameters respectively 0.5 mm and 1.0 mm, below the surface of water in a beaker. The air pressure in the upper part is increased until the meniscus in one limb is in level with the water outside. Find the height of water in the other limb. Density of water = 1000 kg m^{-3} and surface tension of water is 7.5 × 10^{-2} Nm $^{-1}$.**

Solution. Fig. 19.33 shows the conditions of the problem. Suppose p is the air pressure inside the U-tube when the meniscus Q is in level with water outside and P is the other meniscus at a height h. Let A be the atmospheric pressure. Then if r_1 (= 0.25 × 10^{-3} m) is the radius at P, then,

$$p-(A-h\rho g) = \frac{2T}{r_1} \qquad ...(i)$$

Since the pressure in the liquid below P is $(A - h\rho g)$,

$\therefore$ Pressure in the liquid below $Q = A$. Therefore for Q,

$$p - A = \frac{2T}{r_2} \qquad ...(ii) \text{ where } r_2 = 0.5 \times 10^{-3} \text{ m}$$

From eqs. (*i*) and (*ii*), we have, $h\rho g = \frac{2T}{r_1} - \frac{2T}{r_2}$

or $$h = \frac{1}{\rho g}\left(\frac{2T}{r_1} - \frac{2T}{r_2}\right)$$

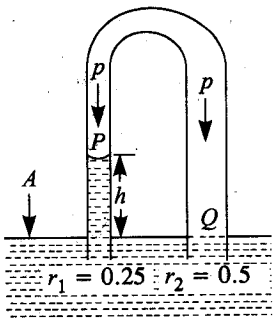


Fig. 19.33

Here $\rho = 1000$ kg m^{-3} ; $g = 9.8$ ms^{-2}; $T = 7.5 \times 10^{-2}$ N m^{-1}.

$$\therefore \quad h = \frac{1}{1000\times9.8}\left[\frac{2\times7.5\times10^{-2}}{0.25\times10^{-3}} - \frac{2\times7.5\times10^{-2}}{0.5\times10^{-3}}\right]$$

$$= 3.1 \times 10^{-2} \text{ m (app.)}$$

Example 19.35. **(*i*) Water rises to a height h inside a clean glass capillary tube of radius 0.2 mm when the tube is placed vertically inside a beaker of water. Calculate h if the surface tension of water is 7.0×10^{-2} Nm^{-1} and the angle of contact is zero.**

(*ii*) The tube is now pushed into water until 4.0 cm of its length is above the surface. Describe and explain what happens.

Density of water= 1000 kg m^{-3}; $g = 10$ ms^{-2}.

Solution. (*i*) Height of water-column in a capillary tube with usual notations is given by ;

$$h = \frac{2T\cos\theta}{r\rho g}$$

Here $T = 7 \times 10^{-2}$ Nm^{-1} ; $\theta = 0°$; $r = 0.2 \times 10^{-3}$ m ; $\rho = 1000$ kg m^{-3} ; $g = 10$ ms^{-2}

$$\therefore \quad h = \frac{2\times7\times10^{-2}\times\cos0°}{0.2\times10^{-3}\times1000\times10} = 7 \times 10^{-2} \text{ m}$$

(*ii*) When the tube is pushed down so that the height h_1 in Fig. 19.34 is 4 cm, the angle of contact at the top of the tube changes from zero to a value θ as shown.

Now $\cos\theta = \frac{r}{R}$

where R is the radius of the meniscus.

Excess pressure, $p_2 - p_1 = \frac{2T}{R}$

where p_2 = atmospheric pressure P_a and $p_1 = P_a - h_1\rho g$.

$$\therefore \quad h_1\rho g = \frac{2T}{R}$$

Also $h\rho g = \frac{2T}{r}$

Dividing, we get, $\frac{h_1}{h} = \frac{r}{R} = \cos\theta$

$$\therefore \quad \cos\theta = \frac{4}{7} \text{ or } \theta = 55°$$

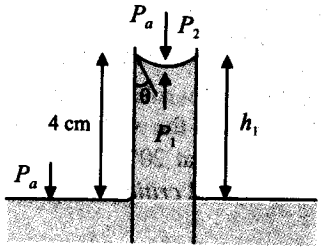


Fig. 19.34

PROBLEMS FOR PRACTICE

1. The radius of a capillary tube is 0.025 mm. It is held vertically in a liquid whose density is 0.8×10^3 kg m^{-3}, surface tension is 3.0×10^{-2} Nm^{-1} and for which the cosine of angle of contact is 0.3. Determine the height to which the liquid will rise in the tube relative to the liquid surface outside. Take $g = 10$ ms^{-2}. **[9 cm]**
2. A capillary tube whose inside radius is 0.5 mm is dipped in water having surface tension 7.5×10^{-2} Nm^{-1}. To what height is the water raised above the normal water level? Angle of contact of water with glass is assumed to be 0°. **[3 cm]**
3. Water rises up in a glass capillary upto a height of 9.0 cm while mercury falls down by 3.4 cm in the same capillary. Assume angles of contact for water-glass and mercury- glass as 0° and 135° respectively. Determine the ratio of surface tensions of mercury and water. Density of mercury = 13.6×10^3 kg m^{-3}, density of water = 10^3 kg m^{-3}. **[7.2 : 1]**

4. A capillary tube of radius 0.4 mm is dipped vertically in water. Find upto what height the water will rise in the capillary. If the capillary is inclined at an angle of 60° with the vertical, how much length of the capillary is occupied by water ? Surface tension of water = 7.0×10^{-2} N m^{-1}, density of water = 10^3 kg m^{-3}. **[3.57 cm ; 7.14 cm]**

5. Liquid rises up in a glass capillary upto a height of 7.0 cm. The radius of the tube is 0.1 mm and density of liquid is 0.8×10^3 kg m^{-3}. If the angle of contact between the liquid and the wall of the tube is zero, determine the surface tension of the liquid. **[27.4×10^{-3} N m^{-1}]**

19.36. FACTORS AFFECTING SURFACE TENSION

The surface tension of a liquid depends upon (*i*) its nature (*ii*) contamination and (*iii*) temperature.

1. **Nature of liquid.** Different liquids have different surface tension. For example, mercury has a very high surface tension of 0.55 N/m while surface tension of water is 0.072 N/m.

2. **Contamination.** In general, impurities in a liquid lower its surface tension. It is because addition of impurities results in liquid spreading out as a thin film which means surface tension is lowered.

   Applications

   (*i*) The cleaning action of soap is due to its ability to lower the surface tension of water, making it possible for the water and soap to penetrate more readily into the pores of the cloth being washed. In other words, soap decreases the forces of adhesion between small dust particles and the cloth. Consequently, dust particles are easily removed.

   (*ii*) Mosquitoes hang their eggs from the surface of water. When a small amount of oil is poured on water, the surface tension of water is reduced. This breaks the elastic film of water surface and mosquitoes eggs are killed by drowning.

3. **Temperature.** The surface tension of a liquid decreases with the increase in temperature. Thus the surface tension of water is about 0.0756 N/m at 0°C but it becomes 0.072 N/m at 20°C. The temperature at which the surface tension of a liquid becomes zero is called *critical temperature*. The variation of surface tension with temperature is given by :

$$T_t = T_0 (1 - \alpha t)$$

where T_0 = Surface tension at 0°C

T_t = Surface tension at t°C

α = Temperature coefficient of surface tension

Applications

(*i*) The surface tension of hot soup is less than that of cold soup. Consequently, hot soup will spread over a large area of the tongue. For this reason, hot soup is tastier than the cold soup.

(*ii*) If you float two matchsticks on the surface of water and touch the surface of water between the matchsticks by a hot needle, the matchsticks will fly apart. It is because hot needle lowers the surface tension of water between the matchsticks and the pull of the water molecules on the outside of matchsticks becomes greater than the pull on the inside. As a result, the matchsticks fly apart.

19.37. ACTION OF DETERGENTS

Detergent means cleansing agent. We clean dirty clothes containing grease and oil stains by adding detergent or soap to water. Washing with water alone does not remove grease stains. It is

because the surface tension in water is so great that individual water molecules are unable to leave the water to pass through the fabric. When we add soap solution to water, surface tension of water is greatly reduced *i.e.* the cohesive force between water molecules is greatly reduced. This allows the water molecules to break apart and penetrate more easily into the fabric of the cloth where it dissolves dirt. If soap solution in water is heated, the surface tension is further reduced. It is for this reason that hot soap solution is a better cleansing agent than the cold one.

Note. Soap solution also decreases the adhesive force between dirt and fabric.

CONCEPTUAL QUESTIONS

Q.1. What is the importance of fluids to living organisms?

Ans. Fluids are of paramount importance to living organisms. The human body is two-third water and liquids in the body not only transport nutrients to its living cells but also carry waste products from those cells. The oxygen we need for combustion of food flows in and out of the respiratory system along with other molecules in the air.

Q.2. Why does not the pressure of atmosphere break windows?

Ans. Atmospheric pressure is exerted on both sides of a window, so no net force is exerted on the window. If for some reason, the pressure is reduced or increased on one side only, then window may break.

Q.3. Can water be used to make a barometer?

Ans. The answer is yes, but the glass tube would have to be much longer — 13.6 times as long. Therefore, a water barometer would have to be atleast = $13.6 \times 0.76 = 10.3$ m — too tall to be practical.

Q.4. What is the maximum height to which water could be drunk through a straw?

Ans. What happens in a barometer is similar to what happens during the process of drinking water through a straw. By sucking, you reduce the pressure in the straw that is placed in water. Atmospheric pressure on water pushes liquid up into the reduced-pressure region. Strictly speaking, the liquid is not sucked up ; it is pushed up by the pressure of atmosphere. Since atmospheric pressure is 10.3 m of water column, the maximum height to which water can be sucked is 10.3 m.

Q.5. A helium-filled balloon rises to a certain height and then halts. Why?

Ans. Initially, the upward buoyant force due to air is greater than the weight of the helium and the balloon. It is because the balloon displaces weight of air greater than its own weight. Since air becomes less dense with altitude, a lesser weight of air is displaced as the balloon rises. At some altitude, the weight of air displaced becomes equal to the total weight of the balloon and it stops rising. We can also say that when the buoyant force on the balloon equals its weight, the balloon will cease rising.

Q.6. Two balloons that have the same weight and volume contain equal amounts of helium. One is rigid and the other is free to expand as outside pressure decreases. When released, which balloon will rise higher?

Ans. The balloon that is free to expand will displace more air as it rises than the balloon that is restrained. Hence the balloon that is free to expand will experience more buoyant force and rise higher.

Q.7. Does Archimedes' principle tell us that if an immersed object displaces 10 N of fluid, then buoyant force on the object is 10 N?

Ans. Yes. Looking at it another way, the immersed object pushes 10 N of fluid aside. The displaced fluid reacts by pushing back on the immersed object with 10 N.

Q.8. A boulder is thrown into a deep lake. As it sinks deeper and deeper into water, does the buoyant force change?

Ans. The buoyant force does not change as the boulder sinks because the boulder displaces the same volume of water at any depth. Since water is practically incompressible, its density is practically the same at all depths; hence the weight of water displaced or the buoyant force is the same at all depths.

Q.9. A 1 litre container completely filled with lead has a mass of 11.3 kg and is completely submerged in water. What is the buoyant force acting on it?

Ans. The buoyant force is equal to the weight of 1 kg (*i.e.*, 9.8 N) because the volume of water displaced is 1 litre which has a mass of 1 kg and a weight of 9.8 N. The 11.3 kg of the lead is irrelevant. Note that 1 litre of anything completely submerged in water will displace 1 litre of water and the upward force will be 9.8 N.

Q.10. Does Archimedes' principle hold in a vessel in a free fall?

Ans. For a free fall, the effective value of acceleration due to gravity is zero. As a result, the buoyant force will not exist. Therefore, Archimedes' principle will not hold good in this situation.

Q.11. A wooden cylinder floats in water in a vessel with its axis vertical. How will the level of water in the vessel change if the cylinder floats with its axis horizontal?

Ans. There will be no change in the level of water in the vessel. It is because in both the positions of the cylinder, the same volume is displaced by the cylinder.

Q.12. A man is sitting in a boat which is floating in a pond. If the man drinks some water from the pond, what will happen to the level of water in the pond?

Ans. The level of water in the pond will not change. The reason is simple. If m is the mass of water drunk by the man and ρ_w is the density of water, then,

$$\text{Decrease in volume of water in the pond} = \frac{m}{\rho_w}$$

On drinking water, the mass of man increases by m and the volume of water displaced by the boat will increase by m/ρ_w. As a result, there will be no change in the level of water in the pond.

Q.13. What is the principle of floatation?

Ans. When a body floats, the upthrust on it must be equal to its weight because it moves neither up nor down. Therefore, according to Archimedes' principle, the weight of body is equal to the weight of liquid displaced.

Upthrust or buoyant force = Weight of body ... Principle of floatation

or Weight of liquid displaced = Weight of body

Q.14. Why is gas not used in a hydraulic lift?

Ans. Although Pascal's principle applies to both liquids and gases, a gas cannot be used as a working fluid in a hydraulic lift. It is because gases are compressible and most of the effort would go into compressing the gas rather than into moving the load.

Q.15. Why does a small quantity of liquid assume spherical shape?

Ans. Due to surface tension forces, the free surface of a liquid tends to have the smallest possible surface area. For a given volume, a sphere has the least surface area. Therefore, in the absence of external forces (*e.g.*, gravitational force etc.), the free surface of a liquid would tend to assume spherical shape. A small liquid drop is spherical because the gravitational force on it is very small and the liquid shape is then mainly due to surface tension forces.

Q.16. A drop of oil placed on the surface of water spreads out, but a drop of water placed on oil contracts. Why?

Ans. The cohesive forces between the oil molecules are less than the adhesive forces between water molecules and oil molecules. Therefore, the drop of oil placed on water surface spreads out. On the other hand, the cohesive forces between water molecules are greater than the adhesive forces between water molecules and oil molecules. So the drop of water placed on oil surface contracts.

Q.17. Oil is sprinkled on sea waves to calm them. Why?

Ans. On sprinkling oil, the breeze spreads the oil on the sea water in its own direction. The surface tension of sea water without oil is greater than the oily water. Therefore, the water without oil pulls the oily water against the direction of breeze. As a result, the sea waves become calm.

Q.18. Why are soap bubbles almost perfect spheres?

Ans. The soap bubbles are almost perfect spheres because they have large surface areas and negligible masses. The effect of gravitational force is almost nil and the shape of the bubbles is mainly due to surface tension forces.

Q.19. Why are falling rain drops spherical?

Ans. The rain drops falling freely under gravity are spherical. It is because every part of the drop is being accelerated to the same extent and the acceleration cannot, therefore, affect the shape of the drop.

Q.20. Molecules on the free surface of a liquid have more potential energy than those in the interior. Why?

Ans. In the interior, no *net* force acts on a molecule but as it approaches the surface, the attractive forces pulling it back dominate. Therefore, work must be done on the molecule to bring it to the surface. So molecules on the free surface of a liquid have more potential energy than those in the interior.

Q.21. What determines the shape of the meniscus?

Ans. The surface of a liquid is generally curved where it is in contact with a solid. Whether the meniscus is concave or convex depends on the liquid concerned and on the solid with which it is in contact. If the adhesive force is large compared with the cohesive force, the liquid tends to stick to the wall and so has a concave meniscus. In this case, the angle of contact in less than 90°. On the other hand, if the adhesive force is small compared with the cohesive force, the liquid surface is pulled away from the wall and the meniscus is convex. In this case, the angle of contact is greater than 90°.

Q.22. Why is it not possible to separate two pieces of paper joined by glue or gum?

Ans. The adhesive force between the glue and paper is greater than the force of cohesion between glue molecules. For this reason, paper pieces stick together with a very large force and it is not possible to separate them.

Q.23. What shape does a liquid take when it weighs nothing?

Ans. When a liquid weighs nothing, surface tension is the only force acting on it. Due to surface tension, the liquid surface tends to have the smallest possible surface area. Since for a given volume, a sphere has the smallest surface area, the liquid assumes spherical shape.

Q.24. Why is soap solution a better cleansing agent than ordinary water?

Ans. A cloth has narrow spaces in the form of fine capillaries. The capillary rise is given by ;

$$h = \frac{2T\cos\theta}{r\rho g}$$

Addition of soap to water reduces the angle of contact θ. This will increase the value of $\cos\theta$ and hence h. This means that soap water will rise more in the narrow spaces in the cloth and clean fabrics better than water can do alone.

Q.25. The antiseptics used for cuts and wounds in human flesh have low surface tension. Why?

Ans. A liquid spreads more on a surface if its own surface tension is less. Since the surface tension of antiseptics is less, they spread more on the cuts and wounds. As a result, the cut or wound is healed quickly.

VERY SHORT ANSWER QUESTIONS

Q.1. What are the SI units and dimensional formula of pressure?

Ans. The SI unit of pressure is Nm^{-2} and its dimensional formula is $[ML^{-1}T^{-2}]$.

Q.2. Is pressure a scalar or vector quantity?

Ans. Pressure is a scalar quantity. Now pressure = Force/area. Although force is a vector quantity, the pressure at any point in a static fluid is exerted equally in all directions. Therefore, it makes no sense to associate direction with pressure.

Q.3. Why can a liquid change its shape easily while a solid cannot?

Ans. It is because the forces of cohesion between molecules in liquids are very small as compared to that of solids.

Q.4. The bags and suitcases are provided with broad handles. Why?

Ans. Pressure = Force/Area. Due to large area of broad handles, the pressure exerted on the hand is small while carrying them.

Q.5. What are the two commonly used units of pressure in the science of meterology?

Ans. The two commonly used units of pressure in meterology are torr and bar.

1 torr = 1 mm of Hg ; 1 bar = 10^5 N m^{-2}

Q.6. Express 1 torr in N m^{-2}.

Ans. 1 torr = Pressure exerted by 1 mm of mercury column

$= h \rho g = (1 \times 10^{-3}) \times 13600 \times 9.8 = 133.3$ Nm^{-2}

Q.7. What is the relation between torr and bar?

Ans. 1 torr = 133.3 $Nm^{-2} = \frac{133.3}{10^5}$ bar.

Q.8. Why do fluids exert pressure on the walls of the containing vessel?

Ans. It is because fluids (liquids and gases) transmit pressure equally in all directions.

Q.9. The atmospheric pressure is about 10^5 Nm^{-2}. What does it mean?

Ans. It means that the weight of air on 1 m^2 surface of earth is about 10^5 N.

Q.10. Why do we not feel the crushing weight of air?

Ans. It is because the pressure inside our bodies equals that of the surrounding air. There is no net force for us to sense.

Q.11. Why is the dam of water reservoir thick at the bottom?

Ans. Pressure due to a liquid column of height h is $P = h \rho g$. Since pressure of water increase with depth, the dam is made thick at the bottom.

Q.12. What are the factors which affect the atmospheric pressure at a place?

Ans. Pressure, $P = h \rho g$. Therefore, the factors that affect the atmospheric pressure at a place are (*i*) height (h) of the atmosphere (*ii*) density (ρ) of the atmosphere and (*iii*) acceleration due to gravity (g).

Q.13. Why does not water come out of a dropper unless the rubber bulb is pressed?

Ans. Water is held inside the dropper against the atmospheric pressure. When we press the rubber bulb, the pressure on water exceeds the atmospheric pressure. Therefore, the water comes out of the dropper.

Q.14. What is thrust?

Ans. The force acting normally on a surface is called thrust. Its SI unit is N and the dimensional formula is $[MLT^{-2}]$

Q.15. It is difficult to stop bleeding from a cut in the body at high altitude. Why?

Ans. At high altitude, the atmospheric pressure P $(= h \rho g)$ is less so that blood pressure becomes greater than the atmospheric pressure. As a result, it becomes difficult to stop the bleeding from a cut in the body.

Q.16. What is a buoyant force?

Ans. The upward thrust exerted by a liquid on a body immersed wholly or partially in it is called buoyant force or buoyancy.

Q.17. What is Archimedes' Principle?

Ans. The buoyant force on a body wholly or partially submerged in a liquid is equal to the weight of liquid displaced by the body.

Q.18. What is the apparent weight of a block of wood floating in a lake?

Ans. The apparent weight of the floating block of wood is zero. It is because the weight of the block acting vertically downward is balanced by the buoyant force (upward thrust) due to water acting on the block.

Q.19. A body floats in a liquid contained in a beaker. If the system falls freely under gravity, what is the buoyant force?

Ans. For a free fall, the effective value of acceleration due to gravity is zero (*i.e.*, $g = 0$). As a result, the buoyant force F_B $(= \rho_l g V_l)$ on the body due to the liquid will be zero.

Q.20. What are the factors upon which buoyant force depends?

Ans. The buoyant force F_B $(= \rho_l g V_l)$ depends upon (*i*) the density of the liquid (*ii*) acceleration due to gravity and (*iii*) volume of the body inside the liquid.

Q.21. What is the principle of floatation?

Ans. Buoyant force = Weight of the body.

Q.22. The buoyant force exerted by a liquid on a body immersed in it is 5 N. What is the weight of liquid displaced by the body?

Ans. 5 N. It is because the weight of liquid displaced by the body is equal to the buoyant force.

Q.23. How do you explain the phenomenon in Q. 22?

Ans. Here the immersed body pushes 5 N of liquid aside. The displaced liquid reacts by pushing back on the immersed body with 5 N.

Q.24. A piece of ice is floating in a vessel containing water. What will be the effect on the level of water when ice melts?

Ans. The level of water will remain the same. The reason is simple. For floating ice, weight of ice is equal to the weight of water displaced by the immersed part of ice. When ice melts, the volume of water formed must be equal to the volume of water displaced by the immersed part of ice.

Q.25. Why is it easier to swim in sea water than in the river water?

Ans. The density (ρ) of sea water is greater than that of the river water. As a result, buoyant force F_B $(= \rho g V)$ on the swimmer is greater in sea water than in river water. For this reason, it is easier to swim in sea water than in river water.

Q.26. What is surface tension?

Ans. It is the property of a liquid at rest by virtue of which its free surface behaves like a stretched membrane and tries to occupy minimum area.

Q.27. Give the SI unit and dimensional formula of surface tension.

Ans. The SI unit of surface tension is Nm^{-1} and its dimensional formula is $[ML^0T^{-2}]$.

Q.28. **What is the effect of temperature on the surface tension of a liquid?**

Ans. The surface tension of a liquid decreases with the increase in temperature.

Q.29. **What is the surface tension of water?**

Ans. The surface tension of water at 20°C is 7×10^{-2} Nm^{-1}.

Q.30. **What is a critical temperature?**

Ans. The temperature at which the surface tension of a liquid becomes zero is called critical temperature.

Q.31. **The addition of flux makes soldering easy. Why?**

Ans. The addition of flux reduces the surface tension of the molten tin so that it spreads easily over the junction.

Q.32. **Why does oil spread over water surface?**

Ans. The surface tension of oil is less than that of water. Therefore, the oil spreads over a large surface of water (large surface tension of water forces the oil to spread).

Q.33. **Why is the surface tension of lubricating oils and paints kept low?**

Ans. Due to low surface tension, the lubricating oils and paints spread over a large area.

Q.34. **On which side of the curved liquid surface, pressure is more?**

Ans. For a curved liquid surface in equilibrium, the pressure on the concave side of the liquid surface is more.

Q.35. **Air is blown into a soap bubble to increase its size. What will be the effect on the air pressure inside it?**

Ans. Excess pressure inside a soap bubble = $4T/R$. When air is blown into the soap bubble, its radius (R) will increase. Therefore, excess pressure inside the bubble will decrease.

Q.36. **What is the work done in forming a soap bubble of radius R?**

Ans. Work done = $2\ (4\pi R^2 \times T) = 8\pi R^2 T$ where T is the surface tension of soap.

Q.37. **What is angle of contact?**

Ans. It is the angle (θ) which the tangent to the liquid surface at the point of contact makes with the wall of the solid within the liquid.

Q.38. **What is the shape of the meniscus if the angle of contact is (*i*) less than 90° (*ii*) more than 90°?**

Ans. (*i*) concave meniscus (*ii*) convex meniscus.

Q.39. **What are the factors on which angle of contact depends?**

Ans. It depends on (*i*) the nature of the liquid and the solid in contact (*ii*) temperature of the liquid.

Q.40. **What is the effect of temperature of liquid on the angle of contact?**

Ans. It increases with the increase of temperature.

Q.41. **One end of a capillary tube is immersed in a liquid whose angle of contact is (*i*) less than 90° (*ii*) more than 90°. What will happen to water level in the tube?**

Ans. (*i*) The liquid in the tube will rise above the level of the liquid in the vessel.

(*ii*) The liquid in the tube will be depressed below the level of the liquid in the vessel.

Q.42. **A mercury barometer always reads less than the actual pressure. Why?**

Ans. Since angle of contact of mercury is obtuse, mercury column suffers capillary depression. For this reason, the reading is less than the actual pressure.

Q.43. **In summer, we prefer cotton dress. Why?**

Ans. The cotton cloth has fine pores. When we sweat in summer, water (sweat) is sucked by the cotton cloth due to capillary action.

Q.44. **Why does the water-proof raincoat not wet in rain?**

Ans. The angle of contact between water and the material of the raincoat is obtuse. Therefore, the raincoat does not wet in the rain.

Q.45. **When wax is rubbed on cloth, the cloth becomes water proof. Why?**

Ans. When wax is rubbed on cloth, the capillaries in the cloth disappear and the cloth becomes water proof.

SHORT ANSWER QUESTIONS

Q.1. The boots for soccer and hockey players have studs on their soles. Why?

Ans. This is to avoid slipping during playing. The reason is simple. The studs decrease the area of contact between the feet of the player and the ground. As a result, pressure of the feet on the ground increases and the player can run about on the ground without slipping.

Q. 2. Three vessels of different shapes are filled with water to the same height. The base area of the vessels is the same. Will the force on the base of vessels be the same?

Ans. Yes. Pressure, $P = h\,\rho\,g$. Since height (h) of water column is the same in the three vessels, the pressure at the base will be the same. Now the vessels have the same base area. Therefore, force ($= P \times$ area) on the base in the three cases will be the same.

Q.3. What is the pressure on a swimmer 10 m below the surface of a lake?

Ans. $P = P_a + h\rho g = 1.01 \times 10^5 + 10 \times 10^3 \times 9.8 = 1.99 \times 10^5\ \text{Nm}^{-2}$.

Q.4. A tank 5 m high is half filled with water and then filled to the top with oil of density 0.85 g/cm^3. What is the pressure at the bottom due to these liquids?

Ans. $P = h_1\,\rho_1\,g + h_2\,\rho_2\,g$

Here $\rho_1 = 10^3\ \text{kg/m}^3$; $\rho_2 = 0.85 \times 10^3\ \text{kg/m}^3$; $h_1 = h_2 = 2.5$ m

$\therefore$ $P = (2.5 \times 10^3 + 2.5 \times 0.85 \times 10^3) \times 9.8 = 45.325 \times 10^3\ \text{Nm}^{-2}$

Q.5. The pressure of water at the ground level is 196000 Nm^{-2} and at the third floor is 98000 Nm^{-2}. What is the height of third floor?

Ans. Pressure difference = 196000 – 98000 = 98000 Nm^{-2}. Let h metres be the height of the third floor. Then,

$$P = h\,\rho\,g \quad \text{or} \quad h = \frac{P}{\rho\,g} = \frac{98000}{10^3 \times 9.8} = 10\ \text{m}$$

Q.6. Estimate the height of atmosphere.

Ans. Atmospheric pressure, $P = h\rho g$. Here $P = 1.01 \times 10^5\ \text{Nm}^{-2}$; $g = 9.8\ \text{ms}^{-2}$ and $\rho = 1.293$ kg/m^3 (assumed constant).

$\therefore$ $$P = h\rho g \quad \text{or} \quad h = \frac{P}{\rho\,g} = \frac{1.01 \times 10^5}{1.293 \times 9.8} \simeq 8000\ \text{m}$$

Q.7. What will be the height of water barometer?

Ans. Atmospheric pressure, $P = 1.01 \times 10^5\ \text{Nm}^{-2}$

Now $$P = h_w\,\rho_w\,g \quad \text{or} \quad h_w = \frac{P}{\rho_w\,g} = \frac{1.01 \times 10^5}{10^3 \times 9.8} = 10.3\ \text{m}$$

Note that tube required for water barometer is very long and it is difficult to hold it in the vertical position. For this reason, mercury is preferred as a barometric substance over water.

Q.8. A boat carrying a number of large stones is floating in a water tank. What will happen to water level if the stones are unloaded into water?

Ans. Let m be the mass of stones and ρ_w and ρ_s the density of water and stones respectively. When the stones are lying in the boat, then volume of water displaced by stones is $V = m/\rho_w$. When the stones are unloaded into water, the volume of water displaced by stones is $V' = m/\rho_g$. Since $\rho_s > \rho_w$, $V' < V$. Therefore, water level falls.

Q.9. A small iron needle sinks in water while a large iron ship floats. Why?

Ans. For a body to float, the condition is

Weight of liquid displaced = Weight of the body

In case of needle, the weight of water displaced is much less than the weight of the needle. Therefore, the needle sinks. In case of a large ship, the weight of water displaced is more than the weight of ship. Therefore, ship floats.

Q.10. The force required by a man to move his limbs immersed in water is smaller than the force for the same movement in air. Why?

Ans. The upthrust (buoyant force) on a body is much more in water than that when immersed in air ($F_B = \rho\, g\, V$). Therefore, the effective weight of limbs of a man is less in water than in air. For this reason, the force required to move his limbs in water is much smaller than the force for the same movement in air.

Q.11. What is the normal blood pressure of a person?

Ans. The blood pressure is measured in mm of mercury. We measure the blood pressure of a person under two conditions *viz* (*i*) when the heart is contracted to its smallest size (*systolic*) and the pumping of blood is hardest (*ii*) when the heart is expanded to its largest size (*diastolic*). For a normal person, the systolic blood pressure is 120 mm of mercury and diastolic blood pressure is 80 mm of mercury.

Q.12. A rectangular tank is 10 m long, 5 m broad and 3 m high. It is filled to the brim with water. What is the pressure at (*i*) the bottom (*ii*) the walls due to water?

Ans. (*i*) $P = h\,\rho\,g = 3 \times (10^3) \times (9.8) = 29.4 \times 10^3\ \text{Nm}^{-2}$

(*ii*) Average pressure on walls $= \dfrac{0 + h\rho g}{2} = \dfrac{0 + 29.4 \times 10^3}{2} = 14.7 \times 10^3\ \text{N m}^{-2}$

Q.13. What happens to surface tension of a liquid when impurity is added to it?

Ans. When impurity is added to a liquid, its surface tension increases.

Q.14. In a capillary tube, water rises by 2 mm. What is the height to which water rises in a capillary tube having one-third radius of the first?

Ans. Capillary rise, $h = \dfrac{2T\cos\theta}{r\rho g}$. Since T, θ, ρ and g are constant, $h \propto 1/r$.

$$\therefore \quad \frac{h_2}{h_1} = \frac{r_1}{r_2} \quad \text{or} \quad \frac{h_2}{2} = \frac{r_1}{r_1/3} \quad \therefore \quad h_2 = 6\ \text{mm}$$

Q.15. What is the work done against surface tension in blowing a soap bubble from a radius 10 cm to 20 cm? The surface tension of soap solution is 25×10^{-3} N/m.

Ans. The soap bubble has two surfaces, external and internal.

Original total surface area, $A_1 = 2 \times 4\pi R_1^2 = 8\pi R_1^2$

Final total surface area, $A_2 = 2 \times 4\pi R_2^2 = 8\pi R_2^2$

Increase in surface area, $\Delta A = 8\pi\left[R_2^2 - R_1^2\right] = 8\pi\left[(0.2)^2 - (0.1)^2\right] = 0.24\,\pi\, m^2$

$\therefore$ Work done $= T \times \Delta A = (25 \times 10^{-3}) \times 0.24\pi = 6\pi \times 10^{-3}$ J

Q.16. Ice floats in water with about nine-tenth of its volume submerged. What is the fractional volume submerged for an iceberg floating on a fresh water lake of a planet whose gravity is ten times that of the earth?

Ans. Consider a body of volume V and density ρ. If this body floats in a liquid of density ρ_l with V' volume of the body submerged in the liquid, then,

$$\frac{V'}{V} = \frac{\rho}{\rho_l} \quad \text{or} \quad V' = V \times \frac{\rho}{\rho_l}$$

It is clear that fraction of volume submerged is independent of the value of acceleration due to gravity (g). Therefore, on the new planet, the iceberg will also float in water with nine-tenth of its volume submerged.

Q.17. Water rises in a capillary tube to a certain height such that the upward force due to surface tension is balanced by 75×10^{-4} N force due to the weight of the liquid. If surface tension of water is 6×10^{-2} N/m, what is the inner circumference of the capillary tube?

Ans. Weight of liquid $= mg = 75 \times 10^{-4}$ N ; $T = 6 \times 10^{-2}$ N/m

Now $2\pi r \times T = mg \quad \therefore \quad 2\pi r = \dfrac{mg}{T} = \dfrac{75 \times 10^{-4}}{6 \times 10^{-2}} = 12.5 \times 10^{-2}$ m

Q.18. A body is floating in a liquid ; their densities being equal. What will happen to the body if it is slightly pressed and released?

Ans. Initially, the weight of the body is equal to the upthrust (buoyant force). When the body is slightly pressed, the downward force becomes greater than the upthrust and the body sinks to the bottom.

Q.19. A body floats with one-third of its volume outside water and 3/4 of its volume outside another liquid. What is the density of the other liquid?

Ans. Let V cm^3 be the total volume of the body.

$$\text{Weight of body} = \text{Weight of water displaced} = \frac{2}{3}V \times 1 \times g \quad ...(i)$$

$$\text{Weight of body} = \text{Weight of liquid displaced} = \frac{1}{4}V \times \rho \times g \quad ...(ii)$$

From eqs. (*i*) and (*ii*), $\frac{1}{4}V \times \rho \times g = \frac{2}{3}V \times 1 \times g \quad \therefore \quad \rho = \frac{8}{3}$ g/cm^3

Q.20. A piece of copper having an internal cavity weighs 264 g in air and 221 g in water. What is the volume of the cavity? Density of copper = 8.8 g/cm^3.

Ans. The density ρ of the body (See Art. 19.21) is given by ;

$$\rho = \frac{\rho_w \times W}{W - W_a} = \frac{1 \times 264}{264 - 221} = \frac{264}{43} \text{ g/cm}^3$$

$$\therefore \quad \text{Volume of body} = \frac{\text{mass}}{\text{density}} = \frac{264}{264/43} = 43 \text{ cm}^3$$

Volume of copper in body = 264/8.8 = 30 cm^3

$\therefore$ Volume of cavity = 43 – 30 = 13 cm^3

Q.21. What is the fractional volume submerged of an ice cube in a pail of water placed in an enclosure which is freely falling under gravity?

Ans. For a free fall, the effective value of acceleration due to gravity is zero (*i.e.*, $g = 0$). Therefore, weight of ice cube as well as upthrust on the cube will be zero. As a result, the ice cube can float with any value of fractional volume submerged.

Q.22. A liquid drop of radius R breaks up into 64 drops each of radius r. What is the increase in surface area?

Ans. $\frac{4}{3}\pi R^3 = 64 \times \frac{4}{3}\pi r^3 \quad \therefore r = \frac{R}{4}$

$\therefore$ Increase in surface area = $64 \times 4\pi r^2 - 4\pi R^2 = [64 \times 4\pi (R/4)^2 - 4\pi R^2] = 12\pi R^2$

Q.23. How is the rise of liquid affected if top of the capillary tube is closed?

Ans. If the top of the capillary tube is closed, there will be a small rise in the capillary tube. The reason is simple. The rise of liquid in the capillary tube due to surface tension is opposed by the downward force exerted by the compressed air above the liquid in the tube. The liquid will rise in the tube till the two forces balance each other. Therefore, there will be small rise of liquid column if the top of the capillary tube is closed.

Q.24. What will happen if the length of the capillary tube is smaller than the height to which the liquid rises?

Ans. The liquid will rise upto the top of the capillary tube but will not overflow. The radius of curvature of the liquid meniscus increases and the meniscus becomes more and more flat. The liquid cannot emerge as a fountain from the upper end of the tube.

Q.25. Why does water wet the glass?

Ans. If the angle of contact between a liquid and solid surface is acute (*i.e.*, $< 90°$), the liquid spreads on the surface. It is because in this case the resultant adhesive force is much larger than the resultant cohesive force. Since angle of contact between water and glass is acute, water wets the glass.

Q.26. Why mercury does not wet the glass?

Ans. The angle of contact between mercury and glass is obtuse (*i.e.*, > 90°) so that the resultant cohesive force is much larger than the resultant adhesive force. Therefore, mercury does not spread over glass ; it forms into little drops when split on the glass.

Q.27. When a piece of chalk is put into water, it emits bubbles in all directions. Why?

Ans. A piece of chalk has a very large number of narrow capillaries. Due to capillary action, water rises and air present in the capillaries is forced out in the form of bubbles.

Q.28. We often wet the end of the thread before putting it through the eye of the needle. Why?

Ans. When we wet the end of the thread, the water film formed around the thread contracts due to surface tension. This reduces the area of X-section of the thread. Therefore, the thread easily goes through the eye of the needle.

Q.29. Water rises in a capillary tube while mercury falls in the same tube. Why?

Ans. Capillary rise, $h = \dfrac{2T\cos\theta}{r\rho g}$

(*i*) For water, θ is acute (*i.e.*, < 90°) so that cosθ is positive. Therefore, water rises in the capillary tube.

(*ii*) For mercury, θ is obtuse (*i.e.*, > 90°) so that cos θ is negative. Therefore, mercury gets depressed in the capillary tube.

Q.30. When two or more drops of mercury are brought into contact, they form a single drop. Why?

Ans. When two or more drops of mercury are brought into contact, they form a single drop so as to acquire minimum potential energy. Note that potential energy of a single drop is less than the total potential energy of small drops that formed the single drop.

LONG ANSWER QUESTIONS

1. State and prove Pascal's law. **[Refer to Art. 19.7]**
2. Write short notes on (*i*) hydraulic lift (*ii*) hydraulic brakes. **[Refer to Arts. 19.8 and 19.9]**
3. Explain the construction and working of a simple barometer. **[Refer to Art. 19.15]**
4. State and prove Archimedes' principle. **[Refer to Arts. 19.19 and 19.20]**
5. Explain the principle of floatation. **[Refer to Art. 19.21]**
6. Derive an expression for the volume of a floating body outside the liquid. **[Refer to Art. 19.22]**
7. What is surface tension? Discuss the molecular theory of surface tension. **[Refer to Art. 19.24]**
8. Derive an expression of excess pressure (*i*) inside a liquid drop (*ii*) inside a liquid bubble. **[Refer to Art. 19.30]**
9. Derive an expression for the rise of liquid in a capillary tube of uniform diameter and sufficient length. **[Refer to Arts. 19.33 and 19.35]**
10. Discuss the rise of liquid in a capillary tube of insufficient length. **[Refer to Art. 19.33]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Both liquids and gases are fluids.
2. In a liquid (or fluid) at rest, the forces exerted by the liquid are always perpendicular to the surfaces in contact with the liquid, regardless of the orientation of the surfaces.
3. In a liquid (or fluid) at rest, the net force on any volume element of the liquid is zero.
4. The force acting normally on a unit area of the surface is called pressure.

$$\text{Pressure, } P = \frac{F}{A} \qquad \ldots\ldots F \text{ acting normally on } A$$

The SI unit of pressure is N/m^2 or Pascal (Pa). 1 N/m^2 = 1 Pa.

5. At any point at a depth h below the surface of a liquid of density ρ, the pressure due to the weight of the fluid is

$$P = h\rho g$$

6. The pressure in a volume of liquid at rest is the same at all points at the same depth.
7. Thrust = Pressure × Area. Since thrust is a force, its SI unit is N.
8. The total pressure P at a point inside a liquid of density ρ at depth h is

$$P = h\rho g + P_a \qquad \text{where } P_a = \text{atmospheric pressure}$$

9. $P - P_a$ is called the gauge pressure (P_G). This is the pressure measured by the gauge.
10. Normal atmospheric pressure P_a = 0.76 m of mercury column

$$= (0.76) \times (13.6 \times 10^3) \times 9.8 = 1.013 \times 10^5 \text{ N/m}^2$$

11. The atmospheric pressure (P_a) varies with temperature and altitude as well as with variations in weather.
12. An external pressure exerted on a static enclosed fluid is transmitted uniformly throughout the volume of the fluid. This is known as *Pascal's principle.*
13. Relative density of a substance $= \dfrac{\text{density of substance}}{\text{density of water}}$
14. When a body is partially or wholly immersed in a liquid, the liquid exerts an upward force on the body. This is called **buoyant force** F_B.
15. When a body is partially or wholly immersed in a liquid (or fluid), it experiences an upward buoyant force F_B equal to the weight of the liquid displaced by the body. This is known as **Archimedes' Principle.**

$$\text{Buoyant force, } F_B = \text{Weight of liquid displaced by the body}$$

16. The apparent weight of a body of mass m immersed in a liquid is

$$\text{Apparent weight } = mg - F_B$$

17. For a body of volume V wholly immersed in a liquid of density ρ_l,

$$F_B = \text{Weight of liquid displaced by volume } V \text{ of the body}$$
$$= \rho_l Vg$$

Therefore, the magnitude of buoyant force F_B depends upon (*i*) the density of the liquid (*ii*) volume of the body inside the liquid and (*iii*) acceleration due to gravity at that place.

18. For a body of volume V wholly immersed in a liquid of density ρ_l,

$$F_B = \rho_l Vg$$

If ρ_b is the density of the body, then, $V = \frac{m}{\rho_b}$.

$\therefore$ Apparent weight of body $= mg - F_B = mg - \rho_l \times \frac{m}{\rho_b} \times g$

$$= mg\left(1 - \frac{\rho_l}{\rho_b}\right)$$

(*i*) If $\rho_b > \rho_l$, there will be *net* downward force and the body will sink.

(*ii*) If $\rho_b < \rho_l$, there will be *net* upward force and the body will float.

19. The condition of a body of mass m floating requires that $F_B = mg$.

20. Consider a body of density ρ_b floating in a liquid of density ρ_l.

Let V = total volume of the body

V' = volume of body immersed in the liquid

For floating, $F_B = mg$

or $\rho_l V' g = \rho_b V g$

$\therefore$ Fraction of the volume that is immersed in the liquid is

$$\frac{V'}{V} = \frac{\rho_b}{\rho_l}$$

How much fraction of the volume of the floating body is immersed in the liquid depends upon the densities of the body and the liquid. Clearly, for floating, ρ_l should be greater than ρ_b.

21. Large ships float in sea water even though they are made of steel and carry dense objects. The answer is that the **average density** of the entire ship, including all the air spaces, is less than the density of sea water.

22. The property due to which a liquid surface tends to contract and occupy the minimum area is called surface tension.

23. Surface tension acts over the free surface of a liquid only and not within the interior of the liquid.

24. Surface tension, $T = \frac{\text{Force}}{\text{Length}} = \frac{\text{Work done}}{\text{Surface area}}$

Its SI unit is Nm^{-1}.

25. For a single spherical surface such as a liquid drop or an air bubble, excess pressure inside the drop or air bubble is

$$p = \frac{2T}{R} \quad \text{... } R \text{ is radius of liquid drop or gas bubble}$$

26. A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble.

27. The excess pressure inside a soap bubble of radius R is

$$p = \frac{4T}{R}$$

28. The work done in forming a soap bubble of radius R is

$$\text{Work done} = 2\,[4\pi R^2 \times T\,] = 8\pi R^2 T$$

29. When a large number of small drops combine to form a single drop, there is decrease in surface area and there is a release of energy.

30. When a big drop is broken into a large number of small drops, there is increase in surface area and energy is needed to do it.

31. The surface tension of a liquid decreases with the increase in temperature.

32. The work done in breaking a big drop of liquid of radius R into n drops each of equal radius = $4\pi R^2 T\ (n^{1/3}-1)$
33. The molecules on the surface of a liquid have more potential energy than those in the interior of the liquid.
34. The potential energy per unit area of the liquid surface is called surface energy of the liquid. Its unit is Jm^{-2}.
35. The surface energy of a liquid has the same numerical value as the surface tension.
36. If the angle of contact is less than 90°, the liquid has a **concave meniscus** and wets the solid. If the angle of contact is greater than 90°, the liquid has a **convex meniscus** and does not wet the solid.
37. It is the angle of contact at which surface tension acts on the walls of the container.
38. Angle of contact (*i*) increases with increase in temperature (*ii*) decreases on adding some soluble impurity to a liquid (*iii*) does not depend on the inclination of the tube.
39. For curved surface, surface tension is tangential to the liquid at every point.
40. In a capillary tube, the free surface of the liquid becomes curved.
    (*i*) If the angle of contact is acute, the surface tension acquires a vertically upward component and the liquid rises in the tube.
    (*ii*) If the angle of contact is obtuse, the surface tension acquires vertically downward component and the level of liquid inside the tube falls below the level outside it.
41. The rise or fall of liquid in a capillary tube is according to law of conservation of energy.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A solid of density D is floating in a liquid of density d. If v is the volume of solid submerged in the liquid and V is the total volume of the solid, then v/V is equal to **[MP PMT 1997]**
   (*a*) $\frac{d}{V}$ (*b*) $\frac{D}{d}$
   (*c*) $\frac{D}{(D+d)}$ (*d*) $\frac{D+d}{D}$
2. A body of volume 100 c.c. is immersed completely in water contained in a jar. The weight of water and jar before immersion of the body was 700 g wt. After immersion, the weight of water and jar will be **[CPMT 1996]**
   (*a*) 700 g wt (*b*) 800 g wt
   (*c*) 500 g wt (*d*) 100 g wt
3. The density of ice is 0.9 g/c.c. and that of sea water is 1.1 g/c.c. An iceberg of volume V is floating in sea water. The fraction of iceberg above water level is **[AMU PMT 1997]**
   (*a*) 1/11 (*b*) 2/11
   (*c*) 3/11 (*d*) 4/11
4. A hydraulic lift is designed to lift cars of maximum mass 3000 kg. The area of cross-section of piston carrying the load is 4.25×10^{-2} m^2. What maximum pressure the smaller piston would have to bear? **[Pb PMT 1999]**
   (*a*) 6.92×10^5 N/m^2 (*b*) 7.82×10^7 N/m^2
   (*c*) 9.63×10^9 N/m^2 (*d*) 13.76×10^{11} N/m^2
5. The weight of a block is 120 N. When immersed completely in water, the weight becomes 80 N. What is the relative density of the material of the block? **[AMU PMT 1999]**
   (*a*) 6/4 (*b*) 6/3
   (*c*) 6 (*d*) 6/2
6. The total weight of a piece of wood is 6 kg. In the floating state in water, its 1/3rd part remains inside the water. On this floating wood, what weight is to be put such that whole of the wood is drowned in water? **[MNR 1999]**
   (*a*) 13 kg (*b*) 14 kg
   (*c*) 12 kg (*d*) 15 kg
7. If a big drop of radius R is formed by 1000 small droplets of water, the radius of small drop is **[Pb PMT 1997]**
   (*a*) $\frac{R}{1000}$ (*b*) $\frac{R}{10}$
   (*c*) $\frac{R}{500}$ (*d*) $\frac{R}{100}$

8. A wooden cube first floats inside water when a 200 g mass is placed on it. When the mass is removed, the cube rises by 2 cm. The side of the cube is **[AFMC 2000]**
(a) 5 cm (b) 15 cm
(c) 20 cm (d) 10 cm

9. A body floats with one-third of its volume outside water and 3/4 of its volume outside another liquid. The density of the other liquid is **[Rajasthan PMT 1997]**
(a) 9/4 g/c.c. (b) 4 g/c.c.
(c) 8/3 g/c.c. (d) 3/8 g/c.c.

10. A vessel contains oil (density = 0.8 g/c.c.) over mercury (density = 13.6 g/c.c.). A uniform sphere floats with half its volume immersed in mercury and the other half in oil. The density of the sphere material in g/c.c. is **[Pb PMT 1995]**
(a) 3 (b) 6.4
(c) 7.2 (d) 12.8

11. The work done in blowing a soap bubble of 10 cm radius is (surface tension of soap solution is 0.03 Nm^{-1}) **[MP PMT 1995]**
(a) 37.68×10^{-4} J (b) 75.36×10^{-4} J
(c) 126.82×10^{-4} J (d) 75.36×10^{-3} J

12. In a capillary tube, water rises by 2 mm. The height of water that will rise in another capillary tube having one-third radius of the first is **[Pb PMT 1996]**
(a) 2 mm (b) 3 mm
(c) 6 mm (d) 9 mm

13. A spherical drop of water has 1 mm radius. If the surface tension of water is 70×10^{-3} N/m, then difference of pressure between inside and outside of the drop is **[AIIMS 2001]**
(a) 35 N/m^2 (b) 70 N/m^2
(c) 140 N/m^2 (d) zero

14. A drop of liquid assumes spherical shape because of **[AFMC-Pune 2001]**
(a) density (b) viscosity
(c) gravitation (d) surface tension

15. Suppose W is the work done when a bubble of volume V is formed from a soap solution. How much work is required to be done to form a bubble of $2V$? **[CPMT 1996]**
(a) W (b) $4^{1/3} W$
(c) $2W$ (d) $2^{1/3} W$

16. The radius of a soap bubble is r and the surface tension of soap solution is T. Keeping the temperature constant, the extra energy needed to double the radius of soap bubble is **[AIIMS 1996]**
(a) $24\pi r^2 T$ (b) $32\pi r^2 T$
(c) $16\pi r^2 T$ (d) $8\pi r^2 T$

17. When two soap bubbles of radii r_1 and r_2 ($r_2 > r_1$) coalesce, the radius of curvature of common surface is **[MP PMT 1996]**
(a) $r_2 - r_1$ (b) $r_2 + r_1$
(c) $\frac{r_2 - r_1}{r_1 r_2}$ (d) $\frac{r_1 r_2}{r_2 - r_1}$

18. The force of cohesion is maximum in **[Pb PMT 1995]**
(a) solids (b) gases
(c) liquids (d) all of these

19. The value of surface tension is 70 dynes/cm. What will be its value in N/m? **[AIIMS 1995]**
(a) 70 N/m (b) 7×10^2 N/m
(c) 7×10^{-2} N/m (d) 7×10^3 N/m

20. The change of surface energy when a drop of surface tension T and radius R splits into 1000 droplets of radius r is **[AIIMS 1994]**
(a) $4\pi R^2 T$ (b) $16\pi R^2 T$
(c) $36\pi R^2 T$ (d) $72\pi R^2 T$

21. Excess pressure of one soap bubble is four times the other soap bubble. Then ratio of volume of first bubble to the other is **[CPMT 1997]**
(a) 1 : 64 (b) 1 : 4
(c) 64 : 1 (d) 1 : 2

22. A soap bubble in vacuum has a radius of 3 cm and another soap bubble has a radius of 4 cm. If the two bubbles coalesce under isothermal conditions, then radius of the new bubble is **[MP PMT 1998]**
(a) 2.3 cm (b) 5 cm
(c) 7 cm (d) 4.5 cm

23. At which of the temperature, the value of surface tension of water will be minimum? **[MP PMT 1998]**
(a) 5°C (b) 20°C
(c) 50°C (d) 75°C

24. If the surface tension of water is 0.06 Nm^{-1}, then capillary rise in a tube of diameter 1 mm is ($\theta = 0°$) **[AFMC 1998]**
(a) 1.22 cm (b) 2.44 cm
(c) 3.12 cm (d) 3.86 cm

25. The rise of liquid in a capillary tube has a meniscus. If the angle of contact is 90°, the shape of the meniscus is **[AMU PMT 2000]**
(a) concave (b) convex
(c) flat (d) none of the above

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*b*)	**2.** (*b*)	**3.** (*b*)	**4.** (*a*)	**5.** (*d*)
6. (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*c*)	**10.** (*c*)
11. (*b*)	**12.** (*c*)	**13.** (*c*)	**14.** (*d*)	**15.** (*b*)
16. (*a*)	**17.** (*d*)	**18.** (*a*)	**19.** (*c*)	**20.** (*c*)
21. (*a*)	**22.** (*b*)	**23.** (*d*)	**24.** (*b*)	**25.** (*c*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Weight of liquid displaced = vdg ; Weight of solid = VDg

According to principle of floatation,

Weight of liquid displaced = Weight of solid

or $$vdg = VDg \quad \therefore \quad \frac{v}{V} = \frac{D}{d}$$

2. Upthrust, $F_B = 100 \times 1 \times g = 100$ gwt.

According to Newton's third law, there will be downward thrust on the jar equal to 100 gwt. Therefore, total weight = 700 + 100 = **800 gwt.**

3. Let v be the volume of iceberg outside the sea water. The volume of iceberg submerged in water = $(V - v)$.

Weight of sea water displaced $= (V - v) \times 1.1 \times g$

Weight of iceberg $= V \times 0.9 \times g$

According to principle of floatation,

$$(V - v) \times 1.1 \times g = V \times 0.9 \times g$$

or $$1.1\,v = 1.1V - 0.9V = 0.2V \text{ or } v/V = 0.2/1.1 = \mathbf{2/11}$$

4. Load on larger piston, $F = mg = 3000 \times 9.8$ N

$$\text{Pressure on larger piston} = \frac{F}{\text{Area}} = \frac{3000 \times 9.8}{4.25 \times 10^{-2}} = 6.92 \times 10^5 \text{ N/m}^2$$

According to Pascal's law, the pressure is transmitted equally in all directions.

Therefore, pressure on smaller piston = **6.92×10^5 N/m^2.**

5. Weight of block in air, $W = 120$ N ; Weight of block in water, $W_a = 80$ N

$$\therefore \text{ Relative density of block} = \frac{W}{W - W_a} = \frac{120}{120 - 80} = \mathbf{6/2}$$

6. Let V be the total volume of wood in m^3. Then,

$$6 \times g = \frac{V}{3} \times 10^3 \times g \qquad \text{...(}i\text{) given}$$

Let m kg be the mass that is to be placed to drown the wooden piece. Then,

$$(6 + m)g = V \times 10^3 \times g \qquad \text{...(}ii\text{)}$$

Dividing eq. (*ii*) by eq. (*i*), we get, $\frac{6+m}{6} = 3 \quad \therefore \quad m = \mathbf{12\ kg}$

7. Let r be the radius of small drop.

$\therefore$ Mass of 1000 droplets = Mass of big drop

or $$1000 \times \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi R^3 \rho \quad \text{or} \quad r^3 = R^3/1000 \quad \therefore \quad r = R/10$$

8. Let l cm be each side of cube. Therefore, $l^2 \times 2 \times 1 \times g = 200 \times g$ or $l = \mathbf{10\ cm}$

9. Let V cm^3 be the total volume of the body.

$$\text{Weight of body} = \text{Weight of water displaced} = \frac{2}{3}V \times 1 \times g \qquad ...(i)$$

$$\text{Weight of body} = \text{Weight of liquid displaced} = \frac{1}{4}V \times \rho \times g \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), we have, $\frac{1}{4}V \times \rho \times g = \frac{2}{3}V \times 1 \times g \quad \therefore \ \rho = \frac{8}{3}$ **g/c.c.**

10. Let V cm^3 be the total volume of sphere and ρ be its density in g/c.c. Then,

$$V\rho g = \frac{V}{2} \times 0.8 \times g + \frac{V}{2} \times 13.6 \times g \quad \text{or} \quad V\rho = 7.2\,V \quad \therefore \quad \rho = \mathbf{7.2\ g/c.c.}$$

11. Work done = Surface tension × Increase in surface area

$$= 0.03 \times 2\,[4\pi R^2] = 0.03 \times 2\,[4\pi \times (0.1)^2] = \mathbf{75.36 \times 10^{-4}\ J}$$

12. Capillary rise, $h = \frac{2T\cos\theta}{r\rho g}$

Since T, θ, ρ and g are constant, $h \propto \frac{1}{r}$.

$\therefore \frac{h_2}{h_1} = \frac{r_1}{r_2}$ or $h_2 = h_1 \times \frac{r_1}{r_2} = 2 \times \frac{r_1}{r_1/3} = \mathbf{6\ mm}$

13. Radius of drop, $R = 1$ mm $= 1 \times 10^{-3}$ m ; Surface tension of water, $T = 70 \times 10^{-3}$ N/m

Difference of pressure $= \frac{2T}{R} = \frac{2 \times 70 \times 10^{-3}}{1 \times 10^{-3}} = \mathbf{140\ N/m^2}$

14. A drop of liquid tends to acquire minimum surface area due to surface tension. For a given volume, the surface area is minimum for a sphere. Therefore, a drop of liquid under no external force (*e.g.* gravity) always assumes spherical shape.

15. $V = \frac{4}{3}\pi R^3$ and $2V = \frac{4}{3}\pi R'^3 \quad \therefore \quad \frac{R'^3}{R^3} = 2$ or $R' = (2)^{\frac{1}{3}} R$

Now $W = T \times 2\,(4\pi R^2)$ and $W' = T \times 2\,(4\pi R'^2)$

$\therefore \quad \frac{W'}{W} = \left(\frac{R'}{R}\right)^2 = \left(2^{1/3}\right)^2 = 4^{1/3} \quad \therefore W' = \mathbf{4^{1/3} W}$

16. Energy required to blow a soap bubble of radius r is

$$E_1 = T \times 2\,(4\pi r^2) = 8\pi r^2 T$$

Energy required to blow a soap bubble of radius $2r$ is

$$E_2 = T \times 2\,[4\pi (2r)^2] = 32\pi r^2 T$$

$\therefore$ Extra energy needed $= E_2 - E_1 = 32\pi r^2 T - 8\pi r^2 T = \mathbf{24\pi r^2 T}$

17. Excess pressure inside soap bubble of radius r_1 is $p_1 = 4T/r_1$

Excess pressure inside soap bubble of radius r_2 is $p_2 = 4T/r_2$

If radius of curvature of interface is r, then excess pressure inside is $p = 4T/r$.

Now $p = p_1 - p_2$ or $\frac{4T}{r} = \frac{4T}{r_1} - \frac{4T}{r_2}$ or $\frac{1}{r} = \frac{1}{r_1} - \frac{1}{r_2} \quad \therefore \ r = \frac{r_1 r_2}{r_2 - r_1}$

18. The molecules of solids are very close to each other. Therefore, molecular forces of attraction or cohesion are maximum in solids. It is for this reason that solids have definite shape and volume.

19. 1 dyne = 10^{-5} N and 1 cm = 10^{-2} m

$$\therefore \text{ 70 dynes/cm} = \frac{70\times10^{-5}}{10^{-2}} = 7 \times 10^{-2} \text{ N/m}$$

20. Volume of large drop = Volume of 1000 droplets

$$\text{or} \quad \frac{4}{3}\pi R^3 = 1000\times\frac{4}{3}\pi r^3 \text{ or } R^3 = (10r)^3 \quad \therefore r = 0.1\,R$$

$$\text{Change in surface area} = 1000\times4\pi r^2 - 4\pi R^2 = 1000\times4\pi\times(0.1\,R)^2 - 4\pi R^2$$

$$= 40\pi R^2 - 4\pi R^2 = 36\pi R^2$$

$$\therefore \text{ Change in surface energy} = 36\pi R^2 \times T = 36\,\pi R^2 T$$

21. $p_1 = 4\,p_2$ or $\frac{4T}{r_1} = 4\times\frac{4T}{r_2} \quad \therefore \frac{r_1}{r_2} = \frac{1}{4}$

$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \left(\frac{r_1}{r_2}\right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64} \quad \therefore V_1 : V_2 = 1 : 64$$

22. $p = \frac{4T}{r}$; $V = \frac{4}{3}\pi r^3$

Since the conditions are isothermal,

$$\therefore \quad \frac{4T}{R}\times\frac{4}{3}\pi R^3 = \frac{4T}{r_1}\times\frac{4}{3}\pi r_1^3 + \frac{4T}{r_2}\times\frac{4}{3}\pi r_2^3$$

$$\text{or} \quad R^2 = r_1^2 + r_2^2 \quad \therefore R = \sqrt{r_1^2 + r_2^2} = \sqrt{3^2 + 4^2} = 5 \text{ cm}$$

23. Surface tension of water decreases with the increase in temperature.

24. Capillary rise, $h = \frac{2T\cos\theta}{r\rho g} = \frac{2\times0.06\times\cos0°}{0.5\times10^{-3}\times10^3\times9.8} = 2.44\times10^{-2}$ m $= 2.44$ cm

25. When the angle of contact is 90°, the meniscus is **flat**.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A hemispherical bowl just floats without sinking in a liquid of density 1.2×10^3 kg /m^3. If the outer diameter and the density of the bowl are 1 m and 2×10^4 kg/m^3 respectively, then the inner diameter of the bowl is **[SCRA 1998]**

(*a*) 0.94 m (*b*) 0.97 m
(*c*) 0.98 m (*d*) 0.99 m

2. A 700 g solid cube having each side 10 cm floats in water. How much volume of the cube is outside the water? Density of water = 1000 kg/m^3. **[CET Karnataka 1999]**

(*a*) 300 cm^3 (*b*) 400 cm^3
(*c*) 600 cm^3 (*d*) 250 cm^3

3. Water is filled in a flask upto a height of 20 cm. The bottom of the flask is circular with radius 10 cm. If atmospheric pressure is 1.013×10^5 N/m^2, what is the force exerted by water on the bottom? Take g = 10 ms^{-2} and density of water = 1000 kg/m^3. **[Rajasthan PET 1996]**

(*a*) 2754 N (*b*) 3246 N
(*c*) 3102 N (*d*) 1710 N

4. We have two different liquids A and B whose relative densities are 0.75 and 1 respectively. If we dip two solid objects P and Q of relative densities 0.6 and 0.9 in these liquids, then, **[SCRA 1998]**

(*a*) *P* floats in *A* and *Q* sinks in *B*
(*b*) *P* sinks in *A* and *Q* floats in *B*
(*c*) *P* floats in *B* and *Q* sinks in *A*
(*d*) *P* sinks in *B* and *Q* floats in *A*

5. At what depth below the surface of water will the pressure be equal to two atmospheres? Take atmospheric pressure = 10 N/cm^2, density of water = 1000 kg/m^3 and g = 9.8 m/s^2. **[MP PET 1995]**
(*a*) 11.3 m (*b*) 9.4 m
(*c*) 13.6 m (*d*) 10.2 m

6. A solid of density *D* has weight *W*. If it is completely immersed in a liquid of density *d*, then apparent weight of the solid is **[MP PET 1998]**
(*a*) $W\left(1-\frac{d}{D}\right)$ (*b*) $W\left(1-\frac{D}{d}\right)$
(*c*) $W\left(1+\frac{d}{D}\right)$ (*d*) $W\left(1+\frac{D}{d}\right)$

7. A sample of metal weighs 210 g in air, 180 g in water and 120 g in an unknown liquid. Then density of the metal is **[MNR 1999]**
(*a*) 3 g/cm^3 (*b*) 7 g/cm^3
(*c*) 6 g/cm^3 (*d*) none of the above

8. A hemisphere portion of radius *R* is removed from the bottom of a cylinder of radius *R* (see Fig. 19.35). The volume of the remaining cylinder is *V* and its mass is *M*. It is suspended by a string in a liquid of density ρ where it stays vertical. The upper surface of the cylinder is at a depth of *h* below the liquid surface. The force on the bottom of the cylinder is **[IIT 2001]**

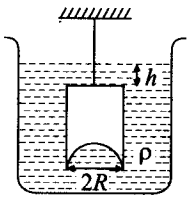


Fig. 19.35

(*a*) Mg (*b*) $Mg - V\rho g$
(*c*) $Mg + \pi R^2 h\rho g$ (*d*) $\rho g(V + \pi R^2 h)$

9. To what height should a rectangular cylinder having square base of length 10 cm be filled so that total force on the bottom is equal to that on the sides? **[CET Karnataka 1996]**
(*a*) 5 cm (*b*) 10 cm
(*c*) 20 cm (*d*) 8 cm

10. A weightless bag is filled with 5 kg of water and then weighed in water. The reading of the spring balance is **[MP PET 1996]**
(*a*) 10 kg wt (*b*) 2 .5 kg wt
(*c*) 1.25 kg wt (*d*) zero

11. A small drop of water of surface tension *T* is squeezed between two clean glass plates so that a thin layer of thickness *d* and area *A* is formed between them. If the angle of contact is zero, the force required to pull the plates apart is **[CET Haryana 2001]**
(*a*) $\frac{8TA}{d}$ (*b*) $\frac{4TA}{d}$
(*c*) $\frac{2TA}{d}$ (*d*) $\frac{TA}{d}$

12. A capillary tube made of glass is dipped into mercury. Then, **[MP PET 1996]**
(*a*) mercury rises in the capillary tube
(*b*) mercury falls in the capillary tube
(*c*) mercury rises and flows out of the capillary tube
(*d*) mercury neither rises nor falls in the tube

13. The excess pressure inside the soap bubble is twice the excess pressure inside a second soap bubble. The volume of the first soap bubble is *n* times the volume of the second soap bubble. The value of *n* is **[CET Karnataka 1998]**
(*a*) 0.125 (*b*) 1
(*c*) 2 (*d*) 4

14. Two capillary tubes of radii 0.2 cm and 0.4 cm are dipped in the same liquid. The ratio of heights through which the liquid will rise in the tubes is **[CEET 1998]**
(*a*) 1 : 2 (*b*) 2 : 1
(*c*) 1 : 4 (*d*) 4 : 1

15. A liquid will not wet the surface of a solid if its angle of contact is **[CEET 1998]**
(*a*) zero (*b*) less than 90°
(*c*) more than 90° (*d*) 90°

16. The work done in increasing the size of a film from 10 cm × 6 cm to 10 cm × 11 cm is 3×10^{-4} J. The surface tension of the film is **[MP PET 1999]**
(*a*) 1.5×10^{-2} N/m (*b*) 3×10^{-2} N/m
(*c*) 6×10^{-2} N/m (*d*) 11×10^{-2} N/m

17. If we dip capillary tubes of different radii in water and water rises to different heights in them, then, **[B.H.U. 2000]**
(*a*) $\frac{h}{r^2}$ = constant (*b*) $\frac{h}{r}$ = constant
(*c*) hr = constant (*d*) hr^2 = constant

18. Work done in splitting a drop of water of 1 mm radius into 64 droplets (S.T. of water = 72×10^{-3} J/m^2) is **[B.H.U. 2000]**
(*a*) 2×10^{-6} J (*b*) 2.7×10^{-6} J
(*c*) 4×10^{-6} J (*d*) 5.4×10^{-6} J

19. Water rises in a capillary tube to a certain height such that the upward force due to surface tension is balanced by 75×10^{-4} N force due to the weight of the liquid. If the surface tension of water is 6×10^{-2} N/m, the inner circumference of the capillary tube is **[CET Haryana 1998]**
(a) 1.25×10^{-3} m (b) 0.5×10^{-2} m
(c) 6.5×10^{-2} m (d) 12.5×10^{-2} m

20. The energy stored in a soap bubble of diameter 4 cm is (T = 0.07 N/m) nearly **[CET Karnataka 1999]**
(a) 7×10^{-4} J (b) 3.5×10^{-4} J
(c) 1.76×10^{-2} J (d) 8.8×10^{-3} J

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (c)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (d)
11. (c)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (c)
16. (b)	**17.** (c)	**18.** (b)	**19.** (d)	**20.** (a)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Let d metre be the inner diameter of the bowl. According to principle of floatation,

Weight of bowl = Weight of liquid displaced

or $$\frac{1}{2}\times\frac{4}{3}\pi\left[\left(\frac{1}{2}\right)^3-\left(\frac{d}{2}\right)^3\right]\times 2\times 10^4\times 9.8 = \frac{1}{2}\times\frac{4}{3}\pi\left(\frac{1}{2}\right)^3\times 1.2\times 10^3\times 9.8$$

On solving, $d = 0.98$m

2. Total volume of cube, $V = (10)^3 = 1000$ cm^3

Let v m^3 volume of the cube be inside water. According to principle of floatation,

Weight of cube = Weight of water displaced

or $$0.7 \times g = v\ \rho\ g \quad \text{or} \quad v = \frac{0.7}{\rho} = \frac{0.7}{1000} = 7\times 10^{-4}\ \text{m}^3 = 700\,\text{cm}^3$$

$\therefore$ Volume of cube outside water = $V - v = 1000 - 700 = 300$ cm^3

3. Pressure at the bottom, $P = P_a + h\rho g = 1.013 \times 10^5 + 0.2 \times 1000 \times 10$

$= 1.033 \times 10^5$ N /m^2

Area of bottom, $A = \pi r^2 = \pi \times (0.1)^2$

$\therefore$ Force at bottom, $F = PA = 1.033 \times 10^5 \times \pi \times (0.1)^2 = 3246$ N

4. The material of density greater than that of liquid sinks in it while the material of density smaller than that of liquid floats on it.

5. Let h metres be the required depth.

$$2P_a = P_a + h\rho g \quad \text{or} \quad h = \frac{P_a}{\rho_g}$$

Here $P_a = 10$ N/cm^2 $= 10 \times 10^4$ N/m^2 $= 10^5$ N/m^2 $\therefore$ $$h = \frac{10^5}{1000\times 9.8} = 10.2\ \text{m}$$

6. Volume of solid = W/Dg ; Weight of liquid displaced = $\frac{W}{Dg} \times d \times g$

$\therefore$ Upthrust, $F_B = \frac{W}{D} \times d$

Apparent weight = $W - F_B = W - W \times \frac{d}{D} = W\left(1-\frac{d}{D}\right)$

7. Relative density of metal $= \dfrac{\text{Weight of metal in air}}{\text{Loss of weight in water}} = \dfrac{210 \times g}{210 \times g - 180 \times g} = 7$

$\therefore$ Density of metal = 7 × Density of water = 7 × 1 g/c.c. = 7 g/c.c.

8. $F_{bottom} - h\,\rho\,g \times \pi R^2 = V\,\rho\,g$ ...Neglecting atmospheric pressure

$\therefore$ $F_{bottom} = V\,\rho\,g + h\,\rho\,g \times \pi R^2 = \rho g\,(V + \pi R^2 h)$

9. Let h cm be the required height. If l is the length of each side of base, base area $= l \times l = l^2$

Pressure at base $= h\,\rho\,g$; Force at base, $F_B = (h\,\rho\,g) \times l^2$

Pressure on walls $= \dfrac{0 + h\rho g}{2} = \dfrac{h\rho g}{2}$; Area of four walls $= 4 \times (l \times h)$

Force on walls, $F_W = (h\rho g/2) \times 4lh$

$\therefore$ $(h\rho g/2) \times 4lh = (h\rho g) \times l^2$ or $h = \dfrac{l}{2} = \dfrac{10}{2} = 5$ cm

10. Since weight of bag with water is equal to the weight of water displaced, the reading of the spring balance will be **zero**.

11. An extremely thin layer of liquid can be regarded as a collection of a large number of semi-spherical drops. Hence excess pressure is T/r instead of $2T/r$ as in case of a spherical drop.

$\therefore$ Excess pressure $= \dfrac{T}{r} = \dfrac{T}{d/2} = \dfrac{2T}{d}$

Force required = Excess pressure × Area $= \dfrac{2T}{d} \times A = \dfrac{2TA}{d}$

12. Capillary rise or fall, $h = \dfrac{2T\cos\theta}{r\rho g}$

Since angle of contact of mercury with glass is obtuse, $\cos\theta$ is negative so that h is negative. This means **mercury falls in the capillary tube below the mercury level outside the tube.**

13. $\dfrac{4T}{r_1} = 2 \times \dfrac{4T}{r_2}$ or $r_2 = 2\,r_1$

Now $\dfrac{4}{3}\pi r_1^3 = n \times \dfrac{4}{3}\pi r_2^3$ or $r_1^3 = n r_2^3 = n(2r_1)^3$ or $1 = 8n$ $\therefore$ $n = 1/8 = 0.125$

14. Capillary rise, $h = \dfrac{2T\cos\theta}{r\rho g}$

Since T, θ, ρ and g are constant, $h \propto \dfrac{1}{r}$.

$\therefore$ $\dfrac{h_1}{h_2} = \dfrac{r_2}{r_1} = \dfrac{0.4}{0.2} = 2$ $\therefore$ $h_1 : h_2 = 2 : 1$

15. The liquid will not wet the surface of a solid if its angle of contact is more than **90°**.

16. Increase in surface area $= 2\,(10 \times 11 - 10 \times 6) \times 10^{-4}\ m^2 = 10^{-2}\ m^2$

Work done = T × Increase in surface area

$\therefore$ $T = \dfrac{\text{Work done}}{\text{Increase in surface area}} = \dfrac{3 \times 10^{-4}}{10^{-2}} = 3 \times 10^{-2}$ N/m

17. Capillary rise, $h = \dfrac{2T\cos\theta}{r\rho g}$

Since T, θ, ρ and g are constant, $h \propto \frac{1}{r}$. $\therefore$ hr = constant

18. Let R and r be the radius of bigger drop and each of smaller drop respectively. Then,

$$\frac{4}{3}\pi R^3 = 64 \times \frac{4}{3}\pi r^3 \qquad \therefore\ r = \frac{R}{4} = \frac{10^{-3}}{4}$$

Increase in surface area $= (64 \times 4\pi r^2 - 4\pi R^2)$

$\therefore$ Work done $= T \times (64 \times 4\pi r^2 - 4\pi R^2)$

$$= 72\times10^{-3}\left[64\times4\pi\times\left(\frac{10^{-3}}{4}\right)^2 - 4\pi\times(10^{-3})^2\right]$$

$$= 2.7 \times 10^{-6}\ \text{J}$$

19. $2\pi r \times T = mg$ $\therefore$ $2\pi r = \frac{mg}{T} = \frac{75\times10^{-4}}{6\times10^{-2}} = 12.5 \times 10^{-2}\text{m}$

20. Increase in surface area $= 2 \times 4\pi R^2 = 2 \times 4\pi \times (2 \times 10^{-2})^2$

Energy stored $= T \times$ increase in surface area

$$= 0.07 \times 2 \times 4\pi \times (2 \times 10^{-2})^2 = 7 \times 10^{-4}\ \text{J}$$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. A tank 5 m high is half filled with water and then filled to the top with oil of density 0.85 g/cm^3. What is the pressure at the bottom of the tank due to these liquids?

Hint. $P = h_1 \rho_1 g + h_2 \rho_2 g$

Here $\rho_1 = 10^3$ kg/m^3 ; $\rho_2 = 0.85 \times 10^3$ kg /m^3 ; $h_1 = h_2 = 2.5$ m

$\therefore$ $P = (2.5 \times 10^3 + 2.5 \times 0.85 \times 10^3)\ 9.8 = \mathbf{45.325 \times 10^3\ N/m^2}$

2. To what height should a cylindrical vessel be filled with a homogeneous liquid to make the force with which the liquid presses on the sides of the vessel equal to the force exerted by the liquid on the bottom of the vessel? [I.I.T.]

Hint. Let r be the radius of the cylindrical vessel and h be the required height of the liquid in the vessel [See Fig. 19.36]. If ρ is the density of the liquid, then,

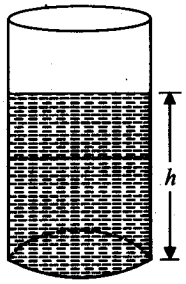


Fig. 19.36

Force acting on the bottom of the liquid

= Weight of liquid filled in the cylinder

$= m g = (\pi r^2 h\rho)\ g$ ...(*i*)

Area of wall in contact with the liquid

$= 2\pi rh$

Average pressure of the liquid on the wall of cylinder

$$= \frac{(\text{Bottom pressure} + \text{Pressure at top})\text{of liquid}}{2} = \frac{h\rho g + 0}{2} = \frac{h\rho g}{2}$$

$\therefore$ Force on the sides of vessel $= \frac{h\rho g}{2} \times 2\ \pi rh = \pi\rho r g h^2$ ...(*ii*)

According to the conditions of the problem, the forces given by eqs. (*i*) and (*ii*) are equal.

$\therefore$ $\pi r^2 h \rho g = \pi \rho g r h^2$ or $h = r$

Therefore, the liquid should be filled in the cylinder upto a height equal to radius of the cylinder.

3. **A cubical block of wood, 40 cm along each side, floats in water with its lower surface 5 cm below the free surface of water as shown in Fig. 19.37. What is the mass of the block in kg? What is the pressure of water in pascal (Pa) on the lower surface of the block? Take $g = 10\ ms^{-2}$ and density of water as 1000 kg/ m^3. [C.E.E.]**

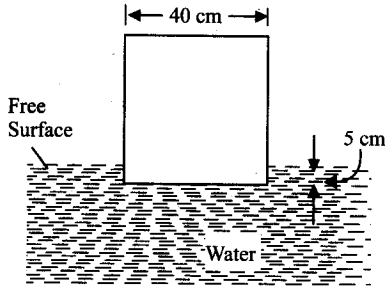


Fig. 19.37

Hint. Figure 19.37 shows the conditions of the problem.

Mass of floating block

= Mass of water displaced

= Volume of water displaced × density of water

$= (0.4 \times 0.4 \times 0.05) \times 10^3 = 8$ kg

If P_0 is the atmospheric pressure (in N/m^2), then pressure at a depth h (= 5 cm = 5×10^{-2} m)

$= P_0 + h\,\rho\,g = P_0 + (5 \times 10^{-2}) \times 10^3 \times (10) = (P_0 + 500)$ Pa

4. **At what depth in an ocean will a tube of air have one-fourth volume it will have on reaching the surface? Atmospheric pressure = 76 cm of Hg and density of Hg = 13.6 g/c.c.**

Hint. Suppose the volume of the bubble is V on reaching the surface. Let the volume of the bubble become $V/4$ at a depth of h cm inside water.

$$\therefore \quad V_1 = V\ ;\ V_2 = V/4$$

Pressure on bubble at the surface, P_1 = 76 cm of Hg

Pressure on bubble at a depth of h cm is

$$P_2 = \left(76 + \frac{h}{13.6}\right) \text{cm of Hg}$$

According to Boyle's law, we have,

$$P_1 V_1 = P_2 V_2$$

or $$76 \times V = \left(76 + \frac{h}{13.6}\right)\frac{V}{4}$$

or $$76 \times 4 = 76 + \frac{h}{13.6} \qquad \therefore \quad h = 3100.8 \text{ cm}$$

5. **A balloon filled with hydrogen has a volume of 1000 litres and its mass is 1 kg. What would be the volume of the block of a very light material which it can just lift? One litre of material has mass of 91.3 g. Density of air is 1.3 g/litre. [I.I.T.]**

Hint. Let V and M be the volume and mass of the balloon respectively. Let v and m be the corresponding values for the block.

Total volume of balloon and block = $V + v$

$\therefore$ Volume of air displaced = $V + v$

If ρ_a is the density of air, then,

Weight of air displaced = volume × density × $g = (V + v)\,\rho_a\,g$

According to the principle of floatation, we have,

Weight of (balloon + block) = Weight of air displaced

or $$(M + m)\,g = (V + v)\,\rho_a\,g$$

If ρ is the density of block, then, $m = v\rho$.

$$\therefore \quad (M + v\rho)\,g = (V + v)\,\rho_a\,g$$

or $$v = \frac{V\rho_a - M}{\rho - \rho_a}$$

Here V = 1000 litres = 1000×10^{-3} m^3 = 1 m^3 ; M = 1 kg ; ρ = 91.3 g/litre = 91.3 kg/m^3 ; ρ_a = 1.3 g/litre= 1.3 kg/m^3

$$\therefore \quad \text{Volume of block, } v = \frac{1\times1.3-1}{91.3-1.3} = \frac{0.3}{90}\text{m}^3 = \frac{0.3}{90} \times 1000 \text{ litres} = \mathbf{3.3 \text{ litres}}$$

6. **A raft of wood (density 600 kg/m^3) of mass 120 kg floats in water. How much weight can be put on the raft to make it just sink?**

Hint. Volume of raft, $V = \dfrac{\text{mass}}{\text{density}} = \dfrac{120}{600} = 0.2 \text{ m}^3$

When raft is fully immersed,

Weight of water displaced = $V \times \rho_w \times g = 0.2 \times 10^3 \times g = 200\ g$ newton

$\therefore$ Additional mass that can be put on the raft

$$= \frac{200g}{g} - 120 = \mathbf{80 \text{ kg}}$$

7. **A queen's gold crown has a mass of 1.30 kg. When it is weighed while it is completely immersed in water, its apparent mass is 1.14 kg. Is crown solid gold?**

Hint. Being made of solid gold would mean that crown would have the density of gold. If it was made of a mixture of materials or of some other uniform material or was hollow, its density would be different from that of pure gold.

The apparent weight W_a of the fully immersed crown is equal to its weight in air (W) minus the buoyant force F_B *i.e.*,

$$W_a = W - F_B = V\rho g - V\rho_w g$$

where V is the volume of the crown, ρ its density and ρ_w density of water.

$$\therefore \quad W_a = V\rho g\left(1-\frac{\rho_w}{\rho}\right) = W\left(1-\frac{\rho_w}{\rho}\right)$$

or
$$\rho = \frac{\rho_w \times W}{W - W_a}$$

Here $\rho_w = 10^3$ kg/m^3 ; $W = 1.30 \times 9.8$ N ; $W_a = 1.14 \times 9.8$ N

$$\therefore \quad \rho = \frac{(10^3)\times(1.30\times9.8)}{(1.3\times9.8-1.14\times9.8)} = 8.13 \times 10^3 \text{ kg/m}^3$$

The density of pure gold (= 19.3×10^3 kg/m^3) is much larger than this value. The crown certainly is not pure gold.

8. **A piece of copper having an internal cavity weighs 264 g in air and 221 g in water. Find the volume of the cavity. Density of copper = 8.8 g/cm^3.** **[I.I.T.]**

Hint. As shown in Example 7, density ρ of the body is given by ;

$$\rho = \frac{\rho_w \times W}{W - W_a}$$

where W is the weight of the body and W_a is its apparent weight in water

$$\therefore \quad \rho = \frac{1\times264}{264-221} = \frac{264}{43}$$

$$\therefore \quad \text{Volume of the body} = \frac{\text{mass}}{\text{density}} = \frac{264}{264/43} = 43 \text{ cm}^3$$

Volume of copper in the body = $\frac{264}{8.8} = 30\ cm^3$

$\therefore$ Volume of cavity = 43 – 30 = **13 cm^3**

9. A large block of ice of thickness 5 m has a vertical hole drilled through it and is floating in the middle of a lake. What is the minimum length of the rope required to scoop up a bucket full of water through the hole? Density of ice = 0.9 g/cm^3.

Hint. Let x metre be the thickness of ice block above the surface of water (See Fig. 19.38). The minimum length of the rope required to scoop up a bucket is equal to the minimum length of rope (= x metre) required to reach the water through the hole. If H (metres) is the thickness of the block, A (m^2) its area of cross-section and ρ the density of ice, then,

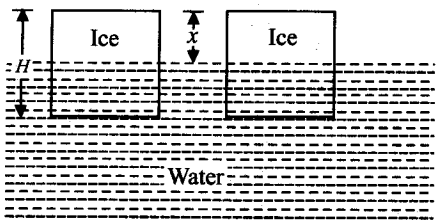


Fig. 19.38

Weight of ice block = $(AH\,\rho)\,g$

Weight of water displaced = $(H - x)\,A\,\rho_w\,g$

According to principle of floatation,

Weight of block = Weight of water displaced

or $(AH\rho)\,g = (H - x)\,A\rho_w\,g$

$\therefore$ $x = H\left(1 - \frac{\rho}{\rho_w}\right)$

Here H = 5 m ; $\rho/\rho_w = 0.9$ $\therefore$ $x = 5\,(1 - 0.9)$ = **0.5 m**

10. Calculate the volume of the balloon filled with hydrogen gas which will be sufficient to lift a load of 25 kg in air. Given that respective densities of air and hydrogen are 0.00129 g/cm^3 and 0.00009 g/cm^3.

Hint. Let the volume of the hydrogen filled be V cm^3. Since the mass of balloon is negligible,

$\therefore$ Upthrust on balloon = Weight of air displaced, $F_B = V \times 129 \times 10^{-5} \times g$

Weight of balloon filled with hydrogen is

$$W = V \times (9 \times 10^{-5}) \times g$$

$\therefore$ Resultant upward force = $F_B - W = V \times 10^{-5}\,(129 - 9) \times g$

The resultant upward force is balanced by the weight of 25 kg.

$\therefore$ $F_B - W = m\,g$

or $V \times 10^{-5} \times (129 - 9) \times g = 25 \times 10^3 \times g$

or $V = \frac{25 \times 10^3}{120} \times \frac{1}{10^{-5}} = 2.083 \times 10^7\ cm^3 =$ **2.083 × 10^4 litres**

11. A piece of sugar of mass 40 g is coated with 5.76 g of wax of specific gravity 0.96. If the mass of the coated composite piece is 14.76 g in water, find the density of sugar. **[I.I.T.]**

Hint. Volume of wax = $\frac{5.76}{0.96} = 6\ cm^3$

Total mass of sugar and wax in air = 40 + 5.76 = 45.76 g

Mass of composite piece in water = 14.76 g

Mass of water displaced = 45.76 – 14.76 = 31g

$\therefore$ Volume of water displaced $= \dfrac{\text{mass}}{\text{density}} = \dfrac{31}{1} = 31\ \text{cm}^3$

$\therefore$ Volume of sugar alone $= 31 - 6 = 25\ \text{cm}^3$

$\therefore$ Density of sugar $= \dfrac{\text{mass}}{\text{volume}} = \dfrac{40}{25} = \mathbf{1.6\ g/cm^3}$

12. **A piece of cork is embedded inside the block of ice which floats on water. What will happen to the level of water when all the ice melts?** **[I.I.T.]**

Hint. Let M= mass of ice block ; m = mass of cork ; V = volume of water displaced
According to principle of floatation,
Mass of ice block plus cork = Mass of water displaced
or $(M + m) = V \times 1 = V$ ...(*i*)
When the ice melts, let it be converted into V' c.c. of water.
$\therefore$ $M = V' \times 1 = V'$
When all ice melts, the piece of cork floats on the surface of water. Let the cork displace a volume V'' c.c. of water.
$\therefore$ $m = V'' \times 1 = V''$
Now $(M + m) = V' + V''$ ...(*ii*)
From eqs. (*i*) and (*ii*), $V = V' + V''$
Therefore, the volume of water displaced before the ice melts is equal to the volume of water displaced when all the ice melts. Hence, there is **no change in the level of water.**

13. **A soap film is on a rectangular wire ring of size 3 cm × 3 cm. If the size of the film is changed to 3 cm × 4 cm, calculate the work done in the process. The surface tension of soap film is 3×10^{-2} N/m.**

Hint. Initial surface area of film $= 3 \times 3 = 9\ \text{cm}^2 = 9 \times 10^{-4}\ \text{m}^2$
Final surface area of film $= 3 \times 4 = 12\ \text{cm}^2 = 12 \times 10^{-4}\ \text{m}^2$
Increase in surface area $= (12 - 9) \times 10^{-4} = 3 \times 10^{-4}\ \text{m}^2$
Since the film has two surfaces, the net increase in surface area is
$\Delta A = 2\,(3 \times 10^{-4}) = 6 \times 10^{-4}\ \text{m}^2$
Work done = Surface tension × Increase in surface area
$= (3 \times 10^{-2}) \times (6 \times 10^{-4}) = \mathbf{1.8 \times 10^{-5}\ J}$

14. **The surface tension of soap solution is 0.07 N/m. How much work is required to form a bubble of 1 cm radius from this solution?**

Hint. A soap bubble has two surfaces, external and internal. The area of both the surfaces is
$A = 2 \times (4\pi r^2) = 2 \times [4\pi \times (1.0 \times 10^{-2})^2] = 2.512 \times 10^{-3}\ \text{m}^2$
Work will have to be done against surface tension in blowing a soap bubble.
Work done = Surface tension × Increase in surface area
$= (0.07) \times (2.512 \times 10^{-3} - 0) = \mathbf{1.76 \times 10^{-4}\ J}$

15. **Calculate the work done against surface tension in blowing a soap bubble from a radius 10 cm to 20 cm. The surface tension of soap solution is 25×10^{-3} N/m.**

Hint. The soap bubble has two surfaces, external and internal.
Original total surface area, $A_1 = 2 \times 4\pi R_1^2$
Final total surface area, $A_2 = 2 \times 4\pi R_2^2$
$\therefore$ Increase in surface area, $\Delta A = 2 \times 4\pi [R_2^2 - R_1^2] = 8\pi[(0.2)^2 - (0.1)^2] = 0.24\pi\,\text{m}^2$
Work done = Surface tension × Increase in surface area
$= (25 \times 10^{-3}) \times (0.24\pi) = \mathbf{6\pi \times 10^{-3}\ J}$

16. Two spherical soap bubbles combine. If V is the change in volume of the contained air, A is the change in total surface area, then show that $3P_aV + 4AT = 0$, where T is the surface tension and P_a is the atmospheric pressure. **[I.I.T.]**

Hint. Let R_1 and R_2 be the radii of the two bubbles before combination and R be the radius of the bubble after combination.

Total pressure inside the first bubble is $P_1 = P_a + \frac{4T}{R_1}$

Total pressure inside the resultant bubble is $P_2 = P_a + \frac{4T}{R_2}$

Total pressure inside the second bubble is $P_3 = P_a + \frac{4T}{R}$

By Boyle's law, $P_1V_1 + P_2V_2 = P_3V_3$

or $$\left(P_a + \frac{4T}{R_1}\right)\frac{4}{3}\pi R_1^3 + \left(P_a + \frac{4T}{R_2}\right)\frac{4}{3}\pi R_2^3 = \left(P_a + \frac{4T}{R}\right)\frac{4}{3}\pi R^3$$

or $$P_a\left(\frac{4}{3}\pi R_1^3 + \frac{4}{3}\pi R_2^3 - \frac{4}{3}\pi R^3\right) + \frac{4T}{3}(4\pi R_1^2 + 4\pi R_2^2 - 4\pi R^2) = 0$$

But $$\frac{4}{3}\pi R_1^3 + \frac{4}{3}\pi R_2^3 - \frac{4}{3}\pi R^3 = V \text{ and } 4\pi R_1^2 + 4\pi R_2^2 - 4\pi R^2 = A$$

$\therefore$ $$P_aV + \frac{4T}{3} \times A = 0 \quad \text{or} \quad 3P_aV + 4AT = 0$$

17. Two separate air bubbles (radii 0.002 m and 0.004 m) formed of the same liquid (surface tension 0.07 N/m) come together to form a double bubble. Find the radius and sense of curvature of the internal film surface common to both the bubbles.

[Roorkee]

Hint. Figure 19.39 shows the conditions of the problem. Let R_A and R_B be the radii of the two bubbles A and B. Let R be the radius of curvature of the common internal film of the double bubble of the two bubbles.

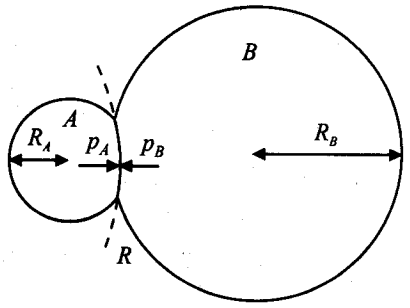


Fig. 19.39

Excess pressure in A, $p_A = \frac{4T}{R_A}$

Excess pressure in B, $p_B = \frac{4T}{R_B}$

Therefore, in the double bubble, the pressure difference between A and B on either side of the common surface is

$$p_A - p_B = \frac{4T}{R_A} - \frac{4T}{R_B} = 4T\left[\frac{1}{R_A} - \frac{1}{R_B}\right] = 4T\left[\frac{1}{0.002} - \frac{1}{0.004}\right] = 1000\,T$$

This pressure difference will be equal to $4T/R$.

$\therefore$ $$\frac{4T}{R} = 1000\,T \quad \text{or} \quad R = 0.004 \text{ m}$$

The pressure inside the smaller bubble is greater than inside the bigger bubble. Since excess pressure is always towards concave surface, the common surface is concave toward the centre of the smaller bubble as shown in Fig. 19.39.

18. An air bubble of radius 0.1 mm is situated just below the surface of water. Calculate the gauge pressure (excess of pressure). Surface tension of water = 7.2×10^{-2} N/m.

Hint. Gauge pressure (excess of pressure) is $p = \dfrac{2T}{R}$

Here $T = 7.2 \times 10^{-2}$ N/m ; $R = 0.1$ mm $= 0.1 \times 10^{-3}$ m

$$\therefore \quad p = \frac{2\times(7.2\times10^{-2})}{0.1\times10^{-3}} = 1440 \text{ N/m}^2$$

19. A glass tube of internal radius 5×10^{-4} m is dipped vertically into a vessel containing mercury such that the lower end of the tube is 10^{-2} m below the surface of mercury. Calculate the gauge pressure of air inside the tube to blow a hemispherical bubble at the lower end of the tube. Surface tension of mercury = 3.5×10^{-2} N/m. Density of mercury = 13.6×10^3 kg/m^3.

Hint. Figure 19.40 shows the conditions of the problem. When the tube is dipped vertically in the liquid to a height h ($=10^{-2}$ m) below the surface of mercury, the pressure just below the meniscus is P_a (same as outside). However, just above the meniscus the pressure is $P_a + (2T/R)$, where T is the surface tension and R is the internal radius of the tube.

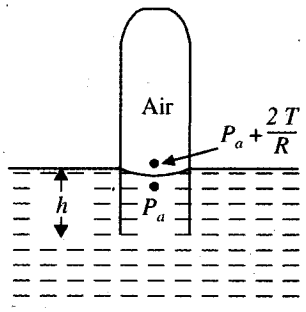


Fig. 19.40

$\therefore$ Gauge pressure required $= \dfrac{2T}{R} + h\rho g$

Here $T = 3.5 \times 10^{-2}$ N/m ; $R = 5 \times 10^{-4}$ m ;

$h = 10^{-2}$ m ; $\rho = 13.6 \times 10^3$ kg/m^3

$$\therefore \text{ Gauge pressure } = \frac{2\times(3.5\times10^{-2})}{5\times10^{-4}} + 10^{-2}\times(13.6\times10^3)\times9.8$$

$$= 140 + 1332 = 1472 \text{ N/m}^2$$

20. Two soap bubbles of radii a and b coalesce to form a single bubble of radius c. If the external pressure is P, show that the surface tension T of soap solution is

$$\boldsymbol{T = \frac{P(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}}$$

Hint. Pressure inside the bubble of radius a, $p_1 = P + \dfrac{4T}{a}$

Volume of the bubble of radius a, $V_1 = \dfrac{4}{3}\pi a^3$

Pressure inside the bubble of radius b, $p_2 = P + \dfrac{4T}{b}$

Volume of the bubble of radius b, $V_2 = \dfrac{4}{3}\pi b^3$

Pressure inside the bubble of radius c, $p_3 = P + \dfrac{4T}{c}$

Volume of the bubble of radius c, $V_3 = \dfrac{4}{3}\pi c^3$

Since temperature remains the same during the change, from Boyle's law, we have,

$$p_1V_1 + p_2V_2 = p_3V_3$$

or $$\left(P + \frac{4T}{a}\right)\times\frac{4}{3}\pi a^3 + \left(P + \frac{4T}{b}\right)\times\frac{4}{3}\pi b^3 = \left(P + \frac{4T}{c}\right)\times\frac{4}{3}\pi c^3$$

or $P\,[a^3 + b^3 - c^3] + 4T\,(a^2 + b^2 - c^2) = 0$

or $4T(a^2 + b^2 - c^2) = P(c^3 - a^3 - b^3)$

$\therefore$ $$T = \frac{P(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}$$

21. The limbs of a manometer consists of uniform capillary tubes of radii 1.44×10^{-3} m and 7.2×10^{-4} m. If the height of the liquid in the narrow tube is 0.2 m more than that in wide tube, calculate the true pressure difference. Density of liquid = 10^3 kg/m^3, surface tension = 72×10^{-3} N/m, g = 9.8 ms^{-2}. [Roorkee]

Hint. Let the pressures in the wide and narrow limbs be P_1 and P_2 respectively. If R_1 and R_2 are the radii of the meniscus in the wide and narrow limb, then,

Pressure just below the meniscus of wide limb $= P_1 - \frac{2T}{R_1}$

Pressure just below the meniscus of narrow limb $= P_2 - \frac{2T}{R_2}$

$\therefore$ Pressure difference is given by ;

$$\left(P_1 - \frac{2T}{R_1}\right) - \left(P_2 - \frac{2T}{R_2}\right) = h\rho g$$

or $$P_1 - P_2 = h\rho g - 2T\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

$$= 0.2 \times 10^3 \times 9.8 - 2 \times 72 \times 10^{-3}\left[\frac{1}{7.2 \times 10^{-4}} - \frac{1}{1.44 \times 10^{-3}}\right]$$

$$= (1.96 \times 10^3) - (0.1 \times 10^3) = \mathbf{1860\ N/m^2}$$

22. There is a soap bubble of radius 2.4×10^{-4} m in an air cylinder which is originally at a pressure of 10^5 N/m^2. The air in the cylinder is now compressed isothermally until the radius of the bubble is halved. Calculate now the pressure of air in the cylinder. The surface tension of soap film is 0.08 Nm^{-1}. [Roorkee Engg.]

Hint. Initial pressure of air in the cylinder is

$$P_0 = 10^5\ \text{N/m}^2$$

Suppose the initial volume of the soap bubble is V_1. If R is the radius of the bubble and T the surface tension of the soap solution, then,

Pressure inside the bubble is given by;

$$P_1 = P_0 + \frac{4T}{R}$$

Let P be the air pressure in the cylinder when the bubble is compressed to radius = $R/2$.

$\therefore$ Pressure inside the bubble is given by ;

$$P_2 = P + \frac{4T}{R/2} = P + \frac{8T}{R}$$

Now volume of the bubble is given by;

$$V_2 = \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{1}{8} \times \frac{4}{3}\pi R^3 = \frac{V_1}{8}$$

Since the temperature remains constant during compression, from Boyle's law, we have,

$$P_1V_1 = P_2V_2$$

or $$\left(P_0 + \frac{4T}{R}\right)V_1 = \left(P + \frac{8T}{R}\right)\frac{V_1}{8}$$

or $$P_0 + \frac{4T}{R} = \frac{P}{8} + \frac{T}{R}$$

$\therefore$ $$P = 8P_0 + \frac{24T}{R} = (8 \times 10^5) + \frac{24 \times (0.08)}{2.4 \times 10^{-4}} = \mathbf{8.08 \times 10^5\ N/m^2}$$

23. A small drop of clean water is squeezed between two glass plates so that a very thin layer of large area is formed. If the thickness of water film is 10^{-6} m and the surface area of it is 40 cm^2, calculate the force required to separate the two glass plates. Surface tension of water is 7×10^{-2} N/m.

Hint. Figure 19.41 shows the conditions of the problem. Suppose *X*, *Y* are the two plates, P_a is the external pressure and *p* is the pressure inside the very thin water layer of thickness *d*. Assuming zero angle of contact, the meniscus shown has a curvature $1/r$ or $1/(d/2)$ which is very much larger than the curvature parallel to the plates. If we neglect the curvature parallel to the plates, the excess pressure is given by T/r in place of $2T/r$ obtained with a spherical meniscus.

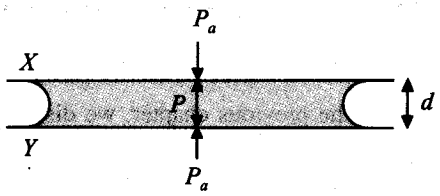


Fig. 19.41

$\therefore$ $$P_a - p = \frac{T}{r} = \frac{T}{d/2} = \frac{2T}{d}$$

Force pushing the two plates together is

$$F = \text{Excess pressure} \times \text{Surface area}$$

$$= \frac{2T}{d} \times \text{area} = \frac{2 \times 7 \times 10^{-2}}{10^{-6}} \times 40 \times 10^{-4} = 560\ \text{N}$$

Therefore, a force of **560 N** is needed to pull the plates apart. Note that as the thickness (*d*) of water layer becomes larger, the force becomes smaller ; the plates are now easier to pull apart.

24. A soap bubble in vacuum has a radius of 3 cm and another soap bubble in the vacuum has a radius of 6 cm. If the two bubbles coalesce under isothermal conditions, calculate the radius of the bubble formed.

Hint. Since the bubbles coalesce under isothermal conditions, the surface tension *T* is constant. Suppose *R* cm (= $R \times 10^{-2}$ m) is the radius of the bubble formed. Then,

$$\text{Work done} = T \times \text{Surface area} = T \times 8\pi R^2 \times 10^{-4}$$

But $$\text{Original work done} = (T \times 8\pi \times 3^2 + T \times 8\pi \times 6^2) \times 10^{-4}$$

$\therefore$ $$T \times 8\pi R^2 \times 10^{-4} = (T \times 8\pi \times 3^2 + T \times 8\pi \times 6^2) \times 10^{-4}$$

or $$R^2 = 3^2 + 6^2 \quad \therefore \quad R = \sqrt{3^2 + 6^2} = \mathbf{6.7\ cm}$$

20

Hydrodynamics (Fluids in Motion)

INTRODUCTION

In the previous chapter, we discussed the properties of a fluid at rest (*i.e.,* hydrostatics). Let us now turn our attention to fluids in motion. *The branch of physics which deals with the study of properties of fluids in motion is called* **hydrodynamics**. The study of hydrodynamics reveals many interesting phenomena occurring in nature. It is not possible to cover all topics of hydrodynamics in a single chapter. Therefore, in this chapter, we shall discuss some topics related to fluids in motion including **viscosity** and **Bernoulli's theorem.**

20.1. VISCOSITY

We have seen that when a solid surface moves over another solid surface, their relative motion is opposed by the friction between the surfaces. This is called *solid friction.* The situation is similar for a moving fluid (liquid or gas). A fluid moves in the form of layers. When a layer of fluid moves over another layer, their relative motion is opposed by the internal friction of the fluid. This property of a fluid is called *fluid friction* or *viscosity.* The obvious effect of fluid friction or viscosity is to retard the motion of the fluid.

The property of a moving fluid (liquid or gas) to oppose the relative motion between its layers is called **viscosity** *of the fluid.*

Thus viscosity is that property of a fluid that indicates its internal friction. The greater the viscosity of a fluid, the greater is the force required to cause one layer of fluid to slide past another. The viscosity of honey is very large as compared to that of water. This means that for the same applied external force, the rate of flow of honey will be very small as compared to that of water.

The viscosity of a fluid not only retards its own motion but it also retards the motion of a solid through it. The greater the viscosity of a fluid, the harder it is for a solid to move through it. Imagine the difference between swimming in water and honey!

20.2. CAUSE OF VISCOSITY

Viscosity is the internal friction of a fluid which opposes the motion of one layer of fluid past another. The forces of attraction between the molecules of a moving fluid determine the viscosity of the fluid. Consider a liquid flowing steadily over a solid horizontal surface *AB* as shown in Fig. 20.1. The layer of the liquid in contact with the surface *AB* is at rest because of the adhesion forces. As we move up, the velocity of layers goes on increasing so that the layer at the top has maximum velocity.

Consider two parallel adjoining layers *PQ* and *MN* as shown in Fig. 20.1. The fast moving layer *MN* tends to accelerate the adjoining slow moving layer *PQ*. It is as if the faster layer *MN* is dragging the slower layer *PQ* with it. Therefore, the average speed of the molecules of the faster

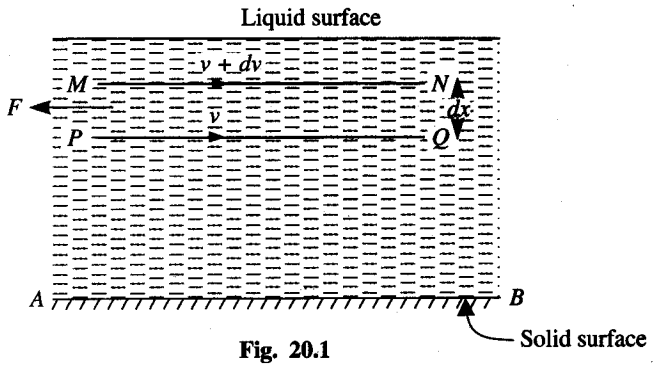


Fig. 20.1

layer is decreased. The result is that a *backward dragging force* called *viscous drag F* is developed which opposes the motion of the liquid. *The viscous drag (or backward retarding force) acts tangentially on the layers.* It is found that viscous drag between the two layers is directly proportional to the area of the layers in contact and the velocity gradient *dv/dx* between them (*i.e.*, change in velocity (*dv*) divided by perpendicular distance (*dx*) between the layers).

Thus when a liquid (or fluid) flows, a backward dragging force or viscous drag is developed between its layers that retards the motion of the liquid. The viscous drag acts tangentially on the layers of the liquid.

In order to maintain the liquid flow, an external force equal to the viscous drag must be applied.

20.3. COEFFICIENT OF VISCOSITY

Fig. 20.2 shows the steady flow of a liquid over a horizontal solid surface *AB*. Consider two parallel layers *PQ* and *MN*, each of area *A*, at a distance x and $x + dx$ from the solid surface *AB* and moving with velocities v and $v + dv$ respectively. *The increase in velocity divided by the distance over which this change occurs (i.e., dv/dx) is called the velocity gradient and is measured perpendicular to the direction of velocity.*

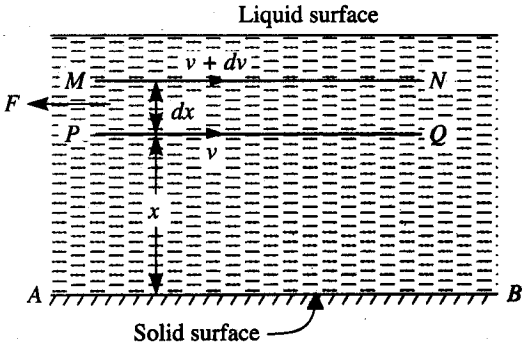


Fig. 20.2

It has been found experimentally that viscous drag F between the two layers varies as under :

$$F \propto A \quad \text{and} \quad F \propto \frac{dv}{dx}$$

$$\therefore \qquad F \propto A\frac{dv}{dx}$$

or

$$F = -\eta A \frac{dv}{dx}$$

where η is a constant of proportionality and is called *coefficient of viscosity.* Its value depends upon the nature of the liquid. Note that negative sign shows that the direction of viscous drag F is opposite to the direction of motion of the liquid.

Definition of η. $$F = \eta A \frac{dv}{dx}$$ ...magnitude

If $A = 1$ and $dv/dx = 1$, then $\eta = F$.

Hence coefficient of viscosity of a liquid is defined as the tangential force required to maintain a unit velocity gradient between two parallel layers each of unit area.

Units of η. $$\eta = \frac{F}{A \times \text{velocity gradient}}$$

The SI unit of F is 1N, the SI unit of A is 1 m^2 and that of velocity gradient is 1 ms^{-1} per m *i.e.,* s^{-1}.

$$\therefore \quad \text{SI unit of } \eta = \frac{1\,N}{1\,m^2 \times s^{-1}} = 1\ Ns\ m^{-2}$$

We find that the SI unit of η is Ns m^{-2}. It is also called **decapoise** or **pascal second** (Pa s).

Hence the coefficient of viscosity of a liquid is **1 decapoise** (*or* **1 Ns m^{-2}**) *if a tangential force of 1 N is required to maintain a velocity gradient of 1ms^{-1}/m between two parallel layers each of area 1 m^2.*

The corresponding C.G.S. unit is dynes-s/cm^2 called **poise** and is in common use. Clearly, 1 decapoise = 10 poise.

Dimensional formula of η. $$\eta = \frac{F}{A \times \text{velocity gradient}}$$

$$\therefore \quad [\eta] = \frac{[M\,LT^{-2}]}{[L^2][LT^{-1}/L]} = [M\,L^{-1}T^{-1}]$$

Discussion. The following points may be noted :

(*i*) The viscosity of an ideal liquid is zero.

(*ii*) The coefficient of viscosity of a liquid decreases with the increase in temperature and *vice-versa.* However, the coefficient of viscosity of gases increases with the increase in temperature.

20.4. FLUID FRICTION (VISCOSITY) VERSUS SOLID FRICTION

Solid friction arises when one solid surface moves over another solid surface. However, fluid friction (or viscosity) comes into play when one layer of a fluid moves over another layer.

Similarities. The similarities between solid friction and viscosity are :

(*i*) Both come into play whenever there is a relative motion.

(*ii*) Both oppose the relative motion.

(*iii*) Both arise from intermolecular forces.

(*iv*) Both depend on nature of surfaces.

Differences. The differences between solid friction and viscosity are :

(*i*) The force of friction between two layers of a fluid depends upon the area of contact and the velocity gradient between the layers. However, the frictional force between the surfaces of two solids is independent of the area of contact and of the relative velocity.

(*ii*) The viscosity of a liquid decreases with the increase in temperature whereas the solid friction is independent of the temperature.

(*iii*) In case of solid friction, heat is generated at the surfaces between the solids. When a fluid moves over a solid, heat is generated within the fluid and not at the interface of the solid and the fluid.

Example 20.1. A square plate of side 20 cm moves parallel to another plate with a velocity of 20 cm s^{-1}; both plates immersed in water. If the viscous force is 4×10^{-3} N and viscosity of water is 0.001 decapoise, what is their distance apart?

Solution. Area of plate, $A = 20 \times 20 = 400\ cm^2 = 400 \times 10^{-4}\ m^2$

Relative velocity, $dv = 20\ cm\ s^{-1} = 0.2\ m\ s^{-1}$

Coefficient of viscosity, $\eta = 0.001$ decapoise

Now $$F = \eta A \frac{dv}{dx}$$

or $$4 \times 10^{-3} = 0.001 \times \frac{400 \times 10^{-4} \times 0.2}{dx}$$

or $$4 \times 10^{-3} = \frac{0.08 \times 10^{-4}}{dx}$$

$\therefore$ $$dx = \frac{0.08 \times 10^{-4}}{4 \times 10^{-3}} = 0.002\ m = \mathbf{0.2\ cm}$$

Example 20.2. The relative velocity between two layers of water is 8 cm/s. If the perpendicular distance between the layers is 0.1 cm, find the velocity gradient.

Solution. Velocity gradient $= \frac{dv}{dx}$

Here $dv = 8$ cm/s ; $dx = 0.1$ cm

$\therefore$ Velocity gradient $= \frac{8}{0.1} = \mathbf{80\ s^{-1}}$

Example 20.3. Calculate the horizontal force required to move a metal plate of area $2 \times 10^{-2}\ m^2$ with a velocity of $4.5 \times 10^{-2}\ ms^{-1}$ when it rests on a layer of oil 1.5×10^{-3} m thick. Coefficient of viscosity of oil = 2 Nsm^{-2}.

Solution. $$F = \eta A \frac{dv}{dx}$$

Here $\eta = 2\ Nsm^{-2}$; $A = 2 \times 10^{-2}\ m^2$; $\frac{dv}{dx} = \frac{4.5 \times 10^{-2}}{1.5 \times 10^{-3}} = 30\ s^{-1}$

$\therefore$ $$F = 2 \times 2 \times 10^{-2} \times 30 = \mathbf{1.2\ N}$$

Example 20.4. A metal plate of area 0.1 m^2 is connected to a 0.01 kg mass via a string that passes over an ideal pulley as shown in Fig. 20.3. A liquid with a film thickness of 0.3 mm is placed between the plate and the table. When released, the plate moves to the right with a constant speed of 0.085 ms^{-1}. Find the coefficient of viscosity of the liquid. The pulley may be considered massless and frictionless.

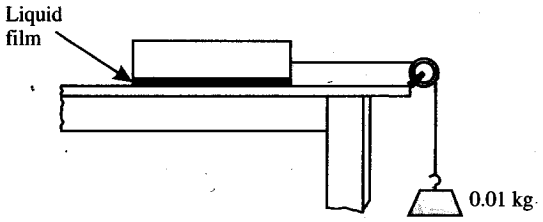


Fig. 20.3

Solution. The plate will move towards the right due to tension T (= mg) in the string.

Now $$T = mg = 0.01 \times 9.8 = 9.8 \times 10^{-2}\ N$$

Since the plate moves with a constant speed (= 0.085 ms^{-1}), the viscous drag (F) is equal to tension in the string *i.e.*, $F = T = 9.8 \times 10^{-2}$ N.

Now $$F = \eta A \frac{dv}{dx}$$

Here $F = 9.8 \times 10^{-2}$ N ; $A = 0.1$ m^2 ; $dv = 0.085$ ms^{-1}; $dx = 3 \times 10^{-4}$ m

$$\therefore \quad 9.8 \times 10^{-2} = \eta \times 0.1 \times \frac{0.085}{3 \times 10^{-4}} \quad \text{or} \quad \eta = 3.46 \times 10^{-3} \text{ Pas}$$

20.5. STOKES' LAW

When a solid body moves through a fluid at rest, the fluid in contact with the solid is dragged with it. This causes relative motion between different layers of the fluid near the solid and a retarding viscous force (viscous drag) acts on the body. In other words, when a solid body moves through a fluid, the fluid exerts a viscous force on the solid body to oppose the motion of the body. The magnitude of the viscous force depends on (*i*) the shape and size of the solid body (*ii*) the speed of the body and (*iii*) the coefficient of viscosity of the fluid.

According to stokes, a spherical body of radius r moving at velocity v (at an instant) through a fluid of viscosity coefficient η will experience a viscous drag F (retarding viscous force) given by ;

$$F = 6\pi \eta r v \qquad ...(i)$$

Eq. (*i*) is called **stokes' law** and is applicable to small spherical bodies only.

Derivation of the formula. Consider a sphere of radius r moving with velocity v through a fluid whose coefficient of viscosity is η. It is desired to find an expression for the viscous force F on the sphere. For this purpose, we shall use dimensional analysis. Stokes observed that in case of a slowly moving small sphere, the viscous force F depends upon :

(*i*) *the radius r of the spherical body*

(*ii*) *the coefficient of viscosity* η *of the fluid*

(*iii*) *the velocity v of the spherical body*

Therefore, we can express the relation as :

$$F \propto r^x \eta^y v^z$$

or $$F = k\, r^x \eta^y v^z \qquad ... (i)$$

where k is dimensionless constant and x, y, z are the unknown indices. Since each side of the equation must have the same dimensions,

$$\therefore \quad [F] = [r^x][\eta^y][v^z]$$

or $$[M\,LT^{-2}] = [L]^x [M\,L^{-1}T^{-1}]^y [LT^{-1}]^z$$

or $$[M\,LT^{-2}] = [M]^y [L]^{x+z-y} [T]^{-y-z}$$

Equating the indices of M, L and T on both sides, we have,

$$1 = y \; ; \quad 1 = x + z - y \; ; \quad -2 = -y - z$$

Solving gives $y = 1$, $z = 1$, $x = 1$. Therefore, relation in eq. (*i*) becomes :

$$F = k\, r \eta v$$

The value of k was found to be 6π (in case of small spheres).

$$\therefore \quad F = 6\pi \eta r v$$

Notes. (*i*) Strictly, the law applies to a fluid of infinite extent.

(*ii*) The law does not hold good if the spherical body is moving so fast that conditions are not streamline.

(*iii*) The spherical body must be smooth and rigid.

20.6. IMPORTANCE OF STOKES' LAW

(*i*) It is used in Millikan's experiment for the measurement of charge on an electron.

(*ii*) It accounts for the formation of clouds.

(*iii*) It explains why large rain drops hurt much more than small ones when they fall on you. It is not just that they are heavier ; they are actually falling faster.

(*iv*) It is used to find the size of small particles.

(*v*) It is useful in the consideration of geological processes in which the rate of sedimentation is important.

20.7. TERMINAL VELOCITY

Consider a small sphere falling freely from rest through a large column of a viscous fluid as shown in Fig. 20.4 (*i*). The forces acting on the sphere are :

(*i*) Weight W of the sphere acting vertically downwards.

(*ii*) Upthrust U equal to the weight of the fluid displaced.

(*iii*) Viscous drag F acting vertically upward *i.e.*, in a direction opposite to motion of the sphere.

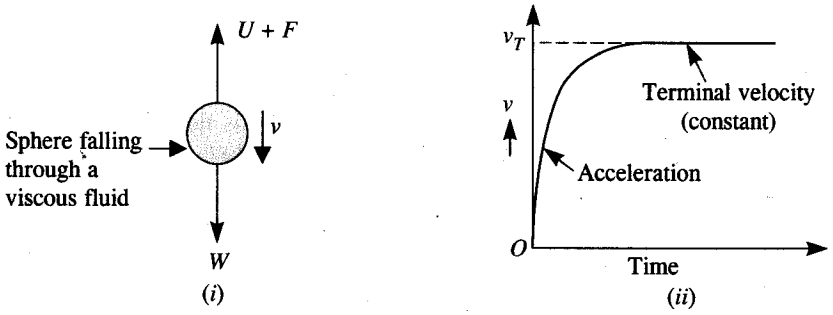


Fig. 20.4

Initially, the downward force W is greater than the upward force, $U + F$, and the sphere accelerates downwards. As the velocity of the sphere increases, the magnitude of velocity dependent viscous force F (Stokes' law) also increases. Eventually, a stage is reached when $U + F = W$. Since the resultant force on the sphere is zero, it now moves downwards with a constant maximum velocity, called *terminal velocity* v_T.

The maximum constant velocity acquired by a body while falling freely through a viscous medium is called **terminal velocity** *of the body.*

Fig. 20.4 (*ii*) shows the variation of the velocity v of the freely falling sphere till the terminal velocity v_T is reached.

Expression for terminal velocity. Consider a small sphere of radius r falling freely through a viscous fluid.

Let ρ = Density of the material of the sphere

σ = Density of the fluid

η = Coefficient of viscosity of the fluid

Weight of sphere, $W = mg = \left(\frac{4}{3}\pi r^3 \rho\right) g = \frac{4}{3}\pi r^3 \rho g$

Upthrust, $U = \frac{4}{3}\pi r^3 \sigma g$

At terminal velocity v_T, $F = 6\pi \eta r v_T$

We know that at the terminal velocity,

$$W = U + F$$

or $$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi \eta r v_T$$

or $$\frac{4}{3}\pi r^3 (\rho - \sigma) g = 6\pi \eta r v_T$$

or Terminal velocity, $$v_T = \frac{2}{9} \times \frac{r^2 (\rho - \sigma) g}{\eta} \quad ...(i)$$

Eq. (*i*) gives the expression for the terminal velocity of a spherical body falling freely through a fluid.

Discussion. The following points are worth noting :

(*i*) The terminal velocity (v_T) of a spherical body falling freely through a viscous fluid is *directly proportional to the square of its radius.* This means that for a given medium, the terminal velocity of a large sphere is greater than that of a small sphere of the same material. For this reason, bigger raindrops fall with greater velocity as compared with smaller ones.

(*ii*) The terminal velocity (v_T) of a spherical body is *directly proportional to the difference in the densities of the body and the fluid* (*i.e.*, $\rho - \sigma$). If the density of the fluid is greater than that of body (*i.e.*, $\sigma > \rho$), then terminal velocity is negative. This means that the body instead of falling, moves upward. This is why bubbles of air rise up in water.

(*iii*) The terminal velocity (v_T) of a spherical body is *inversely proportional to the coefficient of viscosity of the fluid.* This means that more viscous the fluid, the smaller the terminal velocity that the spherical body will acquire.

(*iv*) If ρ, σ and η are known, measuring v_T can provide information regarding the size of the spherical particle. R.A. Millikan employed this technique in his historic measurement of charge on an electron.

20.8. IMPORTANCE OF VISCOSITY

The knowledge of viscosity is very important as is evident from the following discussion.

(*i*) The viscosity of blood depends upon the concentration of red blood corpuscles. Therefore, viscosity of blood can be used to detect blood corpuscle deficiency.

(*ii*) Viscosity of air or liquid is used in providing damping torque in measuring instruments.

(*iii*) Oil used as a lubricant should have proper value of viscosity.

(*iv*) The viscosity of oil helps in applying brakes.

(*v*) Blood circulation through arteries depends upon the viscosity of blood.

Example 20.5. A drop of water of radius 0.01 mm is falling through a medium whose density is 1.21 kg/m^3 and coefficient of viscosity is 1.8×10^{-5} Nsm^{-2}. Find the terminal velocity of the drop and viscous force on the drop.

Solution. Radius of drop, $r = 0.01 \text{ mm} = 0.01 \times 10^{-3} \text{ m}$

Density of drop, $\rho = 1000 \text{ kg/m}^3$

Density of medium, $\sigma = 1.21 \text{ kg/m}^3$

Viscosity of medium, $\eta = 1.8 \times 10^{-5} \text{ Ns m}^{-2}$

The terminal velocity v_T of the drop is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

$$= \frac{2}{9}\times\frac{(0.01\times10^{-3})^2\times(1000-1.21)\times9.8}{1.8\times10^{-5}} = 1.2\times10^{-2}\ \text{ms}^{-1}$$

Viscous force on drop, $F = 6\pi\,\eta\,r\,v_T$

$$= 6\pi\times(1.8\times10^{-5})\times(0.01\times10^{-3})\times1.2\times10^{-2} = 4.07\times10^{-11}\ \text{N}$$

Example 20.6. A metallic sphere of radius 1.0×10^{-3} m and density 1.0×10^4 kg/m^3 enters a tank of water after a free fall through a height h in earth's gravitational field. If its velocity remains unchanged after entering water, determine the value of h. Given : coefficient of viscosity of water = 1.0×10^{-3} Nsm^{-2}; g = 10 ms^{-2} and density of water = 1.0×10^3 kg/m^3.

Solution. The velocity acquired by the sphere in falling freely through a height h is

$$v = \sqrt{2gh}$$

As per the conditions of the problem, this is the terminal velocity of the sphere in water *i.e.*, Terminal velocity of sphere in water is

$$v_T = \sqrt{2gh} \qquad ...(i)$$

By Stokes' law, the terminal velocity v_T of sphere in water is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

$$= \frac{2}{9}\times\frac{(1.0\times10^{-3})^2\times(1.0\times10^4-1.0\times10^3)\times10}{1.0\times10^{-3}} = 20\ \text{ms}^{-1}$$

From eq. (*i*), $h = \frac{v_T^2}{2g} = \frac{(20)^2}{2\times10} = 20$ m

Example 20.7. Two identical drops of water are falling through air with a steady velocity of 10 cm s^{-1}. If the drops combine to form a single drop, what would be the terminal velocity of the single drop?

Solution. Terminal velocity $\propto$ (radius)^2.

Suppose r is the radius of each of the small drop and R is the radius of bigger drop. Let v_T and v_T' be the terminal velocities of the drops before and after combination respectively. Then,

$$\frac{v_T}{v_T'} = \frac{r^2}{R^2}$$

Now $$2\times\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3 \quad \text{or} \quad \left(\frac{r}{R}\right)^2 = \frac{1}{(2)^{2/3}}$$

$\therefore$ $$v_T' = v_T\left(\frac{R}{r}\right)^2 = 10\times(2)^{2/3} = 15.9\ \text{cm s}^{-1}$$

Example 20.8. With what terminal velocity will an air bubble 0.8 mm in diameter rise in a liquid of viscosity 0.15 Ns m^{-2} and specific gravity 0.9? Density of air is 1.293 kg m^{-3}.

Solution. The terminal velocity of the bubble is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

Here $r = 0.4\ \text{mm} = 4\times10^{-4}$ m ; $\sigma = 0.9\times10^3$ kg m^{-3} ; $\rho = 1.293$ kg m^{-3} ; $\eta = 0.15$ Ns m^{-2}

$\therefore$ $$v_T = \frac{2}{9}\times\frac{(4\times10^{-4})^2\times(1.293-0.9\times10^3)\times9.8}{0.15}$$

$$= -0.0021\ \text{ms}^{-1} = -0.21\ \text{cm s}^{-1}$$

The negative sign shows that the bubble will rise up.

Example 20.9. **In Millikan's experiment for determining electronic charge, an oil drop of density 0.95 g/cm^3 falls with a terminal velocity 1.142 × 10^{-2} cm/s through air of density 0.0013 g/cm^3 and viscosity 181 × 10^{-6} g cm^{-1} s^{-1}. Find the radius of the drop.**

Solution. $$v_T = \frac{2}{9} \times \frac{r^2(\rho - \sigma)g}{\eta}$$

or $$r^2 = \frac{9}{2} \times \frac{\eta v_T}{(\rho - \sigma)g}$$

Here $\rho = 0.95$ g/cm^3 ; $\sigma = 0.0013$ g/cm^3 ; $v_T = 1.142 \times 10^{-2}$ cm/s ; $\eta = 181 \times 10^{-6}$ g cm^{-1} s^{-1}

$\therefore$ $$r^2 = \frac{9}{2} \times \frac{(181 \times 10^{-6}) \times (1.142 \times 10^{-2})}{(0.95 - 0.0013) \times 981} = 10^{-8}$$

or $$r = \sqrt{10^{-8}} = 10^{-4} \text{ cm}$$

PROBLEMS FOR PRACTICE

1. The terminal velocity of a copper ball of radius 2.0 mm falling through a tank of oil at 20°C is 6.5 cm/s. Compute the viscosity of the oil at 20°C. Density of oil = 1.5 × 10^3 kg/m^3, density of copper = 8.9 × 10^3 kg/m^3. **[0.992 N s m^{-2}]**
2. Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of 0.0150 ms^{-1}. What would be the terminal velocity if these two drops were to coalesce to form a large spherical drop? **[0.238 ms^{-1}]**
3. Calculate the terminal velocity of a raindrop of radius 0.3 mm. Take coefficient of viscosity of air = 1.83 × 10^{-5} Pa s, density of air = 1.3 kg m^{-3}, density of water = 1000 kg m^{-3} and g = 9.8 m s^{-2}. **[10.69 ms^{-1}]**

20.9. FLUID FLOW

The study of fluids in motion is called fluid dynamics. The general flow of a fluid can be very complicated. For example, consider the flow of a river in flood or the rise of smoke from a burning cigarette. While each drop of water or each smoke particle is governed by Newton's laws of motion, the resulting equations can be very complex. Fortunately, many features of fluid motion can be understood with certain simplifying assumptions. While discussing fluid flow, we *generally* make the following assumptions :

(*i*) The fluid is *nonviscous i.e.,* there is no internal friction between the adjacent layers of the fluid.

(*ii*) The fluid is *incompressible.* This means that density of the fluid is constant.

(*iii*) The fluid motion is *steady.* The steady flow of a fluid means that the velocity, density and pressure at each point in the fluid do not change with time.

The above assumptions permit us to study the flow of fluid in a simple way. In the sections to follow, *we shall be discussing liquid flow only.*

20.10. TYPES OF LIQUID FLOW

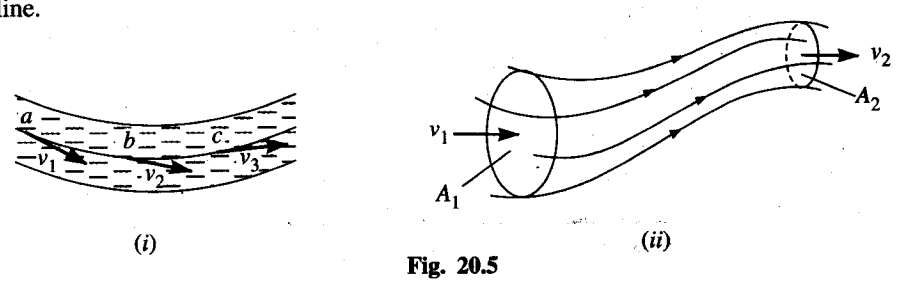
The liquid flow is of two main types *viz.*

(*i*) Streamline flow or steady flow

(*ii*) Turbulent flow.

(*i*) **Streamline flow or steady flow.** *The flow of a liquid is said to be* **streamline flow** *or* **steady flow** *if all the liquid particles that pass any given point follow the same path at the same speed (i.e., they have the same velocity).*

For example, consider a liquid flowing through a pipe as shown in Fig. 20.5 (*i*). The flowing liquid will have a certain velocity v_1 at *a*, a velocity v_2 at *b* and so on. If with the passage of time, the velocity of whatever liquid particle happens to be at *a* is still v_1, that at *b* is still v_2, then the flow is said to be steady or streamline flow. Every particle starting at *a* will follow the same path *abc*. The line *abc* is called *streamline*. At each point, the direction of velocity is tangent to the streamline.



(*i*) (*ii*)

Fig. 20.5

A **streamline** *is a curve whose tangent at any point is along the direction of the velocity of the liquid particle at that point.*

Streamlines cannot cross. If they were to do so, particles reaching the intersection would not have a unique velocity at that point in space. A bundle of streamlines such as those shown in Fig. 20.5 (*ii*) is called a **tube of flow**. Since the streamlines represent the paths of particles, we see that no liquid can flow into or out of the sides of a tube of flow. In a steady flow, the velocity, density and pressure at each point in the liquid do not change with time.

Laminar flow. It is a special case of streamline or steady flow in which the liquid flows as a series of parallel layers (laminae) and no one layer crosses another layer. *Thus smooth streamline flow is known as* **laminar flow**.

(*ii*) Turbulent flow. *The flow of a liquid is said to be* **turbulent flow** *or* **disorderly flow** *if the speed and direction of the liquid particles passing any point change with time.*

Fig. 20.6 shows the turbulent flow of a liquid moving through a narrow constriction. Instead of following a given path, the molecules (particles) of the liquid form whirlpool patterns called *eddies*. Generally, turbulent flow occurs in liquids moving at high speeds. When the velocity of the liquid is small, the flow is laminar or steady. However, if the velocity of liquid exceeds certain limiting value called *critical velocity,* the steady flow changes into turbulent flow. We will be mainly concerned with situations involving streamline flow.

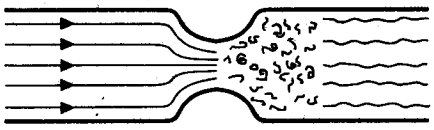


Fig. 20.6

20.11. RATE OF FLOW

The volume of a liquid that passes the cross-section of a vessel (e.g., pipe) in one second is called the rate of flow. It is denoted by the symbol *Q*.

If *V* is the volume of a liquid that passes the cross-section of a pipe in time *t*, then rate of flow (*Q*) of the liquid is

$$\text{Rate of flow, } Q = \frac{V}{t}$$

The SI unit of rate of flow of liquid is m^3/s.

Consider a pipe of uniform cross-sectional area 'a' as shown in Fig. 20.7. If the liquid is flowing at an average velocity of v, then distance l through which the liquid moves in time t is vt. This may be regarded as the length of an imaginary cylinder of the liquid that has passed the section S in time t. Then volume of liquid that has passed section S in time t is $V = a \times v\, t$.

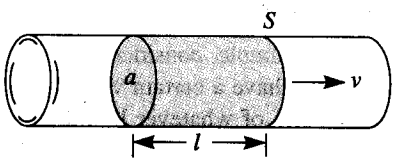


Fig. 20.7

$$\therefore \qquad \text{Rate of flow, } Q = \frac{V}{t} = \frac{a \times v\, t}{t}$$

or $$Q = a\, v$$

This is called **discharge equation**. It is clear that for constant rate of flow, velocity v is inversely proportional to the area of cross-section of the pipe. *If the cross-sectional area of the pipe decreases, the velocity of the liquid increases and vice-versa.*

20.12. REYNOLDS NUMBER

When a liquid flows in a pipe with a small velocity, the flow is smooth streamline or laminar flow. As the velocity is gradually increased, a stage is reached when the liquid flow becomes turbulent. Whether the liquid flow is laminar or turbulent is decided by Reynolds number (N_R). The expression for Reynolds number is given by ;

$$\text{Reynolds number, } N_R = \frac{\rho v D}{\eta} \qquad ...(i)$$

where
ρ = Density of the liquid
v = Average velocity of flow.
D = Diameter of the tube or pipe
η = Coefficient of viscosity of the liquid

Experiments show that if N_R is less than 2000, the flow is laminar (or steady). If N_R is greater than 3000, the flow is turbulent. If N_R is between 2000 and 3000, the flow is unstable. It may change from laminar to turbulent and *vice-versa.*

Discussion. The following points may be noted :

(*i*) N_R is a pure number (all of its units cancel) that determines the nature of flow of the liquid through a pipe.

(*ii*) The value of N_R is independent of the system of units used for the measurement of various quantities in eq. (*i*).

20.13. CRITICAL VELOCITY

When the velocity of the liquid flowing through a pipe is small, the flow is steady. As the velocity is increased, a stage is reached when the flow becomes turbulent.

That velocity of liquid flow upto which its flow is steady and above which its flow becomes turbulent is called **critical velocity**.

Experiments show that for cylindrical pipes critical velocity is given by ;

$$v_c \simeq \frac{1100\eta}{r\rho}$$

where
η = Coefficient of viscosity of the liquid
r = Radius of pipe
ρ = Density of the liquid

Derivation of formula. We can derive the formula for critical velocity by dimensional analysis. The critical velocity v_c of a liquid flowing through a pipe depends upon (*i*) coefficient of viscosity η of the liquid (*ii*) density ρ of the liquid and (*iii*) the radius r of the pipe. Therefore, v_c can be expressed as :

$$v_c = k\,\eta^a\,\rho^b\,r^c \qquad ...(i)$$

where k is a dimensionless constant and a, b and c are the unknown numbers. In dimensional form, eq. (*i*) can be written as under :

$$[M^0 L T^{-1}] = [M\,L^{-1}\,T^{-1}]^a\,[M\,L^{-3}]^b\,[L]^c$$

or $$[M^0 L T^{-1}] = [M^{a+b}][L^{-a-3b+c}][T^{-a}]$$

Equating powers of M, L and T on both sides of above equation, we have,

$$a + b = 0 \quad \text{or} \quad a = -b \qquad ...(ii)$$

$$-a - 3b + c = 1 \quad \text{or} \quad -a + 3a + c = 1 \text{ or } 2a + c = 1 \qquad ...(iii)$$

$$-a = -1 \text{ or } a = 1 \qquad ...(iv)$$

From eqs. (*ii*), (*iii*) and (*iv*), $a = 1$; $b = -1$; $c = -1$

Putting the values of a, b and c in eq. (*i*), we have,

$$v_c = k\,\eta^1\,\rho^{-1}\,r^{-1} = \frac{k\eta}{r\rho}$$

By experiment, the value of $k \simeq 1100$.

$$\therefore \qquad v_c = \frac{1100\,\eta}{r\rho}$$

20.14. EQUATION OF CONTINUITY

Consider the steady flow of a liquid through a pipe of varying cross-sectional area as shown in Fig. 20.8.

Let a_1 and a_2 = Cross-sectional areas of the pipe at A and B respectively

ρ_1 and ρ_2 = Densities of the liquid at A and B respectively

v_1 and v_2 = Velocities of the liquid at A and B respectively

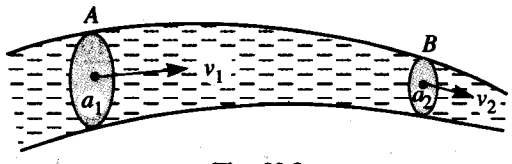


Fig. 20.8

The mass m_1 of the liquid passing through cross-section a_1 in one second is

$$m_1 = Q_1\,\rho_1 = a_1 v_1 \rho_1$$

The mass m_2 of the liquid passing through cross-section a_2 in one second is

$$m_2 = Q_2\,\rho_2 = a_2 v_2 \rho_2$$

Since the flow is steady, the mass of liquid passing through cross-section a_1 must be equal to the mass of liquid passing in the same time through cross-section a_2 *i.e*, $m_1 = m_2$.

$$\therefore \qquad a_1 v_1 \rho_1 = a_2 v_2 \rho_2$$

Since liquids are practically incompressible, $\rho_1 = \rho_2$.

$$\therefore \qquad a_1 v_1 = a_2 v_2 \qquad ...(i)$$

The expression (*i*) is called the **equation of continuity** for the steady flow of an incompressible liquid. Note that velocity of flow is large where the area of cross-section is small and *vice-versa.*

The equation of continuity states that the mass of a liquid flowing through a pipe of varying cross-section is constant when the density of the liquid does not change. Therefore, equation of continuity is a statement of conservation of mass.

Example 20.10. Water flows through a horizontal pipe of varying cross-section at the rate of 10 m^3/minute. Determine the velocity of water at a point where the radius of the pipe is 10 cm.

Solution. Rate of discharge, $Q = 10\ m^3/\text{minute} = \frac{1}{6}\ m^3/s$

Cross-sectional area, $a = \pi r^2 = \pi(0.1)^2 = 0.0314\ m^2$

Now, $Q = a\,v$

$\therefore$ Velocity of water, $v = \frac{Q}{a} = \frac{1/6}{0.0314} = 5.3\ ms^{-1}$

Example 20.11. How fast can a raindrop having a diameter 3.0 mm fall before the flow of air around it becomes turbulent? Critical Reynolds number for laminar flow around a sphere is 10. The viscosity of air = 0.019 × 10^{-3} Pas and density of air = 1.29 kg/m^3.

Solution. We can consider the raindrop as a sphere. As it falls at speed v, the air flows past it with the same speed. The Reynolds number is

$$N_R = \frac{\rho v D}{\eta}$$

The critical velocity above which turbulence will occur ($N_R = 10$) is

$$v_{max} = \frac{\eta N_R}{\rho D} = \frac{10 \times 0.019 \times 10^{-3}}{1.29 \times 3.0 \times 10^{-3}} = 4.9 \times 10^{-2}\ ms^{-1} = 4.9\ cms^{-1}$$

Example 20.12. The flow rate of water from a tap of diameter 1.25 cm is 3 litres per minute. The coefficient of viscosity of water is 10^{-3} Pa-s. Characterise the flow.

Solution. Reynolds number, $N_R = \frac{\rho v D}{\eta}$

Here $\rho = 10^3\ kg/m^3$; $V = 3$ litres/min $= \frac{3 \times 10^3}{60}$ c.c./s $= 50 \times 10^{-6}\ m^3/s$; $D = 1.25 \times 10^{-2}$ m

Velocity of water, $v = \frac{V}{\text{Area}} = \frac{50 \times 10^{-6}}{\pi/4 \times (1.25 \times 10^{-2})^2} = 0.4074\ ms^{-1}$

$\therefore$ $N_R = \frac{10^3 \times 0.4074 \times (1.25 \times 10^{-2})}{10^{-3}} = 5092.5$

Since the value of N_R is greater than 3000, the flow of water from the tap is turbulent.

Example 20.13. Water flows through a pipe of internal diameter 20 cm at the speed of 1 ms^{-1}. What should the diameter of the nozzle be if the water is to emerge at the speed of 4 ms^{-1} ?

Solution. Here $d_1 = 20$ cm $= 0.2$ m ; $v_1 = 1\ ms^{-1}$; $v_2 = 4\ ms^{-1}$; $d_2 = ?$

Now $a_1 v_1 = a_2 v_2$

or $\left(\frac{\pi}{4} d_1^2\right) v_1 = \left(\frac{\pi}{4} d_2^2\right) v_2$

or $\frac{d_2^2}{d_1^2} = \frac{v_1}{v_2} = \frac{1}{4}$

or $\frac{d_2}{d_1} = \frac{1}{2}$

$\therefore$ $d_2 = \frac{d_1}{2} = \frac{0.2}{2} = 0.1\text{m} = 10\ \text{cm}$

PROBLEMS FOR PRACTICE

1. Water is flowing through a horizontal pipe of varying cross-section at the rate of 20 litres per minute. Find the velocity of water at a point where diameter of the pipe is 2 cm. **[10.61 ms^{-1}]**

2. A water pipe is 10 cm in diameter and has a constriction of 2 cm diameter. If the velocity of flow in the main pipe is 0.84 m/s, calculate (*i*) the velocity of flow in the constriction (*ii*) rate of discharge of water through the pipeline. **[(*i*) 21 ms^{-1} (*ii*) 0.0066 m^3/s]**

3. The cylindrical tube of a spray pump has a cross-section of 8.0 cm^2, one end of which has 40 fine holes, each of diameter 0.1 mm. If the liquid flow inside the tube is 1.5 m^3/minute, what is the speed of ejection of the liquid through the holes? **[0.637 ms^{-1}]**

20.15. TOTAL ENERGY OF A MOVING LIQUID

A moving liquid can possess the following types of energies :

(*i*) Kinetic energy due to its motion.

(*ii*) Potential (gravitational) energy due to its position *w.r.t.* a chosen reference level.

(*iii*) Pressure energy due to the pressure of the liquid.

Fig. 20.9 shows the flow of liquid of density ρ from a tank to a pipe ; the water level *AB* in the tank being kept constant.

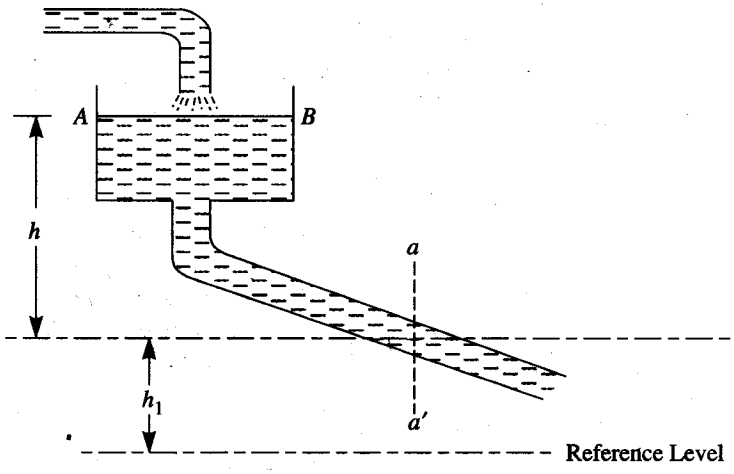


Fig. 20.9

Let at the section $a - a'$ of the pipe,

m = Mass of the liquid passing at any instant

v = Velocity of the liquid

h_1 = Height of the pipe above the chosen reference level

P = Pressure of liquid = $h\rho g$

∴ At the section $a - a'$ of the pipe, the liquid has various energies as under :

$$\text{Kinetic energy (K.E.)} = \frac{1}{2}mv^2$$

$$\text{Potential energy (P.E.)} = mgh_1$$

$$\text{Pressure energy} = mgh$$

$$= mg \times \frac{P}{\rho g} \qquad (\because\ P = h\rho g)$$

$$= \frac{mP}{\rho}$$

$$\therefore \quad \text{Total energy at section } a - a' = \frac{1}{2}mv^2 + mgh_1 + \frac{mP}{\rho}$$

$$\text{Total energy per unit mass at } a - a' = \frac{1}{2}v^2 + gh_1 + \frac{P}{\rho}$$

$$\text{Total energy per unit volume at } a - a' = \frac{1}{2}\rho v^2 + \rho g h_1 + P$$

20.16. BERNOULLI'S THEOREM

Bernoulli's theorem is simply a statement of law of conservation of energy applied to a liquid in motion.

This theorem states that for the steady flow of an ideal liquid, the total energy (i.e., sum of pressure energy, potential energy and kinetic energy) per unit volume remains constant throughout the flow.

Mathematically, we can write Bernoulli's theorem as :

$$P + \rho gh + \frac{1}{2}\rho v^2 = \text{Constant} \qquad \ldots (i)$$

where P = Pressure energy per unit volume

ρgh = Potential energy per unit volume

$\frac{1}{2}\rho v^2$ = Kinetic energy per unit volume

Thus Bernoulli's theorem tells us that the combination of the above three quantities evaluated at a point in the flow has the same value as at any other point. *The importance of this theorem lies in the fact that it can be used to determine the variation of pressure in liquids as a function of velocity of the liquid and the elevation of the pipe through which the liquid is flowing.*

20.17. PROOF OF BERNOULLI'S THEOREM

In order to derive Bernoulli's theorem, we shall use work-energy theorem equation *i.e.,* work done on a liquid as it flows from one place to another is equal to the change in its mechanical energy.

Fig. 20.10 (*i*) shows an ideal liquid flowing through a pipe that changes in elevation and cross-sectional area. We shall apply the work-energy theorem to the liquid contained initially between points 1 and 2 and after short time, this liquid will have moved along the pipe, and will be contained between points 1′ and 2′.

Let P_1 and P_2 = Pressures at 1 and 2

v_1 and v_2 = Velocities at 1 and 2

A_1 and A_2 = Areas of cross-section at 1 and 2

h_1 and h_2 = Average heights at 1 and 2 from chosen reference level

Let 1′ be close to 1 so that each of the parameters listed above has the same value at 1′ as at 1. Let 2′ be close to 2 with a similar consequence. Since the liquid is undergoing steady flow, the

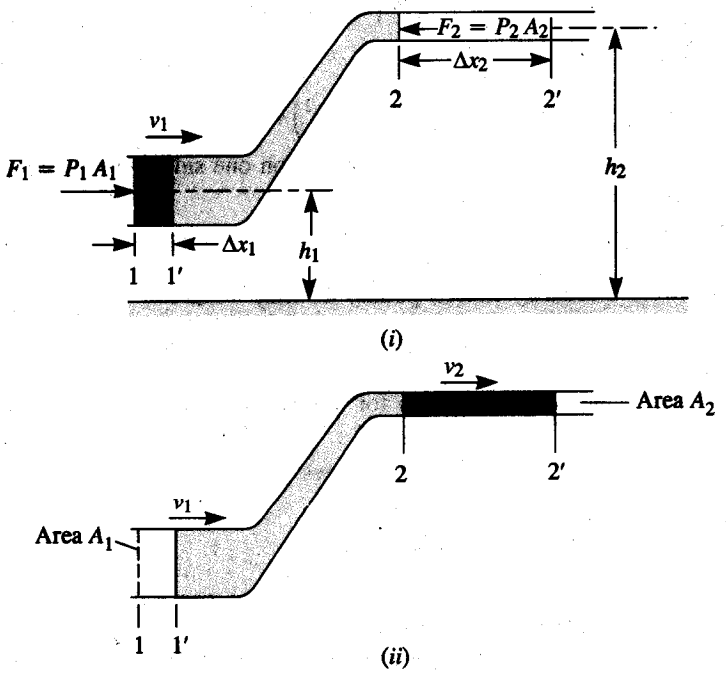


Fig. 20.10

mass of liquid that was originally between 1 and 1′ is equal to the mass of the liquid which is now between 2 and 2′. Let this mass be m.

$\therefore$ Volume between 1 and 1′ = $A_1 \Delta x_1 = \dfrac{m}{\rho}$

Similarly, Volume between 2 and 2′ = $A_2 \Delta x_2 = \dfrac{m}{\rho}$

Here ρ is the constant density of the liquid (incompressible).

Work done. The liquid moves from left to right because the force F_1 (= $P_1 A_1$) is greater than the force F_2 (= $P_2 A_2$). The force F_1 moves the liquid a distance Δx_1 and the liquid moves a distance Δx_2 against the force F_2. Therefore, the net work done on the liquid is

Net work done, $W = F_1 \Delta x_1 - F_2 \Delta x_2$

or $W = (P_1 A_1)\,\Delta x_1 - (P_2 A_2)\,\Delta x_2$

or $W = P_1 \times \dfrac{m}{\rho} - P_2 \times \dfrac{m}{\rho}$ $\left(\because A_1 \Delta x_1 = A_2 \Delta x_2 = \dfrac{m}{\rho}\right)$

Change in mechanical energy. We have seen that mass m of liquid which originally had velocity v_1 and average height h_1 has been replaced by an equal mass with velocity v_2 and average height h_2. Therefore,

Change in kinetic energy = $\dfrac{1}{2} m\left(v_2^2 - v_1^2\right)$

Change in potential energy = $mg\ (h_2 - h_1)$

None of the work done on the liquid has been used to overcome internal friction because the liquid is non-viscous. Therefore, according to principle of conservation of energy,

Net work done = Change in K.E. + Change in P.E.

or $$P_1 \times \frac{m}{\rho} - P_2 \times \frac{m}{\rho} = \frac{1}{2} m \left(v_2^2 - v_1^2\right) + m g \left(h_2 - h_1\right)$$

Dividing throughout by m/ρ, we get,

$$P_1 - P_2 = \frac{1}{2} \rho \left(v_2^2 - v_1^2\right) + \rho g \left(h_2 - h_1\right)$$

When we collect all the quantities with subscript 2 on one side and those with subscript 1 on the other, this equation becomes :

$$P_1 + \rho g h_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2} \rho v_2^2 \quad ...(i)$$

Since 1 and 2 were arbitrarily chosen points, we may write :

$$P + \rho g h + \frac{1}{2} \rho v^2 = \text{Constant} \quad ..(ii)$$

Eq. (*ii*) is known as ***Bernoulli's theorem*** for steady, non-viscous flow of an incompressible liquid.

Flow in a horizontal pipe. When the liquid flows through a horizontal pipe ($h_1 = h_2$), eq. (*i*) becomes :

$$P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2$$

or $$P + \frac{1}{2} \rho v^2 = \text{Constant} \quad ...(iii)$$

Thus for a liquid moving horizontally, if velocity (v) increases at a point, then pressure (P) decreases at that point and vice-versa. Fig. 20.11 shows water moving with streamline flow through a horizontal pipe. Because the diameter of central portion is less than that at either end, water flows faster through the narrow part of the pipe. The vertical tubes act as pressure gauges, filling with liquid until the pressure due to the weight of the liquid in the tube equals that of the moving liquid.

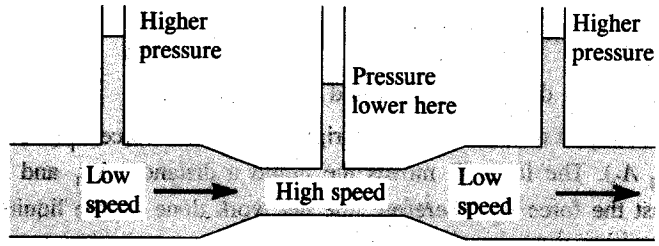


Fig. 20.11

These gauges show that the pressure is higher where the speed of the liquid is lower. Where the speed of the liquid is higher, the pressure it exerts is lower. Note that in eq. (*iii*), P is called **static pressure** and $\rho v^2/2$ is called **dynamic pressure**. Both terms have the dimensions of pressure.

20.18. DIFFERENT FORMS OF BERNOULLI'S THEOREM

We can write Bernoulli's theorem in three different forms :

(*i*) $$P + \rho g h + \frac{1}{2} \rho v^2 = \text{Constant} \quad ...(i)$$

In this equation, all the three terms carry the dimensions of pressure (*i.e.*, energy per unit volume).

(*ii*) If we divide the above equation by ρ, we get,

$$\frac{P}{\rho}+gh+\frac{1}{2}v^2 = \text{Constant} \qquad ...(ii)$$

In this equation, all the three terms carry the dimensions of energy per unit mass.

(*iii*) If we divide eq. (*ii*) by g, we get,

$$\frac{P}{\rho g}+h+\frac{v^2}{2g} = \text{Constant} \qquad ...(iii)$$

In this equation, all the three terms carry the dimensions of length. For this reason, $P/\rho g$, h and $v^2/2g$ are respectively called **pressure head, elevation (gravitational) head** and **velocity head.**

20.19. LIMITATIONS OF BERNOULLI'S THEOREM

In the derivation of Bernoulli's equation, certain assumptions are made which are not justified in actual practice. As a result, Bernoulli's equation has the following limitations :

(*i*) In the derivation of Bernoulli's equation, it is assumed that the liquid is nonviscous *i.e.*, the liquid has zero viscosity (no friction). However, a real liquid does have some viscosity so that a part of mechanical energy is lost to overcome liquid friction. This fact is not taken into account in this equation.

(*ii*) In the derivation of Bernoulli's equation, it is assumed that rate of flow of liquid is constant (*i.e.*, $a_1v_1 = a_2v_2$). But this is not correct in actual practice. Thus in the case of a liquid flowing through a pipe, the velocity of flow is maximum at the centre and goes on decreasing towards the walls of the pipe. Therefore, we should take average velocity of the liquid.

(*iii*) In the derivation of Bernoulli's equation, it is assumed that there is no loss of energy when liquid is in motion. In practice, this is not true. For example, a part of K.E. of flowing liquid is converted into heat and is lost forever.

(*iv*) If the liquid is flowing along a curved path, the energy due to centrifugal force must be considered.

Example 20.14. At what speed will the velocity head of a stream of water be equal to 40 cm?

Solution. Velocity head, $\frac{v^2}{2g} = h \quad \therefore \quad v=\sqrt{2gh}$

Here $g = 9.8 \text{ m s}^{-2}$; $h = 40 \text{ cm} = 0.40 \text{ m}$

$$\therefore \qquad v = \sqrt{2\times9.8\times0.40} = 2.8 \text{ ms}^{-1}$$

Example 20.15. **A pipe is running full of water. At a certain point *A*, it tapers from 60 cm diameter to 20 cm diameter at *B* ; the pressure difference between *A* and *B* is 100 cm of water column. Find the rate of flow through the pipe.**

Solution.
$$\frac{a_A}{a_B} = \left(\frac{d_A}{d_B}\right)^2 = \left(\frac{0.6}{0.2}\right)^2 = 9$$

$$P_A - P_B = h\rho g = (1)\times(1000)\times9.8 = 9800 \text{ N/m}^2$$

Now
$$v_A a_A = v_B a_B \quad \therefore \quad v_B = v_A\left(\frac{a_A}{a_B}\right) = 9\, v_A$$

Using Bernoulli's theorem for a horizontal pipe,

$$\frac{P_A}{\rho}+\frac{1}{2}v_A^2 = \frac{P_B}{\rho}+\frac{1}{2}v_B^2$$

or
$$\frac{1}{\rho}(P_A - P_B) = \frac{1}{2}\left(v_B^2 - v_A^2\right)$$

or $$P_A - P_B = \frac{\rho}{2}\left(v_B^2 - v_A^2\right)$$

or $$9800 = \frac{1000}{2}\left(81 v_A^2 - v_A^2\right) \qquad (\because\ v_B = 9\, v_A)$$

or $$9800 = 40000\, v_A^2$$

$\therefore$ $$v_A = \sqrt{\frac{9800}{40000}} = 0.495 \text{ ms}^{-1}$$

$\therefore$ Rate of discharge, $Q = v_A\, a_A = 0.495 \times \frac{\pi}{4}(0.6)^2 = 0.14 \text{ m}^3/\text{s}$

Example 20.16. The reading of a pressure meter attached to a closed pipe is 2.5×10^5 Nm^{-2}. On opening the valve of the pipe, the reading of the pressure meter reduces to 2.0×10^5 Nm^{-2}. Calculate the speed of water flowing through the pipe.

Solution. $P_1 = 2.5 \times 10^5 \text{ Nm}^{-2}$; $v_1 = 0$ ($\because$ Initially the pipe was closed)

$P_2 = 2.0 \times 10^5 \text{ Nm}^{-2}$; $v_2 = ?$; $\rho = 1000$ kg m^{-3} (for water)

According to Bernoulli's theorem for a horizontal pipe,

$$P_1 + \frac{1}{2}\rho\, v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

or $$P_1 = P_2 + \frac{1}{2}\rho v_2^2 \qquad (\because\ v_1 = 0)$$

or $$v_2^2 = \frac{2(P_1 - P_2)}{\rho} = \frac{2\left(2.5\times10^5 - 2.0\times10^5\right)}{1000} = 100$$

$\therefore$ $$v_2 = \sqrt{100} = 10 \text{ ms}^{-1}$$

Example 20.17. Water enters a house through a pipe 2.0 cm in inside diameter at an absolute pressure of 4×10^5 Pa. The pipe leading to second-floor bathroom 5 m above is 1.0 cm in diameter. When the flow velocity at the inlet pipe is 4 m s^{-1}, find the flow velocity and pressure in the bathroom.

Solution. Radius of inlet pipe, $r_1 = 2.0/2 = 1.0$ cm $= 1.0 \times 10^{-2}$ m

Radius of bathroom pipe, $r_2 = 1.0/2 = 0.5$ cm $= 0.5 \times 10^{-2}$ m

Velocity of water at inlet pipe, $v_1 = 4 \text{ ms}^{-1}$

Absolute pressure at inlet pipe, $P_1 = 4 \times 10^5$ Pa

Let v_2 and P_2 be the velocity and absolute pressure in the bathroom.

According to equation of continuity, we have,

$$v_1 a_1 = v_2 a_2$$

or $$v_2 = v_1 \times \frac{a_1}{a_2} = 4 \times \frac{\pi(1.0\times10^{-2})^2}{\pi(0.5\times10^{-2})^2} = 16 \text{ ms}^{-1}$$

Using Bernoulli's theorem, we have,

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

or $$P_2 = P_1 - \rho g\left(h_2 - h_1\right) - \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

$$= 4\times10^5 - 10^3 \times 9.8 \times 5 - \frac{1}{2}\times10^3\left(256 - 16\right)$$

$$= 2.3 \times 10^5 \text{ Pa}$$

We note that when the water is turned off, the third term on the right-hand side becomes zero and the pressure rises to 3.5×10^5 Pa.

Example 20.18. A pitot tube is mounted on an aeroplane to measure the speed of the plane. The tube contains alcohol and shows a level difference of 40 cm. What is the speed of the plane relative to air? Given that specific gravity of alcohol = 0.8 and density of air = 1 kg /m^3.

Solution. According to Bernoulli's theorem,

Kinetic energy per unit volume = Pressure energy per unit volume

or $$\frac{1}{2}\rho v^2 = h d g$$

or $$v = \sqrt{\frac{2hdg}{\rho}}$$

Here, h = 40 cm = 0.4 m ; $d = 0.8 \times 10^3$ kg/m^3 ; ρ = 1 kg/m^3

$\therefore$ $$v = \sqrt{\frac{2\times0.4\times0.8\times10^3\times9.8}{1}} = 79.2 \text{ ms}^{-1}$$

Example 20.19. Calculate the speed at which the velocity head of a stream of water is equal to 0.50 m of Hg.

Solution. Velocity head = 0.50 m of Hg = 0.5 × 13.6 m of water

But velocity head = $\frac{v^2}{2g}$

or $$0.5 \times 13.6 = \frac{v^2}{2g}$$

$\therefore$ $$v = \sqrt{0.50\times13.6\times2\times9.8} = 11.54 \text{ ms}^{-1}$$

PROBLEMS FOR PRACTICE

1. Water enters a horizontal pipe of non-uniform cross-section with a velocity of 0.6 ms^{-1} and leaves the other end with a velocity of 0.4 ms^{-1}. At the first end, pressure of water is 1600 N/m^2. Calculate the pressure of water at the other end. Density of water = 1000 kg m^{-3}. **[1700 N/m^2]**
2. The reading of a pressure meter attached with a closed water pipe is 4×10^5 N/m^2. On opening the valve of the pipe, the reading of pressure meter is reduced to 3.2×10^5 N/m^2. Calculate the speed of water flowing in the pipe. **[12.64 ms^{-1}]**
3. At a certain point in a pipeline, the velocity is 1 m/s and the gauge pressure is 3×10^5 N/m^2. Find the gauge pressure at a second point in the line 20 m lower than the first if the cross-section at the second point is one-half that at the first. The liquid in the pipe is water. **[4.95 × 10^5 N/m^2]**
4. Water flows through a constricted pipe at a uniform rate. At one point, where the pressure is 2.5×10^4 Pa, the diameter is 8.0 cm. At another point 0.5 m higher, the pressure is 1.5×10^4 Pa and the diameter is 4.0 cm. Find (*i*) the speed of flow in the lower and upper sections (*ii*) the rate of flow through the pipe.
[(*i*) 0.83 ms^{-1} (lower) ; 3.3 ms^{-1} (upper) (*ii*) 4.15 × 10^{-3} m^3/s]
5. Water is flowing through a horizontal pipe of non-uniform cross-section. The speed of water is 30 cm/s at a place where pressure is 10 cm of water. Calculate the speed of water at the other place where pressure is half of that at the first place. **[103.4 cm s^{-1}]**

20.20. APPLICATIONS OF BERNOULLI'S THEOREM

It follows from Bernoulli's theorem that whenever a flowing fluid speeds up, there is a corresponding decrease in the pressure and/or the potential energy of the fluid. If the flow is horizontal, the whole of the velocity increase is accounted for by a decrease in pressure. This fact has a large number of practical applications and we are discussing below a few of them by way of illustration.

(*i*) **Aerofoil lift.** An aerofoil (*e.g.*, an aircraft wing) is shaped so that air flows faster along the top surface than the lower one as shown in Fig. 20.12. This is shown by the closeness of the streamlines above the aerofoil compared with those below. From Bernoulli's theorem, the pressure of the air below is greater than that above. It is this difference in pressure that provides a net upward force. This is called **dynamic lift.** Actually Bernoulli's theorem is only one aspect of the lift on a wing. Wings are usually tilted slightly upward so that air striking the bottom surface is directed downward. The change in momentum of the rebounding air molecules results in an additional upward force on the wing.

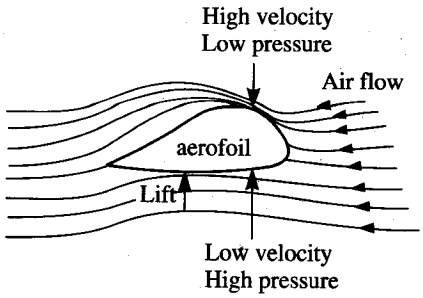


Fig. 20.12

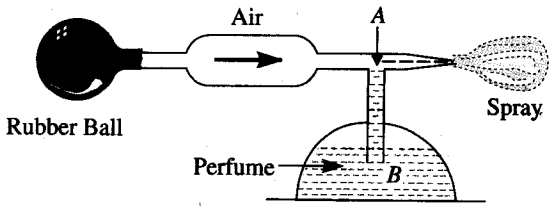


Fig. 20.13

(*ii*) **Atomiser or sprayer.** Fig. 20.13 shows a perfume atomiser. Similar devices are used to spray paint or insecticide. When the bulb in the atomiser is squeezed, air rushes through the narrow neck of the atomiser. This reduces the pressure in the narrow channel (point *A*) so that atmospheric pressure pushes the perfume up the tube leading to the narrow channel. The perfume is then dispersed into a fine spray of droplets.

(*iii*) **Blowing off roof in wind storm.** When wind flows with a high velocity above the roof of a house (See Fig. 20.14), it causes a lowering of pressure above the roof. However, pressure below the roof is still atmospheric. If the wind blows fast enough, the internal air pressure (atmospheric) can blow the roof off. Let us calculate the upward force on the roof during wind storm. Air above the roof pushes down with a force $F_1 = P_1 A$ where P_1 is the air pressure above the roof and A is the area of the roof. The air inside the house pushes up with a force $F = P_a A$ where P_a is the atmospheric pressure of the air inside the house. The net upward force is

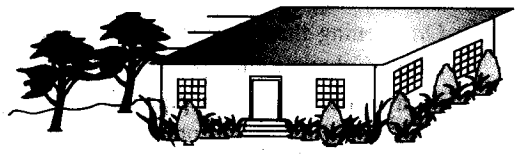


Fig. 20.14

$$F_{net} = F - F_1 = P_a A - P_1 A = A (P_a - P_1)$$

A small pressure difference over a large area can produce a large upward force.

(*iv*) **A suction effect** is experienced by a person standing close to the platform at a station when a fast train passes. The fast-moving air between the person and the train produces a decrease in pressure and the excess air pressure on the other side pushes the person towards the train.

(*v*) **Why does smoke go up a chimney?** It is partly because hot air rises (*i.e.*, density decreases). But Bernoulli's theorem also plays a role. Because the wind blows across the top of

a chimney, the pressure is less there than inside the house. Hence air and smoke are pushed up the chimney. Even on an apparently still night, there is usually enough ambient air flow at the top of a chimney to allow upward flow of smoke.

20.21. VENTURIMETER

A venturimeter (named after its inventor G.B. Venturi) is a device that is used to measure the flow speed (and hence flow rate) of a liquid through a pipe. Its working is based on Bernoulli's theorem.

Construction. Fig. 20.15 shows the various parts of a venturimeter. It consists of two identical wide tubes A and B connected by a narrow tube C. The small narrow tube C is called *constriction* or *throat.* The liquid enters through the wide tube A and after passing through the narrow tube C (the throat) leaves through the other wide tube B. The entire arrangement is horizontal so that liquid flow through the venturimeter is horizontal. Two vertical tubes are connected to measure the pressure at the wide tube A and narrow tube C.

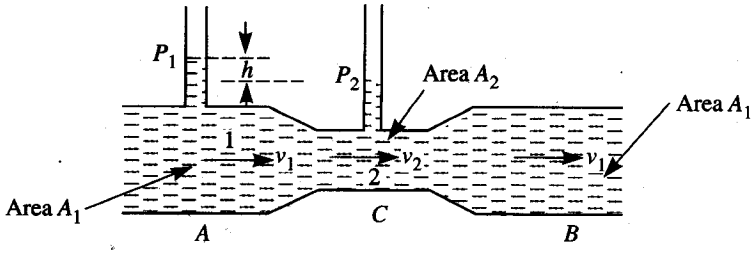


Fig. 20.15

Working. The wide tube A or B has cross-sectional area A_1 while the narrow tube C (throat) has smaller cross-sectional area A_2. The flow speed of liquid is v_1 in tube A or B and v_2 in tube C. As the liquid passes through the narrow tube C, its speed (v_2) increases according to the equation of continuity ($A_1v_1 = A_2v_2$). Therefore, according to Bernoulli's equation, pressure in tube C decreases *i.e.*, $P_1 > P_2$. If $P_1 - P_2$ ($= h\,\rho\,g$) is known, we can find the speed of flow (v_1 or v_2) and hence the rate of flow Q ($= A_1v_1 = A_2v_2$). To do so, let us apply Bernoulli's theorem to points 1 and 2 (both points situated at the axis of the venturimeter) in Fig. 20.15. Since the liquid flow is horizontal ($h_1 = h_2$), Bernoulli's theorem applied to points 1 and 2 gives :

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

From equation of continuity, $A_1v_1 = A_2v_2$ or $v_2 = \dfrac{A_1}{A_2} \times v_1$

$$\therefore \quad P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho \left(\frac{A_1}{A_2}\right)^2 v_1^2$$

or
$$P_1 - P_2 = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2 - A_2^2}{A_2^2}\right)$$

Now $P_1 - P_2 = \rho\, g\, h$ so that, we have,

$$v_1 = A_2\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}} = A_2\sqrt{\frac{2\rho\, g\, h}{\rho(A_1^2 - A_2^2)}}$$

or
$$v_1 = A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$

Thus the speed v_1 of liquid flow can be found out. We can also find v_2 from the equation $A_1v_1 = A_2v_2$.

$$\text{Rate of flow, } Q = A_1v_1 = A_1A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$

Example 20.20. A venturimeter is 37.5 cm in diameter in the mains and 15 cm diameter in the throat. The difference between the pressure of water in the mains and the throat is 23 cm of mercury. Find the rate of discharge of water from the venturimeter.

Solution. X-sectional area of mains, $A_1 = \frac{\pi}{4}\times(37.5)^2 = 1104.5 \text{ cm}^2$

X-sectional area of throat, $A_2 = \frac{\pi}{4}\times(15)^2 = 176.7 \text{ cm}^2$

$\therefore$ Velocity at the throat, $v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$

$\therefore$ Rate of discharge, $Q = A_2v_2 = A_1A_2\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$

Here $P_1 - P_2 = 23$ cm of Hg $= 23 \times 13.6 \times 980$ dynes/cm^2 ; $\rho = 1$ g/cm^3

$\therefore$ $Q = 1104.5\times176.7\times\sqrt{\frac{2(23\times13.6\times980)}{1[(1104.5)^2 - (176.7)^2]}} = \mathbf{1.402 \times 10^5}$ **cm^3/s**

20.22. TORRICELLI'S THEOREM

This theorem applies to a liquid flowing from a drum with a horizontal opening near the base and may be stated as under :

This theorem states that if the difference in levels between the hole and the upper liquid surface in a drum is h, then the velocity with which the liquid emerges from the hole is $\sqrt{2gh}$.

Note that this is the same velocity which a freely falling object will acquire in falling from rest through a vertical distance h.

Proof. Suppose an ideal liquid flows through a hole H at the bottom of a wide drum as shown in Fig. 20.16. Let ρ be the density of the liquid and h the depth of the hole below the free surface of the liquid. According to Bernoulli's theorem, at any point of the liquid, we have,

$$P + \rho g h + \frac{1}{2}\rho v^2 = \text{Constant} \quad ...(i)$$

We select points 1 and 2 for applying Bernoulli's theorem. The point 1 is at the surface of the liquid in the drum and point 2 is the place where liquid leaves the hole. Suppose we measure the height from the hole level.

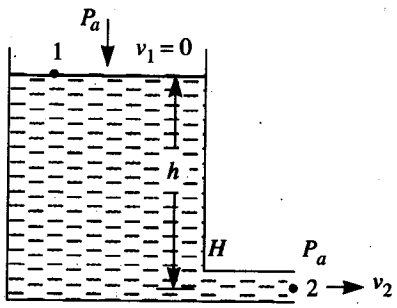


Fig. 20.16

At point 1. $P = P_a$; $\rho gh = \rho gh$; $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho(0)^2 = 0$

It is because the drum is wide and, therefore, $v_1 = 0$. Therefore, eq. (*i*) evaluated at point 1 becomes :

$$P_a + \rho g h = \text{Constant} \quad ...(ii)$$

At point 2. $P = P_a$; $\rho gh = 0$; $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho v_2^2$

Therefore, eq. (*i*) evaluated at point 2 becomes :

$$P_a + \frac{1}{2}\rho v_2^2 = \text{Constant} \qquad ...(iii)$$

$$\therefore \quad P_a + \rho g h = P_a + \frac{1}{2}\rho v_2^2$$

or $$v_2 = \sqrt{2gh}$$

Although it is seen to be a special case of Bernoulli's theorem, it was discovered by Torricelli about 100 years before Bernoulli gave his theorem. *Note that velocity* ($v_2 = \sqrt{2gh}$) *of the liquid emerging from the hole depends only upon the depth (h) of the hole below the free surface of the liquid.*

Horizontal range. The liquid flows out of the hole H in the form of a parabolic jet (See Fig. 20.17) and strikes the ground at a distance R from the base of the drum. The distance R is the horizontal range of the liquid coming out of the hole. At the hole H, the velocity v_2 of the emerging liquid is along the horizontal direction.

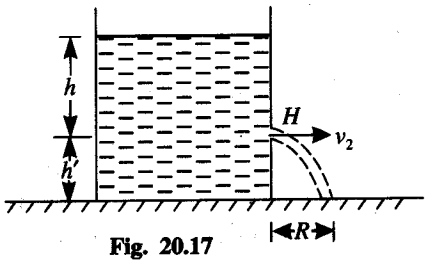


Fig. 20.17

$$\therefore \quad R = v_2 t = \sqrt{2gh} \times t$$

Here t is the time taken by the parabolic jet to strike the ground after emerging from the hole H.

Let h' = Height of hole above the bottom of the drum

Therefore, the vertical distance covered by the jet in time t is h'. Now at point H, vertical velocity is zero.

$$\therefore \quad h' = 0 + \frac{1}{2} g t^2 \text{ or } t = \sqrt{\frac{2h'}{g}}$$

$$\therefore \text{ Horizontal range, } \quad R = \sqrt{2gh} \times \sqrt{\frac{2h'}{g}} = 2\sqrt{hh'}$$

Example 20.21. Water flows out of a small hole in the wall of a large tank near its bottom. What is the speed of efflux of water when the height of water level in the tank is 5m?

Solution. According to Torricelli's theorem, velocity of efflux of a liquid through an orifice at depth h from the liquid surface is

$$v = \sqrt{2gh}\,. \text{ Here } g = 9.8 \text{ ms}^{-2} ; \; h = 5 \text{ m}$$

$$\therefore \quad v = \sqrt{2 \times 9.8 \times 5} = \mathbf{9.9 \text{ ms}^{-1}}$$

Example 20.22. At what velocity does water emerge from an orifice in a tank in which gauge pressure is 3×10^5 N/m^2 before the flow starts? Density of water = 1000 kg/m^3.

Solution. $$P = h\rho g \quad \text{or} \quad h = \frac{P}{\rho g} = \frac{3 \times 10^5}{1000 \times 9.8} = 30.6 \text{ m}$$

$$\therefore \quad \text{Velocity of efflux, } \quad v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 30.6} = \mathbf{24.5 \text{ ms}^{-1}}$$

Note. The word efflux means liquid outflow.

Example 20.23. Water is maintained at a height of 10 m in a tank. Calculate the diameter of the circular hole needed at the base of the tank to discharge water at the rate of 26.4 m^3/minute.

Solution. According to Torricelli's theorem, velocity (v) of the efflux is

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 10} = 14 \text{ ms}^{-1} \qquad (\because h = 10 \text{ m})$$

If Q is the volume of water flowing out per second through the circular hole, then,

$$Q = 26.4/60 = 0.44 \text{ m}^3/\text{s}$$

Now $Q = av = \frac{\pi}{4} d^2 \times v$ ...d is diameter of hole

or $0.44 = \frac{\pi}{4} d^2 \times 14 \quad \therefore \quad d = 0.2 \text{ m} = 20 \text{ cm}$

20.23. POISEUILLE'S FORMULA

When liquid flows through a horizontal pipe with a steady flow, the layer in contact with wall of the pipe remains at rest while the velocity of layers goes on increasing towards the centre of the pipe. As a result, the rate of flow of the liquid is slowest near the pipe walls and fastest in the centre of the pipe. Poiseuille studied the liquid flow through horizontal pipes and concluded that the rate of flow Q of a liquid (*i.e.,* volume of liquid flowing per second) through a pipe varies as under :

(*i*) $Q \propto p$ ... p is pressure difference across ends of the pipe

(*ii*) $Q \propto R^4$ ... R is the radius of the pipe

(*iii*) $Q \propto 1/l$ ... l is length of the pipe

(*iv*) $Q \propto 1/\eta$ ... η is viscosity coefficient of liquid

Combining these factors, we get,

$$Q \propto \frac{pR^4}{\eta l} \text{ or } Q = k\frac{pR^4}{\eta l}$$

Here k is a constant of proportionality and is found to be $\pi/8$.

$$\therefore \quad Q = \frac{\pi p R^4}{8\eta l} \qquad ...(i)$$

Eq. (*i*) is known as *Poiseuille's formula. It is true only for the steady flow of a liquid through a horizontal pipe.* Note that rate of flow Q of the liquid is directly proportional to the fourth power of the radius of the pipe carrying the liquid. If the radius of the pipe carrying the liquid is reduced by a factor 2, the rate of flow will be reduced by a factor of 16 ($= 2^4$).

20.24. DERIVATION OF POISEUILLE'S FORMULA (BY DIMENSIONAL ANALYSIS)

Consider a viscous liquid undergoing steady flow through a horizontal pipe of circular cross-section as shown in Fig. 20.18. Because of viscous drag, the velocity *varies from a maximum at the centre of the pipe to zero at the walls. We shall use dimensional analysis to derive an expression for the rate of flow of liquid Q (= volume /time = V/t) through the pipe. It is reasonable to suppose that the rate of flow of liquid through the pipe depends on

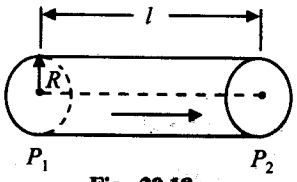


Fig. 20.18

(*i*) the coefficient of viscosity η

(*ii*) the radius R of the pipe

(*iii*) the pressure gradient p/l, where p ($= P_1 - P_2$) is the pressure difference between the ends of the pipe and l is its length [See Fig. 20.18].

We can express the rate of flow Q ($= V/t$) of the liquid as :

$$\frac{V}{t} \propto \eta^x R^y \left(\frac{p}{l}\right)^z$$

or $$\frac{V}{t} = k\eta^x R^y \left(\frac{p}{l}\right)^z \qquad ... (i)$$

* The liquid in contact with the walls is at rest and successive thin cylindrical layers move with increasing speeds. The liquid at the centre has maximum speed.

where k is a dimensionless constant and x, y and z are unknown indices. Since each side of the equation must have the same dimensions,

$$\therefore \quad \left[\frac{V}{t}\right] = [\eta^x][R^y]\left[\left(\frac{p}{l}\right)^z\right]$$

or $$[L^3\ T^{-1}] = [ML^{-1}T^{-1}]^x\ [L]^y\ [\ ML^{-2}\ T^{-2}]^z$$

or $$[L^3\ T^{-1}] = M^{x+z}\ L^{-x+y-2z}\ T^{-x-2z}$$

Equating the indices of M, L and T on both sides, we have,

$$0 = x + z\ ;\ 3 = -x + y - 2z\ ;\ -1 = -x - 2z$$

On solving, $x = -1\ ;\ y = 4\ ;\ z = 1$

$\therefore$ Eq. (*i*) becomes $\frac{V}{t} = k\frac{R^4}{\eta}\left(\frac{p}{l}\right)$

or $$Q = k\frac{R^4}{\eta}\left(\frac{p}{l}\right)$$

The value of k cannot be found by using dimensional analysis. Mathematical analysis shows that value of k is $\pi/8$.

$$\therefore \quad Q = \frac{\pi p R^4}{8\eta l} \qquad ...(ii)$$

This formula for Q is called Poiseuille's formula.

Discussion. The following points may be noted :

(*i*) The above formula indicates that high viscosity leads to low flow rates (*i.e.*, less Q). This is quite reasonable.

(*ii*) The formula also shows that flow rate is proportional to R^4 which is somewhat surprising. Thus, rate of flow is extremely dependent on the radius of the pipe.

Example 20.24. **In giving a patient blood transfusion, the bottle is set up so that the level of blood is 1.3 m above the needle, which has an internal diameter of 0.36 mm and is 3 cm in length. If 4.5 cm^3 of blood passes through the needle in one minute, calculate the viscosity of blood. The density of blood is 1020 kgm^{-3}.**

Solution. Volume of blood flowing per second is

$$Q = \frac{\text{Total volume}}{\text{Time}} = \frac{4.5}{60} = 0.075\ \text{cm}^3\text{s}^{-1}$$

Length of needle, l = 3 cm

Radius of needle, $R = \frac{0.36}{2}$ mm = 0.18 mm = 0.018 cm

Density of blood, ρ = 1020 kg m^{-3} = 1.02 g cm^{-3}

Pressure difference, p = 1.3 m of column of blood

$= h\rho g = (1.3 \times 100) \times (1.02) \times (980)$ dynes cm^{-2}

Now $$Q = \frac{\pi p R^4}{8 \eta l}$$

$\therefore$ Viscosity of blood, $\eta = \frac{\pi p R^4}{8 Q l}$

$$= \frac{\pi \times (1.3 \times 100) \times (1.02) \times 980 \times (0.018)^4}{8 \times 0.075 \times 3} = 0.0238 \text{ poise}$$

Example 20.25. Water is conveyed through a horizontal tube 0.08 m in diameter and 4 km length at the rate of 20 litres per second. Assuming only viscous resistance, calculate the pressure difference required to maintain the flow ($\eta = 10^{-3}$ Nsm^{-2}).

Solution. According to Poiseuille's formula,

$$\text{Rate of flow, } Q = \frac{\pi p R^4}{8\eta l}$$

$$\therefore \text{Required pressure difference, } p = \frac{8Q\eta l}{\pi R^4}$$

Here Q = 20 litres/s = 2×10^{-2} m^3/s ; $\eta = 10^{-3}$ Nsm^{-2} ; l = 4 km = 4000 m ; R = 0.04 m

$$\therefore \quad p = \frac{8\times(2\times10^{-2})\times10^{-3}\times4000}{\pi\times(0.04)^4} = 7.96 \times 10^4 \text{ N/m}^2$$

Example 20.26. Two capillaries of same length and radii in the ratio of 1 : 2 are connected in series and a liquid flows through this system under streamline conditions. If the pressure across the two extreme ends of the combination is 1 m of water, what is the pressure difference across the first capillary?

Solution. Let l and R be the length and radius of the first capillary respectively. Then the corresponding values for the second capillary are l and $2R$. Let p_1 and p_2 be the pressure differences across the first and second capillary respectively. Since the capillaries are connected in series, the rate of flow of liquid through them is the same *i.e.*,

$$\frac{\pi p_1 R^4}{8\eta l} = \frac{\pi p_2 (2R)^4}{8\eta l}$$

or $$p_1 = 16\, p_2$$

It is given that $p_1 + p_2 = 1$ m.

$$\therefore \quad p_1 + \frac{p_1}{16} = 1 \text{ m}$$

$$\therefore \quad p_1 = \frac{16}{17}\text{m} = \mathbf{0.94\ m}$$

CONCEPTUAL QUESTIONS

Q.1. The diameter of ball A is half that of ball B. What will be their ratio of terminal velocities in water?

Ans. The terminal velocity is directly proportional to the square of radius of the ball. Therefore, the ratio of terminal velocities will be 1 : 4.

Q.2. Why do the clouds seem floating in the sky?

Ans. The terminal velocity of a raindrop is directly proportional to the square of radius of the drop. When falling, large drops have high terminal velocities while small drops have small terminal velocities. The small drops fall so slowly that cloud seems to be floating.

Q.3. Why a parachute descends slowly whereas a stone dropped from the same height falls rapidly?

Ans. The open parachute has a large surface area. Therefore, the viscous force of air on the parachute is larger than that on a falling stone. For this reason, parachute descends slowly *i.e.*, it has a small terminal velocity.

Q.4. What is viscosity?

Ans. Viscosity is fluid friction caused (in liquids) by cohesive forces between molecules and (in a gas) by collisions between molecules. The viscosity of liquids decreases as temperature rises, but the opposite is true for gases.

Q.5. What is the principle on which continuity equation is based?

Ans. The continuity equation is based on the principle of conservation of mass. We know that fluid is conserved ; as a fluid moves and deforms, new fluid is neither created nor destroyed. The continuity equation is a mathematical statement of this fact that fluid (*i.e.*, matter itself) is conserved.

Q.6. What is the principle on which Bernoulli's theorem is based?

Ans. Bernoulli's theorem is based on the principle of law of conservation of energy. The work done on a fluid as it flows from one place to another is equal to the change in its mechanical energy.

Q.7. What are the implications of Bernoulli's theorem when the fluid is at rest?

Ans. Bernoulli's theorem states that :

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2 \qquad ...(i)$$

The equations of hydrostatics are special cases of Bernoulli's theorem when the velocity is zero everywhere. Thus when v_1 and v_2 are zero, eq. (*i*) becomes :

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

or $$P_1 - P_2 = \rho g (h_2 - h_1)$$

Q.8. Name three physical principles which apply to both liquids and gases.

Ans. The three principles which apply to both liquids and gases are : Bernoulli's theorem, Pascal's principle and Archimedes' principle.

Q.9. What causes turbulent flow in a fluid?

Ans. The turbulent flow in a fluid may be caused due to too great a speed, abrupt change in areas or direction of flow.

Q.10. When a fluid flows through a narrow constriction, its speed increases. How does it get the energy for this extra speed?

Ans. When a fluid flows through a narrow constriction, its speed increases. This is easily noticed by the increased speed of a river when it flows through the narrow parts. The fluid must speed up in the constricted region if the flow is to be continuous. Bernoulli wondered how the fluid got energy for this extra speed. He rightly reasoned that it is acquired at the expense of a lowered internal pressure.

Q.11. Why does a flag flutter when strong winds are blowing on a certain day?

Ans. When strong winds blow on the surface of a flag, the velocity of the wind is different at the different points on the flag. According to Bernoulli's theorem, the pressure will also be different at these points. This pressure difference causes the flag to flutter.

Q.12. According to Bernoulli's theorem, the pressure of water should remain uniform in a pipe of uniform cross-section. But actually it goes on decreasing. Why?

Ans. It is due to the viscosity of water. When water flows, work is done against the viscous force. This work done is taken from the pressure energy. Hence pressure of water falls as it flows through a pipe.

Q.13. If two ships are moving parallel and close to each other, they experience an attractive force. Why?

Ans. When the two ships come close to each other, the air velocity between the narrow-gap increases and so pressure decreases. The pressure on the outer surfaces of the ships is then greater than the pressure in the gap. Therefore, the ships are pulled towards each other and sometimes they may collide.

Q.14. The accumulation of snow on an aeroplane wing may reduce the lift. Explain.

Ans. Due to accumulation of snow on the wings of the aeroplane, the structure of the wings no longer remains as that of aerofoil. As a result, the net upward force (*i.e.*, lift) is decreased.

Q.15. Show that the quantity $\frac{\rho v D}{\eta}$ (Reynolds number) is dimensionless.

Ans. Dimensions of $\rho = \left[M L^{-3}\right]$; Dimensions of $v = \left[LT^{-1}\right]$

Dimensions of $D = [L]$; Dimensions of $\eta = \left[ML^{-1}T^{-1}\right]$

$$\therefore \text{Dimensions of } \frac{\rho v D}{\eta} = \frac{\left[M L^{-3}\right]\left[LT^{-1}\right][L]}{\left[M L^{-1} T^{-1}\right]} = \frac{\left[M L^{-1} T^{-1}\right]}{\left[M L^{-1} T^{-1}\right]} = \left[M^0 L^0 T^0\right]$$

Therefore, Reynolds number is dimensionless.

Q.16. A liquid flowing out of a small hole in a vessel results in a backward thrust on the vessel. Why?

Ans. Since the area of cross-section of the hole is small, the speed of the liquid flowing out of the vessel will be large. Therefore, the fluid possesses large momentum at the hole. As no external force acts on the system, in order to conserve linear momentum, the vessel acquires a velocity in the backward direction *i.e.* a backward thrust is exerted on the vessel.

Q.17. Water is coming out of a hole made in the wall of tank filled with fresh water. If the size of the hole is increased, will the velocity of efflux of water change?

Ans. Velocity of efflux, $v = \sqrt{2gh}$

Since the velocity of efflux is independent of the area of hole, it will remain the same.

Q.18. Water flows faster than honey. Why?

Ans. According to Poiseuille's formula, the liquid flowing per second (Q) through a pipe is given by ;

$$Q = \frac{\pi p R^4}{8 \eta l}$$

It is clear that for given P, R and l, $Q \propto 1/\eta$ where η is the coefficient of viscosity. Since η for water is less compared to honey, Q for water is greater than for honey. For this reason, water flows faster than honey.

VERY SHORT ANSWER QUESTIONS

Q.1. What is viscosity?

Ans. The internal friction between the layers of a fluid (liquid or gas) which opposes the motion of the fluid is called viscosity.

Q.2. What is the (*i*) SI unit (*ii*) dimensional formula of viscosity?

Ans. (*i*) Decapoise (*ii*) $[M L^{-1} T^{-1}]$

Q.3. Define one decapoise.

Ans. The coefficient of viscosity of a liquid is *one decapoise* if a tangential viscous drag of 1 N/m^2 acts between two parallel liquid layers moving with a velocity gradient of 1 ms^{-1} per metre.

Q.4. If liquid in a flask is shaken violently and kept on a table, the liquid comes to rest after some time. Why?

Ans. It is due to the internal friction between the layers of the liquid (*i.e.*, viscosity).

Q.5. Hotter liquids flow faster than cold ones. Why?

Ans. The coefficient of viscosity of a liquid decreases with the increase in temperature. Therefore, a hot liquid will flow faster than the cold one.

Q.6. If water in one flask and glycerine in the other are violently shaken and kept on the table, then which one will come to rest earlier?

Ans. The glycerine comes to rest earlier because the viscosity of glycerine is greater than that of the water.

Q.7. How will you compare the viscosity of two liquids?

Ans. Take two identical measuring cylinders. Fill each cylinder with a different liquid to the same height. Allow identical ball-bearings to fall through each cylinder. The ball-bearing will fall more slowly through a liquid of higher viscosity.

Q.8. What is the effect of temperature on the viscosity of (*i*) liquids (*ii*) gases?

Ans. (*i*) The viscosity of liquids decreases with the increase in temperature. (*ii*) The viscosity of gases increases with the increase in temperature.

Q.9. What is the (*i*) viscosity (*ii*) compressibility of an ideal liquid?

Ans. (*i*) zero (*ii*) zero.

Q.10. Why are machine parts jammed in winter?

Ans. Due to low temperature in winter, the viscosity of the lubricants used in machine parts increases. As a result, the machine parts get jammed.

Q.11. Why does a bigger rain drop fall faster than a smaller rain drop?

Ans. The terminal velocity of a rain drop (sphere) is directly proportional to the square of radius of the drop (See Art. 20.7). For this reason, a bigger rain drop falls faster than a smaller rain drop.

Q.12. What is the weight of a body falling with a terminal velocity through a viscous medium?

Ans. Zero. It is because when the body is falling through a viscous medium with the terminal velocity, the upward force (upthrust + viscous drag) is equal to the weight of the body.

Q.13. A body is falling through a viscous medium with terminal velocity. What is the acceleration of the body?

Ans. Zero. It is because when a body attains terminal velocity, it continues to move with the same velocity throughout the viscous medium.

Q.14. Why is parachute used while jumping from an aeroplane?

Ans. Viscous drag, $F = 6\pi\eta rv$. The open parachute has a large surface area (*i.e.*, large r) and experiences a large viscous force while descending. As a result, the terminal velocity becomes very small and the person does not get hurt.

Q.15. What is the terminal velocity of a body in a freely falling system?

Ans. Zero. It is because terminal velocity $v_T \propto g$ and for a freely falling system, $g = 0$.

Q.16. What is steady or streamline flow?

Ans. In a streamline or uniform flow, all the particles of the fluid that pass any given point follow the same path at the same speed (*i.e.*, they have the same velocity).

Q.17. What is turbulent flow?

Ans. In a turbulent flow (also known as disorderly flow), the speed and direction of the fluid particles passing any point vary with time.

Q.18. What is meant by critical velocity of a liquid?

Ans. The velocity of a liquid flow upto which its flow is streamline (steady) and above which its flow becomes turbulent is called critical velocity of the liquid.

Q.19. What is pressure head?

Ans. The pressure head at a location = $P/\rho g$ and has the dimensions of length.

Q.20. What is velocity head?

Ans. The velocity head at a location in a liquid flow = $v^2/2g$ and has the dimensions of length.

Q.21. What is the principle on which equation of continuity for steady flow of fluid is based?

Ans. It is based on the law of conservation of mass.

Q.22. What is Bernoulli's theorem?

Ans. It states that for the steady flow of an incompressible and non-viscous fluid, the total energy (pressure energy, potential energy and kinetic energy) of the fluid remains constant throughout the fluid flow.

Q.23. Upon which principle is Bernoulli's theorem based?

Ans. It is based on the principle of conservation of energy.

Q.24. What is the importance of Reynolds number?

Ans. It determines the nature of liquid flow (laminar or turbulent) through a pipe. It is a pure number (*i.e.*, dimensionless).

Q.25. What is an incompressible fluid?

Ans. An incompressible fluid is that in which changes in pressure produce no change in the density of the fluid.

SHORT ANSWER QUESTIONS

Q.1. What are the differences between solid friction and fluid friction?

Ans. (*i*) Solid friction is independent of the area of contact between the two surfaces. However, fluid friction (viscosity) is directly proportional to the area of liquid layers in contact.

(*ii*) Solid friction between two surfaces is independent of relative velocity between them. However, fluid friction is directly proportional to the relative velocity between the two liquid layers.

(*iii*) Solid friction is directly proportional to the normal reaction between the surfaces in contact. However, fluid friction is independent of the normal reaction between two layers of the liquid.

Q.2. What are the similarities between solid friction and fluid friction?

Ans. (*i*) Both come into play whenever there is a relative motion (*ii*) Both oppose the relative motion (*iii*) Both arise from intermolecular forces.

Q.3. What is the effect of (*i*) density (*ii*) pressure on the viscosity of liquids and gases?

Ans. (*i*) The viscosity of liquids increases with the increase in density while for gases, it decreases with the increase in density.

(*ii*) The viscosity of liquids (except water) increases with the increase in pressure while it is almost independent of pressure for gases. Note that viscosity of water decreases with the increase in pressure.

Q.4. The relative velocity between two parallel layers of water is 8 cms^{-1} and the perpendicular distance between them is 0.1 cm. What is the velocity gradient?

Ans. Velocity gradient $= \frac{dv}{dx} = \frac{8\ \text{cms}^{-1}}{0.1\ \text{cm}} = 80\ \text{s}^{-1}$

Q.5. Two hail stones with radii in the ratio of 1 : 2 fall from a great height through the atmosphere. What is the ratio of their terminal velocities?

Ans. Terminal velocity, $v \propto r^2$.

$$\therefore \frac{v_1}{v_2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad \therefore \quad v_1 : v_2 = 1 : 4$$

Q.6. Why are oils of different viscosities used in automobiles in different seasons?

Ans. The viscosity of lubricating oils (liquids) decreases with the increase in temperature. Therefore, lubricating oil used in summer may not be suitable in winter. For this reason, oils of different viscosities are used as lubricants in different seasons.

Q.7. Why rain drops falling under gravity do not acquire very high velocity?

Ans. The rain drops falling under gravity acquire terminal velocity v_T (constant velocity) when viscous drag plus upthrust becomes equal to the weight of rain drop. Now v_T is directly proportional to the square of the radius of the drop (spherical body). Since the radius of rain drop is small, it does not acquire very high velocity.

Q.8. An object is thrown horizontally through air. What will be its terminal velocity?

Ans. Zero. It is because in this case, only retarding force due to air friction acts on the object. Therefore, the velocity of the object along the horizontal will ultimately become zero.

Q.9. What is terminal velocity? Upon which factors it depends?

Ans. When a spherical body falls through a fluid (liquid or gas), its velocity increases till the viscous drag plus upthrust becomes equal to the weight of the body. From now onwards, the body moves with constant maximum velocity called terminal velocity v_T. It is given by ;

$$v_T = \frac{2}{9}\frac{r^2(\rho-\sigma)}{\eta}g$$

It is clear that terminal velocity depends upon radius (r) and density (ρ) of the body, density (σ) of the fluid and coefficient of viscosity (η) of the fluid.

Q.10. Why do air bubbles in water rise up?

Ans. The terminal velocity of a spherical body is directly proportional to the difference in the densities of the body and the fluid (*i.e.*, $\rho - \sigma$). Now here density of water (σ) is greater than the density of air (ρ), the terminal velocity is *negative*. For this reason, air bubbles rise up in water.

Q.11. Two identical drops of water are falling through air with a steady velocity v. If the drops combine to form a single drop, what would be the terminal velocity of the single drop?

Ans. Volume of bigger drop = Volume of 2 small drops

or $$\frac{4}{3}\pi R^3 = 2\times\frac{4}{3}\pi r^3 \text{ or } R = (2)^{1/3}\times r$$

We know that terminal velocity $\propto$ (radius)^2.

$$\therefore \quad \frac{v_T'}{v_T} = \frac{R^2}{r^2} = \frac{(2)^{2/3}r^2}{r^2} = (2)^{2/3}$$

$$\therefore \quad v_T' = (2)^{2/3}\, v_T = (2)^{2/3} v$$

Q.12. The velocity of a small ball of mass m and density ρ when dropped in a container filled with glycerine becomes constant after some time. If the density of glycerine is σ, what is viscous force acting on the ball?

Ans. Since the ball has attained terminal velocity,

$\therefore$ Viscous force = Weight of ball − upthrust

$$= mg - \frac{m}{\rho}\times\sigma\times g = mg\left(1-\frac{\sigma}{\rho}\right)$$

Q.13. Distinguish between streamline and turbulent flow of a liquid.

Ans. (*i*) In streamline flow, all the particles of the liquid that pass any given point have the same velocity. In turbulent flow, the velocity of the liquid particles passing any point varies with time.

(*ii*) In streamline flow, the velocity of the liquid is less than the critical velocity of the liquid. In turbulent flow, liquid moves with a velocity greater than its critical velocity.

Q.14. Why two streamlines cannot cross each other?

Ans. The tangent at any point on a streamline gives the direction of flow of liquid at that point. If two streamlines cross each other at a point, then two tangents can be drawn at that point. It means that the liquid has two velocities along two different directions at this point. This is against the definition of streamline flow. Therefore, two streamlines cannot cross each other.

Q.15. Why does velocity increase when water flowing in broader pipe enters a narrow pipe?

Ans. For a steady flow, the equation of continuity tells us that av = constant. Here a is the area of cross-section of the pipe and v is the velocity of liquid flow. When water enters a narrow pipe, the area of cross-section (a) decreases and hence velocity (v) of water flow increases.

Q.16. **Water flowing through a cylindrical pipe has radius $2R$ at point A and radius R at point B farther along the direction of flow. If the velocity of water at point A is v, then what is its velocity at point B?**

Ans. According to equation of continuity, av = Constant.

$$\therefore \quad \pi(2R)^2 \times v_A = \pi(R)^2 \times v_B \quad \text{or} \quad v_B = 4\, v_A = 4\, v$$

Q.17. **Deep water runs slow. Why?**

Ans. According to equation of continuity, av = Constant *i.e.*, $v \propto 1/a$. For deep water, area of X-section (a) becomes large so that v is small. For this reason, deep water runs slow.

Q.18. **Under what conditions Bernoulli's theorem can be used?**

Ans. (*i*) The fluid is incompressible *i.e.*, density remains constant (*ii*) The fluid is non-viscous (*iii*) The flow is steady and the velocity of the fluid is less than the critical velocity.

Q.19. **Why does the speed of a liquid increase and its pressure decrease when a liquid passes through constriction in a pipe?**

Ans. When a liquid passes through a horizontal pipe, according to Bernoulli's theorem,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Therefore, where pressure is high, velocity of liquid flow is low and *vice-versa*. When liquid passes through a constriction, the velocity increases ($\because$ av = constant) and pressure decreases.

Q.20. **If the velocity head of a stream of water is equal to 10 cm, what is its speed of flow?**

Ans. Velocity head, $h = \dfrac{v^2}{2g}$. Here $h = 10 \times 10^{-2}$ m ; $g = 9.8$ ms^{-2}

$$\therefore \quad \text{Speed of flow, } v = \sqrt{2gh} = \sqrt{2\times 9.8\times 10\times 10^{-2}} = 1.4 \text{ ms}^{-1}$$

Q.21. **At what speed will the velocity head of a stream of water be equal to 40 cm of Hg?**

Ans. Velocity head, $h = v^2/2g$ or $v = \sqrt{2gh}$

Here $g = 980$ cms^{-2}; $h = 40$ cm of Hg $= 40 \times 13.6$ cm of water

$$\therefore \quad v = \sqrt{2\times 980\times 40\times 13.6} = 1032.6 \text{ cms}^{-1} = 10.326 \text{ ms}^{-1}$$

Q.22. **Air streams horizontally pass an air-plane. The speed over the top surface is 60 ms^{-1} and that under the bottom surface is 45 ms^{-1}. What is the difference in pressure? (Density of air = 1.293 kg m^{-3}.)**

Ans. According to Bernoulli's theorem for horizontal flow,

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$$

$$\therefore \quad P_2 - P_1 = \frac{1}{2}\rho(v_1^2 - v_2^2) = \frac{1}{2}\times 1.293[(60)^2 - (45)^2] = 1018 \text{ Nm}^{-2}$$

Q.23. **Water is coming out of a hole made in the wall of a water tank. The level of water in the tank is being kept constant. If the size of the hole is increased, what will be the (*i*) change in velocity of efflux of water (*ii*) volume of water coming out per second?**

Ans. (*i*) The velocity of efflux will remain the same because it does not depend on the area of cross-section of the hole ($v = \sqrt{2gh}$).

(*ii*) Volume of water coming out of hole per second = av. Since area of X-section of the hole is increased, the volume of water coming out per second will increase.

Q.24. **A tank of height 5 m is full of water. There is a hole of cross-sectional area of 1 cm^2 in its bottom. Find the initial volume of water that will come out of this hole per second ($g = 10$ ms^{-2}).**

Ans. $$\text{Velocity of water, } v = \sqrt{2gh} = \sqrt{2\times 10\times 5} = 10 \text{ ms}^{-1}$$

Volume of water flowing per second from the hole is

$$Q = av = (1 \times 10^{-4}) \times 10 = 10^{-3} \text{ m}^3/\text{s}$$

Q.25. Under a given pressure head, the rate of flow of liquid through a pipe is Q. If the length of the pipe is doubled and the diameter of the pipe is halved, what will be the rate of flow?

Ans. According to Poiseuille's formula, rate of flow, $Q = \dfrac{\pi p R^4}{8\eta l}$.

For first case, $Q = \dfrac{\pi p R^4}{8\eta l}$; For second case, $Q_1 = \dfrac{\pi p (R/2)^4}{8\eta (2l)} = \dfrac{\pi p R^4}{8 \times 32 \eta l}$

$\therefore \quad \dfrac{Q_1}{Q} = \dfrac{1}{32}$ or $Q_1 = \dfrac{Q}{32}$

Q.26. Two capillaries of the same length and radii in the ratio 1 : 2 are connected in series and a liquid flows through this system under streamline conditions. If the pressure across the two extreme ends of the combination is 1 m of water, what is the pressure difference across the first and second capillary?

Ans. Since the capillaries are connected in series, the rate of flow of liquid through them is the same *i.e.*,

$$\frac{\pi p_1 R^4}{8\eta l} = \frac{\pi p_2 (2R)^4}{8\eta l} \quad \therefore \quad p_1 = 16\, p_2 \qquad ...(i)$$

It is given that $p_1 + p_2 = 1$ m ...(*ii*)

From eqs. (*i*) and (*ii*), we have, $p_1 = 16/17$ m ; $p_2 = 1/17$ m

LONG ANSWER QUESTIONS

1. Explain viscosity. Discuss its cause. **[Refer to Arts. 20.1 and 20.2]**
2. Define co-efficient of viscosity. Give its units and dimensional formula. **[Refer to Art. 20.3]**
3. State and establish Stokes' law. **[Refer to Art. 20.5]**
4. Derive an expression for the terminal velocity attained by a spherical body falling through a viscous medium. **[Refer to Art. 20.7]**
5. Write short note on (*i*) streamline flow (*ii*) laminar flow and (*iii*) turbulent flow. **[Refer to Art. 20.10]**
6. What is critical velocity of a liquid? Using method of dimensions, find the expression for the critical velocity. **[Refer to Art. 20.13]**
7. State and prove Bernoulli's theorem. **[Refer to Arts. 20.16 and 20.17]**
8. State and prove Torricelli's theorem. **[Refer to Art. 20.22]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Both liquids and gases are fluids.
2. The internal frictional force within a fluid is known as viscosity of the fluid.
3. The greater the viscosity, the less easy it is for the fluid to flow.
4. The fluid frictional force F (or viscous drag) between two layers of liquid each of area A having velocity gradient dv/dx is
$$F = -\eta A\frac{dv}{dx}$$
where η is the coefficient of viscosity of the liquid. This equation is sometimes called **Newton's law of viscosity.**
5. Unlike frictional force within a fluid, the frictional force between the surfaces of two solids is independent of the area of contact and of the relative velocity.
6. For given values of A and dv/dx, F is large for those liquids which have high values of η. Therefore, η is a measure of viscosity of fluids.
7. A spherical object of radius r moving at velocity v through a fluid of viscosity coefficient η will experience a viscous force F given by ;
$$F = 6\pi\eta r v$$
This equation is called **Stokes' law.**
8. In liquids, the viscosity is due to intermolecular forces of attraction. In gases, the viscosity is due to the interchange of molecules that takes place between the different layers of the flowing gas.
9. When a spherical body falls through a fluid (liquid or gas), its velocity increases till the viscous drag plus upthrust becomes equal to the weight of the body. From now onwards, the body moves with constant maximum velocity called *terminal velocity* v_T.
10. The terminal velocity v_T of a spherical body of radius r, density ρ falling through a fluid of density σ is given by ;
$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)}{\eta}g$$
where η is the coefficient of viscosity of the fluid.
11. In **steady flow** (also known as *streamline flow* or *orderly flow* or *uniform flow*), all the particles of the fluid that pass any given point follow the same path at the same speed (*i.e.* they have the same velocity).
12. In **turbulent flow** (also known as *disorderly flow*), the speed and direction of the fluid particles passing any point vary with time.
13. A **streamline** is a curve whose tangent at any point is along the direction of the velocity of the fluid at that point. Streamlines never cross.
14. Laminar flow is a special case of steady flow in which the velocities of all the particles on any given streamline are the same, though the particles of different streamlines may move at different speeds. Fig. 20.19 shows the laminar flow in an open channel of uniform cross-section.
15. An **incompressible fluid** is that in which changes in pressure produce no change in the density of the fluid.

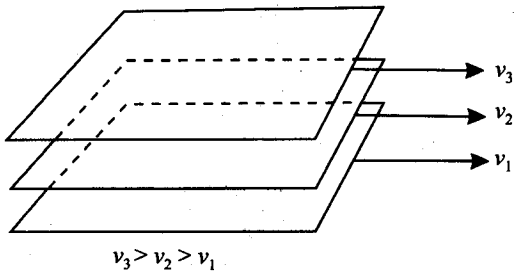


Fig. 20.19

16. If a liquid (or fluid) is undergoing steady flow through a pipe, then, $a\ v$ = constant where a is the area of cross-section of the pipe and v is the velocity of the liquid. This is known as **equation of continuity.** Clearly, $v \propto 1/a$ *i.e.* deeper liquid runs slow.

17. The **critical velocity** is that velocity of liquid flow upto which the flow of liquid is steady and above which the flow becomes turbulent. The critical velocity of a liquid flowing through a pipe is given by ;

$$v_c = \frac{1100\eta}{r\rho}$$

where ρ and η are the density and coefficient of viscosity of the liquid (or fluid) and r is the radius of the pipe.

18. Reynolds number determines the nature of liquid flow through a pipe. It is a pure number *i.e.* dimensionless.

$$\text{Reynolds number, } N_R = \frac{\rho v D}{\eta}$$

where ρ is the density of liquid, v is the average velocity of liquid flow, D is the diameter of pipe and η is the coefficient of viscosity of the liquid. If N_R is less than 2000, the **laminar flow** occurs and if N_R is greater than 3000, the **turbulent flow** occurs.

19. According to Bernoulli's theorem :

(*i*) $$P + \rho g h + \frac{1}{2}\rho v^2 = \text{Constant}$$

(*ii*) $$\frac{P}{\rho} + g h + \frac{1}{2}v^2 = \text{Constant}$$

(*iii*) $$\frac{P}{\rho g} + h + \frac{v^2}{2g} = \text{Constant}$$

20. If the liquid flows in a horizontal pipe, then Bernoulli's theorem becomes :

$$P + \frac{1}{2}\rho v^2 = \text{Constant}$$

It follows that whenever velocity of a flowing liquid through a horizontal pipe increases, there is a corresponding decrease in pressure and *vice-versa. P* is called *static pressure* and $\rho v^2/2$ is called *dynamic pressure.*

21. According to the equation of continuity ($a\ v$ = constant), the liquid speeds up at the constrictions and, therefore, there is a decrease in pressure at the constrictions.

22. The **venturimeter** is a device which introduces a constriction into a pipe carrying liquid in order that the velocity of the liquid can be measured by measuring the resulting drop in pressure.

23. The **static pressure** at a point in a flowing liquid is the actual pressure at that point.

24. The **dynamic pressure** ($\rho v^2/2$) is not a true pressure, it is simply a quantity which has the dimensions of pressure.

25. The rate of steady flow Q of a viscous liquid flowing through a horizontal pipe of length l, radius R under a pressure difference p at its ends is

$$Q = \frac{\pi p R^4}{8\eta l}$$

where η is the coefficient of viscosity of the liquid. This is called **Poiseuille's formula.**

26. The velocity of liquid flowing out of a narrow hole at the bottom of a tank is given by ;

$$v = \sqrt{2gh}$$

where h is the depth of the hole below the free surface of the liquid. Note that v is independent of the density of the liquid and area of the hole.

MCQ FOR MEDICAL ENTRANCE EXAMINATIONS

1. The terminal velocity v_T of a spherical ball of lead of radius r falling through a viscous liquid varies with r such that

   (*a*) $\frac{v_T}{r}$ = constant (*b*) $v_T r$ = constant

   (*c*) v_T = constant (*d*) $\frac{v_T}{r^2}$ = constant

2. The relative velocity between two parallel layers of water is 8 cms^{-1} and the perpendicular distance between them is 0.1 cm. The velocity gradient is

   (*a*) 80 s^{-1} (*b*) 0.8 s^{-1}

   (*c*) 160 s^{-1} (*d*) 40 s^{-1}

3. Two hail stones with radii in the ratio of 1 : 2 fall from a great height through the atmosphere. Then their terminal velocities are in the ratio of

   (*a*) 1 : 2 (*b*) 2 : 1

   (*c*) 1 : 4 (*d*) 4 : 1

4. A metal plate 5 cm × 5 cm rests on a layer of castor oil 1 mm thick whose coefficient of viscosity is 1.55 Nsm^{-2}. The horizontal force required to move the plate with a speed of 2 cms^{-1} is

   (*a*) 0.02 N (*b*) 0.0775 N

   (*c*) 0.0428 N (*d*) 0.0345 N

5. A drop of radius 2×10^{-5} m and density 1.2×10^3 kg/m^3 falls through air. The viscosity of air is 1.8×10^{-5} Ns m^{-2}. Neglecting buoyancy due to air, the terminal speed of the drop is

   (*a*) 2.8 cm/s (*b*) 3.6 cm/s

   (*c*) 5.8 cm/s (*d*) 4.9 cm/s

6. In Bernoulli's theorem, which of the following is conserved? **[AMU PMT 1996]**

   (*a*) linear momentum

   (*b*) angular momentum

   (*c*) energy

   (*d*) none of above

7. A tank of height 5 m is full of water. There is a hole of cross-sectional area 1 cm^2 in its bottom. The initial volume of water that will come out of this hole per second is **[CPMT 1997]**

   (*a*) 10^{-3} m^3/s (*b*) 10^{-4} m^3/s

   (*c*) 10 m^3/s (*d*) 10^{-2} m^3/s

8. The gale blows over a house. The force due to gale on the roof is **[Rajasthan PMT 1998]**

   (*a*) in the direction downward

   (*b*) in the direction upward

   (*c*) zero

   (*d*) in the horizontal direction

9. In a streamline flow if the gravitational head is h, then kinetic and pressure heads are **[AMU PMT 1998]**

   (*a*) v^2/g and P/ρ

   (*b*) $v^2/2g$ and $P/\rho g$

   (*c*) $v^2/2$ and P/ρ

   (*d*) $v^2/2$ and $P/g\rho$

10. To calculate the rate of flow of liquid, which of the following is used? **[CPMT 1999]**

    (*a*) Stokes' law

    (*b*) Bernoulli's theorem

    (*c*) Poiseuille's law

    (*d*) Conservation of pressure

11. An incompressible fluid that flows through a cylindrical pipe has radius $2R$ at point A and radius R at point B farther along the direction of flow. If the velocity at point A is v, then velocity at point B is **[MNR 1999]**

    (*a*) $2v$ (*b*) v

    (*c*) $v/2$ (*d*) $4v$

12. The reading of a manometer fitted to a closed tap is 3.5×10^5 N/m^2. If the valve is opened, the reading of manometer falls to 3×10^5 N/m^2. The velocity of water is **[Pb PMT 1997]**

    (*a*) 1 m/s (*b*) 10 m/s

    (*c*) 100 m/s (*d*) 0.1 m/s

13. If the velocity head of a stream of water is equal to 10 cm, then its speed of flow is **[MP PMT 1996]**

    (*a*) 10 m/s (*b*) 140 m/s

    (*c*) 1.4 m/s (*d*) 0.1 m/s

14. The blood is flowing at the rate of 100 cm^3/s. If the area of cross-section of capillaries is 0.25 m^2, then the average velocity of blood through capillaries is **[AMU PMT 1999]**

    (*a*) 0.4 mm/s (*b*) 4 mm/s

    (*c*) 25 mm/s (*d*) 400 mm/s

15. Three tubes *A*, *B* and *C* are connected to a horizontal pipe in which liquid is flowing as shown in Fig. 20.20. The radii of the pipe at points *A*, *B* and *C* are 2 cm, 1 cm and 2 cm respectively. The height of the liquid **[Pb PMT 1997]**

(*a*) in *A* is maximum
(*b*) in *A* and *B* is equal
(*c*) is same in all the three
(*d*) in *A* and *C* is equal

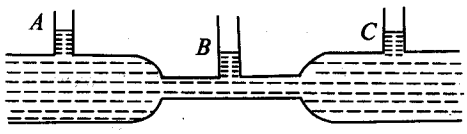


Fig. 20.20

ANSWERS TO MCQ FOR MEDICAL ENTRANCE EXAMINATIONS				
1. (*d*)	**2.** (*a*)	**3.** (*c*)	**4.** (*b*)	**5.** (*c*)
6. (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)
11. (*d*)	**12.** (*b*)	**13.** (*c*)	**14.** (*a*)	**15.** (*d*)

HINTS TO MCQ FOR MEDICAL ENTRANCE EXAMINATIONS

1. Terminal velocity, $v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)}{\eta}g$

$$\therefore \quad \frac{v_T}{r^2} = \frac{2}{9}\times\frac{(\rho-\sigma)g}{\eta} = \text{constant}$$

2. Velocity gradient $= \frac{dv}{dx} = \frac{8\ \text{cms}^{-1}}{0.1\ \text{cm}} = 80\ \text{s}^{-1}$

3. Terminal velocity, $v_T \propto r^2 \quad \therefore \frac{v_1}{v_2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ or $v_1 : v_2 = 1 : 4$

4. Force required, $F = \eta A\frac{dv}{dx}$. Here $\eta = 1.55\ \text{Nsm}^{-2}$; $A = 25\times10^{-4}\ \text{m}^2$;

$$dv = 2\times10^{-2}\ \text{ms}^{-1};\ dx = 10^{-3}\text{m}$$

$$\therefore \quad F = 1.55\times25\times10^{-4}\times\frac{2\times10^{-2}}{10^{-3}} = 0.0775\ \text{N}$$

5. Neglecting buoyancy due to air, we have,

$$\text{Terminal speed, } v_T = \frac{2}{9}\times\frac{r^2\rho g}{\eta} = \frac{2}{9}\times\frac{(2\times10^{-5})^2\times1.2\times10^3\times9.8}{1.8\times10^{-5}}$$

$$= 5.8\times10^{-2}\ \text{ms}^{-1} = 5.8\ \text{cm s}^{-1}$$

6. According to Bernoulli's theorem, for an incompressible, non-viscous liquid (or fluid) undergoing steady flow, the total energy of liquid at all points is constant.

7. Velocity of water, $v = \sqrt{2gh} = \sqrt{2\times10\times5} = 10\,\text{ms}^{-1}$

Volume of water flowing out per second from the hole

$$= av = \left(1\times10^{-4}\right)\times10 = 10^{-3}\ \text{m}^3/\text{s}$$

8. When gale blows over a house, the velocity of gale above the roof is large as compared to that below the roof. According to Bernoulli's theorem, the pressure above the roof is less than that below the roof. Therefore, the force acts on the roof in the upward direction.

9. According to Bernoulli's theorem,

$$h + \frac{P}{\rho g} + \frac{v^2}{2g} = \text{Constant}$$

Here h is elevation (gravitational) head, $P/\rho g$ is pressure head (static) and $v^2/2g$ is kinetic (dynamic) head.

10. In order to find the rate of flow of a liquid (*i.e.* volume of liquid flowing per second) through a horizontal pipe, we use **Poiseuille's law**.

11. From equation of continuity,

$a \times v = \text{Constant}$ or $\pi(2R)^2 \times v_A = \pi(R)^2 \times v_B \quad \therefore v_B = 4v_A = \mathbf{4\ v}$

12. According to Bernoulli's theorem,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

or $\quad 3.5\times10^5 + \frac{1}{2}\rho\times(0)^2 = 3\times10^5 + \frac{1}{2}\rho v_2^2 \qquad (\because v_1 = 0)$

or $\quad \frac{1}{2}\rho v_2^2 = (3.5-3)\times10^5$

$\therefore \quad v_2 = \sqrt{\frac{2(3.5-3)\times10^5}{\rho}} = \sqrt{\frac{2(3.5-3)\times10^5}{10^3}} = \mathbf{10\ m/s}$

13. Velocity head, $h = \frac{v^2}{2g} \quad \therefore$ Speed of flow, $v = \sqrt{2gh} = \sqrt{2\times9.8\times10\times10^{-2}} = \mathbf{1.4\ m/s}$

14. Volume of blood flowing/sec = $a\ v$ or $100 \times 10^{-6} = 0.25 \times v$

$\therefore \quad v = \frac{100\times10^{-6}}{0.25} = 4\times10^{-4}$ m/s $= \mathbf{0.4\ mm/s}$

15. According to equation of continuity, $a\ v$ = Constant.

Since area of cross-section of tubes A and C is the same, the velocity of liquid at these points is the same. According to Bernoulli's theorem, the pressure at these points is the same. Therefore, height of liquid is the **same in tubes *A* and *C***.

MCQ FOR ENGINEERING ENTRANCE EXAMINATIONS

1. A drop of radius 2×10^{-5} m moves with $v_T = 5.8$ cm s^{-1}. Then the viscous force on the drop at the terminal speed is ($\eta_{air} = 1.8 \times 10^{-5}$ Ns m^{-2})

(*a*) 3.9×10^{-10} N (*b*) 2.5×10^{-8} N

(*c*) 1.8×10^{-9} N (*d*) 1.6×10^{-7} N

2. After terminal velocity is reached, the acceleration of a body falling through a fluid is

(*a*) equal to g (*b*) zero

(*c*) less than g (*d*) more than g

3. A metallic sphere of mass m falls through glycerine with a terminal velocity v. If we drop a sphere of mass $8m$ of the same material into a column of glycerine, the terminal velocity of this sphere will be

(*a*) $2v$ (*b*) $4v$

(*c*) $8v$ (*d*) $16v$

4. Eight rain drops each of radius 1 mm are falling through air at a terminal velocity of 5 cm s^{-1}. If they coalesce to form a bigger drop, the terminal velocity of bigger drop is

(*a*) 10 cm/s (*b*) 5 cm/s

(*c*) 15 cm/s (*d*) 20 cm/s

5. A small spherical ball of steel falls through a viscous medium with terminal velocity v. If a ball of twice the radius of the first one but of the same mass is dropped through the same medium, it will fall with a terminal velocity (neglect buoyancy)

(*a*) $\frac{v}{2}$ (*b*) $\frac{v}{\sqrt{2}}$ (*c*) v (*d*) $2v$

6. Rate of flow of liquid through a horizontal pipe of length l and radius R is Q. Then rate of flow of the same liquid through another pipe of length $2l$ and radius $2R$ is **[CET Haryana 2001]**

(*a*) $8\ Q$ (*b*) $16\ Q$

(*c*) $2\ Q$ (*d*) $4\ Q$

7. A tank is filled with water upto a height H. Water is allowed to come out of a hole P in one of the walls at a depth h below the free surface of water (See Fig. 20.21). The horizontal distance X in terms of H and h is **[MP PET 1999]**

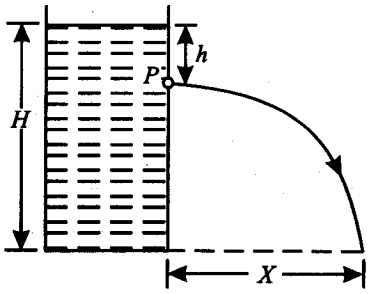


Fig. 20.21

(*a*) $X=\sqrt{h(H-h)}$ (*b*) $X=\sqrt{\frac{h}{2}(H-h)}$

(*c*) $X=2\sqrt{h(H-h)}$ (*d*) $X=4\sqrt{h(H-h)}$

8. Two pipes P and Q having diameters 2×10^{-2} m and 4×10^{-2} m respectively are joined in series with the main supply line of water. The velocity of water flowing in pipe P is **[Roorkee 1998]**

(*a*) 4 times that of Q

(*b*) 2 times that of Q

(*c*) 1/2 times that of Q

(*d*) 1/4 times that of Q

9. A large open tank has two holes in the wall. One is a square hole of side L at a depth y from the top and the other is a circular hole of radius R at a depth of $4y$ from the top. When the tank is completely filled with water, the quantities of water flowing per second from both holes are the same. Then R is equal to **[IIT Screening 2000]**

(*a*) $L/\sqrt{2\pi}$ (*b*) $2\pi L$

(*c*) L (*d*) $L/22$

10. A cylindrical vessel of height 90 cm is kept filled upto the brim. It has four holes 1, 2, 3 and 4 (See Fig. 20.22) which are respectively at heights of 20 cm, 30 cm, 40 cm and 50 cm from the horizontal floor PQ. The water falling at the maximum horizontal distance from the vessel comes from **[CET Karnataka 1999]**

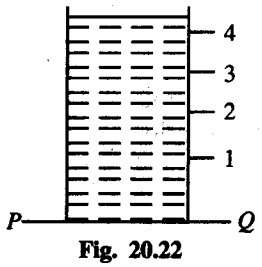


Fig. 20.22

(*a*) hole number 4 (*b*) hole number 3

(*c*) hole number 2 (*d*) hole number 1

11. A liquid is kept in a cylindrical vessel which is rotated along its axis. The liquid rises at the sides [see Fig. 20.23]. If the radius of the vessel is 0.05 m and the speed of rotation is 2 rev/s, the difference in the height of the liquid at the centre of the vessel and its sides is **[CET Haryana 1996]**

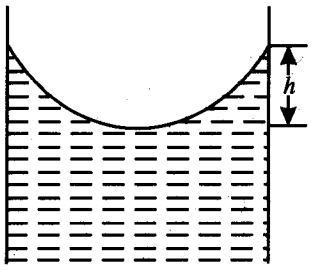


Fig. 20.23

(*a*) 4 cm (*b*) 20 cm

(*c*) 0.2 cm (*d*) 2 cm

12. Under a pressure head, the rate of flow of liquid through a pipe is Q. If the length of the pipe is doubled and the diameter of the pipe is halved, the rate of flow would become **[CET Karnataka 2000]**

(*a*) $Q/4$ (*b*) $Q/16$

(*c*) $Q/32$ (*d*) $Q/8$

13. A cylinder is filled with a non-viscous liquid of density d to a height h_0 and a hole is made at a height h_1 from the bottom of the cylinder. The velocity of the liquid passing out of the hole is **[MP PET 1998]**

(*a*) $\sqrt{2gh_0}$ (*b*) $\sqrt{2g(h_0-h_1)}$

(*c*) $\sqrt{dgh_1}$ (*d*) $\sqrt{dgh_0}$

14. Air streams horizontally past an air-plane. The speed over the top surface is 60 ms^{-1} and that under the bottom surface is 45 ms^{-1}. The difference in pressure is (Density of air = 1.293 kg m^{-3}) **[CET Karnataka 1996]**

(*a*) 516 N/m^2 (*b*) 1140 N/m^2

(*c*) 2250 N/m^2 (*d*) 1018 N/m^2

15. The difference in pressure between the lower and upper surface of an air-plane **[CET Rajasthan 1997]**

(*a*) depends on altitude of plane

(*b*) independent of altitude of plane

(*c*) Insufficient data

(*d*) none of above

ANSWERS TO MCQ FOR ENGINEERING ENTRANCE EXAMINATIONS

1. (*a*)	**2.** (*b*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)
6. (*a*)	**7.** (*c*)	**8.** (*a*)	**9.** (*a*)	**10.** (*a*) and (*b*)
11. (*d*)	**12.** (*c*)	**13.** (*b*)	**14.** (*d*)	**15.** (*a*)

HINTS TO MCQ FOR ENGINEERING ENTRANCE EXAMINATIONS

1. Viscous force, $F = 6\pi\eta r v_T = 6\pi\times1.8\times10^{-5}\times2\times10^{-5}\times5.8\times10^{-2} = 3.9\times10^{-10}$ N

2. Once terminal speed is attained by the body, the net force on the body is zero. Therefore, acceleration of the body is zero.

3. $m=\frac{4}{3}\pi r^3\rho$; $8m=\frac{4}{3}\pi R^3\rho$ $\therefore \frac{R^3}{r^3}=8$ or $R=2r$

Terminal velocity $\propto$ (radius)^2 $\therefore \frac{v_2}{v_1}=\left(\frac{R}{r}\right)^2=(2)^2=4$

or $v_2 = 4v_1 = 4v$

4. Volume of bigger drop = Volume of 8 small drops

or $\frac{4}{3}\pi R^3 = 8\times\frac{4}{3}\pi r^3$ $\therefore R = 2r = 2\times0.1 = 0.2$ cm

Terminal velocity of each small drop is

$$v_T=\frac{2}{9}\times\frac{r^2}{\eta}(\rho-\sigma)g$$

Terminal velocity of bigger drop is

$$V_T=\frac{2}{9}\times\frac{R^2}{\eta}(\rho-\sigma)g$$

$\therefore$ $\frac{V_T}{v_T}=\frac{R^2}{r^2}$ or $V_T = v_T\times\frac{R^2}{r^2}=5\times\frac{(0.2)^2}{(0.1)^2}$ = **20 cm/s**

5. $\frac{4}{3}\pi r^3\rho = \frac{4}{3}\pi(2r)^3\ \rho_1$ $\therefore \rho_1 = \rho/8$

Neglecting buoyancy, the terminal velocity is

$$v=\frac{2}{9}\times\frac{r^2\rho g}{\eta} \quad \text{or} \quad v\propto\rho r^2$$

$\therefore$ $\frac{v_1}{v}=\frac{\rho_1 r_1^2}{\rho r^2}$ or $\frac{v_1}{v}=\frac{\rho/8\times(2r)^2}{\rho\times r^2}=\frac{1}{2}$ or $v_1 = v/2$

6. $Q=\frac{\pi p R^4}{8\eta l}$ and $Q'=\frac{\pi p(2R)^4}{8\eta(2l)}$

$\therefore$ $\frac{Q'}{Q}=\frac{(2R)^4}{2l}\times\frac{l}{R^4}=8$ or $Q'=8Q$

7. The velocity of water, $v=\sqrt{2gh}$; Height of hole from ground = $H-h$

Time taken by water to reach the ground is

$$t=\sqrt{\frac{2(H-h)}{g}}$$

$\therefore$ Horizontal range, $X = vt = \sqrt{2gh} \times \sqrt{\dfrac{2(H-h)}{g}} = 2\sqrt{h(H-h)}$

8. According to equation of continuity,

$$\frac{\pi}{4}d_p^2 \times v_p = \frac{\pi}{4}d_Q^2 \times v_Q \quad \therefore \frac{v_p}{v_Q} = \left(\frac{d_Q}{d_p}\right)^2 = \left(\frac{4\times10^{-2}}{2\times10^{-2}}\right)^2 = 4 \quad \text{or} \quad v_p = 4v_Q$$

9. As per the problem, $a_1v_1 = a_2v_2$

or $$L^2 \times \sqrt{2gy} = \pi R^2 \times \sqrt{2g \times 4y} \quad \text{or} \quad \boldsymbol{R = L / \sqrt{2\pi}}$$

10. Horizontal range, $R = 2\sqrt{h(H-h)}$

H = total height of liquid column

h = depth of hole from free surface of the liquid

Now h for holes 1, 2, 3 and 4 is 70 cm, 60 cm, 50 cm and 40 cm respectively.

For hole 1, $R_1 = 2\sqrt{70(90-70)} = 74.8$ cm

For hole 2, $R_2 = 2\sqrt{60(90-60)} = 84.8$ cm

For hole 3, $R_3 = 2\sqrt{50(90-50)} = 89.4$ cm

For hole 4, $R_4 = 2\sqrt{40(90-40)} = 89.4$ cm

11. At the ends, the velocity is higher so that pressure is lower (Bernoulli's theorem). Therefore, liquid rises to a height h to compensate for the drop of pressure.

Now $$\rho g h = \frac{1}{2}\rho v^2 = \frac{1}{2}\rho (r\omega)^2 = \frac{1}{2}\rho r^2\omega^2$$

$\therefore$ $$h = \frac{r^2\omega^2}{2g} = \frac{r^2(2\pi N)^2}{2g} = \frac{r^2 \times 4\pi^2 \times (2)^2}{2g}$$

$$= \frac{(0.05)^2 \times 4\pi^2 \times 4}{2\times9.8} = 0.02 \text{ m} = 2 \text{ cm}$$

12. $Q = \dfrac{\pi p R^4}{8\eta l}$ and $Q_1 = \dfrac{\pi p (R/2)^4}{8\eta (2l)} = \dfrac{\pi p R^4}{8\times32\eta l}$

$\therefore$ $\dfrac{Q_1}{Q} = \dfrac{1}{32}$ or $\boldsymbol{Q_1 = \dfrac{Q}{32}}$

13. Velocity of liquid, $v = \sqrt{2gh}$

where h = depth of hole below the free surface of liquid = $h_0 - h_1$

$\therefore$ $\boldsymbol{v = \sqrt{2g(h_0 - h_1)}}$

14. According to Bernoulli's theorem,

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$$

or $$P_2 - P_1 = \frac{1}{2}\rho(v_1^2 - v_2^2) = \frac{1}{2}\times1.293\times\left[(60)^2 - (45)^2\right] = 1018 \text{ N/m}^2$$

15. According to Bernoulli's theorem,

$$\frac{P_1}{\rho}+\frac{1}{2}v_1^2=\frac{P_2}{\rho}+\frac{1}{2}v_2^2$$

$$\therefore \quad P_2-P_1=\frac{1}{2}\rho\left(v_1^2-v_2^2\right)$$

At high altitudes, the density of air is considerably low. Therefore, pressure difference depends upon the altitude of the air-plane.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **There is 1 mm thick layer of glycerine between a flat plate of area 100 cm^2 and a big plate. If the coefficient of viscosity of glycerine is 1.0 kg m^{-1} s^{-1}, then how much force is required to move the plate with a velocity of 7 cm/s?**

Hint. $$F = \eta A \frac{dv}{dx}$$

Here $\eta = 1.0$ kg m^{-1} s^{-1} ; $A = 100\ cm^2 = 10^{-2} m^2$; $dv = 7 \times 10^{-2}$ ms^{-1} ; $dx = 1$ mm $= 1 \times 10^{-3}$ m

$$\therefore \quad F = 1.0 \times 10^{-2} \times \frac{7\times10^{-2}}{1\times10^{-3}} = \mathbf{0.7\ N}$$

2. **An oil drop falls through air with a terminal velocity of 5×10^{-4} ms^{-1}. Calculate (*i*) radius of the drop (*ii*) the terminal velocity of a drop of half of this radius. Viscosity of air = 1.8×10^{-5} Nsm^{-2}; density of oil = 900 kg m^{-3}. Neglect the density of air as compared to that of oil.**

Hint. By Stokes' law, the terminal velocity v_T of oil drop of radius r is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

where ρ is the density of oil, σ is the density of air and η is the coefficient of viscosity of air. Since σ is negligible,

$$\therefore \quad v_T = \frac{2}{9}\times\frac{r^2\rho g}{\eta}$$

(*i*) $$\therefore \quad r = \left[\frac{9}{2}\times\frac{v_T\eta}{\rho g}\right]^{1/2}$$

Here $v_T = 5 \times 10^{-4}$ ms^{-1} ; $\eta = 1.8 \times 10^{-5}$ Nsm^{-2} ; $\rho = 900$ kg m^{-3} ; $g = 9.8$ ms^{-2}

$$\therefore \quad r = \left[\frac{9}{2}\times\frac{(5\times10^{-4})\times(1.8\times10^{-5})}{900\times9.8}\right]^{1/2} = \mathbf{2.14 \times 10^{-6}\ m}$$

(*ii*) Terminal velocity $\propto$ (radius)^2

Let v_T' be the terminal velocity of the drop when its radius is halved (*i.e.*, $r' = r/2$).

$$\therefore \quad \frac{v_T'}{v_T} = \frac{r'^2}{r^2} \quad \text{or} \quad v_T' = v_T \times \frac{r'^2}{r^2} = 5\times10^{-4}\times\left(\frac{r/2}{r}\right)^2 = \mathbf{1.25 \times 10^{-4}\ ms^{-1}}$$

3. **An air bubble of diameter 2 cm is allowed to rise through a long cylindrical column of a viscous liquid and travels at the rate of 0.21 cm/s. If the density of the liquid is 1.47×10^3 kg/m^3, find the coefficient of viscosity of the liquid. Given that $g = 9.8$ ms^{-2} and density of air to be neglected.**

Hint. The weight of air bubble is negligible. By Stokes' law, the terminal velocity of air bubble is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}.$$ Here $\sigma = 0$ so that :

$$v_T = \frac{2}{9}\times\frac{r^2\rho g}{\eta}$$

or $$\eta = \frac{2}{9}\times\frac{r^2\rho g}{v_T} = \frac{2}{9}\times\frac{(1\times10^{-2})^2\times(1.47\times10^3)\times9.8}{0.21\times10^{-2}}$$

$$= 1.52 \times 10^2 \text{ kg m}^{-1}\text{s}^{-1} = \mathbf{1.52 \times 10^3 \text{ poise}}$$

4. **An aluminium sphere of radius 1.0 mm is dropped into a bottle of glycerine at 20°C. What is the terminal velocity of the sphere? At 20°C, density of aluminium is 2.7×10^3 kg/m³ and that of glycerine is 1.26×10^3 kg/m³. The coefficient of viscosity of glycerine is 1.49 Nsm^{-2} at 20°C.**

Hint. The terminal velocity v_T of aluminium sphere of radius r is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

Here $r = 1.0 \times 10^{-3}$ m ; $\rho = 2.7 \times 10^3$ kg/m³; $\sigma = 1.26 \times 10^3$ kg/m³; $\eta = 1.49$ Nsm^{-2}

$$\therefore \quad v_T = \frac{2}{9}\times\frac{(1.0\times10^{-3})^2\,(2.7-1.26)\times10^3\times9.8}{1.49} = 2.1\times10^{-3} \text{ m/s} = \mathbf{2.1 \text{ mm/s}}$$

5. **A sphere is dropped under gravity through a fluid of viscosity η. Taking average acceleration as half of the initial acceleration, show that the time taken to attain terminal velocity is independent of fluid density.**

Hint. When the sphere enters the fluid, it first moves with maximum acceleration a. As the velocity increases, the viscous force also increases. After some time, the acceleration becomes zero and the sphere now moves with (constant) terminal velocity v_T.

$\therefore$ Average acceleration = $a/2$

If t is the time taken by the sphere to attain terminal velocity v_T, then,

$$v_T = 0 + \frac{at}{2} \quad \text{or} \quad v_T = \frac{at}{2}$$

Net downward force, F = Weight of sphere – Weight of liquid displaced

$$= \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\sigma g = \frac{4}{3}\pi r^3 g(\rho-\sigma)$$

$$\therefore \quad \text{Acceleration, } a = \frac{F}{m} = \frac{\frac{4}{3}\pi r^3 g(\rho-\sigma)}{\frac{4}{3}\pi r^3\rho} = \frac{g(\rho-\sigma)}{\rho}$$

$$\therefore \text{ Terminal velocity, } v_T = \frac{at}{2} = \frac{g(\rho-\sigma)t}{2\rho}$$

or $$t = \frac{2v_T\rho}{(\rho-\sigma)g}$$

But $$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

$\therefore$ $$t = \frac{2\rho}{(\rho-\sigma)g}\times\frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

or $$t = \frac{4r^2\rho}{9\eta}$$

Thus, the time taken by the sphere to attain terminal velocity is independent of the density σ of the fluid.

6. Approximately what volume of water per second can flow through a pipe 2.0 cm in diameter before turbulent flow will occur? The critical value of Reynolds number is 2000. Viscosity of water is 0.801×10^{-3} Pa s.

Hint. The Reynolds number is $N_R = \dfrac{\rho v D}{\eta}$

The critical value of N_R will give you the maximum flow speed. Therefore, the maximum speed for laminar flow is

$$v_{max} = \frac{N_R \eta}{\rho D} = \frac{(2000)\times(0.801\times10^{-3})}{10^3\times(2.0\times10^{-2})} = 0.0801 \text{ ms}^{-1} = 8.01 \text{ cm/s}$$

$$\text{Area of pipe, } a_1 = \frac{\pi}{4}D^2 = \frac{\pi}{4}(2.0\times10^{-2})^2 = 3.14\times10^{-4} \text{ m}^2 = 3.14 \text{ cm}^2$$

$\therefore$ Maximum volume rate of laminar flow is

$$Q_{max} = v_{max} \times a = 8.01 \times 3.14 = 25.2 \text{ cm}^3/\text{s}$$

7. A horizontal pipe of diameter 20 cm has a constriction of diameter 4 cm. The velocity of water in the pipe is 2 ms^{-1} and pressure is 10^7 N/m^2. Calculate the velocity and pressure at the constriction.

Hint. According to the equation of continuity, $a_1 v_1 = a_2 v_2$ or $v_2 = \dfrac{a_1 v_1}{a_2}$

Here $a_1 = \pi \times (0.1)^2 \text{ m}^2$; $v_1 = 2 \text{ ms}^{-1}$; $a_2 = \pi \times (0.02)^2 \text{ m}^2$

$$\therefore \quad v_2 = \frac{\pi\times(0.1)^2\times2}{\pi\times(0.02)^2} = 50 \text{ ms}^{-1}$$

According to Bernoulli's theorem for the horizontal pipeline, we have,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\therefore \quad P_2 = P_1 - \frac{1}{2}\rho (v_2^2 - v_1^2)$$

Here $P_1 = 10^7 \text{ N/m}^2$; $v_2 = 50 \text{ ms}^{-1}$; $v_1 = 2 \text{ ms}^{-1}$; $\rho = 10^3 \text{ kg/m}^3$

$$\therefore \quad P_2 = 10^7 - \frac{1}{2}\times10^3[(50)^2 - (2)^2] = 10^7 - 12.48\times10^5 = 8.752\times10^6 \text{ N/m}^2$$

8. In a horizontal pipeline of uniform area of cross-section, the pressure falls by 5 N/m^2 between two points separated by a distance of 1 km. What is the change in kinetic energy per kg of the oil flowing at these points? Density of oil = 800 kg/ m^3.

Hint. According to Bernoulli's theorem,

$$\frac{P}{\rho} + g\,h + \frac{1}{2}v^2 = \text{Constant}$$

For a horizontal pipe, $\dfrac{P}{\rho} + \dfrac{1}{2}v^2 = \text{Constant}$

or $$\frac{P_1}{\rho} + \frac{v_1^2}{2} = \frac{P_2}{\rho} + \frac{v_2^2}{2}$$

$\therefore$ $$\frac{P_1 - P_2}{\rho} = \frac{1}{2}\left(v_2^2 - v_1^2\right)$$

$$\text{Change in K.E. per kg} = \frac{1}{2}(v_2^2 - v_1^2) = \frac{P_1 - P_2}{\rho}$$

Here $P_1 - P_2 = 5\ \text{N/m}^2$; $\rho = 800\ \text{kg/m}^3$

$$\therefore \quad \text{Change in K.E. per kg} = \frac{5}{800} = \mathbf{6.25 \times 10^{-3}\ J/kg}$$

9. Water is flowing through a horizontal pipe having different cross-sections at two points *A* and *B*. The diameters of the pipe at *A* and *B* are 0.6 m and 0.2 m respectively. The pressure difference between points *A* and *B* is 1 m column of water. Calculate the volume of water flowing per second.

Hint. Figure 20.24 shows the conditions of the problem. Since the pipe is horizontal, Bernoulli's equation applied to points *A* and *B* gives :

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

From the equation of continuity, we have,

$$a_1 v_1 = a_2 v_2 \text{ or } v_1 = \frac{a_2}{a_1} \times v_2$$

$$\therefore \quad P_1 + \frac{1}{2}\rho\left(\frac{a_2}{a_1}\right)^2 v_2^2 = P_2 + \frac{1}{2}\rho v_2^2$$

or

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2\left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{1}{2}\rho v_2^2\left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

$$\therefore \quad v_2 = a_1\sqrt{\frac{2(P_1 - P_2)}{\rho(a_1^2 - a_2^2)}}$$

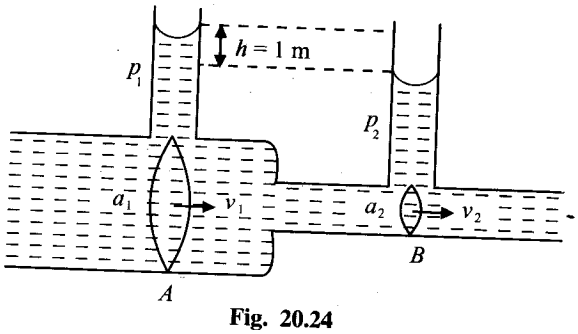


Fig. 20.24

$$\text{Rate of flow, } Q = a_2 v_2 = a_1 a_2\sqrt{\frac{2(P_1 - P_2)}{\rho(a_1^2 - a_2^2)}}$$

Here $a_1 = \pi \times \frac{(0.6)^2}{4} = 0.09\pi\ \text{m}^2$; $a_2 = \pi \times \frac{(0.2)^2}{4} = 0.01\pi\ \text{m}^2$

Now $P_1 - P_2 = h\rho g = 1 \times 10^3 \times 9.8\ \text{N/m}^2$; $a_1^2 - a_2^2 = 8 \times 10^{-3} \times \pi^2$

$$\therefore \quad \text{Rate of flow, } Q = (0.09\pi \times 0.01\pi) \times \sqrt{\frac{2 \times 9.8 \times 10^3}{10^3 \times 8 \times 10^{-3}\pi^2}} = \mathbf{0.1397\ m^3/s}$$

10. A large bottle is fitted with a siphon made of capillary glass tubing. Compare the coefficients of viscosity of water and petrol if the time taken to empty the bottle in the two cases is in the ratio 2 : 5. Specific gravity of petrol = 0.8.

Hint. $$Q_1 = \frac{\pi p_1 R^4}{8\eta_1 l} = \frac{V}{t_1}; \quad Q_2 = \frac{\pi p_2 R^4}{8\eta_2 l} = \frac{V}{t_2}$$

$$\therefore \quad \frac{Q_1}{Q_2} = \frac{t_2}{t_1} = \frac{5}{2}. \quad \text{Also } \frac{Q_1}{Q_2} = \frac{p_1}{p_2}\frac{\eta_2}{\eta_1}$$

But $$\frac{p_1}{p_2} = \frac{\rho_1}{\rho_2} = \frac{10^3}{0.8\times10^3} = 1.25$$

$$\therefore \quad \frac{5}{2} = 1.25\times\frac{\eta_2}{\eta_1}$$

or $$\frac{\eta_1}{\eta_2} = 1.25\times\frac{2}{5} = 0.5 \quad \therefore \quad \eta_1 : \eta_2 = 1 : 2$$

11. Three capillaries of same length but internal radii $3R$, $4R$ and $5R$ are connected in series and a liquid flows through them under streamline conditions. If the pressure across the third capillary is 8.1 mm of liquid, find the pressure across the first capillary.

Hint. According to Poiseuille's formula,

Rate of flow through capillary, $Q = \frac{\pi p R^4}{8\eta l}$

$\therefore$ Rates of flow through the individual capillaries are :

$$Q_1 = \frac{\pi p_1 (3R)^4}{8\eta l}; \quad Q_2 = \frac{\pi p_2 (4R)^4}{8\eta l}; \quad Q_3 = \frac{\pi p_3 (5R)^4}{8\eta l}$$

Since the capillaries are connected in series, the rate of flow through them is the same *i.e.*,

$$Q_1 = Q_2 = Q_3$$

or $$\frac{\pi p_1 (3R)^4}{8\eta l} = \frac{\pi p_2 (4R)^4}{8\eta l} = \frac{\pi p_3 (5R)^4}{8\eta l}$$

or $$81\, p_1 = 256\, p_2 = 625\, p_3$$

$\therefore$ $$p_1 = \frac{625}{81} p_3$$

It is given that $p_3 = 8.1$ mm of liquid.

$\therefore$ $$p_1 = \frac{625}{81} \times 8.1 = 62.5 \text{ mm} = \mathbf{6.25 \text{ cm of liquid}}$$

12. A cylindrical tank 1 m in radius rests on a platform 5 m high. Initially, the tank is filled with water to a height of 5 m. A plug whose area is 10^{-4} m^2 is removed from an orifice on the side of the tank at the bottom. Calculate (*i*) initial speed with which the water flows from the orifice (*ii*) initial speed with which water strikes the ground (g = 10 ms^{-2}). **[Roorkee Engg.]**

Hint. Figure 20.25 shows the conditions of the problem. Let P_a be the atmospheric pressure and ρ the density of water.

(*i*) Let A be a point on water surface and B at the orifice at a depth h (= 5 m) below water surface. The velocity of water at point A = 0. Let v be the initial velocity of efflux of water coming out of the orifice. Applying Bernoulli's equation to points A and B, we have,

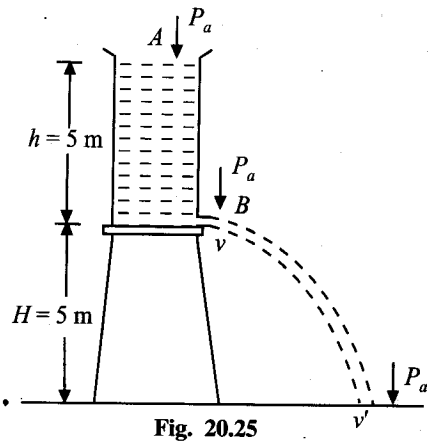


Fig. 20.25

$$P_a + \frac{1}{2}\rho\,(0)^2 + \rho g h = \rho_a + \frac{1}{2}\rho v^2 + \rho g\,(0)$$

or $$\rho g h = \frac{1}{2}\rho v^2$$

$\therefore$ $$v = \sqrt{2gh} = \sqrt{2\times 10\times 5}$$

$$= 10 \text{ ms}^{-1}$$

(*ii*) Let v' be the initial velocity with which the water strikes the ground. Then applying Bernoulli's equation to point A and ground level, we have,

$$P_a + \frac{1}{2}\rho\,(0)^2 + \rho g\,(H+h) = P_a + \frac{1}{2}\rho\,(v')^2 + \rho g\,(0)$$

or $$\rho g\,(H+h) = \frac{1}{2}\rho\,(v')^2$$

$\therefore$ $$v' = \sqrt{2g(H+h)} = \sqrt{2\times 10(5+5)} = 14.1 \text{ ms}^{-1}$$

13. A horizontal tube A of length 50 cm and radius 0.1 mm is joined to another horizontal tube B of length 40 cm and radius 0.2 mm. If a liquid passing through the tube enters A at a pressure of 85 cm of mercury and leaves B at a pressure of 76 cm of mercury, what is the pressure at the junction of the tubes?

Hint. According to Poiseuille's formula, the rate of flow of liquid in tubes A and B is given by ;

$$Q_1 = \frac{\pi p_1 R_1^4}{8\eta l_1} \; ; \quad Q_2 = \frac{\pi p_2 R_2^4}{8\eta l_2}$$

Since the tubes are connected in series, the rate of flow through them is the same *i.e.*,

$$Q_1 = Q_2$$

or $$\frac{\pi p_1 R_1^4}{8\eta l_1} = \frac{\pi p_2 R_2^4}{8\eta l_2}$$

or $$\frac{p_1}{p_2} = \frac{l_1}{l_2}\times\left(\frac{R_2}{R_1}\right)^4$$

Let x cm of mercury be the pressure at the junction of the tubes. Then,

$p_1 = (85 - x)$ cm of Hg ; $p_2 = (x - 76)$ cm of Hg

$\therefore$ $$\frac{85-x}{x-76} = \frac{l_1}{l_2}\times\left(\frac{R_2}{R_1}\right)^4$$

or $$\frac{85-x}{x-76} = \frac{50}{40}\times\left(\frac{0.02}{0.01}\right)^4$$

or $$x = \mathbf{76.43 \text{ cm of Hg}}$$

14. (*i*) Water flows steadily along a horizontal pipe at a rate of 8×10^{-3} m^3/s. If the area of cross-section of the pipe is 40×10^{-4} m^2, calculate the flow velocity of water.

(*ii*) Find the total pressure in the pipe if the static pressure in the horizontal pipe is 3×10^4 Pa. Density of water is 1000 kg/m^3.

(*iii*) What is the new flow velocity if the total pressure is 3.6×10^4 Pa?

Hint. (*i*) $$\text{Velocity of water} = \frac{\text{Rate of flow}}{\text{Area of cross-section}} = \frac{8\times10^{-3}}{40\times10^{-4}} = 2 \text{ ms}^{-1}$$

(*ii*) $$\text{Total pressure} = \text{Static pressure} + \frac{1}{2}\rho v^2$$

$$= 3 \times 10^4 + \frac{1}{2} \times 1000 \times (2)^2 = 3.2 \times 10^4 \text{ Pa}$$

(*iii*) $$\frac{1}{2}\rho v^2 = \text{Total pressure} - \text{Static pressure}$$

or $$\frac{1}{2} \times 1000 \times v^2 = 3.6 \times 10^4 - 3 \times 10^4 = 0.6 \times 10^4$$

$$\therefore \quad v = \sqrt{\frac{2\times0.6\times10^4}{1000}} = 3.5 \text{ ms}^{-1}$$

15. A non-viscous liquid of constant density 1000 kg/m^3 flows in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in Fig. 20.26. The area of cross-section of the tube at two points *P* and *Q* at heights 2 m and 5 m are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at point *P* is 1 ms^{-1}. Find the work done per unit volume by the gravity and pressure forces as the liquid flows from point *P* to point *Q* (g = 9.8 ms^{-2}). **[I.I.T.]**

Hint. The velocity of the liquid at point Q can be found by using equation of continuity *i.e.*,

$$a_P \times v_P = a_Q \times v_Q$$

$$\therefore \quad v_Q = \frac{a_P \times v_P}{a_Q} = \frac{(4\times10^{-3})\times(1)}{8\times10^{-3}} = 0.5 \text{ ms}^{-1}$$

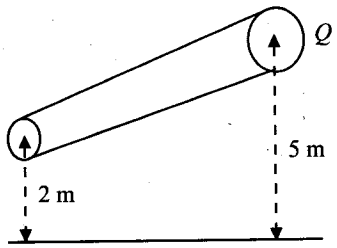


Fig. 20.26

Since the liquid flows from P to Q, the work done per unit volume against gravity

= mass per unit volume × $g \times (h_2 - h_1)$

= $1000 \times 9.8 \times (5 - 2) = 2.94 \times 10^4$ J/m^3

$\therefore$ Work done per unit volume by gravity

$$W_g = \mathbf{-2.94 \times 10^4 \text{ J/m}^3}$$

Suppose W_P is the work done per unit volume by the pressure force. Then according to work-energy theorem, we have,

$$W_P + W_g = \text{Change in kinetic energy per unit volume} = \frac{1}{2}\rho\,(v_Q^2 - v_P^2)$$

or $$W_P = \frac{1}{2}\rho\,(v_Q^2 - v_P^2) - W_g$$

$$= \frac{1}{2} \times 1000\,[(0.5)^2 - (1)^2] - [-2.94 \times 10^4] \simeq \mathbf{2.94 \times 10^4 \text{ Jm}^{-3}}$$

21

Thermal Properties of Matter

INTRODUCTION

Over the past few centuries, scientists have put forward some strange theories about the nature of heat. One of these was *caloric theory.* According to the caloric theory, heat is some kind of fluid called *caloric.* When this fluid (caloric) is added to a body, it becomes hot and when caloric is removed from a body, it becomes cold. When a hot body is placed in contact with a cold body, caloric flows from the hot body to the cold body till the two bodies attain the same temperature. This strange caloric theory failed to explain many heat phenomena and was rejected. But a common unit of heat that is still used today is **calorie** named after caloric.

With the development of kinetic theory, it is now well-known that heat is a form of energy called *thermal energy.* When a hot body is placed in contact with a cold body, heat (energy) flows from the hot body to the cold body till the two bodies attain the same temperature. The composition of a body is an important factor when dealing with thermal phenomena. For example, liquids and solids expand very slightly when heated. On the other hand, a gas will tend to expand appreciably when heated. Certain substances may melt, boil or burn, depending upon their composition and structure. Thus the thermal behaviour of a substance is closely related to its structure.

21.1. CONCEPT OF HEAT

A body is made up of a large number of particles called *molecules.* The *heat* or *thermal energy* of a body is due to the kinetic energy of its molecules. A molecule may possess kinetic energy due to its translational motion, vibrational motion and rotational motion. Each type of motion provides some kinetic energy to the molecule.

Thus **thermal energy** *of a body is the total kinetic energy of all the molecules of the body.*

The greater the kinetic energy of the molecules of a body, the greater is the thermal energy possessed by the body. Thus when a metal surface is struck with a hammer, it becomes hot. It is because the hammer blow causes the molecules in the metal to move faster and the kinetic energy of the molecules increases.

When a hot body is placed in contact with a cold body, heat (energy) is transferred from the hot body to the cold body. This transfer of energy between two bodies as a result of temperature difference is called heat.

Heat *is the energy that is transferred from one body to another due to temperature difference between the two bodies.*

Units of heat. There are two commonly used units of heat *viz calorie* and *joule.*

(*i*) Calorie. *One calorie is the amount of heat required to raise the temperature of 1 gm of water through 1°C.*

Thus if 5 g of water is heated from 50°C to 60°C, then the amount of heat required is 5 × 10 = 50 calories. The bigger unit is kilocalorie (kcal). 1 kcal = 1000 calories.

(*ii*) **Joule.** Since heat is a form of energy, it should be measured in joules—the SI unit of energy. It has been found experimentally that 1 calorie is equal to 4.186 joules *i.e.,*

1 calorie = 4.186 joules

Note that calorie is not the SI unit of heat. However, calorie is still used in many practical applications. For example, the energy value of food is specified in kilocalories (kcal).

21.2. CONCEPT OF TEMPERATURE

Heat and temperature are closely related to each other but the two quantities are different. We can define temperature in several ways.

(*i*) A simple way to define the temperature of a body is as under :
The temperature of a body is its degree of hotness or coldness.
The concept of temperature springs from the qualitative ideas of "hot" and "cold" based on our sense of touch. A hot oven is said to have a high temperature, whereas a cold tray of ice is said to have a low temperature. However, the bodily sensation of warmth and cold is not reliable. For example, if the hand has been in hot water, lukewarm water will feel cold, whereas if the hand has been in cold water, the same lukewarm water will feel warm.

(*ii*) *The temperature of a body is an indicator of the average thermal energy (kinetic energy) of the molecules of the body.*
The greater the average thermal energy of the molecules of a body, the greater is the temperature of the body. Let us now distinguish between heat and temperature. Suppose the ocean and a glass of water are at the same temperature. Then the average thermal energy of one molecule in the ocean is the same as that of the water molecule in the glass. But the ocean contains much more thermal energy because water molecules are much more in the ocean than in the glass.

(*iii*) When two bodies at different temperatures are brought in contact, heat flows from a body at higher temperature to the body at lower temperature till the temperature of the two bodies is the same. Thus temperature of the two bodies decides the direction of heat flow when the two bodies are brought in contact.

Hence when two bodies are brought in contact, the direction of heat flow determines which body is at higher temperature. Heat will flow from a body at a higher temperature to the body at lower temperature.

Note. Two bodies are said to be in **thermal equilibrium** if no transfer of heat takes when they are brought in contact. Clearly, the two bodies are at the same temperature.

21.3. MEASUREMENT OF TEMPERATURE

An instrument that measures temperature is called a *thermometer. It uses some property of matter that changes with temperature to measure the unknown temperature.*

One of the most familiar thermometers is the mercury thermometer. Fig. 21.1 shows Celsius mercury thermometer. This thermometer makes use of the fact that liquids expand with increasing temperature. In this thermometer, mercury is sealed into a fine capillary tube having a bulb at one end. When temperature increases, thermal expansion causes both the volume of the glass bulb and the volume of the liquid to increase. Since mercury expands at a greater rate than the glass does, the mercury is forced to expand in the tube as the temperature increases. In practice, we use a relatively large bulb and a narrow capillary tube. Therefore, the rate of expansion of the mercury with changes in temperature is constant. The distance between the ice point and steam point is divided into 100 equal

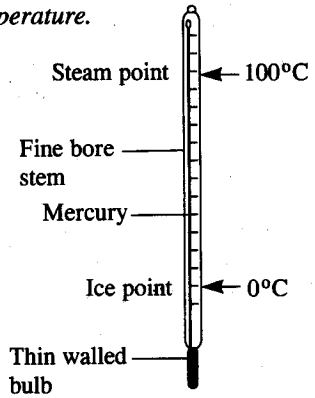


Fig. 21.1

parts. This scale was originally known as centigrade scale because it has one hundred divisions between the principal reference marks. The present name (Celsius scale) was adopted to honour the Swedish astronomer Ander Celsius who suggested this scale.

21.4. SCALES OF TEMPERATURE

In order to measure temperature quantitatively, some sort of numerical scale must be defined. The most common scales of temperature are :

(*i*) **Celsius scale.** On this scale, the ice point (melting point of pure ice) is marked as 0°C and the steam point (boiling point of water) is marked as 100°C ; both taken at normal atmospheric pressure. The interval between these points is divided into 100 equal parts. Each of these divisions is called one degree Celsius and is written as 1°C. If the thermometer is needed to measure temperatures outside the ice and steam points, then the scale markings are continued below the 0°C mark and above the 100°C mark. The distance between the consecutive marks on the scale remains the same.

(*ii*) **Fahrenheit scale.** On this scale, the ice point is marked as 32°F and the steam point is marked as 212°F. The interval between these two reference points is divided into 180 equal parts. Each division is called degree Fahrenheit and is written as °F.

Relationship between above two scales. On Fahrenheit scale, the steam point and ice point are separated by 180°F while on the Celsius scale, these same two points are separated by only 100°C. Therefore, one Fahrenheit degree represents only 100/180 or 5/9 as great a temperature change as one degree Celsius.

To convert from Celsius to Fahrenheit, we note that a Celsius temperature t_C is the number of celsuis degrees above the ice-point ; to obtain the number of Fahrenheit degrees above the ice point, we must multiply this by 9/5. But the ice point on the Fahrenheit scale is at 32°F. Therefore, to obtain the actual Fahrenheit temperature, we must first multiply Celsius value by 9/5 and then add 32°. Symbolically,

$$t_F = \frac{9}{5} t_C + 32$$

To convert Fahrenheit to Celsius, we solve this equation for t_C to get,

$$t_C = \frac{5}{9}(t_F - 32)$$

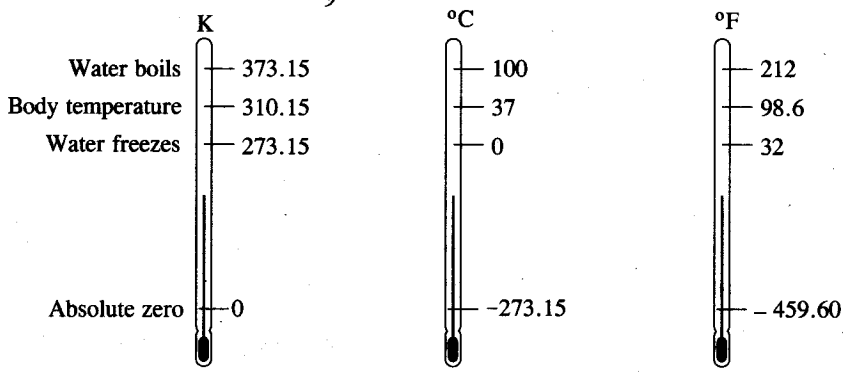


Fig. 21.2

(*iii*) **Reaumer scale.** On this scale, the ice point is marked as 0°R and steam point is marked as 80°R. The interval between these two reference points is divided into 80 equal parts. Each division is called degree Reaumer and is written as °R.

If t_C, t_F and t_R are the temperature values on Celsius scale, Fahrenheit scale and Reaumer scale respectively, then,

$$\frac{t_C - 0}{100} = \frac{t_F - 32}{180} = \frac{t_R - 0}{80}$$

(*iv*) **Kelvin or absolute temperature scale.** The Kelvin or absolute temperature scale is the standard temperature scale adopted for scientific measurements. The Kelvin or absolute temperature is denoted by the symbol T and is measured in *kelvin,* symbol K. In this scale, the lowest possible temperature attainable by the object is assigned the value zero *i.e.*, 0 K. At this temperature, the random motion of the atoms and molecules in a substance is at a minimum and the temperature cannot be lowered further by decreasing this motion. The lowest possible temperature is called **absolute zero** of temperature. *The temperature scale which begins at absolute zero is called Kelvin or absolute temperature scale.* Absolute zero is equivalent to – 273.15°C or –459.67°F. Temperature intervals on the Kelvin scale are the same as on the Celsius scale. The scale is an absolute scale in the sense that its zero (–273.15°C) is the lowest attainable temperature. The melting point of ice on the Kelvin scale is 273.15 K and the boiling point of water is 100 degrees higher *i.e.*, 373.15 K. Fig. 21.2 shows the relation among the three scales.

Note. $\frac{t_C - 0}{100} = \frac{t_F - 32}{180} = \frac{t_R - 0}{80} = \frac{T_K - 273.15}{100}$

Example 21.1. If the temperature of a room is 20°C, what is its temperature in degrees Fahrenheit?

Solution. $t_F = \frac{9}{5} t_C + 32 = \frac{9}{5} \times 20 + 32 = 68°F$

Example 21.2. Calculate the temperature which has the same value on (*i*) the Celsius and Fahrenheit (*ii*) Fahrenheit and Kelvin scales.

Solution. (*i*) Let the required temperature be $x°$.

Now $$t_F = \frac{9}{5} t_C + 32$$

or $$5\ t_F = 9\ t_C + 160$$

or $$5\ x = 9\ x + 160$$

$\therefore$ $$x = \frac{160}{-4} = -40°$$

$\therefore$ $$-40°C = -40°F$$

(*ii*) Let the required temperature be $x°$.

Now $$\frac{t_F - 32}{180} = \frac{T_K - 273.15}{100}$$

$\therefore$ $$\frac{x - 32}{180} = \frac{x - 273.15}{100}$$

On solving, we get, $x = 574.6°$

Example 21.3. A thermometer has its fixed points marked as 5 and 95. What is the correct temperature in °C when this thermometer reads 59°C?

Solution. The interval between the fixed points on the given thermometer = 95 – 5 = 90 ; the lower fixed point is 5. If t_C is the correct temperature in °C, then,

$$\frac{\text{Reading} - \text{Lower fixed point}}{\text{Interval between fixed points}} = \frac{t_C}{100}$$

or $$\frac{59 - 5}{90} = \frac{t_C}{100} \quad \therefore\ t_C = 60°C$$

Example 21.4. In an arbitrary scale of temperature, water freezes at 40°C and boils at 290°C. Find the boiling point of a liquid in this scale if it boils at 62°C.

Solution. Let t be the boiling point of the liquid on this scale. Then,

$$\frac{t-40}{290-40} = \frac{C-0}{100-0}$$

or $$\frac{t-40}{290-40} = \frac{C}{100}$$

The boiling point of the liquid on the Celsius scale is 62°C.

$\therefore$ $$\frac{t-40}{290-40} = \frac{62}{100}$$

$\therefore$ t = 195°C on new scale

PROBLEMS FOR PRACTICE

1. An object has a temperature of 50°F. What is its temperature in degrees celsius and in kelvin? **[10°C ; 283 K]**
2. The temperature of two bodies differ by 1°C. How much they differ on Fahrenheit scale? **[1.8°F]**
3. The distance between the upper and lower fixed points is 80 cm. Find the temperature on the Celsius scale if the mercury level rises to a height 10.4 cm above the lower fixed point. **[13°C]**

21.5. KELVIN OR ABSOLUTE SCALE OF TEMPERATURE

In principle, there is no *upper limit of temperature. In contrast, there is a definite limit of lower temperature. The lowest possible temperature attainable by an object is –273.15°C. Lord Kelvin suggested a temperature scale which has its zero at –273.15°C and temperature intervals on this scale are the same as on the Celsius scale. This temperature scale is called Kelvin or absolute scale of temperature. The lowest possible temperature (*i.e.*, –273.15°C) is called **absolute zero** of temperature and is written as 0 K. The absolute temperature is represented by the symbol T. By definition, the absolute temperature T and Celsius temperature t_C are related as :

$$T = 273.15 + t_C$$

For example, when t_C = 100°C, $T = t_C + 273.15 = 100 + 273.15 = 373.15$ K

A temperature scale that has its zero at –273.15°C and the temperature intervals same as on the Celsius scale is called **Kelvin** *or* **absolute temperature scale**.

Let us discuss the significance of absolute zero (*i.e.*, –273.15°C). Fig. 21.3 shows the graph between pressure P and temperature (°C) at constant volume for three ideal gases. In each case, the pressure-temperature graph is a straight line *i.e.*, pressure varies linearly with temperature. This fact is supported by experiments. Note that individual straight lines intersect the ordinate at different values of pressure at 0°C, *but each line intersects the abscissa* at the same point, namely at –273.15°C. A similar fact is found experimentally for volume-temperature graph at constant pressure for different ideal gases (See Fig. 21.4).

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* As temperature is increased, a solid object first melts and then vapourises ; as the temperature is further increased, molecules break up into atoms and atoms lose some or all of their electrons, thereby forming a cloud of electrically charged particles — a *plasma*. This situation exists in stars where the temperature is many millions of degrees Celsius.

Figs. 21.3 and 21.4 show an interesting feature : all linear plots extrapolate to the same temperature intercept, –273.15°C. This unique temperature is called *absolute zero.* We cannot obtain data near this temperature for most gases because they condense at considerably higher temperatures. Even so, the fact that such a unique temperature exists indicates that –273.15°C is the lowest temperature attainable. In short, Kelvin (or absolute) temperature scale is the one with degree intervals of the same size as the Celsius degree, but with zero at –273.15°C.

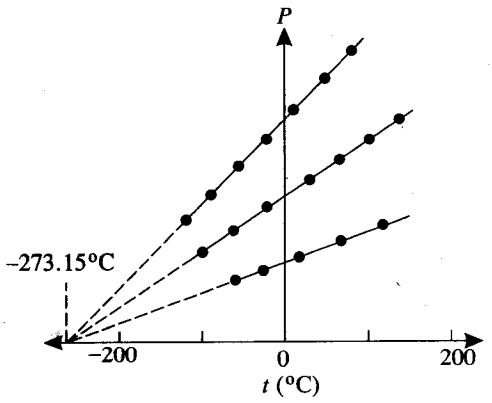


Fig. 21.3

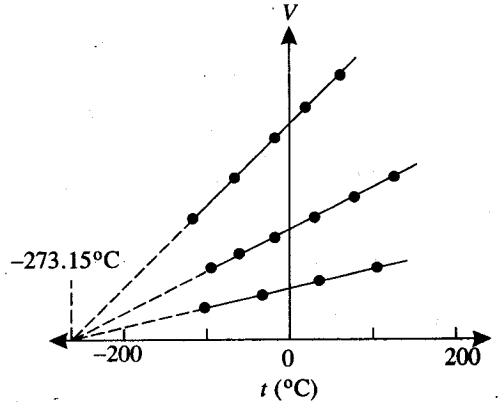


Fig. 21.4

21.6. TRIPLE POINT OF WATER

In the early days, the gas thermometers made use of ice point and steam point as standard temperatures. However, these points are difficult to duplicate experimentally since they are very sensitive to dissolved impurities in water. This difficulty has been overcome by fixing *triple point of water* as the single reference point for constructing the temperature scale. It is because triple point of water can be reproduced.

The **triple point of water** *is that point where water in the solid, liquid and gas states can coexist in equilibrium and this occurs only at a unique temperature and pressure.*

The pressure at the triple point of water is 4.58 mm of Hg and the temperature is 273.16 K (or 0.01°C). The absolute or Kelvin temperature T at any point is then defined, using a constant-volume gas thermometer for an ideal gas as :

$$T = 273.16 \times \frac{P}{P_{tp}} \qquad \text{[Ideal gas ; constant volume]}$$

In this relation, P_{tp} is the pressure in the thermometer at the triple point temperature of water and P is the pressure in the thermometer when it is at the point where T is being measured. Note that if we let $P = P_{tp}$ in this relation, T = 273.16 K as it must.

Example 21.5. When in thermal equilibrium at the triple point of water, the pressure of He in a constant volume gas thermometer is 1020 Pa. The pressure of the He is 288 Pa when the thermometer is in thermal equilibrium with liquid nitrogen at its normal boiling point. What is the normal boiling point of nitrogen as measured using this thermometer?

Solution. Normal boiling point of nitrogen is

$$T = 273.16 \times \frac{P}{P_{tp}}$$

Here $P = 288$ Pa ; $P_{tp} = 1020$ Pa

$$\therefore \quad T = 273.16 \times \frac{288}{1020} = \mathbf{77.1\ K}$$

21.7. THERMAL EXPANSION OF SOLIDS

Most solids expand when heated and contract when cooled. When the temperature of a solid is increased, its atoms and molecules vibrate with greater amplitude. Because of their increased motion, the average separation between atoms and molecules increases. As a result, a solid expands on heating. The thermal expansion of solids is usually quite small, but it can be an important effect. For example, architects and engineers must consider thermal expansion and contraction of metal beams when designing tall buildings and bridges. Dentists must use filling materials that expand and contract with temperature change at the same rate as the tooth ; otherwise filling will become loose. The thermal expansion of solids is of three types *viz.*, linear expansion, superficial expansion and cubical expansion. Experiments show that in each type of expansion, the increase in dimension is directly proportional to the original dimension and the increase in temperature.

(*i*) **Linear expansion.** Suppose the length of a rod of some solid material is l at some initial temperature T. Then at temperature $T + \Delta T$, the length will increase to $l + \Delta l$. Experiments show that increase in length of the rod (*i.e.*, Δl) depends upon the following factors :

(*a*) The increase in length is directly proportional to the original length *i.e.*,

$$\Delta l \propto l \qquad ...(i)$$

(*b*) The increase in length is directly proportional to the rise in temperature *i.e.*,

$$\Delta l \propto \Delta T \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), we have, $\Delta l \propto l \Delta T$

or $$\Delta l = \alpha l \Delta T \qquad ...(iii)$$

where α is a constant of proportionality and is called *coefficient of linear expansion.* Its value depends upon the nature of the material. From expression (*iii*), we see that :

$$\alpha = \frac{1}{l}\frac{\Delta l}{\Delta T}$$

Hence **coefficient of linear expansion** *of a solid may be defined as the fractional change in length* ($\Delta l / l$) *per degree change in temperature.*

The SI unit of α is /°C or / K. Its value is different for different solid materials. For example, α for aluminium is 2.4×10^{-5}/°C whereas for brass, its value is 2.0×10^{-5}/°C. Note that the change in temperature ΔT will be the same whether it is measured on the Celsius scale or on the Kelvin scale : ΔT°C = ΔT K.

New length, $l' = l + \Delta l = l + \alpha l \Delta T$

$\therefore$ $$l' = l(1 + \alpha \Delta T) \qquad ...(iv)$$

(*ii*) **Superficial expansion.** Suppose the surface area of a rectangular metal plate is S at some initial temperature T. Then at temperature $T + \Delta T$, the surface area will increase to $S + \Delta S$. Experiments show that increase in surface area is directly proportional to the original surface area and the rise in temperature *i.e.*,

$$\Delta S \propto S \Delta T$$

or $$\Delta S = \beta S \Delta T \qquad ...(v)$$

where β is a constant of proportionality and is called *coefficient of superficial expansion.* Its value depends upon the nature of material. From expression (*v*), we see that :

$$\beta = \frac{1}{S}\frac{\Delta S}{\Delta T}$$

Hence **coefficient of superficial expansion** *of a solid may be defined as the fractional change in surface area* ($\Delta S / S$) *per degree change in temperature.* Its SI unit is also /°C or /K.

Note that change in temperature ΔT will be the same whether it is measured on the Celsius scale or on the Kelvin scale.

New surface area, $S' = S + \Delta S = S + \beta S \Delta T$

$\therefore$ $$S' = S(1 + \beta \Delta T) \quad ...(vi)$$

(iii) **Cubical expansion.** Suppose the volume of a solid is V at some initial temperature T. Then at temperature $T + \Delta T$, the volume will increase to $V + \Delta V$. Experiments show that the increase in volume is directly proportional to the original volume and the rise in temperature *i.e.,*

$$\Delta V \propto V \Delta T$$

or $$\Delta V = \gamma V \Delta T \quad ...(vii)$$

where γ is a constant of proportionality and is called *coefficient of cubical expansion.* Its value depends upon the nature of material. From exp. *(vii)*, we see that :

$$\gamma = \frac{1}{V}\frac{\Delta V}{\Delta T}$$

Hence **coefficient of cubical expansion** *of a solid may be defined as the fractional change in volume* ($\Delta V/V$) *per degree change in temperature.* Its SI unit is /°C or /K.

New volume, $V' = V + \Delta V = V + \gamma V \Delta T$

$\therefore$ $$V' = V(1 + \gamma \Delta T) \quad ...(viii)$$

21.8. RELATION BETWEEN COEFFICIENTS OF EXPANSION

We shall now show that for solids, the approximate relations between α, β and γ are :

$$\beta = 2\alpha \text{ and } \gamma = 3\alpha$$

(i) **Relation between β and α.** Consider a square plate of side l_0 at 0°C and l_t at t°C.

$\therefore$ $$l_t = l_0(1 + \alpha t)$$

Area of plate at 0°C, $A_0 = l_0^2$

Area of plate at t°C, $A_t = l_t^2 = l_0^2(1 + \alpha t)^2 = A_0(1 + \alpha t)^2$

Also Area of plate at t°C, $A_t = A_0(1 + \beta t)$

$\therefore$ $$A_0(1 + \alpha t)^2 = A_0(1 + \beta t)$$

or $$1 + \alpha^2 t^2 + 2\alpha t = 1 + \beta t$$

Since the value of α is small, the term $\alpha^2 t^2$ may be neglected.

$\therefore$ $$\beta = 2\alpha$$

The result is altogether general because any flat surface can be regarded as a collection of small squares.

(ii) **Relation between γ and α.** Consider a cube of side l_0 at 0°C and l_t at t°C.

$\therefore$ $$l_t = l_0(1 + \alpha t)$$

Volume of cube at 0°C, $V_0 = l_0^3$

Volume of cube at t°C, $V_t = l_t^3 = l_0^3(1 + \alpha t)^3 = V_0(1 + \alpha t)^3$

Also Volume of cube at t°C, $V_t = V_0(1 + \gamma t)$

$\therefore$ $$V_0(1 + \alpha t)^3 = V_0(1 + \gamma t)$$

or $$1 + 3\alpha t + 3\alpha^2 t^2 + \alpha^3 t^3 = 1 + \gamma t$$

Since the value of α is small, we can neglect the higher powers of α.

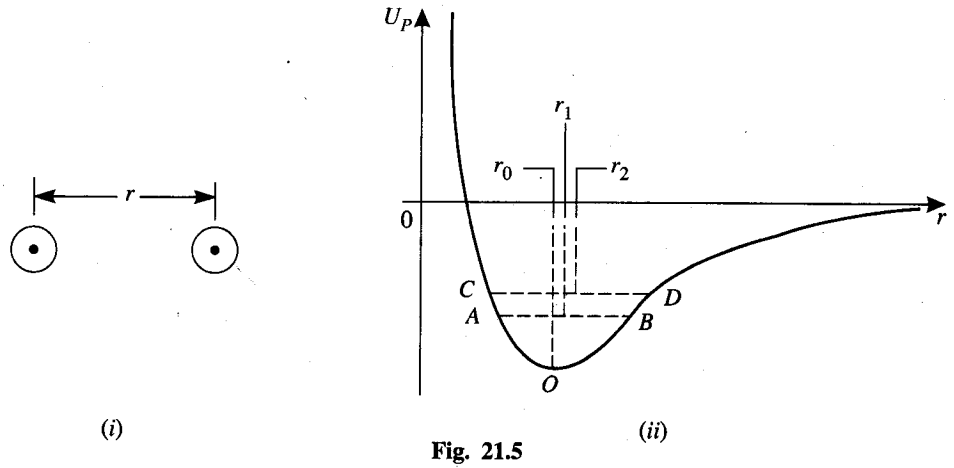
$\therefore$ $$3\alpha t = \gamma t$$

or $$\gamma = 3\alpha$$

Again, the result is general because any solid can be regarded as a collection of small cubes.

21.9. THERMAL EXPANSION FROM MOLECULAR THEORY

We can understand thermal expansion on the basis of intermolecular forces. Fig. 21.5 (*ii*) shows a typical potential energy diagram which represents the potential energy U_p of two molecules (See Fig. 21.5(*i*)) versus their separation r. We take U_p to be zero when the molecules are infinitely apart and as r decreases, P.E. decreases, indicating an attractive force as already discussed. For r less than r_0 (the equilibrium position), the P.E. curve rises, indicating a repulsive force. If the two molecules were left to themselves and if their kinetic energy were zero, they would have a separation distance $r = r_0$. If separation r of the molecules is slightly increased from r_0, the attractive force between them will restore the molecules to their equilibrium position after the external force is removed. If the separation is decreased from r_0, the repulsive force will restore the molecules to their equilibrium position after the external force is removed. At equilibrium position, they have a separation distance r_0.



(*i*) (*ii*)

Fig. 21.5

Molecules remain stationary at absolute zero since their thermal energy is then zero. This corresponds to point O of the energy curve. At a higher temperature, the molecules have some energy above the minimum value. Hence they oscillate between points such as A and B. Note that horizontal lines AB and CD represent the conditions for two different temperatures T_1 and T_2 respectively where $T_2 > T_1$. Because the P.E. curve is not symmetrical, the average separation for molecules is greater for the higher temperature as shown. Thus molecular separation increases with increasing temperature and this accounts for the increase in the dimensions of the object. Note that thermal expansion is due to the nonsymmetry of the potential energy curve. If the P.E. curve were symmetrical, there would be no thermal expansion at all.

21.10. THERMAL STRESSES

In some situations, the ends of a rod or slab of material are rigidly fixed which prevents expansion or contraction. *If temperature changes, large compressive or tensile stresses will be set up ; these are called thermal stresses.* These stresses may become large enough to stress the rod beyond its elastic limit or even beyond its breaking strength. Therefore, in the design of any structure that is subject to changes in temperature, provision must be made for expansion. In a long steam pipe, this is accomplished by the insertion of expansion joints.

Thermal stress in a rod. It is a simple matter to calculate the thermal stress in a rod or slab of material. Suppose that rod of length l and cross-sectional area A has its ends rigidly fixed while the temperature is increased by an amount ΔT, causing compressive stress. If the increase in length is Δl, then,

$$\Delta l = \alpha l \Delta t \qquad ...(i)$$

where α = Coefficient of linear expansion

The compressive force F can be calculated as under :

$$Y = \frac{\text{Stress}}{\text{Strain}} = \frac{F/A}{\Delta l/l} = \frac{Fl}{A\Delta l}$$

or $$\Delta l = \frac{Fl}{AY} \quad ...(ii)$$

Putting the value of Δl from eq. (*i*) into eq. (*ii*), we have,

$$\alpha\, l \Delta T = \frac{F\, l}{AY}$$

or $$\alpha \Delta T = \frac{F}{AY}$$

$\therefore$ $$\text{Compressive stress} = \frac{F}{A} = \alpha Y \Delta T$$

and $$\text{Compressive force, } F = \alpha Y A \Delta T$$

If ΔT represents a decrease in temperature, then F/A and F are tensile stress and tensile force respectively.

Note. When the temperature of a gas enclosed in a vessel of rigid material is increased, then thermal stress is equal to the increase in pressure (Δp) and is given by : $\Delta p = K\gamma\, \Delta T$

where K = Bulk modulus of gas

γ = Coefficient of cubical expansion

ΔT = Increase in temperature

Proof. $$V' = V(1+\gamma\Delta T)$$

or $$V' - V = V\gamma\Delta T \quad \text{or} \quad \Delta V = V\gamma\, \Delta T$$

Now $$K = \frac{V\Delta p}{\Delta V} = \frac{V\Delta p}{V\gamma\Delta T}$$

$\therefore$ $$\Delta p = K\gamma\, \Delta T$$

21.11. APPLICATIONS OF THERMAL EXPANSION OF SOLIDS

There are a large number of important practical applications of thermal expansion of solids. However, we shall brief only a few of them by way of illustration.

(*i*) *While laying the railway tracks, a small gap is left between the successive lengths of the rails.* This gap is provided to allow for the expansion of the rails during summer. If no gap is left, the expansion will cause the rails to buckle.

(*ii*) *When the iron tyre is to be put on a wheel, the tyre is made slightly smaller in diameter than that of wheel.* The iron tyre is first heated uniformly till its diameter becomes more than that of the wheel and is then slipped over the wheel. On cooling, the tyre contracts and makes a tight fit on the wheel.

(*iii*) *In bridges, one end is rigidly fastened to its abutment while the other rests on the rollers.* This provision allows the expansion and contraction to take place during changes in temperature.

(*iv*) *The fact that a solid expands on heating and contracts on cooling is utilised in rivetting e.g.,* rivetting two metal plates together, joining steel girders etc. For joining two steel plates, holes are drilled between them. The rivets (small rods) are made red hot and inserted in the holes in the plates. The ends of the rivets are hammered into the head. After some time, the rivets contract on being cooled and hold the plates very tightly.

(v) *The concrete roads and floors are always made in sections and enough space is provided between the sections.* This provision allows expansion and contraction to take place due to changes in temperature.

21.12. THERMAL EXPANSION OF LIQUIDS

Like solids, liquids also *expand on heating. Since liquids do not have a shape of their own but take up the shape of the vessel in which they are kept, we can only talk of *volume expansion* in case of liquids. When the liquid is heated in a container, the container also expands alongwith the liquid. If the container were not to expand, we will observe the *real expansion* of the liquid. Since the container also expands on heating the liquid, what we observe is the *apparent expansion* of the liquid.

Coefficient of real expansion. *It is defined as the real increase in volume of the liquid per unit original volume per °C rise in temperature.* It is denoted by γ_r.

$$\therefore \quad \gamma_r = \frac{\text{real increase in volume}}{\text{original volume} \times \text{rise in temperature}}$$

Coefficient of apparent expansion. *It is defined as the apparent increase in volume of the liquid per unit original volume per °C rise in temperature.* It is denoted by γ_a.

$$\therefore \quad \gamma_a = \frac{\text{apparent increase in volume}}{\text{original volume} \times \text{rise in temperature}}$$

It is clear that $\gamma_r > \gamma_a$ and both are measured in $°C^{-1}$. It can be shown that :

$$\gamma_r = \gamma_a + \gamma_g$$

where γ_g is the coefficient of cubical expansion of glass (or material of the container).

21.13. VARIATION OF DENSITY OF SOLIDS AND LIQUIDS WITH TEMPERATURE

When a solid or liquid is heated, its volume increases while its mass remains the same. Therefore, the density (= mass/volume) of a solid or liquid decreases on heating. In case of solids, the change in density is very small ($\because$ change in volume is very small) but in case of liquids, the change in density is considerable ($\because$ change in volume is large).

Let V = Volume of solid or liquid at temperature T

V' = Volume of solid or liquid at temperature $T + \Delta T$

ρ = Density of solid or liquid at temperature T

ρ' = Density of solid or liquid at temperature $T + \Delta T$

γ = Coefficient of cubical expansion of solid/liquid

$$\therefore \quad V' = V(1+\gamma\,\Delta T)$$

Now $\rho = M/V$ or $V = M/\rho$

and $\rho' = M/V'$ or $V' = M/\rho'$

$$\therefore \quad \frac{M}{\rho'} = \frac{M}{\rho}(1+\gamma\Delta T)$$

or $$\rho' = \frac{\rho}{1+\gamma\Delta T} = \rho(1+\gamma\,\Delta T)^{-1}$$

* Although there are some notable exceptions to this rule over small temperature ranges (*e.g.* water contracts in going from 0°C to 4°C), it is generally true that liquids expand with increasing temperature.

Expanding by Binomial theorem and neglecting terms containing higher powers of $\gamma\Delta T$,

$$\rho' = \rho(1-\gamma\,\Delta T)$$

It is clear that $\rho' < \rho$ *i.e.*, the density of a solid or liquid decreases with increase in temperature. Note that this formula is applicable if ΔT is small.

Example 21.6. A glass flask of volume 200 cm^3 is just filled with mercury at 20°C. How much mercury will overflow when the temperature of the system is raised to 100°C? The coefficient of volume expansion of glass is 1.2 × 10^{-5}/°C and that of mercury is 18 × 10^{-5}/°C.

Solution. The increase in the volume of the flask is

$$\Delta V = \gamma_g V\Delta T = (1.2\times10^{-5})\times(200)\times(100-20) = 0.19 \text{ cm}^3$$

The increase in the volume of the mercury is

$$\Delta V' = \gamma_m V\Delta T = (18\times10^{-5})\times(200)\times(100-20) = 2.88 \text{ cm}^3$$

$\therefore$ The volume of mercury that will overflow

$$= \Delta V' - \Delta V = 2.88 - 0.19 = \mathbf{2.69 \text{ cm}^3}$$

Example 21.7. Blocks of concrete 10 m long are placed end to end with no space in-between them to allow for expansion. If the blocks were placed at a temperature of 10°C, what will be the force of compression when the temperature reaches 40°C? The contact area of each block is 0.20 m^2. Will fracture occur? The coefficient of linear expansion for concrete is 12 × 10^{-6}/°C and Young's modulus for concrete is 20 × 10^9 N/m^2.

Solution. The force of compression F is given by ;

$$F = \alpha Y A\Delta T$$

Here α = 12 × 10^{-6}/°C ; Y = 20 × 10^9 N/m^2; A = 0.20 m^2 ; ΔT = 40 − 10 = 30°C

$\therefore$ F = (12 × 10^{-6}) × (20 × 10^9) × (0.20) × 30 = **1.4 × 10^6 N**

$\therefore$ Compressive stress $= \dfrac{F}{A} = \dfrac{1.4\times10^6}{0.20} = 7.0\times10^6$ N/m^2

This stress is far from ultimate strength of concrete (= 20 × 10^6 N/m^2) under compression and exceeds it for tensile stress (= 2 × 10^6 N/m^2) and shear (= 2 × 10^6 N/m^2). Therefore, assuming the concrete is not perfectly aligned, part of the force will act in shear and fracture is likely.

Example 21.8. An iron ring is to fit snugly on a cylindrical iron rod. At 20°C, the diameter of the rod is 6.453 cm and the inside diameter of the ring is 6.420 cm. To what temperature must the ring be brought if its hole is to be large enough so it will slip over the rod? The coefficient of linear expansion of iron is 12 × 10^{-6}/°C.

Solution. The hole in the ring must be increased from a diameter of 6.420 cm to 6.453 cm. The ring must be heated since the hole diameter will *increase linearly with temperature.

Now $\Delta l = \alpha l\,\Delta T$

$\therefore$ The required change in temperature of the ring is

$$\Delta T = \frac{\Delta l}{\alpha l}$$

Here Δl = 6.453 − 6.420 = 0.033 cm ; α = 12 × 10^{-6}/°C ; l = 6.420 cm

$\therefore$ $$\Delta T = \frac{0.033}{12\times10^{-6}\times6.420} = 428.35°\text{C}$$

$\therefore$ The temperature of the ring must be raised to = 428.35 + 20 = **448.35°C**

* Note that the material does not expand to fill the hole. In a solid, without a hole, all sections of the object — the ring — increase with temperature ; the presence of the hole does not affect this. So the diameter increases with increased temperature.

Example 21.9. A circular hole of radius 1 cm in a brass sheet is kept at 293 K. What will be the diameter of this hole when the sheet is heated to 393 K? Coefficient of linear expansion for brass = 20×10^{-6}/°C.

Solution. On heating the brass sheet, the diameter of the hole will increase linearly with temperature.

$\therefore$ Increase in the diameter of the hole is

$$\Delta l = \alpha l \Delta T$$

Here $\alpha = 20 \times 10^{-6}$/°C ; $l = 2 \times 1 = 2$ cm ; $\Delta T = 393 - 293 = 100$ K

$\therefore$ $\Delta l = (20 \times 10^{-6}) \times (2) \times (100) = 0.004$ cm

$\therefore$ Diameter of hole = 2 + 0.004 = **2.004 cm**

Example 21.10. A sheet of brass is 40 cm long and 8 cm broad at 0°C. If the surface area at 100°C is 320.1 cm^2, find the coefficient of linear expansion of brass.

Solution. Surface area of sheet at 0°C, $A_0 = 40 \times 8 = 320$ cm^2

Surface area of sheet at 100°C, $A_{100} = 320.1$ cm^2

Rise in temperature, $\Delta T = 100 - 0 = 100$°C

Increase in surface area, $\Delta A = A_{100} - A_0 = 320.1 - 320 = 0.1$ cm^2

Coefficient of superficial expansion β is given by ;

$$\beta = \frac{\Delta A}{A_0 \times \Delta T} = \frac{0.1}{320 \times 100} = 31 \times 10^{-7} / °C$$

$\therefore$ Coefficient of linear expansion, $\alpha = \frac{\beta}{2} = \frac{31 \times 10^{-7}}{2} =$ **15.5×10^{-7}/°C**

PROBLEMS FOR PRACTICE

1. Slabs of concrete 20 m long are laid end to end to form a roadway. How large a gap should be allowed between adjacent slabs at –20°C so that they do not buckle when temperature reaches +50°C? Coefficient of linear expansion for concrete is 10×10^{-6}/°C. **[1.4 cm]**
2. An iron railway line is 30 m long. By how much will its length increase if its temperature is raised by 50°C? The coefficient of linear expansion of iron is 1.2×10^{-5}/°C. **[18 mm]**
3. An iron sphere has a radius of 10 cm at a temperature of 0°C. Calculate the change in volume of the sphere if it is heated to 100°C. The coefficient of linear expansion of iron is 11×10^{-6}/°C. **[13.8 cm^3]**
4. An iron chain is 100 m long at 10°C. It is used to measure a distance of 2 km in a day when the temperature rises to 40°C. Calculate the error in measuring the distance. The coefficient of linear expansion of iron is 1.2×10^{-5}/°C. **[0.72 m *less*]**
5. The coefficient of volume expansion of glycerine is 49×10^{-5}/°C. What is the fractional change in its density for a 30°C rise in temperature? **[1.5×10^{-2}]**
6. During a summer night when the air temperature is 20°C, your house contains 453 m^3 of air. What volume of air leaves the house through an open window if the air warms to 40°C on a very hot summer day? Assume that dimensions of the house experience a negligible change and all other conditions are constant. The volume expansion of air is 3670×10^{-6}/°C. **[33 m^3]**
7. A precise steel tape measure has been calibrated at 20°C. At 40°C, (*i*) will it read high or low and (*ii*) what will be the percentage error? The coefficient of linear expansion for steel is 1.2×10^{-5}/°C. **[(*i*) low (*ii*) 0.024%]**

8. On heating a glass block of 10,000 cm^3 from 25°C to 40°C, its volume increases by 4 cm^3. Calculate the coefficient of linear expansion of glass. **[8.89 × 10^{-6}/°C]**
9. The metal of a pendulum clock has coefficient of linear expansion of 2 × 10^{-5}/°C. If the period is 2 s at 15°C, calculate the loss or gain of time in 10 hours when the temperature rises to 25°C. **[3.6 s *loss*]**
10. A steel wire 8 m long and 4 mm in diameter is fixed to two rigid supports. Calculate the increase in tension when the temperature falls by 10°C. The coefficient of linear expansion of steel = 12 × 10^{-6}/K ; Young's modulus for steel = 2 × 10^{11} N/m^2. **[300 N]**

21.14. SPECIFIC HEAT OR SPECIFIC HEAT CAPACITY

When we supply heat to a solid substance (or liquid), its temperature increases. It is found that amount of heat Q absorbed by the solid substance (or liquid) is

(*i*) directly proportional to the mass (m) of the substance *i.e.*,

$$Q \propto m$$

(*ii*) directly proportional to the rise in temperature (ΔT) *i.e.*,

$$Q \propto \Delta T$$

Combining the two factors, we have,

$$Q \propto m\,\Delta T$$

or
$$Q = c\,m\,\Delta T \qquad ...(i)$$

where c is a constant of proportionality and is called *specific heat capacity* or simply *specific heat* of the substance. From eq. (*i*), we have,

$$c = \frac{Q}{m\,\Delta T} \qquad ...(ii)$$

If m = 1 kg and ΔT = 1°C, then $c = Q$.

Hence the specific heat of a solid (or liquid) may be defined as the amount of heat required to raise the temperature of 1 kg of solid (or liquid) through 1°C (or 1 K). It is clear from eq. (*ii*) that SI unit of specific heat is J kg^{-1} $°C^{-1}$ or J kg^{-1} K^{-1}.

For example, the specific heat of water is 4186 J kg^{-1} $°C^{-1}$. It means that 4186 J of heat energy is required to raise the temperature of 1 kg of water through 1°C.

Note. In C.G.S. system, the unit of c is cal g^{-1} $°C^{-1}$.

$$1 \text{ cal } g^{-1} °C^{-1} = 4.186 \text{ J } g^{-1} °C^{-1} = 4186 \text{ J } kg^{-1} °C^{-1}$$

21.15. MOLAR SPECIFIC HEAT

The molar specific heat of a solid (or liquid) is defined as the amount of heat required to raise the temperature of 1 mole of the solid (or liquid) through 1°C (or 1 K). It is denoted by the symbol C. Therefore, the amount of heat Q required to raise the temperature of n moles of a solid (or liquid) through a temperature change ΔT is given by ;

$$Q = n\,C\,\Delta T$$

It is clear that SI unit of C is J mol^{-1} K^{-1}. For any material of mass m and molecular mass M, the number of moles $n = m/M$.

$$\therefore \qquad Q = \frac{m}{M} C\,\Delta T$$

Also
$$Q = m\,c\,\Delta T$$

$$\therefore \qquad \frac{m}{M} C\,\Delta T = m\,c\,\Delta T$$

or
$$C = M\,c \qquad ...(i)$$

Eq. (*i*) gives the relation between molar specific heat C and the ordinary specific heat c.

21.16. DULONG AND PETIT LAW

Dulong and Petit gave an empirical relation between the specific heat of a solid and its atomic weight.

According to this law, for heavy solid substances (atomic weight above 35), the product of specific heat and the atomic weight is nearly constant and has a value of 6 cal g^{-1} $°C^{-1}$ i.e.,

Atomic weight × Specific heat = Constant = 6 cal g^{-1} $°C^{-1}$

Though it seems strange that two such apparently dissimilar properties as specific heat and atomic weight should have the same product for many dissimilar elements, a little thought will show that this relationship is actually what should be expected.

Consider one gram of element *A*, composed of atoms each of which weighs W_A. The number of calories that must be added to raise its temperature by 1 degree is its specific heat c_A. If we take one gram of another element *B*, whose atomic weight is twice the atomic weight of *A*, the gram will contain just half as many atoms as the gram of *A*. Temperature is a measure of the average kinetic energy *per atom* (or molecule) ; so to raise the temperature of 1 g of *B* by one degree will require only half the heat energy needed for 1g of *A*. In other words, $c_B = c_A/2$. Thus the product of atomic weight and specific heat for element *A* is the same as that for element *B*.

Applying Dulong and Petit rule to bismuth,

Atomic weight of Bi × specific heat of Bi = 6

or $$\text{Atomic weight of Bi} = \frac{6}{\text{specific heat of Bi}} = \frac{6}{0.0294} = 204$$

This is a rough value of the atomic weight of bismuth. This rule is only approximate but serves to show that atomic weight of bismuth is about 204.

21.17. SPECIFIC HEAT OF WATER

As with solids and liquids, the *specific heat of water is the amount of heat required to raise the temperature of 1 g of water through 1°C.* This is, by definition, equal to 1 calorie.

$\therefore$ Specific heat of water, c_w = 1 cal g^{-1} $°C^{-1}$ = 1 cal g^{-1} K^{-1} = 4.2 J g^{-1} K^{-1}
= 4200 J kg^{-1} K^{-1}

Water has a large specific heat compared with other liquids. This makes it useful as a coolant.

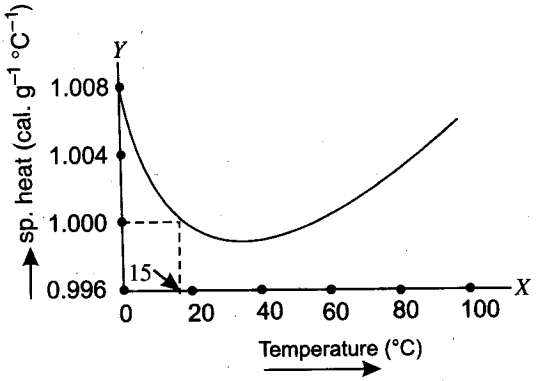


Fig. 21.6

In general, the specific heat of a liquid changes with temperature. But the specific heat variation of water with temperature shows a peculiar behaviour. Fig. 21.6 shows the variation of specific heat of water with temperature. The following points are worth noting :

(*i*) At 15°C, the specific heat of water is 1 cal g^{-1} $°C^{-1}$.

(*ii*) At 0°C, the specific heat of water is maximum and is equal to 1.008 cal g^{-1} $^0C^{-1}$.

(*iii*) At 30°C, the specific heat of water has the lowest value.

(*iv*) Beyond 30°C, the specific heat of water increases slightly with the increase in temperature.

21.18. HEAT CAPACITY AND WATER EQUIVALENT

1. Heat capacity. Sometimes, engineers prefer to talk about the heat capacity of the whole body rather than just 1 kg as we have done above. For example, the heat capacity of the body might be 100 J/°C. This means that the body needs 100 J to raise its temperature by 1°C.

The **heat capacity** *of a body is defined as the amount of heat required to raise its temperature by 1°C.* It is also known as thermal capacity of the body. Suppose a body has mass m and specific heat c.

$$\text{Heat capacity} = \text{Heat required to raise the temperature of the body by 1°C}$$

$$= mc \times 1 = mc$$

$$\therefore \quad \text{Heat capacity} = mc \qquad ...(i)$$

Hence heat capacity of a body (solid or liquid) is equal to the product of its mass and specific heat. Clearly, the SI unit of heat capacity is J/°C or J/K. The greater the mass of a body, the greater is its heat capacity.

(*i*) From eq. (*i*), we have,

$$c = \frac{\text{Heat capacity}}{m}$$

Hence the **specific heat** *of a body (solid or liquid) may be defined as its heat capacity per unit mass.* Note that specific heat capacity (or specific heat) is for 1 kg while heat capacity is for the whole body.

(*ii*) If C is the molar specific heat of a body (solid or liquid) having n moles, then its heat capacity is given by ;

$$\text{Heat capacity} = n\ C \times 1 = nC$$

or

$$C = \frac{\text{Heat capacity}}{n}$$

Hence the **molar specific heat** *of a body (solid or liquid) may be defined as its heat capacity per mole.*

2. Water Equivalent. *The* **water equivalent** *of a body is defined as the mass of water that will absorb or lose the same amount of heat as the body for the same rise or fall in temperature.*

The water equivalent of a body is measured in kg in SI units and in g in C.G.S. units. Suppose the water equivalent of a body is 10 kg. It means that if the body is heated through, say 10°C, it will absorb the same amount of heat as absorbed by 10 kg of water when heated through 10°C.

Consider a body of mass m and specific heat c. The amount of heat required to raise the temperature of the body through ΔT is

$$Q = c\ m\ \Delta T \qquad ...(i)$$

Suppose w is the water equivalent of this body. Then, by definition, Q is given by ;

$$Q = w\ \Delta T \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), we have,

$$w\ \Delta T = c\ m\ \Delta T$$

or

$$w = m\ c$$

Thus the water equivalent of a body is numerically equal to the product of the mass of the body and its specific heat. Note that mc is the heat capacity of the body. **Therefore, we may conclude that water equivalent and heat capacity of a body are numerically equal.**

21.19. DIFFERENT STATES OF A SUBSTANCE

Most substances can exist in solid, liquid or gaseous states. For example, chemical substance H_2O exists in the *solid state* as ice, in the *liquid state* as water and in the *gaseous state* as steam. We can change solid state into liquid state or liquid state into gaseous state by supplying heat to the substance. It is found that temperature of the substance does not change as long as the change of state takes place. The heat supplied to the substance during the change of state is used up in increasing the intermolecular distances. This increases the potential energy of the molecules but their kinetic energy remains the same. Therefore, the temperature of the substance remains constant during the change of state of the substance. The heat required to change the state of a unit mass of the substance is called **latent heat** of the substance. Note that latent means hidden *i.e.,* latent heat is hidden from thermometer.

21.20. LATENT HEAT

The amount of heat required to change the unit mass of a substance completely from one state to another at constant temperature is called **latent heat** *of the substance.*

If mass m of a substance requires heat Q to change completely from one state to another at constant temperature, then,

$$\text{Latent heat, } L = \frac{Q}{m}$$

The SI unit of latent heat of a substance is J kg^{-1}. There are two types of latent heats *viz.* latent heat of fusion and latent heat of vaporisation.

(*i*) **Latent heat of fusion.** We know that a solid changes into liquid at a constant temperature called melting point.

The amount of heat required to change the unit mass of a solid into its liquid state at constant temperature is called **latent heat of fusion** *of the solid.*

For example, latent heat of fusion of ice is 334 kJ/kg. It means that to change 1 kg of ice at 0°C into liquid water at 0°C, we must supply 334 kJ of heat.

(*ii*) **Latent heat of vaporisation.** We know that a liquid changes into gaseous state at a constant temperature called boiling point.

The amount of heat required to change the unit mass of a liquid into its gaseous state at constant temperature is called **latent heat of vaporisation** *of the liquid.*

For example, latent heat of vaporisation of water is 2257 kJ/kg. It means that to convert 1 kg of water at 100°C into steam at 100°C, 2257 kJ of heat must be supplied to it.

21.21. LAW OF HEAT EXCHANGE

When a hot body is mixed or kept in contact with a cold body, the hot body loses heat and its temperature falls. On the other hand, the cold body gains heat and its temperature rises. The final temperature of the mixture will lie between the original temperatures of the hot body and the cold body. If the system is completely isolated, no energy can flow into and out of the system. Therefore, according to the law of conservation of energy, the heat lost by one body is equal to the heat gained by the other body *i.e.,*

$$\text{Heat lost} = \text{Heat gained}$$

This is known as *law of heat exchange.*

Illustration. Suppose the hot body A having mass m_A, specific heat c_A and temperature θ_A is brought in contact with a cold body B having mass m_B, specific heat c_B and temperature θ_B. Heat will flow from body A to body B until the two bodies are at the same temperature. Let the final common temperature be θ. It is clear that the value of θ will lie between the values of θ_A and θ_B. According to the law of heat exchange,

Heat lost by body A = Heat gained by body B

or $$m_A\, c_A\, (\theta_A - \theta) = m_B\ c_B\, (\theta - \theta_B)$$

Note. Although heat lost by body A = heat gained by body B, the fall in temperature of body A may not be equal to the rise in temperature of body B.

21.22. CALORIMETRY

Calorimetry is an experimental technique for the quantitative measurement of heat exchange. To make such measurements, a *calorimeter* is used. Fig. 21.7 shows a simple water calorimeter. It consists of cylindrical vessel made of copper and provided with a stirrer and a lid. The calorimeter is well-insulated to prohibit the transfer of heat into or out of the calorimeter.

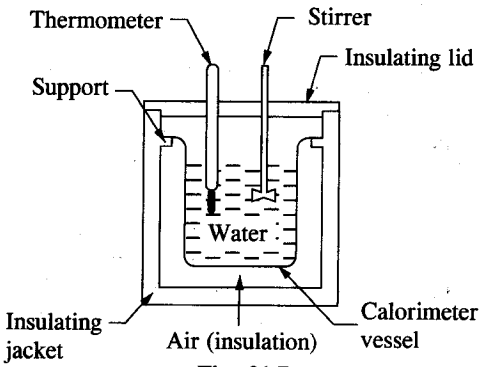


Fig. 21.7

One important use of calorimeter is to determine the specific heat of a substance. In the technique known as the "method of mixtures", a sample of the substance is heated to a high temperature which is accurately measured. The substance is then placed quickly in the calorimeter containing known mass of water at known temperature. The contents are stirred constantly until the mixture attains the final common temperature. The heat lost by the substance will be gained by the water and the calorimeter. The specific heat of the substance can by calculated as under :

Let m_1 = Mass of water in the calorimeter
w = Water equivalent of the calorimeter
θ_1 = Initial temperature of water and calorimeter
m_2 = Mass of the substance
θ_2 = Initial temperature of the substance
θ = Final temperature of the mixture
c = Specific heat of the substance
c' = Specific heat of water

Heat lost by the substance = $m_2 \times c \times (\theta_2 - \theta) = m_2\, c\, (\theta_2 - \theta)$

Heat gained by water and calorimeter = $(m_1 + w) \times c' \times (\theta - \theta_1) = (m_1 + w)\, c'\, (\theta - \theta_1)$

According to the law of heat exchange, we have,

Heat lost = Heat gained

or $$m_2\, c\ (\theta_2 - \theta) = (m_1 + w)\ c'\ (\theta - \theta_1)$$

$\therefore$ $$c = \frac{(m_1 + w)c'(\theta - \theta_1)}{m_2(\theta_2 - \theta)}$$

Since the values of m_1, m_2, θ_1, θ, θ_2, c' and w are known, the specific heat c of the substance can be calculated.

Notes. (*i*) The inner and outer surfaces of the calorimeter are polished to reduce the loss of heat due to radiation.

(*ii*) The space between the calorimeter and the insulating jacket is filled with some poor conductor (*e.g.* air, cotton) to avoid heat loss by conduction.

(*iii*) The purpose of insulating lid is to avoid heat loss by convection.

(*iv*) The insulating lid has two holes, one for the stirrer (for proper mixing of the contents of the calorimeter) and the other for the thermometer (to measure the temperature of the contents of the calorimeter).

Example 21.11. **When a piece of metal of mass 48.3 g at 10.7°C was immersed in a current of steam at 100°C, 0.762 g of steam was found to condense. Calculate the specific heat of the metal.**

Solution. Let c be the specific heat of metal in cal g^{-1} $°C^{-1}$.

Heat gained by metal in raising its temperature from 10.7°C to 100°C

$$= m\,c(\theta_2 - \theta_1) = 48.3 \times c \times (100 - 10.7) \text{ cal}$$

Heat lost by steam = Mass of steam condensed × Latent heat of steam

$$= 0.762 \times 536 \text{ cal}$$

Now according to the principle of heat exchange,

Heat gained = Heat lost

or $48.3 \times c \times (100 - 10.7) = 0.762 \times 536$

$$\therefore \quad c = \frac{0.762 \times 536}{48.3 \times (100 - 10.7)} = 0.095 \text{ cal } g^{-1} °C^{-1}$$

Example 21.12. **A 0.20 kg aluminium block at 80°C is dropped in a copper calorimeter of mass 0.05 kg containing 200 cm^3 of ethyl alcohol at 20°C. What is the final temperature of the mixture? Density of ethyl alcohol = 0.81 g cm^{-3}; specific heat of ethyl alcohol = 0.6 cal g^{-1} $°C^{-1}$; specific heat of copper = 0.094 cal g^{-1} $°C^{-1}$; specific heat of aluminium = 0.22 cal g^{-1} $°C^{-1}$.**

Solution. Let θ °C be the final temperature of the mixture. Mass of ethyl alcohol = volume × density = 200 × 0.81 = 162 g.

Heat lost by aluminium block = mass × specific heat × fall in temperature

$$= (0.20 \times 10^3) \times 0.22 \times (80 - \theta) = 44\,(80 - \theta)$$

Heat gained by ethyl alcohol and calorimeter

$$= 162 \times 0.6 \times (\theta - 20) + 0.05 \times 10^3 \times 0.094\ (\theta - 20)$$

$$= 101.9\,(\theta - 20)$$

But Heat lost = Heat gained

or $44\,(80 - \theta) = 101.9\,(\theta - 20)$

or $3520 - 44\,\theta = 101.9\,\theta - 2038$

$$\therefore \quad \theta = \frac{3520 + 2038}{101.9 + 44} = \frac{5558}{145.9} = 38.1°C$$

Example 21.13. **A geyser heats water flowing at the rate of 3.0 litres per minute from 27°C to 77°C. If the geyser operates on a gas burner, what is the rate of combustion of the fuel if its heat of combustion is 4.0 × 10^4 J g^{-1}?**

Solution. Mass of 3 litres of water = 3 kg

$\therefore$ Mass of water flowing per minute, m = 3 kg = 3000 g min^{-1}

Rise of temperature, $\Delta\theta$ = 77 − 27 = 50°C

Heat absorbed by water/minute = $m\,c\,\Delta\theta$ = 3000 × 1 × 50 cal

$$= 3000 \times 1 \times 50 \times 4.2 \text{ J min}^{-1} = 630000 \text{ J min}^{-1}$$

$\therefore$ Heat supplied by gas burner = 630000 J min^{-1}

Heat of combustion of fuel = 4.0 × 10^4 Jg^{-1}

$$\therefore \quad \text{Rate of combustion of fuel} = \frac{630000}{4.0 \times 10^4} = 15.75 \text{ g min}^{-1}$$

Example 21.14. Calculate the difference in temperature between the water at the top and bottom of a waterfall 200 m high. Specific heat of water = 4200 J kg^{-1} $°C^{-1}$.

Solution. Potential energy of water at height h is

$$Q = mgh = m \times 9.8 \times 200 = 1960 \times m \text{ J}$$

This potential energy is converted into heat. If ΔT is the rise in temperature of water, then,

$$Q = m\, c\, \Delta T$$

or
$$\Delta T = \frac{Q}{mc} = \frac{1960 \times m}{m \times 4200} = \mathbf{0.467°C}$$

Example 21.15. 5 gm of water at 30°C and 5 gm of ice at –20°C are mixed together in a calorimeter. Find the final temperature of the mixture and mass of ice melted. Water equivalent of calorimeter is negligible. Specific heat of ice = 0.5 cal/gm °C and latent heat of ice = 80 cal/gm.

Solution. Here heat is lost by water and is gained by ice. The heat available from water is *insufficient to melt the whole ice. Therefore, final temperature remains at 0°C.

Heat lost by water in cooling from 30°C to 0°C

$$= m\, c\, (\theta_2 - \theta_1) = 5 \times 1 \times (30 - 0) = 150 \text{ cal}$$

Heat required by ice to raise its temperature from –20°C to 0°C

= mass of ice × specific heat of ice × rise in temperature

$$= 5 \times 0.5 \times 20 = 50 \text{ cal}$$

Thus, we find that out of 150 cal available, 50 cal is used up for increasing the temperature of ice from –20°C to 0°C. The remaining 100 cal of heat is used up in melting some mass of ice. If the mass melted is m gm, then,

$$m \times 80 = 100 \quad \text{or} \quad m = 100/80 = \mathbf{1.25 \text{ gm}}$$

Example 21.16. A mixture of 250 g of water and 200 g of ice at 0°C is kept in a calorimeter which has a water equivalent of 50 g. If 200 g of steam at 100°C is passed through this mixture, calculate the final temperature and the mass of contents of the calorimeter.

Solution. In this case, heat is given by steam and is taken by calorimeter, ice and water. The available heat is sufficient to raise the temperature of the mixture and calorimeter to 100°C. Therefore, the final temperature remains at **100°C**.

Amount of heat required to raise the temperature of ice, calorimeter and water from 0°C to 100°C

$$= (200 \times 80) + (50 + 200 + 250)\, 100 = 66000 \text{ cal}$$

$$\text{Mass of steam condensed} = \frac{66000}{540} = 122.22 \text{g}$$

Final mass of contents of calorimeter

$$= 200 + 250 + 122.22 = \mathbf{572.22 \text{ g}}$$

PROBLEMS FOR PRACTICE

1. What is the specific heat of a metallic substance if 36 kcal of heat is needed to raise 4.5 kg of the metal from 20°C to 42°C? **[0.36 cal g^{-1} $°C^{-1}$]**
2. We wish to determine the specific heat of a new alloy. A 0.150 kg sample of the alloy is heated to 540°C. It is then quickly placed in 400 g of water at 10.0°C which is contained in a 200 g aluminium calorimeter cup. The final temperature of the mixture is 30.5°C. Calculate the specific heat of the alloy. Specific heat of aluminium = 0.22 cal g^{-1} $°C^{-1}$. **[0.12 cal g^{-1} $°C^{-1}$]**

* To melt ice from –20°C to water at 0°C, heat required

$$= 5 \times 0.5 \times 20 + 5 \times 80 = 50 + 400 = 450 \text{ cal}$$

3. If we take 15 g of water and add 60 cal of heat to it, what is the rise in temperature? **[4°C]**
4. A piece of lead gives out 1200 cal of heat when it is cooled from 90°C to 10°C. What is its water equivalent? **[15 g]**
5. Calculate the amount of heat required to heat 10 g of a solid from –10°C to 10°C ; its specific heat being 0.1 cal g^{-1} K^{-1}. **[20 cal]**
6. A platinum ball of mass 100 g was placed in a furnace. After a few minutes, it was removed and dropped into 400 g of water at 10°C. It raised the temperature of water to 20°C. How hot was the furnace? Specific heat of platinum = 0.032 cal g^{-1} $°C^{-1}$. **[1270 °C]**
7. Determine the resultant temperature when 150 g of ice is mixed with 300 g of water at 50°C. Heat of fusion of ice = 336 cal g^{-1}. **[6.67 °C]**
8. A calorimeter contains 70.2 g of water at 15.3°C. If 143.7 g of water at 36.5°C is mixed with it, the common temperature is 28.7°C. Find the water equivalent of the calorimeter. **[13.4 g]**
9. A piece of iron of mass 16 g is dropped at a temperature of 112.5°C into a cavity in a block of ice of which it melts 2.5 g. If the latent heat of fusion of ice is 80 cal g^{-1}, find the specific heat of iron. **[0.11 cal g^{-1} $°C^{-1}$]**
10. 100 g of ice at 0°C are added to 1000 g of water at 40°C. Find the resulting temperature. **[29.1°C]**

21.23. SPECIFIC HEAT OF A GAS

We defined the specific heat of a substance as the amount of heat required to raise the temperature of unit mass of the substance by 1°C (or 1 K). This definition is all right for solids and liquids. It is because when the temperature of a solid (or liquid) changes, the variation in pressure and volume are small and are normally neglected. However, the situation with gases is quite different. For example, when a gas is heated, there may be large changes in pressure and volume. Therefore, in order to define the specific heat of a gas, it is necessary to specify the particular conditions of pressure and volume. *So a gas can have any value of specific heat depending upon how heating is carried out.* This fact is illustrated below.

Consider a gas of mass m enclosed in a cylinder fitted with an air-tight and frictionless piston. If Q is the heat supplied and ΔT is the change in temperature, then specific heat of gas,

$$c = \frac{Q}{m\Delta T}$$

We now show that the value of c depends upon the particular condition of pressure and volume.

(*i*) Suppose the gas is compressed *suddenly*. No heat is supplied from outside (*i.e.*, $Q = 0$). However, the temperature of the gas increases due to compression.

$$\therefore \quad c = \frac{Q}{m\Delta T} = 0$$

(*ii*) If the gas is heated in such a way that its temperature is constant (*i.e.*, $\Delta T = 0$), then,

$$c = \frac{Q}{m\Delta T} = \frac{Q}{m \times 0} = \infty$$

Similarly, there can be many conditions of heating and the value of c will be different depending upon the condition of heating.

21.24. TWO PRINCIPAL SPECIFIC HEATS OF A GAS

Out of many values of specific heat of a gas, the following two are of practical importance :

(*i*) Specific heat of a gas at constant volume (c_v)

(*ii*) Specific heat of a gas at constant pressure (c_p)

(*i*) **Specific heat of a gas at constant volume.** *It is defined as the amount of heat required to raise the temperature of 1 g of the gas through 1°C at constant volume.* It is denoted by c_v and is measured in cal g^{-1} K^{-1} or J kg^{-1} K^{-1}.

In the above case, if we consider 1 mole of the gas instead of 1 g, then it is called molar specific heat of the gas at constant volume.

Thus **molar specific heat of a gas at constant volume** *is defined as the amount of heat required to raise the temperature of 1 mole of the gas through 1°C at constant volume.* It is denoted by C_v. As shown earlier,

$$C_v = M\, c_v \qquad \text{where } M = \text{Molecular mass of the gas}$$

The C_v is measured in cal mol^{-1} K^{-1} or J mol^{-1} K^{-1}.

(*ii*) **Specific heat of a gas at constant pressure.** *It is defined as the amount of heat required to raise the temperature of 1 g of the gas through 1°C at constant pressure.* It is denoted by c_p and is measured in cal g^{-1} K^{-1} or J kg^{-1} K^{-1}.

Again if we consider 1 mole of the gas instead of 1 g, then it is called molar specific heat of the gas at constant pressure.

Hence **molar specific heat of a gas at constant pressure** *is defined as the amount of heat required to raise the temperature of 1 mole of the gas through 1°C at constant pressure.* It is denoted by C_p. As shown earlier,

$$C_p = M\, c_p \qquad \text{where } M = \text{Molecular mass of the gas}$$

The C_p is measured in cal mol^{-1} K^{-1} or J mol^{-1} K^{-1}.

Note. The following relations may be noted :

(*i*) The ratio C_p/C_v of a gas is represented by γ *i.e.*,

$$\gamma = \frac{C_p}{C_v} = \frac{M\, c_p}{M\, c_v} = \frac{c_p}{c_v}$$

The value of γ depends upon the atomicity of the gas.

(*ii*) $C_p - C_v = \dfrac{R}{J}$ = Gas constant for 1 gram mole of the gas

(*iii*) $c_p - c_v = \dfrac{r}{J}$ where $r = \dfrac{R}{M}$ = Gas constant for 1 g of gas and M is molecular mass of the gas

(*iv*) For monatomic gas : $C_v = \dfrac{3}{2}R$; $C_p = \dfrac{5}{2}R$

For diatomic gas : $C_v = \dfrac{5}{2}R$; $C_p = \dfrac{7}{2}R$

Note that R = 8.31 J mol^{-1} K^{-1} = 1.98 cal mol^{-1} K^{-1} and J = 4.18 J/cal.

21.25. C_p IS GREATER THAN C_v

We now make use of kinetic theory to show that C_p is greater than C_v. Suppose an ideal gas is heated through ΔT first at constant volume and then at constant pressure.

(*i*) When the *gas is heated at constant volume,* no work is done ($\because$ Change in volume, $\Delta V = 0$) and all the heat goes into raising the internal energy and thus the temperature of the gas.

(*ii*) When the *gas is heated at constant pressure,* it expands and does work so that only a part of heat is used up in increasing the internal energy and hence temperature of the gas. Therefore, in a constant-pressure process, more heat is needed to achieve a given temperature change than that of constant-volume process.

From the above discussion, it follows that the amount of heat required to raise the temperature of a gas at constant-pressure is greater than that required to raise its temperature by the same amount at constant volume. In other words, C_p is greater than C_v.

Example 21.17. How much heat is required to raise the temperature of an ideal monoatomic gas by 10 K if the gas is maintained at constant pressure?

Solution. The heat required is given by ;

$$Q = n\, C_p\ \Delta T$$

Here $n = 1; \Delta T = 10$ K ; $C_p = \frac{5}{2} R = \frac{5}{2} \times 8.3$ J mol^{-1} K^{-1}

$$\therefore \qquad Q = 1 \times \frac{5}{2} \times 8.3 \times 10 = \mathbf{207.5\ J}$$

Example 21.18. Assuming that the density of air at N.T.P. = 0.0013 g/cc and c_p = 0.239 cal g^{-1} K^{-1} and the ratio c_p / c_v = 1.40, calculate the mechanical equivalent of heat.

Solution. $c_p = 0.239$ cal g^{-1} K^{-1} ; $c_p / c_v = 1.40$; $\therefore\ c_v = \frac{c_p}{1.40} = \frac{0.239}{1.40} = 0.171 \text{cal g}^{-1}\text{K}^{-1}$

Volume of 1 g of air at N.T.P. $= \frac{1}{0.0013}\text{cc} = \frac{10^{-6}}{0.0013}\text{m}^3$

Volume of 1 kg (= 1000 g) of air at N.T.P., $V = \frac{10^{-6}}{0.0013} \times 1000 = \frac{10^{-3}}{0.0013}\text{m}^3$

Normal pressure, $P = h\rho g = 0.76 \times 13600 \times 9.8 = 101292.8$ Nm^{-2}

Normal temperature, $T = 273$ K

Gas constant r for 1 kg of air is given by ;

$$r = \frac{PV}{T} = 101292.8 \times \frac{10^{-3}}{0.0013} \times \frac{1}{273} = 285.4 \text{ J kg}^{-1} \text{ K}^{-1}$$

Note that $r = c_p - c_v$. The above value of r means that if 1 kg of air is heated through 1°C (or 1K) first at constant pressure and then at constant volume, then extra heat is needed for constant-pressure process to do a work of 285.4 J *i.e.,* W = 285.4 J.

Heat supplied to do this work is

$$\begin{aligned} Q &= 1000 \text{ g} \times c_p \times 1\text{K} - 1000 \text{ g} \times c_v \times 1\text{K} \\ &= 1000 \text{ g} \times 0.239 \times 1\text{K} - 1000 \text{ g} \times 0.171 \times 1\text{K} \\ &= 239 - 171 = 68 \text{ cal} \end{aligned}$$

Now $W = J\,Q$

$$\therefore \qquad J = \frac{W}{Q} = \frac{285.4}{68} = \mathbf{4.2\ J/cal}$$

Example 21.19. Calculate the amount of heat necessary to raise the temperature of 2 moles of He gas from 20°C to 50°C using (*i*) constant-volume process and (*ii*) constant-pressure process. For He, c_v = 1.5 R and c_p = 2.49 R.

Solution. (*i*) The amount of heat required for constant-volume process is

$$Q_v = n\, c_v\ \Delta T$$

Here $n = 2$ moles ; $c_v = 1.5\ R = 1.5 \times 8.314$ J mol^{-1} °C^{-1}; $\Delta T = 50 - 20 = 30$°C

$$\therefore \qquad Q_v = 2 \times (1.5 \times 8.314) \times 30 = \mathbf{748\ J}$$

(*ii*) The amount of heat required for constant-pressure process is

$$Q_p = n\, c_p \Delta T$$

Here n = 2 moles ; $c_p = 2.49\, R = 2.49 \times 8.314$ J mol^{-1} °C^{-1} ; $\Delta T = 30$°C

$\therefore$ $Q_p = 2 \times (2.49 \times 8.314) \times 30 =$ **1242 J**

Since the temperature rise is the same in the two cases, the change in internal energy is the same *i.e.,* 748 J. However, in constant-pressure process excess heat supplied = 1242 – 748 = 494 J. This extra heat of 494 J went into the work of expansion of the gas.

Example 21.20. **Specific heat of argon at constant pressure is 0.125 cal g^{-1} K^{-1} and at constant volume is 0.075 cal g^{-1} K^{-1}. Calculate the density of argon at N.T.P. Given that J = 4.2 J/cal.**

Solution. $c_p = 0.125$ cal g^{-1} K^{-1} = $0.125 \times 4.2 \times 1000$ J kg^{-1} K^{-1} = 525 J kg^{-1} K^{-1}

$c_v = 0.075$ cal g^{-1} K^{-1} = $0.075 \times 4.2 \times 1000$ J kg^{-1} K^{-1} = 315 J kg^{-1} K^{-1}

The gas constant r for 1 kg of gas is given by ;

$$r = c_p - c_v = 525 - 315 = 210 \text{ J kg}^{-1} \text{ K}^{-1}$$

Normal pressure, $P = h\rho g = 0.76 \times 13600 \times 9.8 = 101292.8$ Nm^{-2}

Normal temperature, $T = 273$ K

Suppose V is the volume of argon in m^3 at N.T.P.

Now $$r = \frac{PV}{T} \quad \therefore \quad V = \frac{rT}{P} = \frac{210 \times 273}{101292.8} = 0.566 \text{ m}^3$$

$\therefore$ Density of argon, $\rho = \dfrac{\text{Mass}}{\text{Volume}} = \dfrac{1 \text{ kg}}{0.556 \text{ m}^3}$ = **1.8 kg m^{-3}**

Examples 21.21. **Calculate the difference between the two principal specific heats of 1 g of helium at S.T.P. Given molecular weight = 4 ; J = 4.186 J cal^{-1} and R = 8.31 J mol^{-1}K^{-1}.**

Solution. Molecular weight, $M = 4$

Now $$c_p - c_v = \frac{r}{J} = \frac{R}{MJ} = \frac{8.31}{4 \times 4.186} = 0.496 \text{ cal g}^{-1} \text{ K}^{-1}$$

Example 21.22. **Two cylinders A and B, fitted with pistons, contain equal amounts of an ideal diatomic gas at 300 K. The piston of A is free to move while that of B is fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in A is 30 K, find the rise in temperature of the gas in B.**

Solution. Let each cylinder contain n moles of an ideal gas and Q be the heat supplied to each cylinder.

Heat supplied to cylinder *A*. Since the piston of the cylinder A is free to move, gas is heated at constant pressure.

$\therefore$ $Q = nC_p\, \Delta T$. For a diatomic gas, $C_p = \dfrac{7}{2} R$

$\therefore$ $$Q = n \times \frac{7}{2} R \times 30 \qquad \ldots(i)$$

Heat supplied to cylinder *B*. Since the piston of cylinder B is fixed, the gas is heated at constant volume.

$\therefore$ $Q = nC_v \Delta T'$. For a diatomic gas, $C_v = \dfrac{5}{2} R$

$\therefore$ $$Q = n \times \frac{5}{2} R \times \Delta T' \qquad \ldots(ii)$$

From eqs. (*i*) and (*ii*), we have,

$$n \times \frac{5}{2} R \times \Delta T' = n \times \frac{7}{2} R \times 30$$

$\therefore$ $$\Delta T' = 42\text{K}$$

PROBLEMS FOR PRACTICE

1. Calculate the amount of heat necessary to raise the temperature of 2 moles of CO_2 from 20°C to 50°C using (*i*) a constant-volume process and (*ii*) constant-pressure process. For CO_2, $C_v = 3.37\ R$ and $C_p = 4.37\ R$. **[(*i*) 1680 J (*ii*) 2180 J]**
2. What amount of heat must be supplied to 2.0×10^{-2} kg of nitrogen at room temperature to raise its temperature by 45°C at constant pressure? Given molecular mass of $N_2 = 28$; $R = 8.3$ J mol^{-1} K^{-1} and $C_p = 7R/2$.. **[933.75 J]**
3. Calculate the ratio of specific heats for nitrogen, given that specific heat at constant pressure is 0.235 cal g^{-1} K^{-1} and density at N.T.P is 1.234 g/litre. **[1.4]**
4. A cylinder of fixed capacity 67.2 litres contains helium gas at standard pressure and temperature. What is the amount of heat needed to raise the temperature of the gas by 20°C ? $R = 8.31$ J mol^{-1} K^{-1}. **[747.9 J]**
5. The specific heat of nitrogen at constant pressure is 0.248 cal g^{-1} K^{-1} and its density at N.T.P. is 1.25×10^{-3} g cm^{-3}. Taking the density of mercury as 13.6 g cm^{-3} and $J = 4.2$ J/cal, calculate the specific heat of nitrogen at constant volume. **[777 J kg^{-1} K^{-1}]**
6. If 820 J are supplied to an ideal monoatomic gas, by how much will the temperature change if the process is carried out (*i*) at constant volume (*ii*) at constant pressure? **[(*i*) 26° C (*ii*) 16°C]**
7. A cylinder contains 3 moles of helium gas at a temperature of 300 K. How much heat must be transferred to the gas to increase its temperature to 500 K if the gas is heated at constant volume? How much heat must be transferred to the gas at constant pressure to raise the temperature to 500 K? Specific heat of helium at constant volume = 2.98 cal mol^{-1} K^{-1} and specific heat of helium at constant pressure = 4.97 cal mol^{-1} K^{-1}. **[1.79×10^3 cal ; 2.98×10^3 cal]**
8. For air, specific heat at constant pressure is 0.273 cal g^{-1} $°C^{-1}$ and specific heat at constant volume is 0.169 cal g^{-1} $°C^{-1}$, density of air at N.T.P. = 0.001293 g cm^{-3}. Calculate the value of *J*. **[4.219 J/cal]**
9. Calculate the ratio of specific heats for nitrogen given that specific heat at constant pressure is 0.236 cal g^{-1} K^{-1} and density at N.T.P. is 0.001234 g cm^{-3}. **[1.435]**
10. Calculate the difference between two specific heats of 1 g of nitrogen. Given that molecular mass of nitrogen = 28 and $J = 4.2$ J/cal. **[4.885 cal mol^{-1} K^{-1}]**

21.26. *PV* DIAGRAM OF A REAL SUBSTANCE

A graph between pressure P and volume V of a substance at constant temperature is called **isothermal**.

Fig. 21.8 shows some isothermals for a real gas (non-ideal gas). At sufficiently high temperature say T_4, the behaviour of the gas is approximately that of an ideal gas *i.e.*, at a given temperature, $P \propto 1/V$. As we lower the gas temperature, its behaviour deviates from the ideal gas model. Eventually, we reach a **critical temperature** T_c, below which the isothermals develop flat portions *i.e.*, below T_c, *P–V* curve becomes flat over a finite range of volumes. Visual inspection of the substance shows that in the flat portion of the isothermal from *B* to *A*, the substance is condensing from vapour to liquid. As compression occurs, more and more substance goes from vapour to liquid, but the pressure does not change. During this process, heat must be removed in order to keep the temperature constant ; this is the heat of fusion.

Thus when a real (non-ideal) gas is compressed at a constant temperature T_2 ($< T_c$), it remains in the gaseous phase until point *B* in Fig. 21.8 is reached. At point *B*, it begins to liquefy and with decrease in volume, more and more substance liquefies, with both pressure and temperature remaining

constant. At point *A*, all the substance is in liquid phase. Further increase in pressure reduces the volume very slightly — liquids are nearly incompressible — so the curve is very steep as shown.

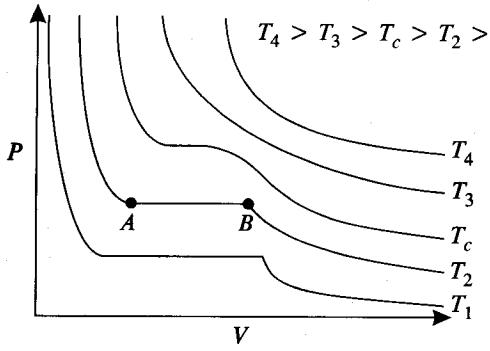


Fig. 21.8

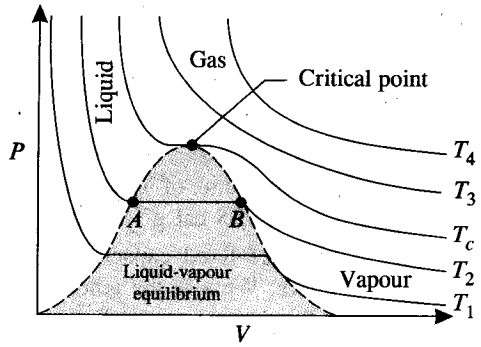


Fig. 21.9

Fig. 21.9 again shows the isothermals with the boundaries between the liquid and gas phases clearly marked. At temperature less than the critical temperature (T_c), a gas will change to the liquid phase if sufficient pressure is applied. Above the critical temperature, no amount of pressure can cause a gas to change phase and become a liquid. What happens instead is that the gas becomes denser and denser as the pressure is increased and gradually it acquires properties resembling a liquid but no liquid surface forms. Scientists tried for many years to liquefy oxygen by pressure alone but without success. It was not until the discovery of the behaviour of substances associated with critical point that it was realised that a gas can be liquefied only if first cooled below its critical temperature. The critical temperature of oxygen is –118°C. So oxygen must be cooled below –118°C to liquefy it by pressure. Air has a critical temperature of – 190°C and nitrogen a critical temperature of –147°C.

The shaded area represents the region where the vapour and liquid phases exist together in equilibrium. As we shall see in the chapter of thermodynamics, the *area* under a *P-V* curve, whether or not it is an isothermal, represents the *work* done by or on the system during a volume change. This work in turn is directly related to heat transfer and changes in the *internal energy* of the system.

Often a distinction is made between the terms "gas" and "vapour". A substance below its critical temperature in the gaseous state is called a *vapour* ; when above the critical temperature, it is called a gas. This is indicated in Fig. 21.9.

Discussion. The following points may be noted :

(*i*) The pressure corresponding to critical temperature is called *critical pressure* (P_c).

(*ii*) Pressure has to be applied to liquefy a gas but it need not be highter than critical pressure.

(*iii*) Temperature lower than T_c (critical temperature) will need reduced pressure ($< P_c$) to be applied to the gas to liquefy it.

Note. The *PV* diagram of Fig. 21.9 is still incomplete for it does not account for the third phase of matter—the solid phase. Modifying our *P-V* diagram to include the solid phase would make it extremely complicated. Instead of trying to plot *P-V* curves for each different temperature, we resort to a three-dimensional diagram with pressure, volume and temperature along the three axes.

21.27. PHASE DIAGRAM

The behaviour of a substance can be diagrammed not only on a *PV* diagram but also on a *PT* diagram. A *PT* diagram, often called a *phase diagram,* is particularly convenient for comparing the different phases of a substance. Fig. 21.10 shows the phase diagram of water.

(*i*) **Vaporisation curve *l*-*v*.** The curve labelled *l*- *v* represents those points where the liquid and vapour phases are in equilibrium. Thus it is a graph of boiling point versus pressure. Note that the curve correctly shows that at a pressure of 1 atm, the boiling point is 100°C and that the boiling point is lowered for a decreased pressure.

(*ii*) **Fusion curve *l*-s.** The curve *l*-*s* represents the points where the solid and liquid phases exist in equilibrium. Thus it is a graph of the freezing point versus pressure. At 1 atm pressure, the freezing point of water is 0°C as shown. Notice also in Fig. 21.10 that at a pressure of 1 atm, the substance is in the liquid phase if the temperature is between 0°C and 100°C but is in the solid or vapour phase if the temperature is below 0°C or above 100°C.

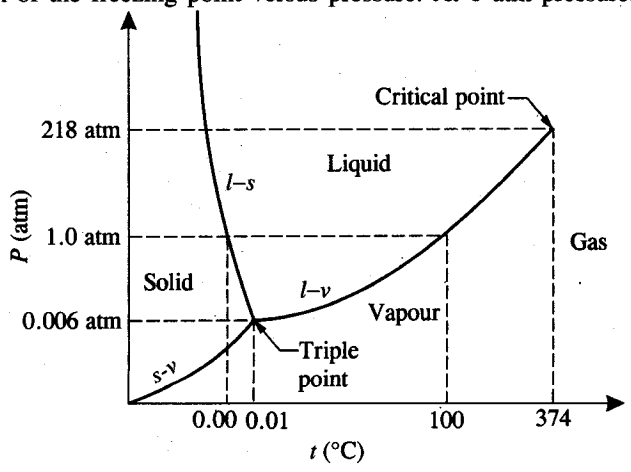


Fig. 21.10

Note that *l*–*s* curve for water slopes upward to the left *i.e.*, fusion curve of water has a slightly negative slope. This is true only of substances that expand upon freezing ; for at a higher pressure, a lower temperature is needed to cause the liquid to freeze. However, most materials contract on freezing. The fusion curve of such materials slopes upward to the right *i.e.*, fusion curve has a positive slope.

(*iii*) **Sublimation curve *s*-*v*.** The curve labelled *s*-*v* is the *sublimation point* versus pressure curve. *Sublimation* refers to the process whereby at low pressures (in case of water at pressure less than 0.0060 atm) a solid changes directly into the vapour phase without passing through the liquid phase. For example, carbon dioxide which in the solid state is called dry ice sublimates even at atmospheric pressure.

(*iv*) **Triple point.** *The intersection of the three curves is the* **triple point**. The triple point represents a unique temperature and pressure and it is only at this point that the three phases can exist together in equilibrium. For water, the triple point occurs at 0.01°C and 4.58 mm of Hg. We can see from Fig. 21.10 that in order for sublimation to occur, the pressure on the substance must be below the triple-point pressure of the substance. For carbon dioxide, the triple point values are –56.6°C and 5.11 atm. We observe the sublimation of carbon dioxide at ordinary pressures since 1 atm is well below the triple-point pressure for CO_2. Because the triple point corresponds to a unique value of temperature and pressure, it is precisely reproducible and is often used as a point of reference. We have already seen that the triple point of water is used in establishing the absolute temperature scale.

21.28. GAS AND VAPOUR

The terms gas and vapour are sometimes used quite indiscriminately. Therefore, it is important to understand the difference between the two. A gas cannot be liquefied by pressure alone, no matter how great the pressure. Scientists tried for many years to liquefy oxygen by pressure alone but without success. In order to liquefy a gas, it must be cooled to a certain temperature. This temperature is called **critical temperature**.

The temperature above which a gas cannot be liquefied, no matter how great the pressure is called the **critical temperature.**

The critical temperature of nitrogen is –147°C (126 K). Therefore, nitrogen must be cooled below –147°C to liquefy it by pressure. Air has a critical temperature of –190°C (83 K) and oxygen has a critical temperature of –118°C (155 K). We can now define gas and vapour.

A substance which is in the gaseous phase and is above its critical temperature is called a **gas.**

The critical temperature of carbon dioxide is 31.1°C. Therefore, carbon dioxide is a gas above 31.1°C.

A substance which is in the gaseous phase and is below its critical temperature is called a **vapour.**

The critical temperature of carbon dioxide is 31.1°C. Therefore, at about 31.1°C, the carbon dioxide exists as a gas, no matter how great the pressure. However, below 31.1°C, carbon dioxide can exist as a vapour or liquid or as a mixture of vapour and liquid. Thus the difference between a vapour and a gas boils down to this :

A *vapour can be liquefied simply by increasing the pressure, a gas cannot.*

The reader may note that difference between a vapour and gas is not really too much. However, this little distinction must be kept in mind. It is clear that vapour must also exert pressure, like a gas.

CONCEPTUAL QUESTIONS

Q.1. What is the difference between heat and temperature?

Ans. Temperature is a measure of the motion of the molecules or atoms within a substance ; more specifically, it is a measure of the average kinetic energy of the molecules or atoms in a substance. Heat is the flow of energy from one body to another as a result of temperature difference. It is important to point out that matter does not contain heat ; it contains molecular kinetic energy and *not heat.* Heat flows and it is the energy that is being transferred. Once heat has been transferred to an object, it ceases to be heat. It becomes internal energy.

Q.2. What do you mean by internal energy of a substance?

Ans. Internal energy is the grand total of all the energies inside a substance. This includes potential energy due to the forces between molecules or atoms and kinetic energy due to movements of molecules and atoms. So a substance does not contain heat—it contains internal energy.

Q.3. A thermometer registers its own-temperature. Discuss.

Ans. Interestingly enough, a thermometer registers its own temperature. When a thermometer is in thermal contact with an object, the temperature of which we wish to measure, energy will flow between the two until their temperatures are equal and thermal equilibrium is established. If we then know the temperature of the thermometer, we also know the temperature of the object.

Q.4. Two thermometers are constructed in the same way except that one has a spherical bulb and the other an elongated cylindrical bulb. Which one will respond quickly to temperature changes?

Ans. The thermometer with cylindrical bulb will respond quickly to temperature changes. It is because the surface area of the cylindrical bulb is greater than that of the spherical bulb.

Q.5. What is absolute zero?

Ans. In principle, there is no upper limit of temperature. In contrast, there is a definite lower limit of the temperature. The lowest possible temperature attainable by an object is –273.15°C. This temperature is called absolute zero.

Q.6. A hole of 1 cm in diameter is drilled through a piece of steel at 20°C. What happens to the diameter of the hole as the steel is heated to 100°C?

Ans. As the steel is heated to 100°C, all the dimensions increase, including the diameter of the hole. A useful analogy to thermal expansion is that of photographic enlargement. Every linear dimension of the object undergoes the same fractional change. Thus increase in temperature will expand the hole, not reduce it.

Q.7. How does the coefficient of linear expansion of a substance per degree compare with its coefficient of linear expansion per degree Fahrenheit?

Ans. Coefficient of expansion per Fahrenheit degree is 5/9 that per celsius degree.

Q.8. Why do solids expand when their temperature is raised?

Ans. When solids are heated, the increased energy increases the amplitude of the molecular vibrations. As a result, intermolecular spacing increases. For this reason, solids expand on heating.

Q.9. How would a thermometer be different if glass expanded more with increasing temperature than mercury?

Ans. The scale would have to be upside down.

Q.10. Suppose you apply a flame to 1 litre of water and its temperature rises by 2°C. If you apply the same flame to 2 litres of water, by how much will its temperature rise?

Ans. Its temperature will rise by only 1°C because there are twice as many molecules in 2 litres of water and each molecule receives only half as much energy on the average. So the average kinetic energy, and thus the temperature, increases by half as much.

Q.11. Why does heat flow from a body at higher temperature to a body at lower temperature?

Ans. A fast moving marble slows when it hits slower-moving marbles. It gives up some of its kinetic energy to the slower ones. The same situation is with the flow of heat. Molecules with more kinetic energy that are in contact with less energetic molecules give up some of their kinetic energy to the less energetic ones. Therefore, the direction of flow of heat is from the hotter to the colder object. However, for both marbles and molecules, the total energy before and after contact is the same.

Q.12. Which has a higher specific heat, water or sand?

Ans. Water has higher specific heat. The temperature of water increases less ($\Delta T = Q/mc$) than the temperature of the sand in the same sunlight. Sand's low specific heat, as evidenced by how quickly the surface warms in the morning sun and how quickly it cools at night, affects local climates.

Q.13. The specific heat of water is more than that of iron. What do you mean by it?

Ans. 1 g of water requires 1 cal of energy to raise its temperature by 1°C. It takes only about one-eighth as much energy to raise the temperature of 1 g of iron by 1°C. Therefore, the specific heat of water is about 8 times that of the iron. In other words, water absorbs more heat than iron for the same increase in temperaure.

Q.14. Why do different materials possess efferent specific heats?

Ans. Different materials absorb energy in different ways. The energy may increase the to-and-fro vibrational motion of molecules which raises the temperature. Or it may increase the amount of internal vibration or rotation within the molecules and go into potential energy, which does not raise the temperature. Generally, there is a combination of both. For this reason, different materials have different specific heat.

Q.15. Why is water a very useful cooling agent?

Ans. Water has a high value of specific heat. It means a relatively small amount of water will absorb a large amount of heat for a correspondingly ($Q = c\,m\,\Delta T$) small temperature rise.

Because of this, water is a very useful cooling agent and is used in the cooling system of automobiles and other engines. If a liquid of lower specific heat were used in cooling systems, its temperature would rise higher for a comparable absorption of heat.

Q.16. Why is specific heat of a gas at constant pressure greater than its specific heat at constant volume?

Ans. When a gas is heated at constant volume, no work is done so that all the heat goes into increasing the internal energy and hence the temperature of the gas. However, when the gas is heated at constant pressure, it expands and does work so that only a part of the heat is used to increase the internal energy and temperature. Therefore, it takes more heat at a constant pressure than at constant volume for a given increase in temperature. In other words, c_p is greater than c_v.

Q.17. How do you justify that when a body is being heated at melting point, the temperature remains constant?

Ans. When a body is being heated below the melting point, the heat supplied increases the potential as well as the kinetic energy of the molecules. Due to the increase in the kinetic energy of the molecules, the temperature increases. But at melting point, heat goes to increase only the potential energy of the molecules.

Q.18. Can a gas be liquefied at any temperature by increasing pressure alone?

Ans. A gas can be liquefied by pressure alone if it is first cooled below its critical temperature. The critical temperature of nitrogen is –147°C. So nitrogen must be cooled below –147°C to liquefy it by pressure.

Q.19. How does the formation of snow or rain affect the temperature of air?

Ans. The formation of snow or rain from water vapour is accompanied by an increase in atmospheric temperature. A gas (in this case water vapour) releases energy in transforming to the liquid and/or solid phase. So it is always a little warmer when it rains or snows than it would be otherwise.

Q.20. What conclusion do you draw from Dulong and Petit's law?

Ans. According to Dulong and Petit's law, the molar specific heats (the heat required to raise the temperature of one mole of a substance by 1 K) of solids are approximately the same and equal to about 25 J mol^{-1} K^{-1}. It follows that the heat required to raise the temperature of a sample of solid depends on *how many* molecules the sample contains and not on the mass of an individual molecule.

Q.21. What is the specific heat of air?

Ans. The specific heat of air is 993 J kg^{-1} K^{-1}. Therefore, it requires a large amount of heat to raise the temperature of 1 kg of air by 1°C. Remember that at normal atmospheric pressure, 1 kg of air has a volume of about 1 m^3.

Q.22. Why is latent heat of vapourisation of a material greater than that of latent heat of fusion?

Ans. When a liquid changes into a gas, there is a large increase in the volume. Therefore, a large amount of work has to be done against the surrounding atmosphere.

VERY SHORT ANSWER QUESTIONS

Q.1. What is the cause of hotness of a body?

Ans. The cause of hotness of a body is the kinetic energy of the molecules constituting the body.

Q.2. What is the SI unit of heat and how is it related to calorie?

Ans. The SI unit of heat is joule (J) and 1 calorie = 4.18 J.

Q.3. What is the basic principle of a thermometer?

Ans. The change of some physical property of a substance with temperature forms the principle of a thermometer.

Q.4. What is the normal temperature of a human body?

Ans. The normal temperature of a human body is 98.4°F or 37°C.

Q.5. What is the value of specific heat of water in SI units? Does it vary with temperature?

Ans. The specific heat of water in SI units is 4180 J kg^{-1} K^{-1}. It varies to a small extent with temperature.

Q.6. What physical changes may occur when a body is heated?

Ans. When a body is heated, the physical changes that may occur are : expansion, change of state, change of electrical properties (resistance, conductivity) etc.

Q.7. Do water and ice have the same specific heats?

Ans. No. For water, c = 1 cal g^{-1} $°C^{-1}$ and for ice, c = 0.5 cal g^{-1} $°C^{-1}$.

Q.8. Can a substance contract on heating?

Ans. Yes. For example, ice contracts on heating.

Q.9. Is the specific heat of water greater than that of sand?

Ans. Yes. As a matter of fact, the specific heat of water is maximum.

Q.10. What is the specific heat of a gas in an isothermal process?

Ans. Infinite. It is because specific heat, $c = \frac{\Delta Q}{m\,\Delta T}$ and for a isothermal process, $\Delta T = 0$.

Q.11. What is the specific heat of a gas in an adiabatic process?

Ans. Zero. It is because specific heat, $c = \frac{\Delta Q}{m\,\Delta T}$ and for an adiabatic process, $\Delta Q = 0$.

Q.12. At what temperature, does the readings of Celsius and Fahrenheit scales coincide?

Ans. $-40°$.

Q.13. Give the values of triple point pressure and triple point temperature of water.

Ans. The triple point pressure of water is 0.46 cm of mercury column and triple point temperature of water is 273.16 K or 0.01°C.

Q.14. What is the principle of calorimetry?

Ans. When a hot body is mixed or kept in contact with a cold body, Heat lost = Heat gained.

Q.15. Tea gets cooled when sugar is added to it. Why?

Ans. Tea loses heat to sugar and the resultant temperature of the mixture is lowered.

SHORT ANSWER QUESTIONS

Q.1. Is J a physical constant or a conversion factor?

Ans. Joule's mechanical equivalent of heat (J) is a conversion factor and not a physical constant. Now 1 calorie = 4.18 joule. This means that 1 calorie of heat is equivalent to 4.18 J of work. Note that both calorie and joule are the units of work (heat).

Q.2. What do you mean by (*i*) specific heat of water is 4180 J kg^{-1} $°C^{-1}$ (*ii*) heat capacity of a body is 100 J $°C^{-1}$?

Ans. (*i*) It means that in order to raise the temperature of 1 kg of water through 1°C, the amount of heat required is 4180 J.

(*ii*) It means that in order to raise the temperature of the body by 1°C, the amount of heat required is 100 J.

Q.3. The water equivalent of a body is 10 kg. What does it mean?

Ans. It means that if a body is heated through say 5°C, it will absorb the same amount of heat as absorbed by 10 kg of water when heated through 5°C.

Q.4. How are thermal capacity and water equivalent related?

Ans. The thermal capacity of a body in joules is equal to its water equivalent in kg. For example, if the thermal capacity of a body is 100 J/°C, then its water equivalent is 100 kg.

Q.5. A cubical block is heated from 0°C to 100°C. The percentage increase in length is 0.2%. What is the percentage increase in its volume?

Ans. $V = L^3 \quad \therefore \quad \frac{\Delta V}{V} \times 100 = \frac{3\Delta L}{L} \times 100 = 3 \times 0.2\% = 0.6\%$

Q.6. A sheet of brass is 50 cm long and 10 cm broad at 0°C. The surface area increases by 1.9 cm^2 at 100°C. What is the coefficient of linear expansion of brass?

Ans. $\Delta S = \beta\, S\, \Delta T \quad \therefore \quad \beta = \frac{\Delta S}{S\,\Delta T} = \frac{1.9}{(50\times10)\times(100-0)} = 3.8 \times 10^{-5}\ °C^{-1}$

$\therefore$ Co-efficient of linear expansion, $\alpha = \frac{\beta}{2} = \frac{3.8\times10^{-5}}{2} = 1.9 \times 10^{-5}\ °C^{-1}$

Q.7. The volume of a block of metal changes by 0.12% when it is heated through 20°C. What is the coefficient of linear expansion of the metal?

Ans. $\frac{\Delta V}{V} = 0.12\% = \frac{0.12}{100}$; $\Delta T = 20°C$

Now $\gamma = \frac{\Delta V}{V\,\Delta T} = \frac{0.12}{100\times20} = 6.0 \times 10^{-5}\ °C^{-1}$

$\therefore \quad \alpha = \frac{\gamma}{3} = \frac{6.0\times10^{-5}}{3} = 2.0 \times 10^{-5}\ °C^{-1}$

Q.8. 540 g of ice at 0°C is mixed with 540 g of water at 80°C. What is the final temperature of the mixture?

Ans. Heat required to melt 540 g of ice at 0°C = 540 × 80 cals. Heat lost by 540 g of water in cooling from 80°C to 0°C = 540 × 80 cals. Since both are equal, the final temperature is 0°C.

Q.9. 70 cals of heat is required to raise the temperature of 2 moles of an ideal gas at constant pressure from 30°C to 35°C. What is the amount of heat required to raise the temperature of the same gas through the same temperature at constant volume?

Ans. $Q = n\, C_p\, \Delta T \quad \text{or} \quad C_p = \frac{Q}{n\,\Delta T} = \frac{70}{2\times(35-30)} = 7\ \text{cals mol}^{-1}\ °C^{-1}$

$\therefore \quad C_v = C_P - R = 7 - 2 = 5\ \text{cals mol}^{-1}\ °C^{-1}$

$\therefore \quad Q = n\, C_v\, \Delta T = 2 \times 5 \times (35 - 30) = 50\ \text{cals}$

Q.10. Water falls from a height of 500 m. What is the rise in temperature of water at the bottom of the whole energy is used up in heating water?

Ans. $mc\ \Delta T = \frac{mgh}{J} \quad \therefore \Delta T = \frac{gh}{Jc} = \frac{980\times(500\times10^2)}{4.2\times10^7\times1} = 1.16°C$

LONG ANSWER QUESTIONS

1. Explain co-efficients of linear expansion, superficial expansion and cubical expansion. **[Refer to Art. 21.7]**
2. Derive the relationship between three coefficients of thermal expansion. **[Refer to Art. 21.8]**
3. Show that on increasing temperature by ΔT, the density (ρ') of a liquid is given by : $\rho' = \rho(1 - \gamma\Delta T)$ where ρ is the initial density of the liquid and γ is the coefficient of cubical expansion. **[Refer to Art. 21.13]**
4. Explain the terms (*i*) specific heat (*ii*) molar specific heat (*iii*) water equivalent. **[Refer to Arts. 21.14, 21.15 and 21.18]**
5. State and explain Dulong and Petit law. **[Refer to Art. 21.16]**
6. State and explain principle of calorimetry. **[Refer to Art. 21.21]**
7. How will you find the specific heat of a substance? **[Refer to Art. 21.22]**
8. Explain the terms (*i*) absolute zero (*ii*) triple point (*iii*) thermal capacity (*iv*) thermal stress. **[Refer to Arts. 21.5, 21.6, 21.18 and 21.10]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. There is no upper limit of temperature.
2. There is a definite limit of lower temperature. The lowest temperature attainable by an object is –273.15°C.
3. The lowest possible temperature (–273.15°C) is called *absolute zero* of temperature and is written as 0 K.
4. A temperature scale that has its zero at –273.15°C and the temperature intervals are the same as on the Celsius scale is called **Kelvin** or **absolute temperature scale.**
5. At no temperature, the Celsius scale can have the same reading as the Kelvin scale.
6. A gas thermometer gives the same reading independent of the gas used.
7. A gas thermometer is not a direct reading thermometer.
8. For measuring very high temperatures, radiation thermometers or pyrometers are used.
9. Heat flows from a body at higher temperature to a body at lower temperature regardless of heat energy possessed by the bodies.
10. Most of the substances (solid, liquid and gas) expand on heating.
11. A few substances do not expand on heating such as water from 0°C to 4°C, silver-iodide from 80°C to 141°C and silica below –80°C.
12. When a substance is heated, the average kinetic energy of its molecules increases and they start moving with increased speed about their mean positions. As a result, the mean distance between the molecules increases. Hence expansion takes place on heating.
13. Co-efficient of linear expansion, $\alpha = \dfrac{\text{increase in length}}{\text{original length} \times \text{rise in temperature}}$

    Co-efficient of superficial expansion, $\beta = \dfrac{\text{increase in surface area}}{\text{original surface area} \times \text{rise in temperature}}$

    Coefficient of cubical expansion, $\gamma = \dfrac{\text{increase in volume}}{\text{original volume} \times \text{rise in temperature}}$
14. $\beta = 2\alpha$ and $\gamma = 3\alpha$.
15. When a substance is heated, its density decreases.
16. Since liquids do not possess any shape, we consider cubical expansion in their case.
17. $V_r = V_a + V_g$ (or $\gamma_r = \gamma_a + \gamma_g$)

    Here V_r = Real expansion of liquid ; V_a = Apparent expansion of liquid ;

    V_g = Volume expansion of glass (or material of container)
18. The amount of heat Q required to change the temperature of a solid (or liquid) of mass m through ΔT is

    $$Q = c\, m\, \Delta T$$

    where c is the specific heat of the solid (or liquid). The SI unit of specific heat is J kg^{-1} $°C^{-1}$ = J kg^{-1} K^{-1}.
19. The heat capacity of a body is defined as the amount of heat required to raise its temperature by 1°C (or 1 K).

    Heat capacity = $m\, c$ $\qquad (\because \Delta T = 1°C)$

    The SI unit of heat capacity is $J°C^{-1} = JK^{-1}$.

20. The water equivalent of a body is the mass of water that will absorb or lose the same amount of heat as the body for the same rise or fall in temperature.

Water equivalent = $m\ c$

Here m = Mass of the body ; c = Specific heat of the body

Note that water equivalent and heat capacity of a body are numerically equal.

21. The molar specific heat of a solid (or liquid) is the amount of heat required to raise the temperature of 1 mole of the solid (or liquid) through 1°C (or 1 K). It is denoted by the symbol C. The SI unit of molar specific heat is J mol^{-1} $°C^{-1}$ = J mol^{-1} K^{-1}.

22. The amount of heat Q required to raise the temperature of n moles of a solid or liquid through ΔT is

$$Q = n\ C\ \Delta T$$

23. For any material of mass m and molecular mass M,

Number of moles, $n = \frac{m}{M}$

24. $$Q = n\ C\ \Delta T = m\ c\ \Delta T$$

or $$\frac{m}{M} C\Delta T = mc\Delta T \quad \therefore \quad C = Mc$$

This is the relation between molar specific heat C and ordinary specific heat c.

25. According to law of heat exchange,

Heat lost = Heat gained

26. According to Dulong and Petit's law, the molar specific heats of solids at room temperature are approximately the same and equal to $3R$ (= 25 J mol^{-1} K^{-1}).

27. A gas can have any specific heat depending upon how heating is carried out.

28. For gases, we consider specific heat at constant pressure and specific heat at constant volume. This is from practical standpoint.

29. Molar specific heat of a gas at constant volume (C_v) is the amount of heat required to raise the temperature of 1 mole of the gas by 1°C at constant volume.

30. Molar specific heat of a gas at constant pressure (C_p) is the amount of heat required to raise the temperature of 1 mole of the gas by 1°C at constant pressure.

31. C_p is greater than C_v.

32. For an ideal gas, $C_p - C_v = R$. Note that R must be expressed in the same units as C_p and C_v.

33. The relation $C_p - C_v = R$ is valid for any gas whether it is monoatomic, diatomic or polyatomic provided it obeys the perfect gas equation $PV = RT$.

34. For an ideal monoatomic gas, $C_p = 5R/2$ and $C_v = 3R/2$ so that :

$$\gamma = \frac{C_p}{C_v} = \frac{5R/2}{3R/2} = 1.67$$

For a diatomic gas, $\gamma = \frac{C_p}{C_v} = \frac{7}{5} = 1.4$

For a *polyatomic gas, $\gamma = \frac{C_p}{C_v} = \frac{4}{3} = 1.33$

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* Unless it happens to consist of molecules all in a straight line.

35. The temperature above which a gas cannot be liquefied, no matter how great the pressure is called the critical temperature.
36. A substance which is in the gaseous phase and is above its critical temperature is called a gas.
37. A substance which is in the gaseous phase and is below its critical temperature is called a vapour.
38. A vapour can be liquefied simply by increasing pressure, a gas cannot.

MCQ FOR MEDICAL ENTRANCE EXAMINATIONS

1. A brass rod 1 m long when heated through 80°C increases in length by 1.60 mm. The coefficient of linear expansion for the brass is
   (*a*) 0.00002 per °C (*b*) 0.0004 per °C
   (*c*) 0.003 per °C (*d*) 0.0006 per °C
2. If two temperatures differ by 25 degrees on Celsius scale, the difference of temperature on Fahrenheit scale is
   (*a*) 65° (*b*) 45°
   (*c*) 38° (*d*) 25°
3. A steel wire of uniform area of 2 mm^2 is heated upto 50°C and is stretched by tying its ends rigidly. The change in tension when temperature falls from 50°C to 30°C is
   ($Y = 2 \times 10^{11}$ N/m^2, $\alpha = 1.1 \times 10^{-5}$/°C)
   [CPMT 1997]
   (*a*) 1.5×10^{10} N (*b*) 5 N
   (*c*) 88 N (*d*) 2.5×10^{10} N
4. A rectangular block is heated from 0°C to 100°C. The percentage increase in length is 0.2%. The percentage increase in its volume is
   (*a*) 0.10% (*b*) 0.2%
   (*c*) 0.4% (*d*) 0.6%
5. The volume of a liquid is 830 m^3 at 30°C and it is 850 m^3 at 90°C. The coefficient of volume expansion of the liquid is
   (*a*) 2×10^{-4} per °C (*b*) 3×10^{-3} per °C
   (*c*) 4×10^{-4} per °C (*d*) 2.5×10^{-4} per °C
6. Calorimeters are made of which of the following? **[AFMC-Pune 2000]**
   (*a*) glass (*b*) metal
   (*c*) wood (*d*) either (*a*) or (*b*)
7. In order that heat flows from one part of a solid to another, what is required?
   [Pb PMT 1999]
   (*a*) uniform density
   (*b*) density gradient
   (*c*) uniform temperature
   (*d*) temperature gradient
8. Water falls from a height of 500 m. What is the rise in temperature of water at the bottom if the whole energy is used up in heating water? **[AFMC-Pune 1997]**
   (*a*) 0.96°C (*b*) 1.02°C
   (*c*) 1.16°C (*d*) 0.25°C
9. Heat capacity of a body is infinite. It means
   [AFMC-Pune 1997]
   (*a*) heat is given out
   (*b*) heat is taken in
   (*c*) no change in temperature whether heat is taken in or given out
   (*d*) All of these
10. 540 g of ice at 0°C is mixed with 540 g of water at 80°C. The final temperature of the mixture is **[CPMT 1997]**
    (*a*) 40°C (*b*) 79.9°C
    (*c*) 80°C (*d*) 0°C
11. For a gas, the difference between two principal specific heats is 4150 J kg^{-1} K^{-1}. What is the specific heat of the gas at constant volume if the ratio of specific heats is 1.4 ?
    [AFMC-Pune 1998]
    (*a*) 5186 J kg^{-1} K^{-1} (*b*) 10375 J kg^{-1} K^{-1}
    (*c*) 1060 J kg^{-1} K^{-1} (*d*) 8475 J kg^{-1} K^{-1}
12. 100 g of ice at 0°C is mixed with 100 g of water at 100°C. The final temperature of the mixture is **[AMU PMT 1999]**
    (*a*) 10°C (*b*) 30°C
    (*c*) 20°C (*d*) 40°C
13. If 70 cals of heat is required to raise the temperature of 2 moles of an ideal gas at constant pressure from 30°C to 35°C, then the amount of heat required to raise the temperature of the same gas through same temperature range at constant volume is
    [MP PMT 1996]
    (*a*) 50 cals (*b*) 70 cals
    (*c*) 60 cals (*d*) 65 cals

14. The quantity of heat required to change the unit mass of a solid substance to its liquid state, while the temperature remains constant, is known as **[AIIMS 1998]**
(a) heat (b) latent heat
(c) specific heat (d) latent heat of fusion

15. The relative humidity on a day, when partial pressure of water vapour is 0.012×10^5 Pa at 12°C (Take vapour pressure of water vapour at this temperature as 0.016×10^5 Pa) is **[AIIMS 1998]**
(a) 25% (b) 50%
(c) 75% (d) 60%

16. For a monoatomic gas, the molar specific heat at constant pressure divided by the molar gas constant R is equal to **[CBSE PMT 1993]**
(a) 2.5 (b) 1.5
(c) 5.0 (d) 3.5

17. If for hydrogen $c_p - c_v = a$ and for oxygen $c_p - c_v = b$ where c_p and c_v refer to specific heats at constant pressure and constant volume per unit mass, then, **[CBSE PMT 1993]**
(a) $a = b$ (b) $a = 16\,b$
(c) $b = 16\,a$ (d) a and b are not related

18. One mole of an ideal gas requires 207 J to raise the temperature by 10 K when heated at constant pressure. If one mole of the same gas is heated at constant volume (R = 8.3 J mol^{-1} K^{-1}) to raise the temperature by 10 K, then heat required is **[CBSE PMT 1992]**
(a) 215.3 J (b) 198.7 J
(c) 29 J (d) 124 J

19. If for gas, $R/C_v = 0.67$, this gas is made up of molecules which are **[CBSE PMT 1992]**
(a) diatomic (b) monoatomic
(c) triatomic (d) cannot say

20. What amount of heat must be supplied to 2.0×10^{-2} kg of nitrogen at room temperature to raise the temperature by 45°C at constant pressure? Given molecular weight of N_2 is 28 g and R = 8.3 J $mol^{-1}K^{-1}$ and C_p (diatomic gases) $= \frac{7}{2}R$. **[MP PMT 1998]**
(a) 516.75 J (b) 933.75 J
(c) 1050.87 J (d) 1412.25 J

ANSWERS TO MCQ FOR MEDICAL ENTRANCE EXAMINATIONS

1. (a)	2. (b)	3. (c)	4. (d)	5. (c)
6. (b)	7. (d)	8. (c)	9. (c)	10. (d)
11. (b)	12. (a)	13. (a)	14. (d)	15. (c)
16. (a)	17. (b)	18. (d)	19. (b)	20. (b)

HINTS TO MCQ FOR MEDICAL ENTRANCE EXAMINATIONS

1. $\alpha = \dfrac{\text{increase in length}}{\text{original length} \times \text{rise in temp.}} = \dfrac{1.6\times10^{-3}}{1\times80} = 0.00002$ per°C

2. $\dfrac{F_1-32}{9} = \dfrac{C_1}{5}$ and $\dfrac{F_2-32}{9} = \dfrac{C_2}{5}$

$\therefore \quad \dfrac{F_1-F_2}{9} = \dfrac{C_1-C_2}{5}$ or $F_1 - F_2 = \dfrac{C_1-C_2}{5}\times9 = \dfrac{25}{5}\times9 = 45°$

3. $\Delta l = \alpha l \Delta T = \dfrac{Fl}{AY} \quad \therefore \quad F = \alpha\,\Delta T\,A\,Y$

Change in tension $= \alpha(50 - 30)\,A\,Y$

$= 1.1 \times 10^{-5} \times (50 - 30) \times 2 \times 10^{-6} \times 2 \times 10^{11} = 88$ N

4. $V = L^3 \quad \therefore \quad \dfrac{\Delta V}{V}\times100 = 3\dfrac{\Delta L}{L}\times100 = 3 \times 0.2\% = 0.6\%$

5. $\gamma = \dfrac{\text{increase in volume}}{\text{original volume} \times \text{rise in temp.}} = \dfrac{850-830}{830(90-30)} = 4 \times 10^{-4}$ per°C

6. According to principle of calorimetry, heat lost by a hot body is equal to the heat gained by the cold body. During this process, the heat exchange must be quick otherwise some heat is lost due to radiation etc. Since the metals are good conductors of heat, the heat exchange is quick. For this reason, calorimeters are made of **metal.**

7. Heat always flows from higher temperature to lower temperature. Therefore, **temperature gradient** (*i.e.,* temperature difference *w.r.t.* length) is required for the heat to flow from one part of the solid to another.

8. $mc\Delta T = mg\dfrac{h}{J} \quad \therefore \Delta T = \dfrac{gh}{Jc} = \dfrac{980\times500\times100}{4.2\times10^7\times1} = 1.16°C$

9. Infinite heat capacity means that there will be no **change in temperature whether heat is taken out or given to the substance.**

10. Heat required to melt 540 g of ice at 0°C = 540 × 80 cal
Heat lost by 540 g of water in cooling from 80° to 0°C = 540 × 80 cal
Since both are equal, the final temperature is **0°C.**

11. $\dfrac{c_p}{c_v} = 1.4$ or $c_p = 1.4\ c_v$
Now $c_p - c_v = 4150$ or $1.4\ c_v - c_v = 4150$
$\therefore \quad c_v = 4150/0.4 = \mathbf{10375\ J\ kg^{-1}\ K^{-1}}$

12. Heat given out by 100 g of water at 100°C to cool to 0°C
= 100 × 100 cals = 10000 cals
Heat used to melt 100 g of ice at 0°C
= 100 × 80 cals = 8000 cals
Amount of heat left, Q = 10000 − 8000 = 2000 cals
Mass of water, m = 100 + 100 = 200 g
$\therefore$ Rise in temperature = $\dfrac{Q}{cm} = \dfrac{2000}{1\times200} = 10°C$

13. $Q = n\ C_p\Delta T$ or $C_p = \dfrac{Q}{n\Delta T} = \dfrac{70}{2\times(35-30)} = 7$ cals $mol^{-1}\ °C^{-1}$
$\therefore \quad C_v = C_p - R = 7 - 2 = 5$ cals $mol^{-1}\ °C^{-1}$
$\therefore \quad Q = nC_v\ \Delta T = 2 \times 5 \times (35 - 30) = \mathbf{50\ cals}$

14. The heat energy required to change the solid substance into its liquid state without any rise in temperature is called **latent heat of fusion.**

15. Relative humidity = $\dfrac{\text{Partial pressure of water vapour}}{\text{Vapour pressure of water vapour}} = \dfrac{0.012\times10^5}{0.016\times10^5} = 0.75 = 75\%$

16. For a monoatomic gas, $\dfrac{C_p}{C_v} = 1.67$
Now $C_p - C_v = R$ or $C_p - \dfrac{C_p}{1.67} = R \quad \therefore C_p\left(1-\dfrac{1}{1.67}\right) = R$
or $C_p \times \dfrac{0.67}{1.67} = R \quad \therefore \quad \dfrac{C_p}{R} = \dfrac{1.67}{0.67} \simeq 2.5$

17. For H_2, $c_p - c_v = \dfrac{R}{2} = a$; For O_2, $c_p - c_v = \dfrac{R}{32} = b$
$\therefore \quad 2a = 32b$ or $a = 16$ b

18. $Q = n\,C_p\,\Delta T$ or $207 = 1 \times C_p \times 10$ $\therefore\ C_p = \dfrac{207}{10} = 20.7$ J mol^{-1} K^{-1}

Now $C_v = C_p - R = 20.7 - 8.3 = 12.4$ J mol^{-1} K^{-1}

Now $Q = nC_v\,\Delta T = 1 \times 12.4 \times 10 = 124$ J

19. $\dfrac{R}{C_v} = 0.67$ or $\dfrac{C_p - C_v}{C_v} = 0.67$

$\therefore\ \dfrac{C_p}{C_v} - 1 = 0.67$ or $C_p/C_v = 1.67$

The gas is made up of monoatomic molecules.

20. Number of moles of the gas is

$$n = \frac{\text{Mass of gas in grams}}{\text{Molecular mass}} = \frac{2.0\times10^{-2}\times10^{3}}{28} = \frac{20}{28}$$

Molar specific heat at constant pressure is

$$C_p = \frac{7}{2}R = \frac{7}{2} \times 8.3 \text{ J mol}^{-1} \text{ K}^{-1}$$

Heat supplied to the gas is

$$Q = nC_p\Delta T = \frac{20}{28}\times\frac{7}{2} \times 8.3 \times 45 = 933.75 \text{ J}$$

MCQ FOR ENGINEERING ENTRANCE EXAMINATIONS

1. A sheet of brass is 50 cm long and 10 cm broad at 0°C. The area of the surface increases by 1.9 cm^2 at 100°C. The coefficient of linear expansion of brass is

(*a*) 1.9×10^{-5} per °C (*b*) 1.4×10^{-4} per °C

(*c*) 2.8×10^{-3} per °C (*d*) 4.0×10^{-4} per °C

2. Railway lines are laid with gaps to allow for expansion. If the gap between the steel rails 66 m long be 3.63 cm at 10°C, then the temperature at which the lines touch (coefficient of linear expansion for steel = 11×10^{-6}°C^{-1}) is

(*a*) 40°C (*b*) 60°C

(*c*) 70°C (*d*) 80°C

3. A hole is drilled in a copper sheet. The diameter of the hole is 4.24 cm at 27°C. The change in the diameter of the hole when the sheet is heated to 227°C is (coefficient of linear expansion of copper 1.7×10^{-5} °C^{-1})

(*a*) 1.22×10^{-3} cm (*b*) 2.4×10^{-2} cm

(*c*) 3.12×10^{-4} cm (*d*) 1.44×10^{-2} cm

4. If α, β and γ are coefficients of linear, superficial and volume expansion of a solid, then,

(*a*) $\alpha:\beta:\gamma = 1:2:3$

(*b*) $\alpha:\beta:\gamma = 3:2:1$

(*c*) $\alpha:\beta:\gamma = 2:3:1$

(*d*) $\alpha:\beta:\gamma = 3:1:2$

5. An iron tyre is to be fitted on to a wooden wheel of diameter 100 cm. The diameter of the tyre is 5 mm smaller than that of the wheel. The temperature by which the iron tyre is to be heated ($\alpha = 12 \times 10^{-6}$°C^{-1}) is

(*a*) 320°C (*b*) 418.8°C

(*c*) 452°C (*d*) 436.6°C

6. The average kinetic energy associated with each degree of freedom is **[CEET 2000]**

(*a*) $\frac{1}{2}kT$ (*b*) $\frac{3}{2}kT$

(*c*) $\frac{1}{3}kT$ (*d*) $\frac{5}{2}kT$

7. During constant temperature, we feel colder on the day when the relative humidity is **[Karnataka CET 2000]**

(*a*) 85% (*b*) 60%

(*c*) 50% (*d*) 25%

8. The number of degrees of freedom for molecules of a monoatomic gas is **[Karnataka CET 1998]**
   (a) 6 (b) 5
   (c) 3 (d) 4

9. 100 g of ice at –10°C is heated till it is converted into water at 30°C. The amount of heat required is (Latent heat of ice = 80 cals/g ; specific heat of ice = 0.5 cal g^{-1} $°C^{-1}$) **[MP PET 1997]**
   (a) 14000 cals (b) 16500 cals
   (c) 25000 cals (d) 11500 cals

10. Which of the following has the highest specific heat? **[MP CEE 1999]**
    (a) copper (b) hydrogen
    (c) water (d) silver

11. In gases of diatomic molecules, the ratio of two specific heats is **[Karnataka CET 1996]**
    (a) 1.4 (b) 1.67
    (c) 1.33 (d) 1.11

12. 0.014 kg of nitrogen is enclosed in a vessel at a temperature of 27°C. How much heat has to be given to the gas to double the r.m.s. speed of its molecules? Take R = 2 cal mol^{-1} K^{-1}. **[Rajasthan PET 1997]**
    (a) 1650 cals (b) 1550 cals
    (c) 2250 cals (d) 2740 cals

13. Half mole of helium is contained in a container at S.T.P. How much heat energy is needed to double the pressure of the gas, keeping the volume of the gas constant? Specific heat of gas = 3 J g^{-1} K^{-1}. **[MP PET 1998]**
    (a) 596 J (b) 1638 J
    (c) 2150 J (d) 1075 J

14. A gas mixture consists of 2 moles of oxygen and 4 moles of argon at temperature T. Neglecting all vibrational modes, the total energy of the system is **[IIT 1999]**
    (a) $7RT$ (b) $15RT$
    (c) $9RT$ (d) $11RT$

15. What is the difference between the two principal specific heats of 1 g of helium gas at S.T.P.? Given molecular weight of helium = 4 and J = 4.186 J cal^{-1} and R = 8.31 J mol^{-1} K^{-1}.
    (a) 0.586 cal g^{-1} K^{-1}
    (b) 0.496 cal g^{-1} K^{-1}
    (c) 0.782 cal g^{-1} K^{-1}
    (d) 0.576 cal g^{-1} K^{-1}

ANSWERS TO MCQ FOR ENGINEERING ENTRANCE EXAMINATIONS

1. (a)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (b)
6. (a)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (b)
11. (a)	**12.** (c)	**13.** (b)	**14.** (d)	**15.** (b)

HINTS TO MCQ FOR ENGINEERING ENTRANCE EXAMINATIONS

1. $\Delta S = \beta S \Delta T \quad \therefore \quad \beta = \dfrac{\Delta S}{S\Delta T} = \dfrac{1.9}{(50\times10)\times(100-0)} = 3.8 \times 10^{-5}\ °C^{-1}$

$$\therefore \qquad \alpha = \frac{\beta}{2} = \frac{3.8\times10^{-5}}{2} = 1.9 \times 10^{-5} \text{ per } °C$$

2. $\Delta l = l\alpha\Delta T$ or $\Delta T = \dfrac{\Delta l}{l\alpha} = \dfrac{3.63\times10^{-2}}{66\times11\times10^{-6}} = 50°C$

   $\therefore$ Final temperature = 50 + 10 = 60 °C

3. When the copper sheet is heated, the diameter of its hole increases in the same manner as the length of a rod.

   Increase in diameter, $\Delta l = l\alpha\Delta T = 4.24\times1.7\times10^{-5}\times(227-27) = 1.44 \times 10^{-2}$ cm

4. $\alpha : \beta : \gamma = \alpha : 2\alpha : 3\alpha = 1 : 2 : 3$

5. $\Delta d = d\,\alpha\,\Delta T \quad \therefore \quad \Delta T = \dfrac{\Delta d}{d\alpha} = \dfrac{0.5}{(100-0.5)\times12\times10^{-6}} = 418.8°C$

6. Average K.E./molecule/degree of freedom = $\frac{1}{2}kT$

7. When the relative humidity is low, the evaporation occurs fast from our body. Therefore, we feel colder.

8. The number of degrees of freedom for the molecules of a monoatomic gas is 3.

9. Amount of heat required to convert 100 g of ice from –10°C to 0°C

$$= m\,c\,\Delta\,T = 100 \times 0.5 \times 10 = 500 \text{ cals}$$

Amount of heat required to convert 100 g of ice at 0°C to water at 0°C

$$= mL = 100 \times 80 = 8000 \text{ cals}$$

Amount of heat required to raise the temperature of 100 g of water from 0°C to 30°C

$$= mc\,\Delta T = 100 \times 1 \times 30 = 3000 \text{ cals}$$

$\therefore$ Total amount of heat required = 500 + 8000 + 3000 = 11500 cals

10. Hydrogen has the highest specific heat (= 14300 J kg^{-1} K^{-1}).

11. For diatomic gases, $C_p = \frac{7}{2}R$ and $C_v = \frac{5}{2}R$. $\therefore$ $\frac{C_p}{C_v} = \frac{7R/2}{5R/2} = \frac{7}{5} = 1.4$

12. Number of moles of the gas is

$$n = \frac{\text{mass of gas in grams}}{\text{molecular mass}} = \frac{0.014 \times 10^3}{28} = 0.5$$

Since nitrogen is a diatomic gas, $C_v = \frac{5}{2}R = \frac{5}{2} \times 2 = 5$ cal mol^{-1} K^{-1}

As r.m.s. speed of gas molecules $\propto$ Absolute temperature of gas,

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} \text{ or } 2 = \sqrt{\frac{T_2}{T_1}} \quad \therefore \quad T_2 = 4T_1$$

$\therefore$ $\Delta T = T_2 - T_1 = 4T_1 - T_1 = 3T_1 = 3(273 + 27) = 900$ K

Heat required, $Q = nC_v\,\Delta T = 0.5 \times 5 \times 900 = 2250$ cals

13. $n = \frac{1}{2}$; $c_v = 3 \text{ J g}^{-1}$; $M = 4$; $T_1 = 273$ K

$\therefore$ Molar specific heat at constant volume is

$$C_v = Mc_v = 4 \times 3 = 12 \text{ J mol}^{-1} \text{ K}^{-1}$$

Since volume is constant, $P \propto T$.

$$\therefore \quad \frac{P_2}{P_1} = \frac{T_2}{T_1} \text{ or } \frac{2P_1}{P_1} = \frac{T_2}{T_1} \quad \therefore \quad T_2 = 2T_1 = 2 \times 273 = 546 \text{ K}$$

Rise in temperature, $\Delta T = T_2 - T_1 = 546 - 273 = 273$ K

Heat required, $Q = n\,C_v\,\Delta T = \frac{1}{2} \times 12 \times 273 = 1638$ J

14. Let N_A be the Avogadro's number.

No. of degrees of freedom of O_2 molecule (diatomic) = 5

No. of degrees of freedom of 2 moles of oxygen = $2N_A \times 5 = 10\,N_A$

No. degrees of freedom of 4 moles of argon (monoatomic)

$$= 4\,N_A \times 3 = 12\,N_A$$

Total degrees of freedom of mixture = $10N_A + 12N_A = 22\,N_A$

Energy associated with each degree of freedom/molecule = $\frac{1}{2}kT$

$\therefore$ Total energy of mixture = $22N_A \times \frac{1}{2} kT = 11kN_AT = 11\ RT$ $(\because kN_A = R)$

15. $c_p - c_v = \frac{r}{J} = \frac{R}{MJ} = \frac{8.31}{4\times4.186} = 0.496$ cal g^{-1} K^{-1}

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **If two temperatures differ by 25 degrees on Celsius scale, what is the difference on Fahrenheit scale?**

Hint. $\frac{F_1-32}{9} = \frac{C_1}{5}$ and $\frac{F_2-32}{9} = \frac{C_2}{5}$

$\therefore$ $\frac{F_1-F_2}{9} = \frac{C_1-C_2}{5}$ or $F_1 - F_2 = \frac{C_1-C_2}{5}\times9 = \frac{25°}{5}\times9 = 45°$

2. **A thermometer has wrong calibration. It reads the melting point of ice as –10°C. It reads 60°C in place of 50°C. What is the temperature of boiling point of water on this scale?**

Hint. The lower fixed point on the wrong scale is –10°C. Let n be the number of divisions between upper and lower fixed points on this scale. If θ is the reading on this scale, then,

$$\frac{C-0}{100} = \frac{\theta-(-10)}{n}$$

Now when C = 50°C, θ = 60°C, then,

$$\frac{50-0}{100} = \frac{60-(-10)}{n} = \frac{70}{n}$$

or $$n = 70\times\frac{100}{50} = 140$$

Now $$\frac{C-0}{100} = \frac{\theta-(-10)}{140}$$

On the Celsius scale, B.P. of water is 100°C

$\therefore$ $$\frac{100-0}{100} = \frac{\theta+10}{140}$$ or $\theta = 140 - 10 = 130°C$

3. **Mercury in a faulty centigrade thermometer stands at 5° mark when placed in melting ice and at 95° mark when in steam at normal pressure. What will be its reading when it is placed in contact with a body at 40°C?**

Hint. Let θ° be the reading.

Now $$\frac{C-0}{100-0} = \frac{\theta-5}{95-5}$$ or $$\frac{40}{100} = \frac{\theta-5}{90}$$

$\therefore$ $$\theta = \frac{40}{100}\times90 + 5 = 36 + 5 = 41°$$

4. **A one litre flask contains some mercury. It is found that at different temperatures, the volume of air inside the flask remains the same. What is the volume of mercury in the flask? Given that coefficient of linear expansion of glass = 9×10^{-6}/°C and coefficient of volume expansion of mercury = 1.8×10^{-4}/°C. [I.I.T]**

Hint. It is given that volume of air in the flask remains the same at different temperatures. This is possible only when the expansion of glass is exactly equal to the expansion of mercury. Coefficient of cubical expansion of glass is

$$\gamma_g = 3\,\alpha_g = 3 \times 9 \times 10^{-6} = 27 \times 10^{-6}/°C$$

Coefficient of cubical expansion of mercury is

$$\gamma_m = 1.8 \times 10^{-4}/°C \text{ (given)}$$

Volume of flask, $V = 1$ litre $= 1000$ cm^3

Let v_m cm^3 be the volume of mercury in the flask.

Now Expansion of flask = Expansion of mercury

or $$V \times \gamma_g \times t = v_m \times \gamma_m \times t$$

$$\therefore \quad \text{Volume of mercury, } v_m = \frac{V \times \gamma_g}{\gamma_m} = \frac{1000 \times 27 \times 10^{-6}}{1.8 \times 10^{-4}} = \mathbf{150\ cm^3}$$

5. What should be the lengths of steel and copper rods so that the length of the steel rodis 5 cm longer than the copper rod at all temperatures? Given α for copper = 1.7×10^{-5}/°C and α for steel = 1.1×10^{-5}/°C. [I.I.T.]

Hint. Let l_s and l_c be the lengths of steel and copper rods respectively. When they are heated through a tempearture t°C, then,

Increase in length of steel rod is $\Delta l_s = l_s\,\alpha_s t$

Increase in length of copper rod is $\Delta l_c = l_c\,\alpha_c t$

It is given that at all temperatures, $l_s - l_c = 5$ cm.

$$\therefore \quad \Delta l_s - \Delta l_c = 0 \quad \text{or} \quad l_c\,\alpha_c t = l_s\,\alpha_s t$$

$$\therefore \quad \frac{l_s}{l_c} = \frac{\alpha_c}{\alpha_s} = \frac{1.7 \times 10^{-5}}{1.1 \times 10^{-5}} = \frac{17}{11}$$

or $$l_s = \frac{17}{11} l_c \qquad ...(i)$$

Also $$l_s - l_c = 5 \qquad ...(ii)$$

Solving eqs. (*i*) and (*ii*), we get, $l_s =$ **14.17 cm**; $l_c =$ **9.17 cm**

6. A zinc scale is graduated at 0°C. The length of a brass rod is 100 cm at 0°C. What would be the observed length at 40°C? Given α for zinc = 29×10^{-6} /K and α for brass = 18×10^{-6}/K.

Hint. True length of the brass rod is 40°C is

$$l_{40} = l_0\,(1 + \alpha t) = 100\,(1 + 18 \times 10^{-6} \times 40) = 100.072 \text{ cm}$$

Now 1 division of zinc scale is 1 cm at 0°C.

$\therefore$ 1 division of zinc scale at 40°C

$$= 1\,(1 + 29 \times 10^{-6} \times 40) = 1.00116 \text{ cm}$$

If l is the observed length at 40°C, then true length = $l \times 1.00116$ cm.

$$\therefore \quad l \times 1.00116 = 100.072$$

or $$l = \frac{100.072}{1.00116} = \mathbf{99.956\ cm}$$

7. A long glass tube of uniform capillary bore contains a thread of mercury which is 1 m long at 0°C. What will be its length at 100°C if the real coefficient of cubical expansion of mercury is 0.000182/°C and the coefficient of cubical expansion of glass is 0.000025/°C?

Hint. Let V_0 be the volume of the mercury at 0°C and V_t at t°C. Further, let a_0 and a_t be the cross-sectional areas of the thread at 0°C and t°C respectively.

Length of thread at 0°C, $l_0 = V_0/a_0$

Volume of thread at t°C, $V_t = V_0 (1+ \gamma_r t)$

Cross-sectional area of thread at t°C is given by ;

$$a_t = a_0 (1 + \beta t) = a_0(1+ 2\alpha t) \qquad [\because \beta = 2\alpha]$$

$\therefore$ Length of the thread at t°C is given by ;

$$l_t = \frac{V_t}{a_t} = \frac{V_0(1+\gamma_r t)}{a_0(1+2\alpha t)} = l_0 (1 + \gamma_r t) (1 + 2\alpha t)^{-1}$$

or $$l_t = l_0(1 + \gamma_r t) (1 - 2\alpha t)^*$$

or $$l_t = l_0 [1 + (\gamma_r - 2\alpha) t]$$

Here $l_0 = 1\ m = 100$ cm ; $t = 100$°C ; $\gamma_r = 0.000182$/°C ;

$$\alpha = \frac{\gamma_g}{3} = \frac{0.000025}{3} = 0.0000083/°C$$

$\therefore$ $$l_t = 100 [1 + (0.000182 - 2 \times 0.0000083) 100] = \mathbf{101.654\ cm}$$

8. A liquid of coefficient of real cubical expansion 300×10^{-6}/°C is kept in a vessel whose coefficient of linear expansion is 12×10^{-6}/°C. Find the coefficient of apparent cubical expansion of the liquid.

Hint. $$\gamma_r = \gamma_a + \gamma_g = \gamma_a + 3\alpha$$

$\therefore$ $\gamma_a = \gamma_r - 3\alpha = (300 \times 10^{-6}) - 3(12 \times 10^{-6}) = \mathbf{26.4 \times 10^{-5}/°C}$

9. The coefficient of cubical expansion of mercury is 0.00018/°C and that of brass is 0.00006/°C. If a barometer having a brass scale were to read 74.5 cm at 30°C, find the true barometric height at 0°C. The scale is supposed to be correct at 15°C.

Hint. The brass scale is true at 15°C and it reads 74.5 cm at 30°C. Therefore, if it is cooled to 15°C, it contracts and shows the actual height l_{30} given by ;

$$l_{30} = 74.5 [1 + \alpha (30 - 15)]$$

Now $$\alpha_{brass} = \frac{\gamma_{brass}}{3} = \frac{0.00006}{3} = 0.00002/°C$$

$\therefore$ $$l_{30} = 74.5 [1 + 0.00002 \times (15)] = 74.522 \text{ cm}$$

Assuming the area of cross-section to be uniform, we have,

$$V_0\rho_0 = V_{30}\rho_{30} \quad \text{or} \quad ah_0\rho_0 = ah_{30}\rho_{30}$$

$\therefore$ $$h_0 = h_{30} \times \frac{\rho_{30}}{\rho_0} = h_{30} \times \frac{^{**}\rho_0}{\rho_0(1+\gamma_m t)} = \frac{h_{30}}{(1+\gamma_m t)} = \frac{74.522}{(1+0.00018\times 30)} = \mathbf{74.122\ cm}$$

10. An iron rod of length 50 cm is joined at an end to an aluminium rod of length 100 cm. All measurements refer to 20°C. Find the length of the composite rod at 100°C and its average coefficient of linear expansion. The coefficient of linear expansion of iron and aluminium are 12×10^{-6}/°C and 24×10^{-6}/°C respectively.

Hint. The length of the iron rod at 100°C is

$$l_i = l_{20} (1 + \alpha t) = 50(1 + 12 \times 10^{-6} \times 80) = 50.048 \text{ cm}$$

The length of the aluminium rod at 100°C is

$$l_a = l_{20} (1 + \alpha t) = 100(1 + 24 \times 10^{-6} \times 80) = 100.192 \text{ cm}$$

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* Using Bionomial theorem and neglecting smaller terms.

** $\rho_t = \dfrac{\rho_0}{1+\gamma t}$

$\therefore$ Length of the composite rod at 100°C is

$$l_{100} = l_i + l_a = 50.048 + 100.192 = \mathbf{150.24\ cm}$$

$\therefore$ Average coefficient of linear expansion of the composite rod is

$$\alpha = \frac{l_{100} - l_{20}}{l_{20} \times (t_2 - t_1)} = \frac{150.24 - 150}{150 \times (100 - 20)} = \frac{0.24}{150 \times 80} = \mathbf{20 \times 10^{-6}/°C}$$

11. It takes for an electric kettle to heat a certain quantity of water from 0°C to boiling point (100°C) in 15 minutes. It requires 80 minutes to turn all the water at 100°C into steam. Find the latent heat of steam.

Hint. Let the mass of water be m grams.

Quantity of heat liberated in 15 minutes

$$= mc\ (\theta_2 - \theta_1) = m \times 1 \times (100 - 0) = 100m \text{ cal}$$

Quantity of heat liberated in 80 minutes $= 100m \times \frac{80}{15} = \frac{8000\,m}{15}$ cal

Let L be the latent heat of steam in cal/g.

$$\therefore \qquad mL = \frac{8000\,m}{15} \quad \text{or} \quad L = \frac{8000}{15} = \mathbf{533.3\ cal/g}$$

12. A 50 g lead bullet, specific heat 0.02 cal g^{-1} $°C^{-1}$ is initially at 30°C. It is fired vertically upward with a speed of 840 ms^{-1} and on returning to the starting level strikes a cake of ice at 0°C. How much ice is melted? Assume that all energy is spent in melting only. Latent heat of ice = 80 cal/g. [Roorkee Engg.]

Hint. The bullet strikes the cake of ice with the same speed *i.e.*, $v = 840$ ms^{-1}.

Heat produced due to kinetic energy of the bullet

$$= \frac{1}{2}mv^2 = \frac{1}{2} \times (50 \times 10^{-3}) \times (840)^2 = 17640 \text{ J}$$

$$= \frac{17640}{4.2} = 4200 \text{ cal}$$

Heat given by bullet due to temperature difference

$$= mc\ (\theta_2 - \theta_1) = 50 \times 0.02 \times (30 - 0) = 30 \text{ cal}$$

Total heat given by bullet = 4200 + 30 = 4230 cal

This entire heat of 4230 cal is taken by the cake of ice. If m gram of ice is melted, then,

$$m \times L = 4230$$

or $$m = \frac{4230}{L} = \frac{4230}{80} = \mathbf{52.88\ g}$$

13. An electric heater rod is dipped in a vessel containing water at 0°C. The electric rod produces heat at the rate of 3000 J/s. The vessel with its contents is maintained at 0°C by adding ice at the rate of 9 g/s. Calculate the latent heat of fusion of ice. If the total heat capacity of water, the vessel and contents is 12000 J/K, at what rate does the temperature start rising when the supply of ice is stopped?

Hint. $H = 3000$ J/s ; $m = 9$ g/s $= 9 \times 10^{-3}$ kg/s

Now $H = m\,L$

$\therefore$ Latent heat of fusion, $L = \frac{H}{m} = \frac{3000}{9 \times 10^{-3}} = \mathbf{3.33 \times 10^5\ J/kg}$

When the supply of ice is stopped, let the rate of rise of temperature be θ K/s.

$$W = 12000 \text{ J/K} \; ; \quad H = 3000 \text{ J/s}$$

Now $H = W\theta$

$$\therefore \quad \theta = \frac{H}{W} = \frac{3000}{12000} = 0.25 \text{ K/s}$$

14. 2000 g of water is contained in a 2.5 kW electric kettle. Assuming that the heat capacity of the kettle is negligible, calculate (*i*) the time taken for the temperature of water to rise from 25°C to its boiling point (100°C) (*ii*) the mass of water evaporated per second from the boiling water. Specific heat of water = 4.2 J/g°C and latent heat of vaporisation of water = 2250 J/g.

Hint. Power of kettle, P = 2.5 kW= 2500 W ; Energy available per second = 2500 J

(*i*) Heat required to raise the temperature of water from 25°C to 100°C is

$$H = m\,c\,(\theta_2 - \theta_1) = 2000 \times 4.2 \times (100 - 25) = 63 \times 10^4 \text{ J}$$

$$\therefore \quad \text{Time taken} = \frac{H}{P} = \frac{63\times10^4}{2500} = 252 \text{ s}$$

(*ii*) Let m grams of water at 100°C be evaporated into steam in one second.

Energy required in one second = 2500 J

$$= m \times 2250 \text{ J}$$

Energy available in one second = 2500 J

$$\therefore \quad m \times 2250 = 2500 \text{ J}$$

or $$m = \frac{2500}{2250} = 1.11 \text{ g}$$

15. 0.75 g of petroleum was burnt in a bomb calorimeter which contained 2000 g of water and had a water equivalent = 750 g. The rise of temperature was 3.08°C. Determine the calorific value of petroleum.

Hint. Heat gained by water = $m\,c\,(\theta_2 - \theta_1)$ = 2000 × 1 × (3.08) = 6160 cal

Heat gained by calorimeter = $w\;(\theta_2 - \theta_1)$ = 750 × 3.08 = 2310 cal

Total heat gained = 6160 + 2310 = 8470 cal

This heat is obtained due to the combustion of 0.75 g of petroleum.

$$\therefore \text{ Calorific value of petroleum} = \frac{8470}{0.75} = 11293 \text{ cal/g}$$

16. A refrigerator converts 50 g of water at 15°C into ice at –20°C in one hour. Determine the quantity of heat removed per minute. Specific heat of ice = 0.5 cal g^{-1} $°C^{-1}$.

Hint. Mass of water, m = 50 g

Heat removed in cooling water from 15°C to 0°C is

$$H_1 = m\,c\,(\theta_2 - \theta_1) = 50 \times 1 \times (15 - 0) = 750 \text{ cal}$$

Heat removed in converting water at 0°C into ice at 0°C is

$$H_2 = m\,L = 50 \times 80 = 4000 \text{ cal}$$

Heat removed in cooling ice from 0°C to – 20°C is

$$H_3 = m \times \text{specific heat of ice} \times \text{fall in temperature}$$
$$= 50 \times 0.5 \times 20 = 500 \text{ cal}$$

$\therefore$ Total heat removed in one hour is given by ;

$$H = H_1 + H_2 + H_3 = 750 + 4000 + 500 = 5250 \text{ cal}$$

$$\text{Heat removed per minute} = \frac{5250}{60} = 87.5 \text{ cal/minute}$$

17. Heat is supplied at a constant rate to 400 g of ice at 0°C in a copper container. All the ice is converted into water in 5 minutes. How much time will be required to (*i*) raise the temperature of water to its boiling point (100°C) and (*ii*) change all the water at 100°C into steam?

Hint. Heat required to melt 400 g of ice is

$$H_1 = m'L = 400 \times 80 = 32000 \text{ cal}$$

Time taken, $t = 5$ minutes $= 5 \times 60 = 300$ s

$$\text{Rate of heat supply} = \frac{H_1}{t} = \frac{32000}{300} = \frac{320}{3} \text{ cal/s}$$

(*i*) Heat required to raise the temperature of water from 0°C to 100°C is

$$H_2 = m\,c\,(\theta_2 - \theta_1) = 400 \times 1 \times (100 - 0) = 40000 \text{ cal}$$

$$\text{Time taken, } t_1 = \frac{H_2}{320/3} = \frac{40000 \times 3}{320} = 375 \text{ s}$$

(*ii*) Heat required to convert water at 100°C into steam at 100°C is

$$H_2 = m\,L = 400 \times 540 = 216000 \text{ cal}$$

$$\text{Time taken, } t_2 = \frac{H_2}{320/3} = \frac{216000 \times 3}{320} = 2025 \text{ s}$$

18. If one mole of a monoatomic gas $\gamma = 5/3$ is mixed with one mole of a diatomic gas $\gamma = 7/5$, what is the value of γ for the mixture?

Hint. The average value of γ is given by ;

$$\gamma = (C_p / C_v)_{av}$$

For monoatomic gas, $C_v = (3/2)R$; For diatomic gas, $C_v = (5/2)R$

$$\therefore \quad (C_v)_{av} = \frac{(3/2)R + (5/2)R}{2} = 2R$$

$$\text{Also} \quad (C_p)_{av} = (C_v)_{av} + R = 2R + R = 3R$$

$$\therefore \quad \gamma_{mix} = (C_p / C_v)_{av} = 3R / 2R = 1.5$$

19. How much heat energy should be added to the gaseous mixture consisting of 1 g of hydrogen and 1 g of helium to raise its temperature from 0°C to 100°C (*i*) at constant volume (*ii*) at constant pressure, assuming both the gases to be ideal. Given $R = 1.99$ cal/mole. K and γ (hydrogen) = 1.41 and γ (helium) = 1.67. [C E E]

Hint. For hydrogen, $C_p/C_v = 1.41$ and $C_p - C_v = R = 1.99$. Therefore, for hydrogen, $C_v = 4.85$ cal /mole.K and $C_p = 6.84$ cal/mole.K. For helium, $C_p/C_v = 1.67$ and $C_p - C_v = 1.99$. Therefore, for helium, $C_v = 2.97$ cal/mole.K and $C_p = 4.96$ cal /mole.K.

(*i*) **Energy added at constant volume.** If m is the mass of the gas and M the molecular mass, then,

$$\text{Mass of gas in moles} = m/M$$

To raise the temperature of m/M mole of a gas by ΔT at constant volume,

$$\text{Energy required} = \frac{m}{M} C_v \Delta T$$

$$\text{Energy given to hydrogen} = \frac{1}{2}\times 4.85\times 100 = 242.5 \text{ cal}$$

$$\text{Energy given to helium} = \frac{1}{4}\times 2.97\times 100 = 74.25 \text{ cal}$$

$\therefore$ Energy given to mixture = 242.5 + 74.25 = **316.75 cal**

(*ii*) **Energy added at constant pressure.** To raise the temperature of a gas by ΔT at constant pressure, energy required

$$= \frac{m}{M} C_p \Delta T$$

$$\text{Energy given to hydrogen} = \frac{1}{2}\times 6.84\times 100 = 342 \text{ cal}$$

$$\text{Energy given to helium} = \frac{1}{4}\times 4.96\times 100 = 124 \text{ cal}$$

$\therefore$ Energy given to mixture = 342 + 124 = **466 cal**

20. The specific heat of an ideal gas at constant pressure is $C_p = 5R/2$. The gas is contained in a closed vessel of volume 0.0083 m^3 at a temperature of 300 K and a pressure of 1.6 × 10^6 N/m^2. It is given 2.49 × 10^4 J of energy. Compute the final temperature and pressure of the gas. Gas constant R = 8.3 J/mole.K. [I.I.T.]

Hint. The conditions of the problem suggest that heating is carried out at constant volume.

Now $$C_v = C_p - R = \frac{5R}{2} - R = \frac{3R}{2}$$

If m is the mass of the gas in the vessel and M is the molecular mass of the gas, then,

$$PV = \frac{m}{M} RT$$

or $$\frac{m}{M} = \frac{PV}{RT} = \frac{(1.6\times 10^6)\times 0.0083}{8.3\times 300} = \frac{16}{3}$$

Let ΔT be the rise in temperature when m/M mole of gas is given heat energy of 2.49 × 10^4 J at constant volume.

$\therefore$ $$Q = \frac{m}{M} C_v \Delta T$$

or $$\Delta T = \frac{Q}{(m/M)\times C_v} = \frac{2.49\times 10^4}{(16/3)\times(3/2)8.3} = 375 \text{ K}$$

$\therefore$ Final temperature of gas, T' = 300 + 375 = **675 K**

If P' is the final pressure of the gas, then,

$$\frac{P'}{T'} = \frac{P}{T}$$

or $$P' = \frac{P}{T}\times T' = \frac{1.6\times 10^6}{300}\times 675 = \mathbf{3.6 \times 10^6 \text{ N/m}^2}$$

22

Transfer of Heat

INTRODUCTION

Heat may be transferred from one point or body to another in three different ways : by **conduction, convection** and **radiation.** Heat passes through solids by conduction. In this process, heat is handed over from molecule to molecule but the molecules do not leave their mean positions. Liquids and gases are mainly heated by convection. In this process, there is a bodily movement of the heated molecules. In order to transfer heat by conduction and convection, a material medium is required. However, transfer of heat by radiation does not need any medium. Heat directly passes from the hot object to the surroundings. The radiation of heat takes place in the same manner and with the same speed as light (*i.e.*, 3×10^8 ms^{-1}). The energy from the sun comes to us by radiation. It may be noted that conduction is a slow process of heat transfer while convection is a rapid process. However, radiation is the fastest process because the transfer of heat takes place at the speed of light. We now discuss each of these processes in turn ; but in practical situations, any two or all three may be operating at the same time.

22.1. CONDUCTION

It is the process by which heat flows from the hot end to the cold end of the solid body without there being any net movement of the particles of the body.

Heat passes through solids by conduction only. When one end of a metal rod is placed in a flame while the other is held in hand, the end being held becomes hotter and hotter, although it is not itself in direct contact with the flame. We say that heat has been conducted from the hot end to the cold end. Heat conduction can be visualised as the result of molecular collisions. As one end of the object is heated, the molecules there vibrate faster and faster. As they collide with their slow moving neighbours, they transfer some of their energy to these molecules whose speeds thus increase. These in turn transfer some of their energy by collision with molecules still farther down the object. Thus the energy of thermal motion is transferred by molecular collision down the object. The transfer of heat continues till the two ends of the object are at the same temperature. This method of heat transfer is called conduction.

Those solid substances which conduct heat easily are called *good conductors of heat e.g.,* silver, copper etc. All metals are good conductors of heat. Those substances which do not conduct heat easily are called *bad conductors of heat e.g.,* wood, cloth, air, paper *etc.* In general, good conductors of heat are also good conductors of electricity. Similarly, bad conductors of heat are also bad conductors of electricity.

22.2. THERMAL CONDUCTIVITY

Thermal conductivity of a solid is a measure of the ability of the solid to conduct heat through it. The greater the thermal conductivity of a solid, the greater is its ability to conduct heat through

it and *vice-versa*. In order to study conduction in more detail, suppose that one end of a metal rod is heated (See Fig. 22.1). The heat flows by conduction from hot end to cold end. As a result, the temperature of every section of the rod starts increasing. Under this condition, the rod is said to be in the *variable state*. After some time, the temperature at each section of the rod becomes steady *i.e.* constant. *Note that temperature of each cross-section of the rod is constant but not the same.* This is called *steady state condition.* Under steady state condition, the temperature at points within the rod decreases *uniformly with distance from the hot end to the cold end. *The fall of temperature with distance between the ends of the rod in the direction of flow of heat is called* **temperature gradient.**

$$\text{Temperature gradient} = \frac{\theta_1 - \theta_2}{x}$$

where θ_1 = Temperature of hot end

θ_2 = Temperature of cold end

x = Length of the rod

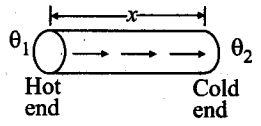

Fig. 22.1

22.3. COEFFICIENT OF THERMAL CONDUCTIVITY

Consider a cube of each side x and each face of cross-sectional area A. Suppose its opposite faces are maintained at temepratures θ_1 and θ_2 ($\theta_1 > \theta_2$) as shown in Fig. 22.2. Experiments show that under steady state condition, the quantity of heat Q that flows from hot face to cold face is

(*i*) *directly proportional to the cross-sectional area A of the face i.e.,*

$$Q \propto A \qquad ...(i)$$

(*ii*) *directly proportional to the temperature difference* ($\theta_1 - \theta_2$) *between the two faces i.e.,*

$$Q \propto \theta_1 - \theta_2 \qquad ...(ii)$$

(*iii*) *inversely proportional to the perpendicular distance x between hot and cold faces i.e.,*

$$Q \propto \frac{1}{x} \qquad ...(iii)$$

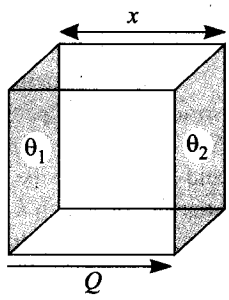

Fig. 22.2

(*iv*) *directly proportional to time t (in seconds) for which heat flows i.e.,*

$$Q \propto t \qquad ...(iv)$$

Combining the above four factors, we have,

$$Q \propto \frac{A(\theta_1 - \theta_2)t}{x}$$

or

$$Q = \frac{kA(\theta_1 - \theta_2)t}{x} \qquad ...(v)$$

where k is a constant of proportionality and is called *coefficient of thermal conductivity.* Its value depends upon the nature of the material.

If A = 1 m^2 ; $\theta_1 - \theta_2$ = 1°C (or 1 K) ; t = 1 s ; x = 1 m, then from eq. (*v*), $Q = k$.

Hence the **coefficient of thermal conductivity** *of a material is defined as the quantity of heat that flows in one second between the opposite faces of a cube of each side 1m, the faces being kept at a temperature difference of 1°C (or 1 K).*

Units of *k*. From eq. (*v*), we have,

$$k = \frac{Q \times x}{A(\theta_1 - \theta_2)t}$$

* Neglecting heat loss from the sides of the rod.

In SI units, Q, x, A, $(\theta_1 - \theta_2)$ and t are measured in J, m, m^2, °C and s respectively.

$$\therefore \quad \text{Unit of } k = \frac{J \times m}{m^2 \times °C \times s} = J\ s^{-1}\ m^{-1}\ °C^{-1} \text{ or } Js^{-1}\ m^{-1}\ K^{-1}$$

Since $J\ s^{-1}$ is watt (W), the unit of k can also be expressed as $W\ m^{-1}°C^{-1}$ (or $Wm^{-1}K^{-1}$).

Rate of heat flow. From eq. (*v*) above, we have,

$$\frac{Q}{t} = \frac{kA(\theta_1 - \theta_2)}{x} \qquad ...(vi)$$

The quantity Q/t is the time rate of heat flow (i.e., heat flow per second) from the hotter face to the colder face and is at right angles to the faces [Unit = W (watt)]. In terms of calculus, the eq. (*vi*) may be written as :

$$\frac{dQ}{dt} = -kA\frac{d\theta}{dx}$$

where $d\theta/dx$ = *temperature gradient (i.e.,* rate of fall of temperature with distance) between the concerned faces (Unit = $°C\ m^{-1}$)

The negative sign indicates that heat flows in the direction of decreasing temperature.

If $A = 1$ and $d\theta/dx = 1$, then $dQ/dt = k$ (numerically).

Hence the **coefficient of thermal conductivity** *of a material may be defined as the rate of flow of heat per unit area per unit temperature gradient when the heat flow is at right angles to the faces of a thin parallel-sided slab of the material.*

The coefficients of thermal conductivity k for a variety of substances are given in the adjoining table. Substances for which k is large, conduct heat rapidly and are said to be good conductors of heat. Substances for which k is small are poor conductors of heat. The relative magnitudes of k can explain simple phenomena such as why a tile floor is much colder on the feet than a rug-covered floor at the same temperature. The tile is a better conductor of heat than the rug. The heat transferred from your foot to the rug is not conducted away rapidly so that the rug quickly heats up on its surface to the temperature of your foot. However, the tile conducts the heat away rapidly and thus can take more heat from your foot so its surface temperature drops.

Substance	*Coefficient of thermal conductivity* ($J\ s^{-1}\ m^{-1}\ °C^{-1}$)
Silver	420
Copper	380
Aluminium	200
Steel	40
Glass	0.84
Brick and concrete	0.84

22.4. THERMAL RESISTANCE

The opposition of a body to the flow of heat through it is called **thermal resistance** *of the body.* It is analogous to electrical resistance which is opposition of a body to the flow of electric current. The greater the thermal conductivity of a material, the smaller is its thermal resistance.

Let l be the length and A the area of X-section of a rod. If θ_1 and θ_2 be the temperatures of the hot and cold ends of the rod in the steady state, then rate of flow of heat is given by ;

$$H = \frac{Q}{t} = \frac{kA(\theta_1 - \theta_2)}{l}$$

or $$\frac{\theta_1 - \theta_2}{H} = \frac{l}{kA} \qquad ...(i)$$

When a current flows through a conductor, the ratio V/I is called the electrical resistance of the conductor where V is the potential difference between the ends of the conductor and I is the current or rate of flow of charge. In eq. (*i*), $\theta_1 - \theta_2$ is the temperature difference between the ends

of the rod and H is the rate of flow of heat. Therefore, $(\theta_1 - \theta_2)/H$ is called thermal resistance R of the material *i.e.*,

$$\text{Thermal resistance, } R = \frac{l}{kA}$$

The unit of thermal resistance is °C-sec/kcal and its dimensional formula is $[M^{-1}L^{-2}T^{3}\theta]$.

22.5. APPLICATIONS OF THERMAL CONDUCTIVITY

Different substances have different coefficients of thermal conductivity. This fact has many practical applications in daily life and we shall discuss a few of them by way of illustration.

(*i*) **Cooking utensils are made of metals but are provided with handles of bad conductors.** Since metals are good conductors of heat, heat can be easily conducted through the base of the utensils. The handles of utensils are made of bad conductors of heat (*e.g.*, wood, ebonite *etc.*) because they will not conduct heat from the utensils to your hand.

(*ii*) **Quilts are filled with fluffy cotton.** Cotton is a bad conductor of heat and the trapped air in it is a still worse conductor. So it provides a good heat insulation *i.e.*, it does not allow the heat to flow from our body to the surroundings. As a result, our body remains warm.

(*iii*) **When hot water is poured in a beaker of thick glass, the beaker cracks.** When we pour hot water in a glass beaker, the inner surface of the glass expands on heating. Since glass is a bad conductor of heat, the heat from inside does not reach the outside surface quickly. Hence the outer surface does not expand and the glass cracks.

(*iv*) **Thick walls are used in the construction of cold storage rooms.** Brick is a bad conductor of heat so that it reduces the flow of heat from the surroundings to the rooms. Still better heat insulation is obtained by using the hollow bricks. Air being a poorer conductor than a brick, it further checks the conduction of heat from outside.

(*v*) **To prevent ice from melting, it is wrapped in gunny bags.** A gunny bag is a poor conductor of heat and reduces the heat flow from outside to ice. Moreover, the air filled in the interspaces of a gunny bag, being very bad conductor of heat, further checks the conduction of heat from outside.

(*vi*) **Woollen clothes are warm because they have fine pores filled with air.** Wool and air are bad conductors of heat. Therefore, woollen clothes check the heat conduction from the body to the surroundings and thus keep the body warm in winter.

(*vii*) **An ice box is a double walled box.** The space between the walls is packed with insulators like cork powder, saw dust etc. Since these materials are bad conductors of heat, they check the flow of heat from outside.

(*viii*) **In cold countries, windows have two panes of glass with a thin layer of air in-between.** Since air is a bad conductor of heat, it checks the conduction of heat from the room to the surroundings. For the same reason, the windows of air conditioned railway coaches have double glass panes with air layer inbetween the two panes.

Example 22.1. **Estimate the rate at which ice melts in a wooden box 2 cm thick of inside measurements 60 × 60 × 60 cm. Assume that external temperature is 27°C and coefficient of thermal conductivity of wood = 0.1674 J $s^{-1}m^{-1}°C^{-1}$. Latent heat of fusion of ice = 336 × 10^3 J kg^{-1}.**

Solution. Area of box, A = Area of six faces = $6 \times (60 \times 60) = 21600$ cm^2 = 2.16 m^2

Thickness of box, x = 2 cm = 0.02 m

Temperature difference, $\theta_1 - \theta_2 = 27 - 0 = 27°C$

Now
$$Q = \frac{k A (\theta_1 - \theta_2) t}{x}$$

$\therefore$ Rate of heat flow, $\frac{Q}{t} = \frac{kA(\theta_1 - \theta_2)}{x} = \frac{0.1674 \times 2.16 \times 27}{0.02} = 488.14$ J

Let the mass of ice melted/second be m kg.

$\therefore$ $m \times 336 \times 10^3 = 488.14$

or $m = \frac{488.14}{336 \times 10^3} = 1.45 \times 10^{-3}$ kg/second $= 1.45$ g/second

Example 22.2. A layer of ice 10 cm thick is formed on a pond. The temperature of air is –10°C. Calculate how long it will take for the thickness of ice to increase by 1 mm. Density of ice = 1 g cm^{-3}, thermal conductivity of ice = 0.005 cal s^{-1} cm^{-1} °C^{-1}, latent heat of ice = 80 cal g^{-1}.

Solution. Let t seconds be the time required to increase the thickness of ice by 1 mm (= 0.1 cm).

Mass of ice required to be formed is given by ;

m = Volume × Density = (A* × 0.1) × 1 = 0.1A gram

In order to form more ice, it is necessary that heat flows from the lower surface to the upper surface of ice and finally into the atmosphere.

$\therefore$ To form the required amount of ice, the quantity of heat that must flow out of pond into atmosphere is

$Q = mL = 0.1A \times 80 = 8A$ cal

But $Q = \frac{kA(\theta_1 - \theta_2)t}{x}$

or $8A = \frac{kA(\theta_1 - \theta_2)t}{x}$ or $t = \frac{8x}{k(\theta_1 - \theta_2)}$

Here x = 10 cm ; k = 0.005 cal cm^{-1} s^{-1} °C^{-1} ; $\theta_1 - \theta_2 = 0 - (-10) = 10$°C

$\therefore$ $t = \frac{8 \times 10}{0.005 \times 10} = 1600$ s

Example 22.3. A rod 0.5 m in length and 10^{-3} m^2 in area of cross-section is heated at one end. The coefficient of thermal conductivity of the material of the rod is 90 cal s^{-1}m^{-1} °C^{-1}. In the steady state, the temperatures of the ends of the rod are 100°C and 0°C. Calculate (*i*) temperature gradient in the rod (*ii*) temperature at a point distant 10.0 cm from the hot end of the rod and (*iii*) rate of heat transmission.

Solution. (*i*) Temperature gradient in the rod

$= \frac{\text{Final temp.} - \text{Initial temp.}}{\text{Distance}} = \frac{\theta_2 - \theta_1}{x} = \frac{0 - 100}{0.5} = -200$°C/m

(*ii*) Change in temperature at a point distant 10 cm (= 0.1 m) from the hot end of rod

$= -200 \times 0.1 = -20$°C

$\therefore$ Temperature at a distance 10 cm from the hot end = 100 – 20 = 80°C

(*iii*) Rate of transmission of heat is given by ;

$H = \frac{Q}{t} = \frac{kA(\theta_1 - \theta_2)}{x}$

* A is the area of upper surface of the layer in cm^2.

Here, $k = 90$ cal m^{-1} s^{-1} $°C^{-1}$ $= 90 \times 10^{-3}$ kcal m^{-1} s^{-1} $°C^{-1}$; $A = 10^{-3}$ m^2 ;
$x = 0.5$ m ; $\theta_1 - \theta_2 = 100 - 0 = 100°C$

$$\therefore \quad H = \frac{(90\times10^{-3})\times10^{-3}\times100}{0.5} = \mathbf{1.8 \times 10^{-2}\ kcal/s}$$

Example 22.4. A slab of stone of area 0.36 m^2 and thickness 0.1 m is exposed on the lower surface to steam at 100°C. A block of ice at 0°C rests on the upper surface of the slab. If in one hour, 4.8 kg of ice is melted, calculate the thermal conductivity of the stone.

Solution. The amount of heat flowing through the slab is

$$Q = \frac{kA(\theta_1 - \theta_2)t}{x}$$

According to the conditions of the problem, $Q = mL$.

$$\therefore \quad mL = \frac{kA(\theta_1 - \theta_2)t}{x}$$

Here $m = 4.8$ kg ; $L = 80$ kcal/kg ; $A = 0.36$ m^2; $\theta_1 - \theta_2 = 100 - 0 = 100°C$;
$t = 1$ hour $= 3600$ s ; $x = 0.1$ m ; $k = ?$

$$\therefore \quad 4.8 \times 80 = \frac{k\times0.36\times100\times3600}{0.1}$$

or Thermal conductivity, $k = \dfrac{4.8\times80\times0.1}{0.36\times100\times3600} = \mathbf{3 \times 10^{-4}\ k\ cal\ s^{-1}m^{-1}{}^{\circ}C^{-1}}$

Example 22.5. As shown in figure 22.3, heat is conducted through a compound plate composed of two parallel plates *A* and *B* made of two different materials. *A* and *B* are respectively 3.6 cm and 4.2 cm thick. The co-efficients of thermal conductivities are 0.32 and 0.14 respectively. If in the steady state the temperatures of the surfaces of *A* and *B* are 96°C and 8°C respectively, find the temperature θ of their interface.

Solution. Figure 22.3 shows the conditions of the problem. Let θ°C be the temperature of the interface of the compound plate. In the steady state, the rate of flow of heat through each plate will be the same, *i.e.*,

$$H = \frac{Q}{t} = \frac{k_1A(\theta_1 - \theta)}{l_1} = \frac{k_2A(\theta - \theta_2)}{l_2}$$

or $$\frac{k_1A(\theta_1 - \theta)}{l_1} = \frac{k_2A(\theta - \theta_2)}{l_2}$$

Here $\theta_1 = 96°C$; $\theta_2 = 8°C$; $l_1 = 3.6$ cm ;
$l_2 = 4.2$ cm ; $k_1 = 0.32$; $k_2 = 0.14$

$$\therefore \quad \frac{0.32\times A\times(96-\theta)}{3.6} = \frac{0.14\times A\times(\theta-8)}{4.2}$$

or $$\frac{8(96-\theta)}{90} = \frac{\theta-8}{30}$$

$$\therefore \quad \theta = 72°C$$

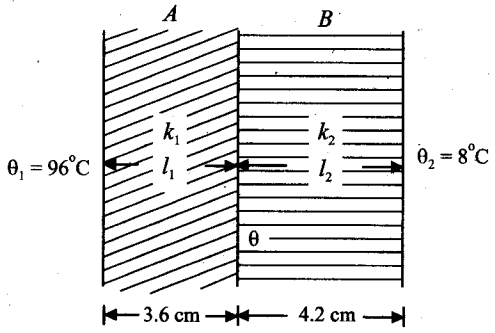

Fig. 22.3

Example 22.6. A room has a 4 m × 4 m × 10 cm concrete roof ($k_1 = 1.26$ W m^{-1} $°C^{-1}$). At some instant, the temperature outside is 46°C and inside is 32°C.

(*i*) Neglecting convection, calculate the amount of heat flowing per second into the room through the roof.

(ii) If bricks (k_2 = 0.65 $Wm^{-1}°C^{-1}$) of thickness 7.5 cm are laid down on the roof, calculate the new rate of heat flow under the same temperature conditions.

Solution. (*i*) Area of roof, $A_1 = 4 \times 4 = 16\ m^2$; Thickness of roof, $x_1 = 10\ cm = 0.1\ m$

Thermal resistance of the roof is given by ;

$$R_1 = \frac{x_1}{k_1 A_1} = \frac{0.1}{1.26 \times 16} = 4.96 \times 10^{-3}\ °CW^{-1}$$

$\therefore$ Rate of heat flow through the roof is

$$H_1 = \frac{Q}{t} = \frac{\theta_1 - \theta_2}{R_1} = \frac{46 - 32}{4.96 \times 10^{-3}} = 2822\ W$$

(*ii*) The thermal resistance of the brick is given by ;

$$R_2 = \frac{x_2}{k_2 A_2} = \frac{7.5 \times 10^{-2}}{0.65 \times 16} = 7.2 \times 10^{-3} °CW^{-1}$$

The equivalent thermal resistance of the roof now is

$$R = R_1 + R_2 = (4.96 + 7.2)\ 10^{-3} = 1.216 \times 10^{-2}\ °C\ W^{-1}$$

$\therefore$ Rate of flow of heat through the roof is

$$H_2 = \frac{Q}{t} = \frac{\theta_1 - \theta_2}{R} = \frac{46 - 32}{1.216 \times 10^{-2}} = 1152\ W$$

PROBLEMS FOR PRACTICE

1. One end of aluminium rod 40 cm long and 5 cm^2 in cross-section is maintained at a temperature of 100°C and the other end at 20°C. Find the amount of heat that will flow through the rod in 2 minutes. Given that coefficient of thermal conductivity of aluminium is 209 $Js^{-1}m^{-1}\ °C^{-1}$. **[2508 J]**
2. A metal rod of length 20 cm and diameter 2 cm is covered with non-conducting substance. One of its end is maintained at 100°C while the other end is put in ice at 0°C. It is found that 25 g of ice melts in 5 minutes. Calculate the coefficient of thermal conductivity of the metal. Specific latent heat of ice = 80 cal g^{-1}. **[0.424 cal $cm^{-1}\ s^{-1}\ °C^{-1}$]**
3. The temperature gradient in a 0.5 m long rod is 80°C/metre. The temperature of the hot end of the rod is 30°C. Find the temperature of the cold end. **[–10°C]**
4. One end of a rod of length 75 cm and cross-sectional area $10^{-2}\ m^2$ is heated. In the steady state, the temperature between its two ends is 90°C. If the thermal conductivity of the metal is 80 cal $s^{-1}\ m^{-1}\ °C^{-1}$, find (*i*) temperature gradient in the rod (*ii*) rate of heat flow. **[(*i*) 120°C/m (*ii*) 96 cal/s]**
5. The opposite faces of a cubical block of iron of cross-section 4 cm^2 are kept in contact with steam and melting ice. If there are no radiation losses, determine the quantity of ice melted in 10 minutes. The coefficient of thermal conductivity of iron = 83.72 J $s^{-1}m^{-1}°C^{-1}$ and latent heat of fusion of ice = 336×10^3 J kg^{-1}. **[0.3 kg]**

22.6. CONVECTION

We have seen that heat is transmitted through solids by conduction wherein energy is transferred from one molecule to another but the molecules themselves do not leave their mean positions. But in convection, heat is transmitted from one point to another by the actual bodily movement of the heated molecules.

The process by which heat is transmitted through a substance from one point to another due to the actual bodily movement of the heated particles of the substance is called **convection.**

In liquids and gases, heat is transmitted by convection because their molecules are quite free to move about. We shall discuss the mechanism of heat transfer by convection in liquids and gases.

(*i*) Consider water being heated in a vessel as shown in Fig. 22.4. The water at the bottom of the vessel is heated first and its density decreases *i.e.*, water molecules at the bottom are separated quite apart. These hot molecules have high kinetic energy and rise upward while the cold molecules come down to take their place. Thus each molecule at the bottom gets heated and rises, then cools and descends. This action sets up the flow of water molecules called *convection currents.* The convection currents transfer heat to the entire mass of water. Note that transfer of heat is by the bodily movement of the heated molecules.

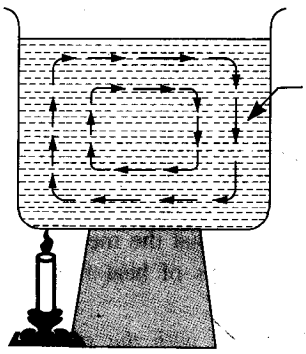

Fig. 22.4

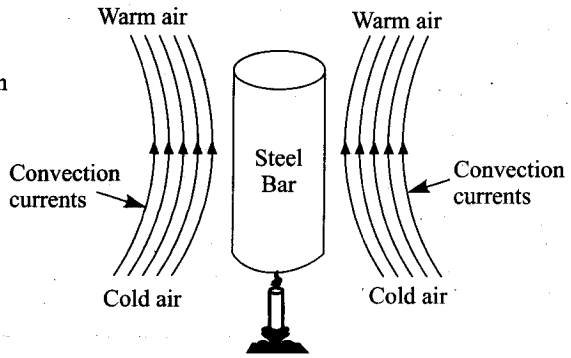

Fig. 22.5

(*ii*) In Fig. 22.5, the molecules of air in contact with the hot steel bar get more energy and rise upward. As the hot air moves upwards, it is replaced by cooler air which, in turn, also warms up. Thus a continuous current of air is set up around the hot bar, removing its heat by convection. We say that convection currents are set up in the air surrounding the bar.

Free and Forced Convection

(*i*) When a hot body is in contact with air under ordinary conditions as in Fig. 22.5, it removes heat from the body by a process called *free* or *natural convection.* Thus the air of the room removes heat of glowing incandescent lamp by free convection.

(*ii*) The convection process can be accelerated by employing a fan to create a rapid circulation of fresh air. This is called *forced convection.* Heat transfer by forced convection is used in most electric motors for efficient cooling.

22.7. APPLICATIONS OF CONVECTION

The mechanism of heat transfer by convection is utilised in many engineering applications. Two such applications are given below by way of illustration :

(*i*) **Heating of rooms.** The mechanism of heating a room by a heater is entirely based on convection. The air molecules in immediate contact with the heater are heated up. These air molecules acquire sufficient energy and rise upward. The cool air at the top being denser moves down to take their place. This cool air is in turn heated and moves upward. In this way, convection currents (See Fig. 22.6) are set up in the room which transfer heat to different parts of the room.

(*ii*) **Cooling of transformers.** Due to currents flowing in the windings of the transformer, enormous heat is produced. Therefore, transformer is always kept in a tank containing oil (transformer oil). The oil in contact with transformer body heats up, creating convection currents which follow the path as shown in Fig. 22.7. The warm oil comes in contact with the cooler tank, gives heat to it and descends to the bottom. It again warms up to rise upward once again. This process is

repeated again and again. The heat of the transformer body is thus carried away by convection to the tank. The tank, in turn, loses its heat by convection to the surrounding air.

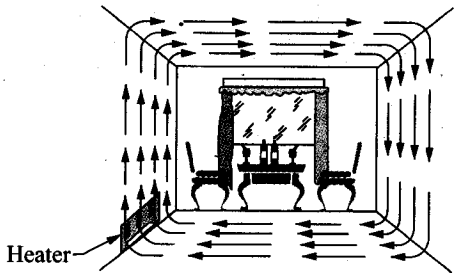

Fig. 22.6

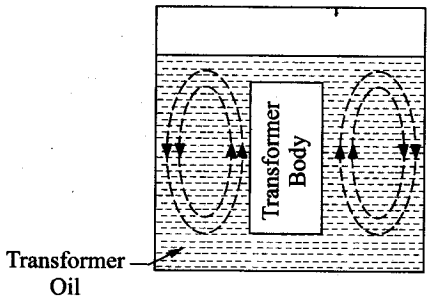

Fig. 22.7

22.8. RADIATION

A person sitting in front of a stove receives heat from the stove, although air in the room is cold. In like manner, sun rays falling on a body will warm it above the temperature of air. Such movement of heat from one place to another is known as transfer of heat by *radiation.* This method of transfer of heat* differs from conduction and convection by the fact that the medium through which transfer occurs is not heated. Thus earth receives great quantities of heat from the sun, although the space between the sun and earth is very cold.

The transfer of heat from one place to another in a straight line with the speed of light without heating the intervening medium is called **radiation**.

For transfer of heat by radiation, molecules are not needed *i.e.*, medium is not required. The fact that earth receives large quantities of heat from the sun shows that this heat can pass through empty space (*i.e.*, vacuum) between the sun and the atmosphere that surrounds the earth. In fact, transfer of heat by radiation has the same properties as light.

A natural question arises as to how heat transfer occurs in the absence of a medium (*i.e.*, molecules). All objects possess thermal energy due to their temperature. The rapidly moving molecules of a hot body emit waves (called *electromagnetic waves*) which travel in straight lines with the velocity of light. These are called *thermal radiations.* These carry energy with them and transfer it to the low-speed molecules of a cold body on which they fall. This results in the increase of molecular motion of the cold body and its temperature rises. Thus transfer of heat by radiation is a two-fold process ; the conversion of thermal energy into waves and reconversion of waves into thermal energy by the body on which they fall.

22.9. THERMAL RADIATION

The radiant energy emitted by a body solely on account of its temperature is called **thermal radiation.**

All objects above 0 K emit thermal radiation continuously. This radiation spans a continuous range of wavelengths and the distribution of energy amongst these wavelengths depends upon the temperature of the emitter and nature of radiating surface of the emitter. At temperatures below about 1000°C, the energy is associated almost entirely with infrared wavelengths (8×10^{-7} m to 4×10^{-4} m). When infrared radiation falls on the skin, it gives the sensation of warmth. It is what we usually have in mind when we speak of heat or thermal radiation and it is the main component

* Heat from the sun passes through space, then through the atmosphere and warms the earth's surface. This heat does not pass through the atmosphere by conduction for air is a poor conductor. Nor does it pass through by convection, for convection begins only after the earth is warmed. We also know that neither convection nor conduction is possible in the empty space between our atmosphere and the sun. We see that heat must be transmitted some other way – by radiation.

of radiation from a hot body. When thermal radiation is incident on a body, some of the radiation may be reflected, some transmitted and some may be absorbed and produce a heating effect.

Properties of thermal radiation. Thermal radiation, like light, is electromagnetic radiation. Therefore, it has all the general properties of electromagnetic waves.

(*i*) *Thermal radiation can travel through vacuum.* The thermal radiation reaches earth after passing through vacuum between the sun and atmosphere.

(*ii*) *Thermal radiation travels in straight lines.* If an obstacle is kept between the source of heat and the receiver, the thermal radiation is cut off in the shadow region of obstacle.

(*iii*) *Thermal radiation obeys the laws of reflection.* We can show the reflection of thermal radiation by two concave mirrors set up as shown in Fig. 22.8. At the focus (F_1) of one mirror, we place an iron ball heated to just below redness. At the focus (F_2) of the other mirror, we put the bulb of a thermometer covered with lamp black to make it a good absorber. The mercury rises in the stem of the thermometer. If we move either the bulb or the ball away from the focus, the mercury falls back ; the bulb has been therefore receiving heat from the ball by reflection at the two mirrors.

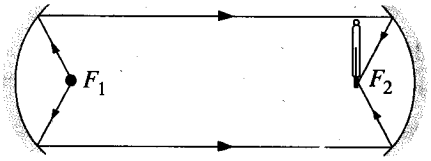

Fig. 22.8

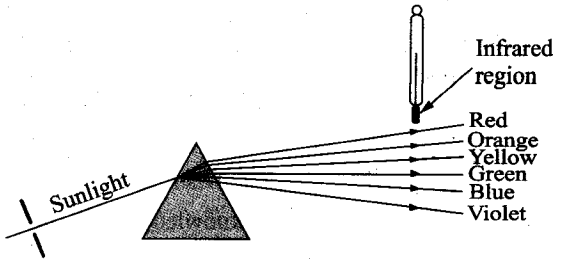

Fig. 22.9

(*iv*) *Thermal radiation can be refracted.* Fig. 22.9 shows the refraction of sunlight by prism. The spectrum of the sunlight can be explored by a sensitive thermometer whose bulb is blackened. The rise of mercury will be maximum in the infrared region. This shows that sun rays consist of thermal radiation as well as light.

(*v*) *Thermal radiation travels with the speed of light.* During solar eclipse, we do not get heat and light of the sun. Once the solar eclipse is over, we get both heat and light simultaneously. This shows that speed of thermal radiation is equal to the speed of light.

(*vi*) *Thermal radiation does not heat the medium through which it passes.* If a piece of paper is placed at the focus of a convex lens and the sun rays are incident on the lens, the paper is burnt. However, the lens does not become hot.

(*vii*) *Thermal radiation obeys inverse square law i.e.,* the intensity of radiation at a point is inversely proportional to the square of the distance of the point from the source of radiation.

(*viii*) *They exhibit the phenomena of interference, diffraction and polarisation.*

Note. Both light and thermal radiation are electromagnetic waves. The main difference between them is in their wavelength (or frequency). The wavelength of thermal radiation lies in the range 8×10^{-7} m to 4×10^{-4} m. However, the wavelength of light lies in the range 4×10^{-7} m to 8×10^{-7} m.

22.10. REFLECTION, TRANSMISSION AND ABSORPTION

Measurements have been made which give the amount of thermal energy approaching the earth from the sun. At the upper limit of our atmosphere, it is about 1340 W m^{-2}. At the surface of the earth, it is less than this because of absorption in the atmosphere. *In general, when thermal energy falls on any surface, a part of it is absorbed, a part is reflected back and a part is transmitted.*

Suppose Q is the thermal energy incident on a surface. Some of the energy is reflected by the surface (R), some is absorbed (A) and some is transmitted (T). The total energy reflected, absorbed and transmitted per metre^2 per second is equal to the energy falling on the surface over the same area and in the same time *i.e.*,

$$R + A + T = Q$$

Dividing both sides by Q, we have,

$$\frac{R}{Q}+\frac{A}{Q}+\frac{T}{Q} = 1$$

or $$r + a + t = 1 \qquad ...(i)$$

where r, a and t are the fractions of the radiant energy which are respectively reflected, absorbed and transmitted by the surface and are called *reflectance* (or *reflecting power*), *absorptance* (or *absorbing power*) and *transmittance* (or *transmitting power*) of the surface.

$$\text{Reflectance, } r = \frac{R}{Q} = \frac{\text{Thermal radiation reflected from the surface in time } t}{\text{Total thermal radiation incident on the surface in time } t}$$

$$\text{Absorptance, } a = \frac{A}{Q} = \frac{\text{Thermal radiation absorbed by the surface in time } t}{\text{Total thermal radiation incident on the surface in time } t}$$

$$\text{Transmittance, } t = \frac{T}{Q} = \frac{\text{Thermal radiation transmitted by the surface in time } t}{\text{Total thermal radiation incident on the surface in time } t}$$

Eq. (i) expresses common knowledge. If a surface does not transmit radiation, $t = 0$ so that :

$$r + a = 1$$

Clearly, if r is more, a is less and *vice-versa.*

Therefore, *good reflectors are bad absorbers of heat and bad reflectors are good absorbers of heat.* If for a body, $r = 0$ and $t = 0$, then $a = 1$. This means that the body will absorb all the radiant energy incident on it and hence will appear black. Such a body is called a *perfectly black body.*

Eq. (i) above is over-simplified. For a body may transmit some wavelengths (colours if visible) and absorb or reflect others. If we let Q denote the radiant energy of a particular wavelength λ, then,

$$r_\lambda + a_\lambda + t_\lambda = 1 \qquad ...(ii)$$

where r_λ, a_λ and t_λ are respectively called monochromatic reflectance, absorptance and transmittance of the surface of the body.

The truth of eq. (ii) is well shown by gold which reflects yellow light better than other colours.

Note. The reflectance (r), absorptance (a) and transmittance (t) depend upon (i) the nature of the surface of the body and (ii) wavelength of incident radiation. However, they do not depend on the nature of material of the body.

22.11. PERFECTLY BLACK BODY

A perfectly black body is one which absorbs completely all the radiation falling on it.

Since a perfectly black body is a perfect absorber, it will also be a perfect radiator. When a perfectly black body is heated to a high temperature, it emits radiation of all possible wavelengths. No known body is a perfectly black body. A surface coated with lamp-black absorbs 96% to 98% of the incident radiation and may be considered as a perfectly black body for all practical purposes. A good black body can be made simply by punching a small hole in the lid of a closed empty tin. The hole looks almost black, although the shinning tin is a good reflector. The hole looks black because the radiation that enters through it is reflected from the inside walls many times and is partly absorbed at each reflection until none remains.

Fery's black body. The simplest and most commonly used black body was designed by Fery. Fig. 22.10 shows Fery's black body. It consists of a hollow double walled metallic sphere having small opening O on one side and conical projection P just opposite to it. The inside of the sphere is coated with lamp black. Any radiation entering the hole is reflected many times at the inner walls of the sphere. After a few reflections, almost entire radiation is absorbed. The conical projection P does not permit any direct reflection.

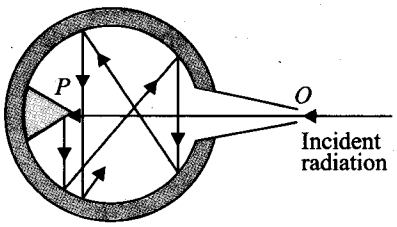

Fig. 22.10

When this black body is heated, the radiation will come out from the hole. Therefore, the hole O now acts as a black body radiator. Since it comes from hole, it is sometimes called *cavity radiation.* It is important to note that only the hole (and not the walls) acts as a black body radiator.

22.12. TERMINOLOGY CONCERNING THERMAL RADIATION

Every object at a temperature greater than 0K emits radiation (known as *thermal radiation)* of all wavelengths within a certain range. The amount of thermal energy radiated for different wavelength intervals is different and depends on temperature and nature of the surface. We now define some terms relating to thermal radiation.

(*i*) **Monochromatic emissive power.** *The* **monochromatic emissive power** *of a body at a given temperature and for a given wavelength* λ *is defined as the radiant energy of that wavelength emitted per second per unit area of its surface within unit wavelength interval around* λ *i.e.* between $(\lambda - 1/2)$ and $(\lambda + 1/2)$. It is denoted by the symbol e_λ.

(*ii*) **Emissive power.** *The* **emissive power** *of a body at a particular temperature is the total energy of all wavelengths radiated per second per unit area of the body.* Since the wavelengths emitted range from 0 to infinity,

$$\therefore \text{ Emissive power, } e = \int_0^\infty e_\lambda \, d\lambda.$$ Its SI unit is $Jm^{-2}\ s^{-1}$ or Wm^{-2}.

(*iii*) **Emissivity.** *The* **emissivity** *of a body is the ratio of the emissive power of the body to the emissive power of the perfectly black body at the same temperature.* It is denoted by the symbol ε.

$$\text{Emissivity, } \varepsilon = \frac{e}{E} \text{ or } e = \varepsilon E$$

where E is the emissive power of the perfectly black body at the same temperature as that of the given body.

The emissivity is a number whose value depends upon the nature of the surface of the body. It ranges in value from 0 to 1. Clearly, for a perfectly black body, $\varepsilon = 1$. Dark, rough surfaces have emissivity close to 1 while bright, shiny surfaces have ε close to zero. For polished copper, $\varepsilon = 0.3$. Generally, good emitters are also good absorbers of radiation.

(*iv*) **Monochromatic absorptive power.** *The* **monochromatic absorptive power** *of a body for a given wavelength* λ *is defined as ratio of thermal energy absorbed by it in a given time to the total thermal energy incident on it for the same time, both in the unit wavelength around* λ. It is denoted by a_λ.

22.13. ENERGY DISTRIBUTION OF BLACK-BODY RADIATION

Figure 22.11 illustrates the way in which the energy radiated by a black body is distributed amongst the various wavelengths. Note that general shape of the curves is the same for all temperatures. These curves provide the following information regarding black body radiation :

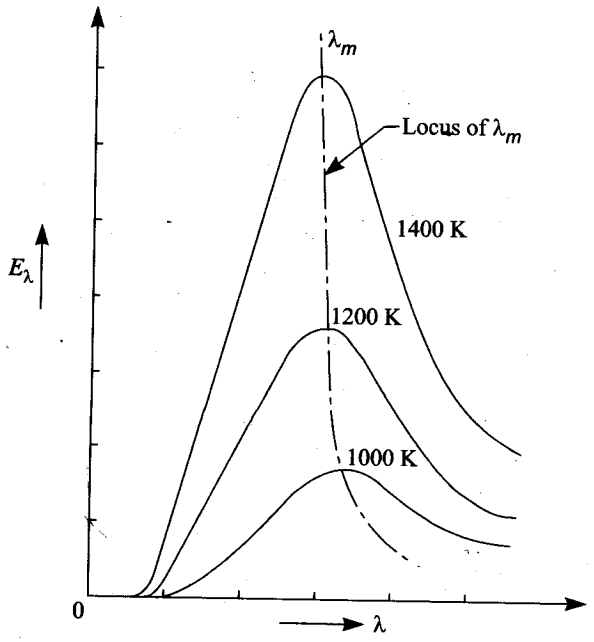

Fig. 22.11

(*i*) The total energy emitted by a black body increases rapidly with increase in temperature for any wavelength.

(*ii*) At a given temperature T (= 1200 K), with increase in wavelength, the energy E_λ first increases, reaches a maximum and then decreases. This means that for a given temperature, the radiant energy emitted by a black body is maximum for a particular wavelength ; this wavelength is represented by λ_m.

(*iii*) As the temperature of the body increases, the peak of the curve shifts toward shorter wavelength *i.e.*, maximum E_λ is obtained at smaller value of λ. In 1896, Wien established the following relation between T and λ_m :

$$\lambda_m T = \text{Constant} \qquad ...(i)$$

where T is the absolute temperature of the body. This relation is known as *Wien's displacement law.* The value of the constant is found by experiment to be 2.9×10^{-3} m K. Eq. (*i*) shows that λ_m is inversely proportional to the absolute temperature T of the body.

(*iv*) The area enclosed by a particular curve represents the total radiant energy (of all wavelengths) per second per unit area emitted by the black body at that temperature. We note from Fig. 22.11 that as the temperature of the body increases, the area enclosed by the curve also increases. When the area enclosed by a curve is measured, it is found to be directly proportional to the fourth power of the corresponding absolute temperature *i.e.*,

$$E \propto T^4$$

This is Stefan's law. Thus the curves also embody Stefan's law.

22.14. WIEN'S DISPLACEMENT LAW

The energy distribution in the spectrum of a black-body radiation shows that for any temperature T, there is one wavelength (λ_m) for which the energy emitted by the black body is maximum. This wavelength λ_m is inversely proportional to the absolute temperature (T) of the black body *i.e.*,

$$\lambda_m \propto \frac{1}{T}$$

or $$\lambda_m T = \text{Constant} = b \qquad ...(i)$$

Eq. (*i*) is known as *Wien's displacement law.* The value of constant b is found by experiment to be 2.9×10^{-3} mK in SI units. Thus Wien's displacement law may be stated as under :

The product of the wavelength (λ_m) at which maximum energy is emitted and the absolute temperature (T) of the black body is always constant.

This means that when a body is heated to ordinary temperature, the maximum emission of energy corresponds to longer wavelength. But as the temperature of the body is raised, the radiation of shorter and shorter wavelength is also emitted in increasing quantity. This law illustrates the well known observation that when iron is heated, it first becomes light-red, then dark-red, then yellow and ultimately it becomes white.

22.15. STEFAN'S LAW

The rate at which objects emit radiant energy depends very strongly on their temperature. It also depends on their surface area and the nature of the surface. According to **Stefan's law,** the radiant energy emitted per second by a body is given by ;

$$\text{Power, } P = \varepsilon \sigma A T^4 \qquad ...(i)$$

where A = Surface area of the body

T = Absolute temperature of the body

σ = A constant called *Stefan-Boltzmann constant* and has value of 5.67×10^{-8} W m^{-2} K^{-4}

ε = A constant called *emissivity* of the surface of the body

The factor ε depends upon the nature of surface of the body. It is a dimensionless number and its value ranges between 0 and 1. For a perfectly black body, $\varepsilon = 1$.

Eq. (*i*) is known as *Stefan's law* and may be stated as under :

The rate at which a body radiates energy (i.e., energy radiated per second) is directly proportional to the fourth power of the absolute temperature of the body. Thus an object at a temperature of 273°C (546 K) radiates 16 times more energy per second as much as it does at 0°C (273 K).

According to Stefan's law, all bodies radiate energy, no matter what their temperature happens to be. Why then do they not lose all their thermal energy by radiation and cool down to 0 K? The answer is that they also absorb radiation from surrounding objects and eventually come to thermal equilibrium with their environment. The book in your hand is radiating, but it is also absorbing radiation from its surroundings. Suppose an object at absolute temperature T_1 is placed in an environment where the absolute temperature of the surroundings is T_0 ($T_1 > T_0$).

Radiant energy emitted per second by the object is

$$P_{emit} = \varepsilon \sigma A T_1^4$$

Radiant energy absorbed per second by the object is

$$P_{abs} = \varepsilon \sigma A T_0^4$$

Net loss of energy per second by the object is

$$P_{net} = \varepsilon \sigma A \ (T_1^4 - T_0^4)$$

Since $T_1 > T_0$, there is a net loss of energy and the object cools. Eventually, the object will come to thermal equilibrium with the environment (*i.e.*, $T_1 = T_0$). Note that we have used the same value of emissivity for absorption as for radiation. This must be correct since the net heat exchange must go to zero when $T_1 = T_0$. *Thus a good radiator is also a good absorber.*

22.16. KIRCHHOFF' S LAW

According to this law, *the ratio of the emissive power to the absorptive power for radiation of a given wavelength is the same for all bodies at the same temperature and is equal to the emissive power of a perfectly black body at that temperature corresponding to same wavelength i.e.*

$$\frac{e_\lambda}{a_\lambda} = E_\lambda \text{ (constant)} \qquad ...(i)$$

Here e_λ and a_λ are the emissive power and absorptive power of a body corresponding to wavelength λ and E_λ is the emissive power of a perfectly black body at the same temperature and corresponding to the wavelength λ.

Applications. Eq. (*i*) reveals an interesting fact. Since E_λ is constant, if e_λ is large, then a_λ is also large. Therefore, if a body emits strongly the radiation of a particular wavelength, it must also absorb the same wavelength strongly. In other words, *good emitters of heat are also good absorbers and vice-versa.*

(*i*) If a piece of glazed China with some dark painting on it is heated to about 1000 K and then taken to a dark room, it will be observed that the dark paintings appear much brighter than the white portion. It is because the dark paintings being better absorber are also better emitter.

(*ii*) A red glass heated in a furnace and then taken out appears green. When cold, red glass appears red because it absorbs green light strongly. When heated, it emits green light strongly.

(*iii*) A large number of dark lines are observed in the solar spectrum. These lines were discovered by Fraunhoffer and are, therefore, called *Fraunhoffer's lines.* The sun has a central hot core which emits a continuous spectrum. Surrounding this central part is a comparatively cooler atmosphere made up of vapours and gases which come from the central part. It contains elements such as H_2, O_2, Na, Fe *etc.* in vapour form. These elements effectively absorb the wavelengths that they themselves emit in the hot core. Therefore, these wavelengths will be missing when solar spectrum is examined. The dark lines in the solar spectrum represent the missing wavelengths.

Example 22.7. The temperature of a furnace is 2324°C and the intensity is maximum in its radiation spectrum nearly at 12000 Å. If the intensity in the spectrum of a star is maximum nearly at 4800 Å, then calculate the surface temperature of the star.

Solution. According to Wien's displacement law, $\lambda_m T$ = Constant

or $$\lambda_{m1} T_1 = \lambda_{m2} T_2$$

or $$T_2 = T_1 \times \left(\frac{\lambda_{m1}}{\lambda_{m2}}\right)$$

Here $\lambda_{m1} = 12000$ Å ; $\lambda_{m2} = 4800$ Å ; $T_1 = 2324°C = 2597$ K

$\therefore$ $$T_2 = 2597 \times \left(\frac{12000}{4800}\right) = 6492.5 \text{ K} = \mathbf{6219.5°C}$$

Example 22.8. The wavelength corresponding to E_{max} for the moon is 14 microns. Estimate the temperature of the moon if $b = 2.884 \times 10^{-3}$ m K.

Solution. According to Wien's displacement law,

$$\lambda_m T = \text{Constant} = b \quad \text{or} \quad T = \frac{b}{\lambda_m}$$

Here, $b = 2.884 \times 10^{-3}$ m K ; $\lambda_m = 14$ microns $= 14 \times 10^{-6}$ m

$$\therefore \quad T = \frac{2.884 \times 10^{-3}}{14 \times 10^{-6}} = \mathbf{206\ K}$$

Example 22.9. Calculate the energy radiated in one minute by a black body of surface area 200 cm^2 maintained at 127°C. Stefan's constant = 5.7×10^{-8} Wm^{-2} K^{-4}.

Solution. The energy radiated per sec by a body is given by ;

$$P = \varepsilon \sigma A \ (T^4 - T_0^4) = \varepsilon \sigma A T^4 \quad (\text{If } T_0 << T)$$

Here, $\varepsilon = 1$ (black body) ; $\sigma = 5.7 \times 10^{-8}$ W m^{-2} K^{-4} ; $T = 127°C = 400$ K ; $A = 200$ cm^2 $= 200 \times 10^{-4}$ m^2

$$\therefore \quad P = 1 \times 5.7 \times 10^{-8} \times 200 \times 10^{-4} \times (400)^4 = 29.184 \text{ J/s}$$

Energy radiated in one minute (= 60 s)

$$= 29.184 \times 60 = \mathbf{1751\ J}$$

Example 22.10. A patient waiting to be seen by his physician is asked to remove all his clothes in an examination room that is at 16°C. Calculate the rate of heat loss by radiation from the patient, given that his skin temperature is 34°C and his surface area is 1.6 m^2. Assume emissivity = 0.80. Stefan's constant = 5.67×10^{-8} Wm^{-2} K^{-4}.

Solution. Temperature of patient's skin, $T = 34°C = 307$ K

Temperature of surroundings, $T_0 = 16°C = 289$ K

The heat loss per second by radiation is

$$P = \varepsilon \sigma A \ (T^4 - T_0^4)$$

Here $\varepsilon = 0.80$; $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} ; $A = 1.6$ m^2 ; $T = 307$ K ; $T_0 = 289$ K

$$\therefore \quad P = 0.80 \times 5.67 \times 10^{-8} \times 1.6 \times [(307)^4 - (289)^4] = 138 \text{ J/s} = \mathbf{138\ W}$$

Example 22.11. The approximate surface temperature of the sun is 6000 K. Taking the sun to be a sphere of radius 7×10^8 m and assuming an emissivity of 0.93, calculate the energy radiated per second from its surface. $\sigma = 5.67 \times 10^{-8}$ $Wm^{-2}K^{-4}$.

Solution. The energy radiated per second by sun's surface is

$$P = \varepsilon \sigma A T^4$$

Here $\varepsilon = 0.93$; $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4} ; $A = 4\pi R^2 = 4\pi (7 \times 10^8)^2 = 6 \times 10^{18}$ m^2 ;

$T = 6000$ K

$$\therefore \quad P = 0.93 \times 5.67 \times 10^{-8} \times 6 \times 10^{18} \times (6000)^4 = \mathbf{4 \times 10^{26}\ W}$$

This enormous power is due to sun's large size and its high temperature.

Example 22.12. Estimate the temperature of the surface of the sun from the following data : average radius of earth's orbit = 1.5×10^8 km; average radius of the sun = 7.0×10^5 km ; solar radiant power on earth at noon = 1400 Wm^{-2}. Assume the sun to be a perfectly black body. Stefan's constant = 5.67×10^{-8} $Wm^{-2}K^{-4}$.

Solution. The energy radiated per second by the sun is

$$P = \varepsilon \sigma A T^4$$

where P = solar radiant power $\times 4\pi r^2$

$$= 1400 \times 4\pi r^2 = 1400 \times 4\pi \times (1.5 \times 10^{11})^2 \text{ W}$$

$$\therefore \quad 1400 \times 4\pi \times (1.5 \times 10^{11})^2 = \varepsilon \sigma A T^4$$

or $$T^4 = \frac{1400 \times 4\pi \times (1.5 \times 10^{11})^2}{\varepsilon \sigma A}$$

Here $\varepsilon = 1$; $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4} ; $A = 4\pi R^2 = 4\pi \times (7.0 \times 10^8)^2$ m^2

$$\therefore \quad T^4 = \frac{1400 \times 4\pi \times (1.5 \times 10^{11})^2}{1 \times 5.67 \times 10^{-8} \times 4\pi \times (7.0 \times 10^8)^2} = 11.34 \times 10^{14}$$

or $$T = \sqrt[4]{11.34 \times 10^{14}} = \mathbf{5.8 \times 10^3 \text{ K}}$$

Example 22.13. An indirectly heated filament is radiating maximum energy of wavelength 2.16×10^{-7} m. Find the net amount of heat energy lost per second per unit area, the temperature of surrounding air is 13°C. Given $b = 2.88 \times 10^{-3}$ mK, $\sigma = 5.77 \times 10^{-8}$ Js^{-1} $m^{-2}K^{-4}$.

Solution. By Wien's law, $\lambda_m T = b$

$$\therefore \quad \text{Temperature of filament, } T = \frac{b}{\lambda_m} = \frac{2.88 \times 10^{-3}}{2.16 \times 10^{-7}} = 13333.3 \text{ K}$$

Temperature of surrounding, $T_0 = 13 + 273 = 286$ K

$\therefore$ Net amount of heat energy lost per second per unit area is

$$E = \sigma (T^4 - T_0^4) = 5.77 \times 10^{-8} [(13333.3)^4 - (286)^4] = \mathbf{18.24 \times 10^8 \text{ J s}^{-1} \text{ m}^{-2}}$$

Example 22.14. A spherical black body of radius 12 cm radiates 450 W power at 500 K. If the radius were halved and the temperature doubled, what would be the power radiated?

Solution. Power radiated, $P = A\sigma T^4 = 4\pi r^2 \sigma T^4 \quad \therefore P \propto r^2 T^4$

For the first case ; $P_1 \propto r^2 T^4$; For the second case, $P_2 \propto (r/2)^2 (2T)^4$

$$\therefore \quad \frac{P_2}{P_1} = \frac{(r/2)^2 (2T)^4}{r^2 T^4} = \frac{(2)^4}{4} = 4$$

or $$P_2 = 4P_1 = 4 \times 450 = \mathbf{1800 \text{ W}}$$

Example 22.15. At what temperature will the filament of a 100 W lamp operate if it is supposed to be a perfectly black body of area 1 cm^2? Given $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}.

Solution. According to Stefan's law, energy radiated per second is given by ;

$$P = \varepsilon \sigma T^4 A$$

or $$T^4 = \frac{P}{\varepsilon \sigma A}$$

Here $P = 100$ W ; $\varepsilon = 1$ (black body) ; $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4} ; $A = 1$ $cm^2 = 10^{-4}$ m^2

$$\therefore \quad T^4 = \frac{100}{1 \times 5.67 \times 10^{-8} \times 10^{-4}} = \left(\frac{100}{5.67}\right) \times 10^{12}$$

or $$T = \left(\frac{100}{5.67}\right)^{1/4} \times 10^3 = \mathbf{2049 \text{ K}}$$

PROBLEMS FOR PRACTICE

1. The surface area of a black body is 5×10^{-4} m^2 and its temperature is 727°C. Calculate the energy radiated by it per minute. Stefan's constant = 5.67×10^{-8} W m^{-2} K^{-4}. **[1.7×10^3 J]**
2. A 20 cm long and 0.01 mm diameter tungsten wire is kept in vacuum at 2500 K. The tungsten wire radiates only 36% heat as compared to a perfectly black body. Calculate the rate of radiation of the tungsten wire in watt. Stefan's constant = 5.67×10^{-8} W m^{-2} K^{-4}. **[5 W]**
3. Calculate the temperature in K at which a perfectly black body radiates at the rate of 5.67×10^4 W m^{-2}. Stefan's constant = 5.67×10^{-8} W m^{-2} K^{-4}. **[1000 K]**
4. Calculate the energy radiated per minute from the filament of an incandescent lamp at 2000 K if the surface area is 5×10^{-5} m^2 and its relative emittance is 0.85. Stefan's constant = 5.7×10^{-8} W m^{-2} K^{-4}. **[2315 J]**
5. The ratio of radiant energies radiated per unit surface by two bodies is 16 : 1. The temperature of the hotter body is 1000 K. Calculate the temperature of the other body. **[500 K]**
6. An indirectly heated filament is radiating maximum energy of wavelength 2.16×10^{-7} m. Find the net amount of heat energy lost per second per unit area if the surrounding temperature is 13°C. Given b = 0.00288 m K ; $\sigma = 5.77 \times 10^{-8}$ W m^{-2} K^{-4}. **[1.824×10^9 Js^{-1} m^{-2}]**
7. A piece of metal loses 255 J heat per second by radiation when its temperature is 1200K and the temperature of the surroundings is 300 K. What will be the rate of loss of heat when the temperature of the metal is 600 K? **[15 J s^{-1}]**
8. A sphere of radius 6.0 cm at 1200°C is suspended in vacuum in an enclosure at 500°C. Find the rate of loss of heat, assuming that it is a black body. Stefan's constant = 5.7×10^{-8} W m^{-2} K^{-4}. **[5.186×10^3 W]**

22.17. NEWTON'S LAW OF COOLING

This law applies when a body is cooling under conditions of natural convection (*i.e.* body cooling in still air) and the temperature difference between the body and surroundings is small (< 30°C). It may be stated as under :

Under conditions of natural convection, the rate of cooling (i.e. rate of loss of heat) is directly proportional to temperature difference between the body and surroundings, provided the temperature difference is small (< 30°C) i.e.

Rate of loss of heat ∝ Temperature difference between body and surroundings

Rate of loss of heat $\propto (\theta - \theta_0)$

where θ = Temperature of the body in °C

θ_0 = Temperature of the surroundings in °C

Proof. We shall prove Newton's law of cooling from Stefan's law. Consider a body maintained at temperature T (kelvin). Let T_0 (kelvin) be the temperature of the surroundings.

According to Stefan's Law,

Heat energy lost per second per unit area, $E = \varepsilon \sigma (T^4 - T_0^4)$

where σ = Stefan's constant, ε = emissivity of surface of the body

Let the small temperature difference between the body and surroundings be ΔT. Therefore,

$$T = T_0 + \Delta T$$

$$\therefore \quad E = \varepsilon \sigma [(T_0 + \Delta T)^4 - T_0^4] = \varepsilon \sigma T_0^4 \left[\left(1 + \frac{\Delta T}{T_0} \right)^4 - 1 \right]$$

Expanding by Binomial theorem, we have,

$$E = \varepsilon \sigma T_0^4 \left[\left(1 + \frac{4\Delta T}{T_0} + \text{terms containing higher powers of } \frac{\Delta T}{T_0} \right) - 1 \right]$$

Since $\Delta T / T_0$ is very small, the terms containing higher powers of $\Delta T / T_0$ can be neglected.

$$\therefore \quad E = \varepsilon \sigma T_0^4 \left[\left(1 + \frac{4\Delta T}{T_0} \right) - 1 \right] = 4\varepsilon \sigma T_0^3 \Delta T = 4\varepsilon \sigma T_0^3 (T - T_0)$$

$$(\because \Delta T = T - T_0)$$

Since the factor $4 \varepsilon \sigma T_0^3$ is constant, we get,

$$E \propto (T - T_0) \quad \text{or} \quad E \propto [(\theta + 273) - (\theta_0 + 273)]$$

$$\therefore \quad E \propto (\theta - \theta_0)$$

Thus Newton's law of cooling stands proved.

Notes. (*i*) Rate of loss of heat $\propto \theta - \theta_0$

or Rate of loss of heat $= k (\theta - \theta_0)$

where k is a constant of proportionality whose value depends on both the nature and the area of body's surface.

(*ii*) Dulong and Petit found that rate of loss of heat was directly proportional to $(\theta - \theta_0)^{5/4}$ if $(\theta - \theta_0)$ is greater than 30°C.

$\therefore$ Rate of loss of heat $\propto (\theta - \theta_0)^{5/4}$ ..... for $(\theta - \theta_0) > 30°C$

Example 22.16. A body cools in 7 minutes from 60°C to 40°C. What will be its temperature after the next 7 minutes? The temperature of the surroundings is 10°C. Assume that Newton's law of cooling holds good throughout the process.

Solution. According to Newton's law of cooling, we have,

$$mc \frac{(\theta_1 - \theta_2)}{t} = k \left(\frac{\theta_1 + \theta_2}{2} - \theta_0 \right)$$

Here $\theta_1 = 60$ °C ; $\theta_2 = 40°C$; $\theta_0 = 10°C$; $t = 7$ min $= 420$ s

$$\therefore \quad mc \frac{(60 - 40)}{420} = k \left(\frac{60 + 40}{2} - 10 \right)$$

or $$mc \frac{20}{420} = k \times 40 \qquad \ldots (i)$$

Suppose after next 7 minutes (= 420s) the temperature of the body is θ °C.

$$\therefore \quad mc \frac{40 - \theta}{420} = k \left(\frac{40 + \theta}{2} - 10 \right) \qquad \ldots (ii)$$

Dividing eq. (*i*) by eq. (*ii*), we get,

$$\frac{20}{40 - \theta} = \frac{40}{\left(\frac{40 + \theta}{2} - 10 \right)}$$

On solving, we get, $\theta = 28°C$

PROBLEMS FOR PRACTICE

1. Two bodies A and B at temperatures 327°C and 127°C respectively are placed in an evacuated enclosure maintained at a temperature of 27°C. Compare their rates of cooling. **[6.94]**
2. A body at 80°C cools to 64°C in 5 minutes and to 52°C in the next 5 minutes. What will be its temperature after another 5 minutes? Also calculate the temperature of the surroundings. **[43°C ; 16°C]**

22.18. SUN

The sun is the central body of the solar system. It is the nearest star and is situated at an average distance of 1.496×10^{11} m from earth. It is the only source of energy (both heat and light) for the whole solar system. From the centre to the surface of the sun, the temperature falls from around 15000,000 K to 6000 K. At such high temperatures, no element can exist in solid or liquid form. The sun is composed predominantly of hydrogen and helium with about 1% of other elements. It generates its energy by nuclear fusion processes.

(*i*) **Structure of Sun.** At the core of the sun, we find a temperature of 15 million kelvin and a pressure of 1 billion atmospheres. In this environment, the body of the sun is gaseous ; in fact the gases (primarily hydrogen and helium) are almost completely ionised. Temperature, pressure and density increase from the sun's surface to the centre where energy is liberated by nuclear fusion processes. The central part of the sun is made of an extremely dense mixture of gases. It is called *photosphere.* The temperature at the core of photosphere is about 15 million kelvin. Outside the photosphere is another spherical region called *chromosphere.* It is at relatively low temperature and contains various elements in the gaseous form.

(*ii*) **Source of Sun energy.** It is known that hydrogen and helium constitute more than 98% of the mass of the sun and the rest are other elements. The sun has been radiating huge amount of energy (about 3.8×10^{26} joules/second) for billions of years. Chemical reactions cannot possibly be the source of sun's energy because the energy released in such reactions is very small. Similarly, it cannot be due to nuclear fission because the amount of heavy elements present in the sun is very small. It is now established that nuclear fusion reactions are responsible for the huge amount of energy of the sun.

At the core of the sun, there is a temperature of 15 million kelvin and a pressure of 1 billion atmospheres. In this environment, the atoms tend to fuse together to cause nuclear fusion. In the process, there occurs a mass defect (Δm) and a huge amount of energy is released according to the following relation :

$$E = (\Delta m)\, c^2 \text{ where } c = 3 \times 10^8 \text{ m s}^{-1}$$

In fact, within the core of the sun, the equivalent of thousands of hydrogen bombs go off every second. It is this release of energy in the sun's core that keeps it in the gaseous state, inspite of the fact that pressure at the core of the sun is 1 billion atmospheres.

(*iii*) **Important parameters.** The two important parameters are (*a*) solar constant and (*b*) solar luminosity.

(*a*) **Solar constant.** *It is the amount of energy received from the sun per second per unit area by a prefectly black body placed on the earth ; the area being perpendicular to the direction of radiation.* It is represented by the symbol S and its value is 1.388×10^3 W/m^2 or 2 cal/cm^2/ minute.

$$S \doteq 1.388 \times 10^3 \text{ W/m}^2 \qquad (\text{or } 2 \text{ cal/cm}^2\text{/minute})$$

This means that at a distance of 1 A.U., the sun's energy could raise the temperature of 1 cm^2 of water by 2 K in one minute.

(*b*) **Solar luminosity.** *It is the amount of energy emitted by the sun per second in all directions.* It is represented by the symbol L_s. If r (= 1 A.U. = 1.496×10^{11}m) is the distance of the sun from earth, then,

$$L_s = 4\pi r^2 S$$

Now $r = 1.496 \times 10^{11}$ m ; $S = 1.388 \times 10^3$ W/m^2

$$\therefore \quad L_s = 4\pi \times (1.496 \times 10^{11})^2 \times 1.388 \times 10^3 = 3.9 \times 10^{26} \text{ W}$$

(*iv*) **Surface temperature.** We can determine the surface temperature of the sun by using Stefan's law. Consider the sun to be a perfect black body of radius R and temperature T. Let r (= 1 A.U. = 1.496×10^{11}m) be the distance of earth from the sun (See Fig. 22.12). If S is the solar constant, then,

$$L_s = 4\pi r^2 S \qquad ...(i)$$

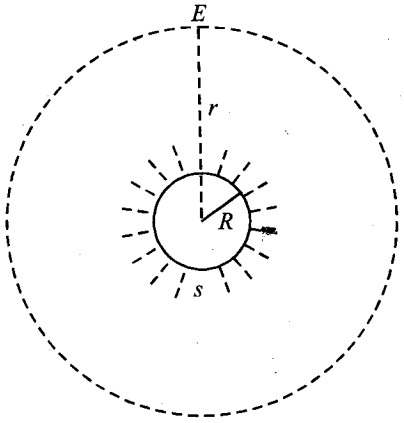

Fig. 22.12

According to Stefan's law, energy radiated by the sun per second is given by ;

$$E = \text{surface area of sun} \times \sigma T^4$$

or $$E = 4\pi R^2 \sigma T^4 \qquad ...(ii)$$

Assuming there is no loss of radiant energy,

exp. (*i*) = exp. (*ii*)

i.e. $$4\pi R^2 \sigma T^4 = 4\pi r^2 S$$

$$\therefore \quad T = \left(\frac{r^2 S}{R^2 \sigma}\right)^{1/4}$$

Since the values of r, S, R and σ are known, the surface tempearture of the sun can be determined.

(*v*) **Mass and density of the sun.** The mass and density of the sun can be determined by studying its gravitational effect on the earth (or on any other planet). Suppose earth of mass m is revolving around the sun of mass M in a circular orbit of radius r.

Gravitational pull of the sun exerted on earth is

$$F = G\frac{Mm}{r^2} \qquad ...(i)$$

This force F causes the earth to accelerate with acceleration a.

$$\therefore \quad F = m a \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $$\frac{GMm}{r^2} = ma$$

or $$M = \frac{ar^2}{G}$$

It is known that value of $a = 0.59 \times 10^{-2}$ m/s^2, $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2} and r = 1 A.U. = 1.496×10^{11} m.

$$\therefore \quad \text{Mass of sun, } M = \frac{(0.59\times10^{-2})\times(1.496\times10^{11})^2}{6.67\times10^{-11}} = 2\times10^{30} \text{ kg}$$

When compared to the earth's mass of 6×10^{24} kg, we see that mass of the sun is approximately 300,000 times that of earth's mass. Since the volume of sun is about 1,300,000 times that of the earth,

$$\therefore \quad \text{Density of sun, } \rho = \frac{\text{Mass of sun}}{\text{Volume of sun}} = \frac{300{,}000 \times \text{Mass of earth}}{1{,}300{,}000 \times \text{Volume of earth}}$$

$$= \frac{1}{4} \times \text{density of earth} = \frac{1}{4} \times 5500 \text{ kg/m}^3 \simeq 1400 \text{ kg/m}^3$$

Thus the sun is approximately one-fourth as dense as the earth. We know of another heavenly body with essentially the same average density — Jupiter. However, there is a significant difference between the structures of Sun and Jupiter. The Jupiter has small solid core encased in a liquid layer which is surrounded by a large amount of gaseous atmosphere. However, within the core of the sun, nuclear fusion reactions are taking place and huge amount of energy is released. It is this release of energy in the sun's core that keeps it in the gaseous state.

Example 22.17. The amount of heat radiation received by the earth from the sun is 1.38×10^3 W/m^2. Suppose all these radiations on the earth are re-emitted by the earth. Calculate the temperature of earth. Stefan's constant = 5.67×10^{-8} W m^{-2} K^{-4}.

Solution. $E = \sigma T^4$

$$\therefore \quad T = \left(\frac{E}{\sigma}\right)^{1/4} = \left(\frac{1.38\times10^3}{5.67\times10^{-8}}\right)^{1/4} = 395 \text{ K}$$

Example 22.18. Surface temperature of the sun is 6000 K. If we consider it as a perfect black body, calculate the energy radiated by the sun per second. Given that radius of the sun = 6.92×10^8 m and $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}.

Solution. Energy radiated by the sun per second, $E = A\sigma T^4$

Here $A = 4\pi R^2 = 4\pi\times(6.92\times10^8)^2 \text{ m}^2$; $\sigma = 5.67\times10^{-8}$ W m^{-2}K^{-4}; T = 6000 K

$\therefore \quad E = 4\pi\times(6.92\times10^8)^2\times5.67\times10^{-8}(6000)^4 = \mathbf{4.42 \times 10^{26}}$ W

22.19. COMPARISON OF CONDUCTION, CONVECTION AND RADIATION

Figure 22.13 shows heat transfer by conduction, convection and radiation. The steel bar is heated by conduction. However, the bar loses heat by convection and radiation.

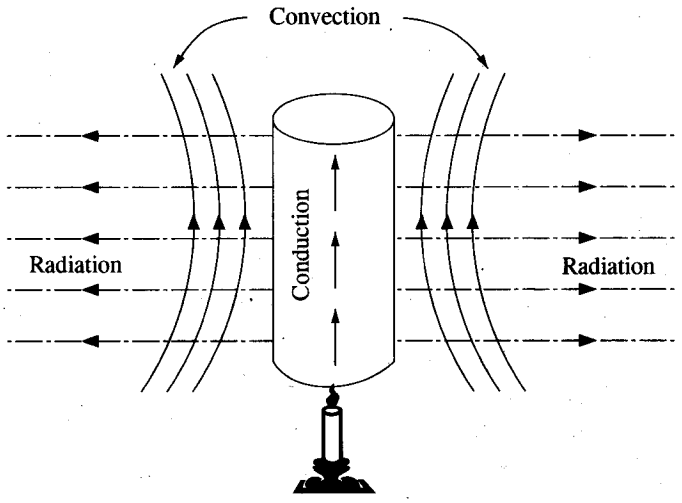

Fig. 22.13

S.No.	Particular	Conduction	Convection	Radiation
1.	*Material medium*	essential	essential	not essential
2.	*Molecules*	Do not leve their mean positions.	Move bodily from one place to another.	Medium does not play any part.
3.	*Transfer of heat*	can be in any direction along any part.	Only vertically upward.	In all directions in straight lines.
4.	*Speed of transfer of heat*	slow	rapid	fastest (3×10^8 m/s).

CONCEPTUAL QUESTIONS

Q.1. The tile floor feels colder than the wooden floor, even though both floor materials are at the same temperature. Why?

Ans. It is because tile is a better heat conductor than wood. The heat transferred from your foot to the wood is not conducted away rapidly. So, the wood quickly heats up on its surface to the temperature of your foot. But the tile conducts the heat away rapidly and thus can take more heat from your foot, so its surface temperature drops.

Q.2. Thermal conductivity of air is less than that of felt but felt is a better heat insulator in comparison to air. Why?

Ans. Free air transmits heat by convection currents. In case of felt, there are fine pores which trap air and do not allow its movement. As a result, convection currents cannot be set up in the felt.

Q.3. Why do two layers of cloth of equal thickness provide warmer covering than a single layer of cloth of double the thickness?

Ans. The air trapped between the two layers of cloth prevents the transmission of heat from our body to the outside. Thus it provides better insulation than a single layer of cloth of double the thickness.

Q.4. A blanket which keeps us warm in winter is able to protect ice from melting. Explain.

Ans. The insulating action of the blanket is due to the air trapped in its fine pores. The air is a bad conductor of heat. Moreover, convection currents cannot be set up in the trapped air. When we cover our body with a blanket, the trapped air prevents heat transmission from our body to the outside. As a result, our body is kept warm. When ice is covered with a blanket, the same trapped air prevents the heat transmission from outside to the ice. Hence ice does not melt.

Q.5. Animals in the forest find shelter from cold in holes in the snow. Why?

Ans. Snow has trapped air (in ice there is no air) which acts as a heat insulator. Therefore, the snow prevents the transmission of heat from the body of the animal to the outside.

Q.6. Why does warm air rise?

Ans. Warm air expands, becomes less dense than the surrounding air and is buoyed upward like a balloon. The buoyancy is in an upward direction because the air pressure below a region of warmed air is greater than the air pressure above. And the warmed air rises because the buoyant force is greater than its weight.

Q.7. Warm air rises up but the atmosphere is cooler with increasing altitude. Why?

Ans. Warmed air is pushed from a region of greater atmospheric pressure at the ground to a region of lesser pressure above. And while it is rising through regions of lesser pressure, it expands. Expansion of air results in cooling.

Q.8. You can hold your fingers beside the candle without harm but not above the flame. Why?

Ans. Heat travels upward by air convection. Since air is a poor conductor, very little heat travels sideways.

Q.9. All objects radiate electromagnetic energy. Why then do they not lose all their thermal energy by radiation and cool down to 0 K?

Ans. The answer is that they also absorb radiation from surrounding objects and eventually come to thermal equilibrium with their environment.

Q.10. On a winter night you feel warmer when clouds cover the sky than when the sky is clear. Why?

Ans. We know that earth absorbs heat in day and radiates at night. When sky is covered with clouds, the heat radiated by earth is reflected back and earth becomes warmer. However, when the sky is clear, the heat radiated by earth escapes into the space.

Q.11. Since a hot cup of coffee loses heat more rapidly than a lukewarm cup of coffee, would it be correct to say that a hot cup of coffee will cool to room temperature before lukewarm cup of coffee?

Ans. No! Although the rate of cooling is greater for the hotter cup, it has farther to cool to reach thermal equilibrium. The extra time is equal to the time it takes to cool to the initial temperature of the lukewarm cup of coffee. Cooling rate and cooling time are not the same thing.

Q.12. Why is the energy of thermal radiation less than that of visible light?

Ans. The energy of an electromagnetic wave is given by ;

$$E = h f$$

Here h = Planck's constant ; f = frequency of wave

Since the frequency of thermal radiation is less than that of visible light, the energy associated with thermal radiation is less than associated with visible light.

Q.13. Why does a good absorber of radiant energy appear black?

Ans. A good absorbing surface removes all visible colours when it converts all the radiant energy incident upon it to heat. It cannot both absorb and reflect the same wavelengths at the same time. Therefore, the absence of reflected light makes it black.

Q.14. In summer, one should not wear black clothes. Why?

Ans. In summer, one should not wear black clothes because black or coloured clothes absorb more heat than white clothes.

Q.15. Why are metals good conductors of heat?

Ans. The ability of a solid material to conduct heat depends on its atomic structure. Metals have a large number of free electrons that can move rather freely throughout the metal. As they move, they carry thermal energy from one part of the metal to another. Hence metals are good conductors of heat because they have large number of free electrons.

Q.16. Why is the pupil of the eye black?

Ans. Any radiation that enters the pupil of the eye is reflected from inside the eye many times and is partly absorbed at each reflection until none remains. For this reason, the pupil of eye is black.

VERY SHORT ANSWER QUESTIONS

Q.1. What is the mode of heat transfer through a solid?

Ans. Heat is transferred through a solid by conduction.

Q.2. Why do electrons in insulators not contribute to its conductivity?

Ans. A material conducts heat because of the *free electrons* present inside it. Since there are no free electrons in insulators, they cannot conduct heat.

Q.3. What is the thermal conductivity of a (*i*) perfect heat conductor (*ii*) perfect heat insulator?

Ans. (*i*) $k = \infty$ and (*ii*) $k = 0$.

Q.4. What are the SI units and dimensional formula of thermal conductivity?

Ans. The SI unit of thermal conductivity is $W\ m^{-1}\ K^{-1}$ and its dimensional formula is $[MLT^{-3}\ K^{-1}]$.

Q.5. If air is a bad conductor of heat, why do we not feel warm without clothes?

Ans. When we are without clothes, the heat is carried away from our body by convection and hence we feel cold.

Q.6. At what temperature does a body stop radiating?

Ans. A body stops radiating at 0 K.

Q.7. Animals curl into a ball when they feel cold. Why?

Ans. The total energy radiated by a body is directly proportional to the surface area of the body. When the animals feel cold, they curl their bodies into the ball so as to decrease the surface area of their bodies. As a result, the loss of heat due to radiation is reduced.

Q.8. A body is at 0°C. Is it radiating heat?

Ans. Yes. A body is radiating heat even at 0°C.

Q.9. Why is it hotter at the same distance over the top of a fire than in front of it?

Ans. It is because in front of fire, heat is received by the process of radiation only while above the fire, heat is received both due to radiation and convection.

Q.10. The temperature of a black body is increased from 50 K to 100 K. How will the rate of radiated energy change?

Ans. The rate of radiated energy $E \propto T^4$. Since temperature is doubled, E will become 16 times.

Q.11. A black body has maximum wavelength λ_m at 2000 K. What will be its wavelength at 3000 K?

Ans. According to Wien's displacement law, $\lambda_m T$ = constant. Therefore, $\lambda_m \propto 1/T$.

$$\therefore \quad \frac{\lambda'_m}{\lambda_m} = \frac{T}{T'} \text{ or } \frac{\lambda'_m}{\lambda_m} = \frac{2000}{3000} \quad \therefore \lambda'_m = \frac{2}{3}\lambda_m$$

Q.12. Sometimes when we shake a bottle vigorously containing hot liquid, its cork blows off. Why?

Ans. On shaking a hot liquid, its temperature increases and some of the liquid is converted into vapours. The vapour pressure inside the bottle may become high enough to blow off the cork.

Q.13. What is a black body?

Ans. A body that absorbs all the radiation incident on it is called a black body. A black body is both an ideal absorber and an ideal emitter.

Q.14. Why are steam pipes wrapped with insulating material?

Ans. To minimise the loss of heat due to radiation.

Q.15. A red glass heated in a furnace and then taken out appears green. Why?

Ans. At low temperature, red glass appears red because it absorbs green light strongly. Therefore, at high temperature, it emits green colour strongly and, therefore, appears green.

SHORT ANSWER QUESTIONS

Q.1. At what temperature would a block of wood and a block of metal feel equally cold or hot on touching?

Ans. At 37°C *i.e.,* the temperature of the human body. The feeling of cold or hot arises due to transfer of heat from or to the human body due to temperature difference. If the human body and blocks are at the same temperature, no heat transfer takes place on touching. Therefore, the wooden block and metal block would feel the same cold or hot.

Q.2. After switching on a heater, its temperature becomes constant after some time, although current remains flowing through it. Why?

Ans. The temperature of the heater becomes constant after some time because thermal equilibrium is reached between the heater and its surroundings. At this stage, the rate at which heat is produced by electric current becomes equal to the rate at which heat is lost to the surroundings by radiation. Therefore, temperature of the heater becomes constant.

Q.3. A body radiates 5 W at a temperature of 400 K. If the temperature is increased to 1200 K, what is the radiated power?

Ans. $\frac{E_2}{E_1} = \frac{T_2^4}{T_1^4} = \left(\frac{1200}{400}\right)^4 = 81 \quad \therefore \quad E_2 = E_1 \times 81 = 5 \times 81 = 405 \text{ W}$

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Heat may be transferred from one point to another by conduction, convection or radiation.
2. A substance is in steady state when the temperatures at all points are steady (*i.e.* constant).
3. Transfer of heat in a solid is mainly by conduction.
4. Liquids and gases transmit heat mainly by convection.
5. Liquids and gases, in general, are poor conductors of heat.
6. Air is a very poor conductor of heat.
7. The ratio of thermal to electrical conductivity is nearly the same for all metals.
8. The heat from sun does not pass through atmosphere by conduction because air is a poor conductor. Nor does it pass through by convection because convection begins only after the earth is warmed.
9. Thermal radiation is an electromagnetic radiation emitted by a body solely on account of its temperature.
10. Objects at low temperatures emit radiation of long-wavelengths.
11. Objects at high temperatures emit radiation of short-wavelengths.
12. A body that absorbs all the radiation incident upon it is called a black body. A black body is both an ideal absorber and an ideal radiator.
13. The rate at which an object radiates or absorbs radiant energy depends upon the nature of its surface, surface area and temperature of the object.
14. The rate at which a black body radiates or absorbs radiant energy depends upon temperature only.
15. Suppose an object is at absolute temperature T and its surroundings is at absolute temperature T_0 ($T > T_0$).
    Net loss of energy per second by the object is :
    $$P = \varepsilon\sigma A(T_1^4 - T_0^4)$$
    where ε = emissivity ; σ = Stefan's constant ; A = surface area of the body
16. For a black body, $\varepsilon = 1$ so that the above relation becomes :
    $$P = \sigma A(T_1^4 - T_0^4)$$
17. All bodies radiate and absorb energy in the form of electromagnetic waves.
18. A body that is hotter than its surroundings radiates more energy than it absorbs. A body that is cooler than its surroundings absorbs more energy than it radiates.

MCQ FROM COMPETITIVE EXAMINATIONS

1. A black body has maximum wavelength λ_m at 2000 K. Its corresponding wavelength at 3000 K will be **[CBSE PMT 2001]**

   (*a*) $\frac{3}{2}\lambda_m$ (*b*) $\frac{2}{3}\lambda_m$

   (*c*) $\frac{16}{81}\lambda_m$ (*d*) $\frac{81}{16}\lambda_m$

2. A sphere is at a temperature of 600 K. Its cooling rate is R' in an external environment of 200 K. If temperature falls to 400 K, then cooling rate R' will be **[CBSE PMT 1999]**

   (*a*) $3R/16$ (*b*) $16R/3$

   (*c*) $9R/27$ (*d*) none of these

3. Two conducting slabs of thermal conductivities K_1 and K_2 are joined as shown in Fig. 22.14. The temperature of ends of slab are θ_1 and θ_2 ($\theta_1 > \theta_2$) . The final temperature of the junction θ_m is **[CBSE PMT 1999]**

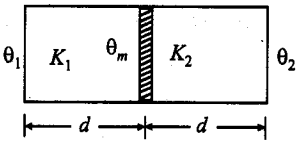

Fig. 22.14

(a) $\frac{K_1\theta_2 + K_2\theta_1}{K_1 + K_2}$ (b) $\frac{K_1\theta_1 - K_2\theta_2}{K_1 + K_2}$

(c) $\frac{K_1\theta_1 + K_2\theta_2}{K_1 + K_2}$ (d) none of the above

4. The radiant energy from the sun incident normally at the surface of earth is 20 kcal/m^2-min. What would have been the radiant energy incident normally on the earth if the sun had a temperature twice the present value? **[CBSE PMT 1999]**

(a) 40 kcal /m^2 – min (b) 80 kcal/m^2–min

(c) 160 kcal/m^2–min (d) 320 kcal/m^2–min

5. If the temperature of a black body increases from 7°C to 287°C, then the rate of energy radiation increases by **[AIIMS 1997]**

(a) $(287/7)^4$ (b) 16

(c) 4 (d) 2

6. Two rods having thermal conductivities in the ratio of 5 : 3 and having equal lengths and equal cross-sections are joined face to face. If the temperature of free end of first rod is at 100°C and the free end of second rod is at 20°C, then temperature of the junction is **[Pb PMT 1996]**

(a) 50°C (b) 90°C

(c) 85°C (d) 70°C

7. A body cools from 60°C to 50°C in 10 min. If room temperature is 25°C, the temperature of the body at the end of next 10 min will be **[MP PMT 1998]**

(a) 38.5°C (b) 42.8°C

(c) 40°C (d) 45°C

8. A black body radiates heat energy at the rate of 2×10^5 J/s/m^2 at a temperature of 127°C. The temperature of black body at which the rate of radiation is 32×10^5 J/s/m^2 is **[AFMC-Pune 1998]**

(a) 527°C (b) 273°C

(c) 873°C (d) 927°C

9. A black body is heated from 27°C to 927°C. What will be the ratio of energies of radiation emitted? **[Pb PMT 1995]**

(a) 1 : 4 (b) 1 : 6

(c) 1 : 256 (d) 1 : 64

10. The temperature of two bodies A and B are respectively 727°C and 327°C. The ratio of H_A: H_B of the rates of radiation by them is **[MP CEE 1999]**

(a) 727 : 327 (b) 5 : 3

(c) 25 : 9 (d) 625 : 81

11. The rectangular surface of area 8 cm × 4 cm of a black body at 127°C emits energy at the rate of E per second. If the length and breadth of the surface are reduced to half of its initial value and the temperature raised to 327°C, the rate of emission of energy will become **[MP PMT 2000]**

(a) $\frac{3}{8}E$ (b) $\frac{81}{16}E$

(c) $\frac{9}{16}E$ (d) $\frac{81}{64}E$

12. A body radiates 5 W at a temperature of 400 K. If the temperature is increased to 1200 K, then it radiates energy at the rate of **[AFMC Pune 2000]**

(a) 81 W (b) 200 W

(c) 405 W (d) 410 W

13. Two rods A and B are of equal length. Each rod has its ends at temperature T_1 and T_2 ($T_1 > T_2$). What is the condition that will ensure equal rates of flow through the rods A and B? **[IIT 1976]**

(a) $\frac{A_1}{A_2} = \frac{K_1}{K_2}$ (b) $\frac{A_1}{A_2} = \frac{K_2}{K_1}$

(c) $\frac{A_1}{A_2} = \frac{K_1^2}{K_2^2}$ (d) none of above

14. Wien's law states that : **[CET 1999]**

(a) $T + \lambda_m$ = constant

(b) $\lambda_m T$ = constant

(c) λ_m / T = constant

(d) T^4 / λ_m = constant

15. If the temperature of the sun is doubled, the rate of energy received on earth will be increased by a factor of **[CET 1999]**

(a) 2 (b) 4

(c) 8 (d) 16

ANSWERS TO MCQ FROM COMPETITIVE EXAMINATIONS

1. (b)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (b)
6. (d)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (d)
11. (d)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (d)

HINTS TO MCQ FROM COMPETITIVE EXAMINATIONS

1. According to Wien's displacement law, $\lambda_m T$ = constant. Therefore $\lambda_m \propto 1/T$.

$$\therefore \quad \frac{\lambda'_m}{\lambda_m}=\frac{T}{T'} \quad \text{or} \quad \frac{\lambda'_m}{\lambda_m}=\frac{2000}{3000} \quad \text{or} \quad \lambda'_m=\frac{2000}{3000}\times\lambda_m=\frac{2}{3}\lambda_m$$

2. According to Stefan's law, $\frac{R'}{R}=\frac{(400)^4-(200)^4}{(600)^4-(200)^4}=\frac{240}{1280}=\frac{3}{16}$. Therefore,

$$R'=(3/16)R = \mathbf{3R/16}.$$

3. In the steady state, the rate of heat transfer in both the slabs is the same *i.e.*,

$$\frac{K_1A(\theta_1-\theta_m)}{d}=\frac{K_2A(\theta_m-\theta_2)}{d} \quad \text{or} \quad (K_1+K_2)\theta_m=K_1\theta_1+K_2\theta_2$$

or $$\boldsymbol{\theta_m=\frac{K_1\theta_1+K_2\theta_2}{K_1+K_2}}$$

4. According to Stefan's law of radiation, energy radiated $\propto T^4$.

$$\therefore \quad \frac{E_2}{E_1}=\frac{(T_2)^4}{(T_1)^4} \quad \text{or} \quad \frac{E_2}{20}=\frac{(2T)^4}{T^4} \quad \text{or} \quad \frac{E_2}{20}=16$$

$$\therefore \quad E_2=20\times16=\mathbf{320\ kcal/m^2\text{–}min}$$

5. $T_1 = 7 + 273 = 280$ K; $T_2 = 287 + 273 = 560$ K. According to Stefan's law of radiation,

$$\frac{E_2}{E_1}=\left(\frac{T_2}{T_1}\right)^4 \quad \text{or} \quad \frac{E_2}{E_1}=\left(\frac{560}{280}\right)^4=2^4=\mathbf{16}$$

6. $K_1 = 5$; $K_2 = 3$; $\theta_1 = 100°C$; $\theta_2 = 20°C$. Let θ_m °C be the temperature of the junction. In the steady state, the rate of heat transfer in both the rods will be the same *i.e.*,

$$\frac{K_1A(\theta_1-\theta_m)}{d}=\frac{K_2A(\theta_m-\theta_2)}{d} \quad \text{or} \quad K_1(\theta_1-\theta_m)=K_2(\theta_m-\theta_2)$$

$\therefore \quad 5(100-\theta_m)=3(\theta_m-20)$. On solving, $\theta_m = \mathbf{70°C}$

7. According to Newtons law of cooling, we have,

$$\frac{mc}{t}(\theta_1-\theta_2)=k\left(\frac{\theta_1+\theta_2}{2}-\theta_0\right) \quad \text{or} \quad mc\left(\frac{60-50}{10}\right)=k\left(\frac{60+50}{2}-25\right) \quad \therefore \quad k=\frac{mc}{30}$$

Now $$mc\left(\frac{50-\theta}{10}\right)=k\left(\frac{50+\theta}{2}-25\right) \quad \text{or} \quad mc\left(\frac{50-\theta}{10}\right)=\frac{mc}{30}\left(\frac{50+\theta}{2}-25\right)$$

$\therefore \quad \theta = \mathbf{42.8°C}$

8. $$\left(\frac{T_2}{T_1}\right)^4=\frac{E_2}{E_1} \quad \text{or} \quad \left(\frac{T_2}{T_1}\right)=\left(\frac{32\times10^5}{2\times10^5}\right)^{1/4}=2$$

$\therefore T_2 = 2T_1 = 2\times(127+273) = 800$ K $= \mathbf{527°C}$

9. $$\frac{E_1}{E_2}=\left(\frac{T_1}{T_2}\right)^4=\left(\frac{27+273}{927+273}\right)^4=\left(\frac{300}{1200}\right)^4=\left(\frac{1}{4}\right)^4=\frac{1}{256} \quad \therefore E_1:E_2=\mathbf{1:256}$$

10. $\frac{H_A}{H_B}=\frac{T_A^4}{T_B^4}=\left(\frac{727+273}{327+273}\right)^4=\left(\frac{1000}{600}\right)^4=\left(\frac{5}{3}\right)^4=\frac{625}{81}$ $\therefore$ $H_A : H_B = 625 : 81$

11. $T_1 = 127 + 273 = 400$ K ; $T_2 = 327 + 273 = 600$ K ; $A_1 = 4A_2$ $\therefore A_2/A_1=\frac{1}{4}$

$$\frac{E_2}{E_1}=\frac{A_2T_2^4}{A_1T_1^4}=\frac{1}{4}\times\left(\frac{600}{400}\right)^4=\frac{81}{64} \qquad \therefore E_2=\frac{81}{64}E$$

12. $\frac{E_2}{E_1}=\frac{T_2^4}{T_1^4}=\left(\frac{1200}{400}\right)^4=81$ $\therefore E_2 = E_1\times 81 = 5\times 81 = 405$ W

13. Let the length of each rod be d. When the rods have same rate of conduction,

$$\frac{K_1A_1(T_1-T_2)}{d}=\frac{K_2A_2(T_1-T_2)}{d} \quad \text{or} \quad K_1A_1=K_2A_2 \quad \text{or} \quad \frac{A_1}{A_2}=\frac{K_2}{K_1}$$

14. According to Wien's law, $\lambda_m T =$ Constant.

15. Rate of energy radiated $\propto T^4$. If the sun temperature is doubled, the rate of energy radiated will increase by a factor $(2)^4 = 16$.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **A copper bar of length 1 m and cross-sectional area 10×10^{-2} m^2 has its one end maintained at 100°C by means 0.4 kW heater. Calculate the temperature of the other end in the steady state. The thermal conductivity of copper is 400 W m^{-1}°C^{-1}.**

Hint. $$Q = \frac{kA(\theta_1-\theta_2)t}{x}$$

$\therefore$ Power of heater, $P = \frac{Q}{t}=\frac{kA(\theta_1-\theta_2)}{x}$

Here P = 0.4 kW = 0.4 × 1000 W ; k = 400 W m^{-1}°C^{-1} ; $A = 10 \times 10^{-2}$ m^2 ;

θ_1 = 100°C ; x = 1 m ; θ_2 = ?

$\therefore$ $$0.4 \times 1000 = \frac{400\times 10\times 10^{-2}(100-\theta_2)}{1} \quad \text{or} \quad \theta_2 = 90°C$$

2. **A room is maintained at 20°C by a heater of resistance 20 ohms connected to 200 V mains. The temperature is uniform throughout the room and the heat is transmitted through a glass window of area 1 m^2 and thickness 0.2 cm. Calculate the temperature outside. Thermal conductivity of glass is 0.2 cal s^{-1} m^{-1} °C^{-1} and mechanical equivalent of heat J = 4.2 J/cal.** **[I.I.T.]**

Hint. The heat generated by the heater per second is

$$H = \frac{V^2}{R}=\frac{(200)^2}{20} = 2000 \text{ J/s} = \frac{2000}{4.2} \text{ cal/s} \qquad ...(i)$$

Heat passing through the glass window per second is

$$H = \frac{Q}{t}=\frac{kA(\theta_1-\theta_2)}{x} \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $\frac{kA(\theta_1-\theta_2)}{x}=\frac{2000}{4.2}$

Here k = 0.2 cal s^{-1} m^{-1}°C^{-1}; A = 1 m^2 ; θ_1 = 20°C; $x = 0.2 \times 10^{-2}$m ; θ_2 = ?

$\therefore$ $$\frac{0.2\times 1\times(20-\theta_2)}{0.2\times 10^{-2}} = \frac{2000}{4.2} \quad \text{or} \quad \theta_2 = 15.24°C$$

3. A cylindrical rod with one end in a steam chamber and the other end in ice results in melting 0.1 g of ice per second. If the rod is replaced by another with half the length and double the radius of the first and if the thermal conductivity of the second rod is one-fourth of the first, find the rate at which ice melts.

[E.A.M.C.E.T.]

Hint. $\frac{Q}{t} = \frac{mL}{t} = \frac{kA(\theta_1 - \theta_2)}{x}$ Here L= latent heat of ice

or $\frac{m}{t} = \frac{kA(\theta_1 - \theta_2)}{xL}$

Now $\frac{m_1}{t_1} = \frac{k_1(\pi r_1^2)(\theta_1 - \theta_2)}{l_1 L}$

and $\frac{m_2}{t_2} = \frac{k_2(\pi r_2^2)(\theta_1 - \theta_2)}{l_2 L}$

$\therefore$ $\frac{m_1}{t_1} \times \frac{t_2}{m_2} = \frac{k_1}{k_2} \times \frac{r_1^2}{r_2^2} \times \frac{l_2}{l_1}$

or $0.1 \times \frac{t_2}{m_2} = \frac{k_1}{(k_1/4)} \times \frac{r_1^2}{(2r_1)^2} \times \frac{l_1/2}{l_1} = \frac{1}{2}$ $\therefore$ $\frac{m_2}{t_2} = 0.1 \times 2 = \mathbf{0.2\ g/s}$

4. An electric heater is used in a room of total wall area 137 m^2 to maintain a temperature of 20°C inside it when the outside temperature is –10°C. The walls have three layers of different materials. The innermost layer is of wood of thickness 2.5 cm, the middle layer is of cement of thickness 1.0 cm and the outermost layer is of brick of thickness 25.0 cm. Find the power of electric heater. Assume that there is no heat loss through the floor and ceiling. The thermal conductivities of wood, cement and brick are 0.125 $Wm^{-1}\,°C^{-1}$, 1.5 $Wm^{-1}°C^{-1}$ and 1.0 $Wm^{-1}°C^{-1}$ respectively.

[I.I.T.]

Hint. Figure 22.15 shows the conditions of the problem. Let l_1, l_2 and l_3 be the thickness of wood, cement and brick layers respectively. Suppose θ_3 and θ_4 are the temperatures of wood-cement and cement-brick interfaces respectively. Let P be the power of the heater.

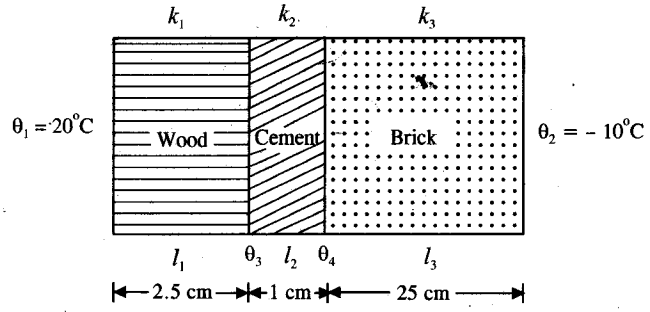

Fig. 22.15

In the steady state, the rate of heat flowing in all the layers is the same, *i.e.*,

$$\frac{Q}{t} = \frac{k_1 A(\theta_1 - \theta_3)}{l_1} = \frac{k_2 A(\theta_3 - \theta_4)}{l_2} = \frac{k_3 A(\theta_4 - \theta_2)}{l_3}$$

or $\frac{k_1 A(\theta_1 - \theta_3)}{l_1} = \frac{k_2 A(\theta_3 - \theta_4)}{l_2}$

or $$\frac{k_1}{k_2}(\theta_1-\theta_3)=\frac{l_1}{l_2}(\theta_3-\theta_4)$$

or $$\frac{0.125}{1.5}(20-\theta_3)=\frac{2.5\times10^{-2}}{1.0\times10^{-2}}(\theta_3-\theta_4)$$

or $$31\,\theta_3-30\,\theta_4=20 \quad ...(i)$$

Also $$\frac{k_1A(\theta_1-\theta_3)}{l_1}=\frac{k_3A(\theta_4-\theta_2)}{l_3}$$

or $$\frac{k_1}{k_3}(\theta_1-\theta_3)=\frac{l_1}{l_3}(\theta_4-\theta_2)$$

or $$\frac{0.125}{1.0}(20-\theta_3)=\frac{2.5\times10^{-2}}{25\times10^{-2}}(\theta_4+10)$$

or $$5\theta_3+4\theta_4=60 \quad ...(ii)$$

Solving eqs. (*i*) and (*ii*), we get, $\theta_3 = 6.86°C$ and $\theta_4 = 6.4°C$.

Power of heater, $P=\frac{Q}{t}=\frac{k_1A(\theta_1-\theta_3)}{l_1}=\frac{0.125\times137(20-6.86)}{2.5\times10^{-2}}=9000\text{ W}=\mathbf{9\ kW}$

5. A cylinder of radius *R* made of a material of thermal conductivity k_1 is surrounded by a cylindrical shell of inner radius *R* and outer radius 2*R* made of material of thermal conductivity k_2. The two ends of the combined system are maintained at two different temperatures. There is no loss of heat across the cylindrical surface and the system is in steady state. Calculate the effective thermal conductivity of the system. [I.I.T.]

Hint. The two conductors are thermally in parallel. Therfore, equivalent thermal resistance *R* is given by ;

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2} \text{ or } \frac{kA}{l}=\frac{k_1A_1}{l_1}+\frac{k_2A_2}{l_2}$$

Here $l_1 = l_2 = l$; $A_1 = \pi R^2$; $A_2 = \pi (2R)^2 - \pi R^2 = 3\pi R^2$; $A = \pi (2R)^2 = 4\pi R^2$

$$\therefore \quad \frac{k\times4\pi R^2}{l}=\frac{k_1\pi R^2}{l}+\frac{k_2\times3\pi R^2}{l} \quad \text{or } k=\frac{\mathbf{k_1+3k_2}}{\mathbf{4}}$$

6. A person walking during a hot day in a dry atmosphere at 40°C will sweat about 1 litre per hour. Assume that all water in the sweat evaporates during that time. Calculate the rate of loss of heat from the body of the person.

Hint. V = 1 litre = 10^{-3} m^3; t = 3600 s ; $\rho = 10^3$ kg/m^3 ; $L = 540 \times 10^3$ cal/kg = $540 \times 10^3 \times 4.2$ J/kg

Rate of loss of heat is given by ;

$$H=\frac{Q}{t}=\frac{mL}{t}=\frac{V\rho L}{t}=\frac{10^{-3}\times10^3\times540\times10^3\times4.2}{3600}\text{ J/s}=630\text{ J/s}=\mathbf{630\ W}$$

Therefore, the rate of cooling is 630 W.

7. One end of a copper rod of uniform cross-section and of length 1.5 m is kept in contact with ice and the other end with water at 100°C. At what point along its length should a temperature of 200°C be maintained so that in the steady state the mass of ice melted be equal to that of the steam produced in the same interval of time? Assume that the whole system is insulated from the surroundings. Latent heat of fusion of ice = 80 cal/g and latent heat of vaporisation = 540 cal/ g. [C.E.E.]

Hint. Figure 22.16 shows the conditions of the problem. Let temperature 200°C (= θ) be maintained at a distance x metres from the end at 0°C. Now heat will flow from this point both toward the ice and water.

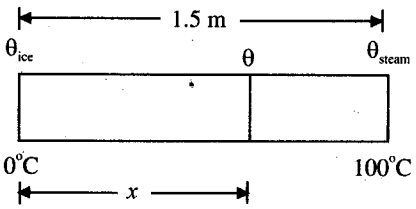

Fig. 22.16

∴ Amount of heat flowing toward ice in time t

$$= \frac{kA(\theta-\theta_{ice})t}{x}$$

Amount of heat flowing toward water at 100°C in time t

$$= \frac{kA(\theta-\theta_{steam})t}{1.5-x}$$

Let m be the mass of ice melted and steam produced in time t. Then,

$$m\,L_{ice} = \frac{kA(\theta-\theta_{ice})\,t}{x} \text{ and } mL_{steam} = \frac{kA(\theta-\theta_{steam})\,t}{1.5-x}$$

$$\therefore \quad \frac{L_{ice}}{L_{steam}} = \frac{\theta-\theta_{ice}}{\theta-\theta_{steam}} \times \frac{1.5-x}{x} \quad \text{or} \quad \frac{80}{540} = \frac{200-0}{200-100} \times \frac{1.5-x}{x}$$

On solving, x = 1.396 m from cold end

8. **An ice box has total wall area (including the lid) of 1.2 m^2 and wall thickness 2.0 cm. The thermal conductivity of the box is about 0.01J $m^{-1}s^{-1}{}^{\circ}C^{-1}$. It is filled with water and ice at 0°C. What is the rate of flow of heat into the box if the outside temperature is 35°C? Calculate the amount of ice that melts in one day. The latent heat of ice = 3.35 $\times 10^5$ J/kg.**

Hint. Rate of heat flow is given by ; $H = \frac{Q}{t} = \frac{kA(\theta_1-\theta_2)}{x}$

Here k = 0.01 J m^{-1} $s^{-1}{}^{\circ}C^{-1}$; A = 1.2 m^2 ; $\theta_1 - \theta_2$ = 35 − 0 = 35°C ; $x = 2 \times 10^{-2}$ m

$$\therefore \quad H = \frac{0.01\times1.2\times(35)}{2\times10^{-2}} = 21 \text{ Js}^{-1}$$

Amount of heat that flows into the box in one day

$$= 21 \times (24 \times 60 \times 60) = 1814400 \text{ J}$$

Let m kg be the mass of ice melted in one day. Then,

$$mL = 1814400 \quad \text{or} \quad m = \frac{1814400}{L} = \frac{1814400}{3.35\times10^5} = 5.42 \text{ kg}$$

9. **Consider two insulating sheets with thermal resistances R_1 and R_2 as shown in Fig. 22.17. Show that :**

$$\theta' = \frac{R_2\theta_1 + R_1\theta_2}{R_1 + R_2}$$

Hint. In the steady state, the rate of heat flow through the two sheets is the same,

i.e., $$\frac{\theta_1-\theta'}{R_1} = \frac{\theta'-\theta_2}{R_2}$$

or $$R_2(\theta_1-\theta') = R_1(\theta'-\theta_2)$$

or $$R_2\theta_1 + R_1\theta_2 = (R_1+R_2)\theta' \quad \therefore \quad \theta' = \frac{R_2\theta_1 + R_1\theta_2}{R_1+R_2}$$

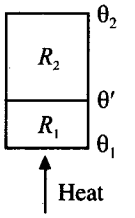

Fig. 22.17

10. Two brass plates, each 0.50 cm thick, have rubber spacer between them which is 0.10 cm thick. The outer side of one brass plate is kept at 0°C while the outside of other plate is at 100°C. Find the temperature of the two sides of the rubber spacer. Given k_{Br} is 490 times k_{ru}.

Hint. Let θ_1°C and θ_2°C be the temperatures at the two rubber-brass junctions. Since rate of flow of heat *(Q/t)* and area *A* is the same in each case,

$$\therefore \quad \frac{k_{Br}(\theta_1 - 0°)}{0.5} = \frac{k_{ru}(\theta_2 - \theta_1)}{0.10} = \frac{k_{Br}(100 - \theta_2)}{0.5}$$

The first and third equations yield $\theta_1 + \theta_2 = 100$ ...(*i*)

The first and second eqs. yield $99\theta_1 - \theta_2 = 0$ ...(*ii*)

Solving eqs. (*i*) and (*ii*), we get, $\theta_1 = 1.0°C$; $\theta_2 = 99°C$

11. A body which has a surface area 5.00 cm^2 and a temperature of 727°C radiates 300 J of energy in one minute. What is its emissivity? Stefan-Boltzmann constant = 5.67×10^{-8} Wm^{-2} K^{-4}.

Hint. The total energy radiated by a body of emissivity ε, surface area *A* at absolute temperature *T* in surroundings of temperature T_0 in time *t* is given by ;

$$\text{Energy, } Q = \varepsilon \sigma (T^4 - T_0^4) At$$

$$= \varepsilon \sigma T^4 At \quad \text{if } T_0 << T$$

Here Q = 300 J ; A = 5.00 cm^2 = 5.00×10^{-4} m^2 ; T = 727 + 273 = 1000 K ; t = 1 min = 60 s.

$$\therefore \quad \text{Emissivity, } \varepsilon = \frac{Q}{\sigma T^4 At} = \frac{300}{5.67\times10^{-8}\times(1000)^4\times5.00\times10^{-4}\times60} = 0.18$$

12. A tungsten wire of length 100 cm and perimeter 0.2 cm behaves as a black body at 2000 K. Calculate the rate at which the wire is radiating energy at 2000 K. Given that Stefan's constant $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}.

Hint. The rate at which the tungsten wire radiates energy is given by ;

$$P = \varepsilon \sigma T^4 A$$

Here $\varepsilon = 1$ (black body) ; $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4} ; T= 2000 K ;

Surface area, A = perimeter × length = $0.2 \times 100 = 20$ $cm^2 = 20 \times 10^{-4}$ m^2

$$\therefore \quad P = 1 \times 5.67 \times 10^{-8} \times (2000)^4 \times 20 \times 10^{-4} = 1814.4 \text{ W}$$

13. Radiation from the moon gives maxima at $\lambda = 4700$ Å and $\lambda = 14 \times 10^{-6}$ m. What conclusions can you draw from this? Given $b = 2.9 \times 10^{-3}$ mK.

Hint. According to Wien's displacement law, $\lambda_m \times T = b$

Temperature corresponding to λ_m = 4700 Å (= 4700×10^{-10} m) is

$$T_1 = \frac{b}{\lambda_m} = \frac{2.9\times10^{-3}}{4700\times10^{-10}} = 6170\,\text{K}$$

Temperature corresponding to $\lambda_m = 14 \times 10^{-6}$ m is

$$T_2 = \frac{b}{\lambda_m} = \frac{2.9\times10^{-3}}{14\times10^{-10}} = 207\,\text{K}$$

The first one is the **temperature of the sun** because sun radiations are reflected from the moon's surface. The other value is the moon's own temperature.

14. An aluminium foil of relative emittance 0.2 is placed between two concentric spheres at temperatures 300 K and 200 K respectively. Calculate the temperature of the foil after the steady state is reached. Also calculate the rate of energy transfer between one of the spheres and the foil. Stefan's constant $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}.

Hint. $T_1 = 300$ K ; $T_2 = 200$ K. In the steady state, let T be the absolute temperature of the foil. Under steady state,

Heat gained per second by foil from first sphere = Heat lost per second by foil to the second sphere

or $$\varepsilon \sigma A (T_1^4 - T^4) = \varepsilon \sigma A (T^4 - T_2^4)$$

or $$T_1^4 + T_2^4 = 2T^4$$

$\therefore$ $$T^4 = \frac{T_1^4 + T_2^4}{2}$$

or $$T = \left[\frac{(300)^4 + (200)^4}{2}\right]^{1/4} = \mathbf{263.9\ K}$$

Rate at which energy is transferred from the first sphere to the foil

$$= \varepsilon \sigma (T_1^4 - T^4) \text{ per } m^2$$

$$= 0.2 \times 5.67 \times 10^{-8} [(300)^4 - (263.9)^4] = \mathbf{36.85\ Wm^{-2}}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 9 : MECHANICAL PROPERTIES OF SOLIDS

Q.1. A steel wire of length 4.7 m and cross-sectional area 3.0×10^{-5} m^2 stretches by the same amount as a copper wire of length 3.5 m and cross-sectional area of 4.0×10^{-5} m^2 under a given load. What is the ratio of the Young's modulus of steel to that of copper?

Ans. $Y = \dfrac{F/A}{\Delta L/L}$ or $Y = \dfrac{FL}{A\,\Delta L}$. Here F (load) and ΔL (elongation), are the same in the two cases.

For steel : $Y_S = \dfrac{F \times 4.7}{3.0 \times 10^{-5} \times \Delta L}$; **For copper :** $Y_C = \dfrac{F \times 3.5}{4.0 \times 10^{-5} \times \Delta L}$

$$\therefore \quad \frac{Y_S}{Y_C} = \frac{4.7 \times 4.0 \times 10^{-5}}{3.5 \times 3.0 \times 10^{-5}} = 1.79$$

Q.2. Figure 9.01 shows the strain-stress curve for a given material. What are (*a*) Young's modulus and (*b*) approximate yield strength for this material?

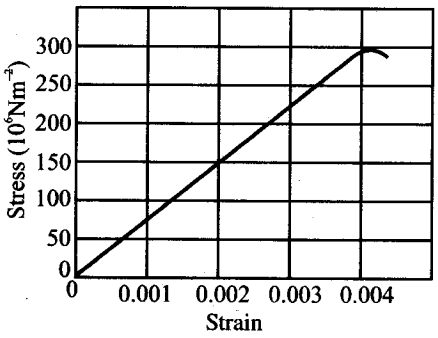

Fig. 9.01

Ans. (*a*) From the stress-strain curve in Fig. 9.01, it is clear that corresponding to stress = 150×10^6 Nm^{-2}, the strain is 0.002.

$$\therefore \quad Y = \frac{\text{Stress}}{\text{Strain}} = \frac{150 \times 10^6}{0.002} = 7.5 \times 10^{10} \text{ Nm}^{-2}$$

(*b*) The approximate yield strength of a material is the maximum stress it can bear without crossing the elastic limit. It is clear from Fig. 9.01 that approximate yield strength = 300×10^6 N m^{-2}.

Q.3. The stress-strain graphs for materials *A* and *B* are shown in Fig. 9.02.

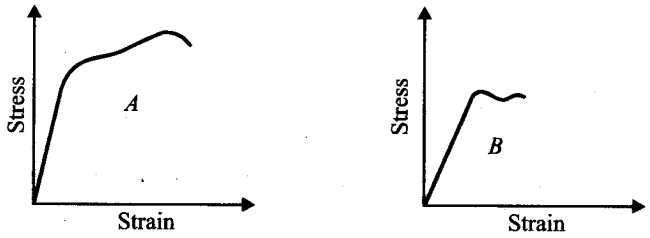

Fig. 9.02

The graphs are drawn to the same scale.

(*a*) Which of the materials has the greater Young's modulus?

(*b*) Which of the two is the stronger material?

Ans. (*a*) Young's modulus, Y = stress/strain. It is clear from Fig. 9.02 that for a given strain, stress for material A is more than that of material B. Therefore, Young's modulus for A is greater than that of B.

(*b*) The strength of a material is measured by the amount of stress it can bear before fracture. It is clear from Fig. 9.02 that material A is stronger than material B.

Q.4. Read the following two statements below carefully and state, with reasons, if it is true or false.

(*a*) The Young's modulus of rubber is greater than that of steel.

(*b*) The stretching of a coil is determined by its shear modulus.

Ans. (*a*) False. Young's modulus, Y = stress/strain. For a given stress, strain produced in the rubber is more than that in steel. Therefore, Y of rubber is less than that of steel and the statement is false.

(*b*) When a coil spring is stretched, neither its length nor its volume changes. There is only change in its shape. Therefore, stretching of coil spring is determined by its shear modulus.

Q.5. Two wires of diameter 0.25 cm, one made of steel and the other made of brass are loaded as shown in Fig. 9.03. The unloaded length of steel wire is 1.5 m and that of brass wire is 1.0 m. Compute the elongations of the steel and the brass wires. Given Young's modulus for steel = 2.0×10^{11} Pa and that for brass = 0.91×10^{11} Pa.

Ans. Fig. 9.03 shows the conditions of the problem.

Steel wire.

$Y = 2.0 \times 10^{11}$ Pa ; $L = 1.5$ m ; $F = 9.8\ (4 + 6) = 98$ N ;

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times(0.25\times10^{-2})^2 \text{ m}^2$$

Now $$Y = \frac{FL}{A\Delta L}$$

$$\therefore \quad \Delta L = \frac{FL}{AY} = \frac{98\times1.5}{\frac{\pi}{4}\times(0.25\times10^{-2})^2\times2.0\times10^{11}} = \mathbf{1.5 \times 10^{-4}\ m}$$

Brass wire.

$Y = 0.91 \times 10^{11}$ Pa ; $L = 1.0$ m ; $F = 6 \times 9.8$ N ; $A = \frac{\pi}{4}\times(0.25\times10^{-2})^2$ m^2

$$\therefore \quad \Delta L = \frac{FL}{AY} = \frac{(6\times9.8)\times1.0}{\frac{\pi}{4}\times(0.25\times10^{-2})^2\times0.91\times10^{11}} = \mathbf{1.3 \times 10^{-4}\ m}$$

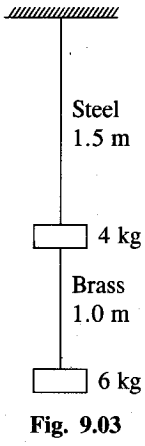

Fig. 9.03

Q.6. The edge of an aluminium cube is 10 cm long. One face of the cube is firmly fixed to a vertical wall. A mass of 100 kg is then attached to the opposite face of the cube. The shear modulus of aluminium is 25 GPa. What is the vertical deflection of this face?

Ans. Shear modulus, $\eta = \frac{FL}{A\Delta L}$

Here, $F = 100 \times 9.8$ N ; $L = 10$ cm $= 0.1$ m ; $A = L^2 = (0.1)^2$ m^2 ; $\eta = 25$ GPa $= 25 \times 10^9$ Pa

$$\therefore \quad 25 \times 10^9 = \frac{(100\times9.8)\times0.1}{(0.1)^2\times\Delta L} \quad \text{or} \quad \Delta L = \mathbf{3.92 \times 10^{-7}\ m}$$

Q.7. Four identical hollow cylindrical columns of mild steel support a big structure of mass 50,000 kg. The inner and outer radii of each column are 30 and 60 cm respectively. Assuming the load distribution to be uniform, calculate the compressional strain of each column. The Young's Modulus of steel is 2.0×10^{11} Pa.

Ans. $R_1 = 30$ cm $= 0.3$ m ; $R_2 = 60$ cm $= 0.6$ m ; $Y = 2 \times 10^{11}$ N/m^2

Area of X-section of each column is

$$A = \pi(R_2^2 - R_1^2) = \pi\left[(0.6)^2 - (0.3)^2\right] = 0.27\ \pi \text{ m}^2$$

The compressional force on each column is given by ;

$$F = \frac{50000\times9.8}{4}\text{N} = 12500 \times 9.8 \text{ N}$$

Now, Young's modulus, $Y = \frac{\text{Stress}}{\text{Strain}}$

$\therefore$ Compressional strain on each column $= \frac{\text{Stress}}{Y} = \frac{F/A}{Y} = \frac{F}{AY}$

$$= \frac{12500\times9.8}{0.27\,\pi\times2\times10^{11}} = \mathbf{7.22 \times 10^{-7}}$$

Q.8. A piece of copper having a rectangular cross-section of 15.2 mm × 19.1 mm is pulled in tension with 44,500 N force, producing only elastic deformation. Calculate the resulting strain. Given the shear modulus for copper is 42×10^9 Nm^{-2}.

Q.13. What is the density of water at a depth where pressure is 80.0 atm, given that its density at the surface is 1.03×10^3 kg m^{-3}? Given that bulk modulus of water = 2.0×10^9 Nm^{-2}. Take 1 atm = 10^5 Nm^{-2}.

Ans. As per the given conditions, mass of 1 m^3 of water at the surface of the ocean is 1030 kg. The mass of water remains the same. Let us calculate the change in volume ΔV of this water (V = 1 m^3) when it is at a pressure of 80 atm.

Now Bulk modulus, $K = -V\dfrac{\Delta p}{\Delta V}$

or $$\Delta V = -\frac{(\Delta p)\,V}{K}$$

Δp = change in pressure = 80 atm – *1 atm = 79 atm = 79×10^5 N/m^2;

$V = 1\ m^3$; $K = 2.0 \times 10^9$ N/m^2

$\therefore$ $$\Delta V = -\frac{(79\times10^5)\times(1)}{2.0\times10^9} = -0.0039\ m^3$$

$\therefore$ Density of water at depth where pressure is 80.0 atm

$$= \frac{m}{V+\Delta V} = \frac{1030}{1-0.0039} = 1034\ kg\ m^{-3}$$

Note that even under this huge pressure, water is only compressed to 2% to 3%. Hence solids and liquids are not easily compressed.

Q.14. Compute the fractional change in volume of a glass slab, when subjected to a hydraulic pressure of 10 atm. Given that bulk modulus of glass = 3.7×10^{10} Nm^{-2} and 1 atm = 1.013×10^5 Nm^{-2}.

Ans. Δp = 10 atm = $10 \times 1.013 \times 10^5$ Nm^{-2} ; $K = 3.7 \times 10^{10}$ Nm^{-2}

Now Bulk modulus, $K = V\dfrac{\Delta p}{\Delta V}$

$\therefore$ Fractional change in the volume of glass slab is

$$\frac{\Delta V}{V} = \frac{\Delta p}{K} = \frac{10\times1.013\times10^5}{3.7\times10^{10}} = 2.74 \times 10^{-5}$$

Q.15. Determine the volume contraction of a solid copper cube, 10 cm on an edge, when subjected to a hydraulic pressure of 7.0×10^6 Pa. Bulk modulus of copper = 140 GPa.

Ans. Here, $V = 10 \times 10 \times 10 = 1000\ cm^3 = 10^{-3}\ m^3$; $\Delta p = 7 \times 10^6$ Pa ; $K = 140 \times 10^9$ Pa

Now $K = V\dfrac{\Delta p}{\Delta V}$ or $140 \times 10^9 = 10^{-3} \times \dfrac{7\times10^6}{\Delta V}$ $\therefore$ $\Delta V = 5 \times 10^{-8}\ m^3 = 50\ mm^3$

Q.16. How much should the pressure on a litre of water be changed to compress it by 0.10%? Bulk modulus of water = 2.2×10^9 Nm^{-2}.

Ans. Here, V = 1 litre = 10^{-3} m^3 ; $\Delta V/V = 0.10\% = 0.1/100 = 10^{-3}$

Now $K = V\dfrac{\Delta p}{\Delta V}$ or $\Delta p = K\dfrac{\Delta V}{V} = 2.2\times10^9\times10^{-3} = 2.2 \times 10^6$ Pa

Q.17. Anvils made of single crystal of diamond, with the shape as shown in Fig. 9.04, are used to investigate behaviour of materials under very high pressures. Flat faces at the narrow end of the anvil have a diameter of 0.50 mm, and the wide ends are subjected to a compressional force of 50,000 N. What is the pressure at the tip of the anvil?

* At the surface, pressure is 1 atm.

Ans. Shear modulus, $\eta = \dfrac{FL}{A\,\Delta L}$. Here $\Delta L/L$ is the shear strain.

Now $\eta = 42 \times 10^9$ Nm^{-2} ; $F = 44500$ N ; $A = 15.2 \times 19.1 \times 10^{-6}$ m^2

$$\therefore \quad \text{Shear strain, } \frac{\Delta L}{L} = \frac{F}{A\eta} = \frac{44500}{(15.2\times19.1\times10^{-6})\times(42\times10^9)} = \mathbf{3.65 \times 10^{-3}}$$

Q.9. A steel cable with a radius of 1.5 cm supports a chairlift at a ski area. If the maximum stress is not to exceed 10^8 N m^{-2}, what is the maximum load the cable can support?

Ans. The maximum load (F_{max}) the cable can support is

F_{max} = Maximum stress × Area of X-section of cable

$= (10^8) \times \pi \times (1.5 \times 10^{-2})^2 = \mathbf{7.07 \times 10^4}$ N

Q.10. A rigid bar of mass 15 kg is supported symmetrically by three wires each 2.0 m long. Those at each end are of copper and the middle one is of iron. Determine the ratios of their diameters if each is to have the same tension. Young's modulus for copper and iron are 11×10^{10} Nm^{-2} and 19×10^{10} N m^{-2} respectively.

Ans. As the bar is supported symmetrically by the three wires, therefore, extension in each wire must be the same.

$$\text{Young's modulus, } Y = \frac{FL}{A\,\Delta L}$$

As per the conditions of the problem, F (tension), L (= 2.0 m) and ΔL (extension) for each wire are the same.

$$\therefore \quad Y \propto \frac{1}{A} \quad \text{or} \quad Y \propto \frac{1}{d^2} \qquad \text{...}d = \text{wire diameter}$$

$$\therefore \quad \frac{d_{copper}}{d_{iron}} = \sqrt{\frac{Y_{iron}}{Y_{copper}}} = \sqrt{\frac{19\times10^{10}}{11\times10^{10}}} = \mathbf{1.314}$$

Q.11. A 14.5 kg mass, fastened to the end of a steel wire of unstretched length 1.0 m, is whirled in a vertical circle with an angular velocity of 2 rev/s at the bottom of the circle. The cross-sectional area of the wire is 0.065 cm^2. Calculate the elongation of the wire when the mass is at the lowest point of its path. Given that Young's modulus of steel is 2.0×10^{11} Nm^{-2}.

Ans. Here $m = 14.5$ kg ; $A = 0.065 \times 10^{-4}$ m^2 ; $f = 2$ *r.p.s.* ; $L = 1$ m (= r)

When the mass is at the lowest point of its circular path, the stretching force is

$$F = mg + mr\,\omega^2 = 14.5 \times 9.8 + 14.5 \times 1 \times (2\pi \times 2)^2 = 2431.85 \text{ N}$$

$$\therefore \quad \text{Elongation, } \Delta L = \frac{FL}{AY} = \frac{2431.85\times1}{(0.065\times10^{-4})\times(2.0\times10^{11})} = 1.87 \times 10^{-3} \text{ m} = \mathbf{1.87 \text{ mm}}$$

Q.12. Compute the bulk modulus of water from the following data : Initial volume = 100.0 litres, Pressure increase = 100.0 atm (1 atm = 1.013×10^5 Pa), Final volume = 100.5 litres. Compare the bulk modulus of water with that of air (at constant temperature). Explain in simple terms why the ratio is so large? Given bulk modulus of air at S.T.P. = 1.0×10^{-4} N m^{-2}.

Ans. Bulk modulus of water, $K_{water} = V\dfrac{\Delta p}{\Delta V}$

Here $V = 100.0$ litres $= 100.0 \times 10^{-3}$ m^3 ; $\Delta p = 100.0$ atm $= 100.0 \times 1.013 \times 10^5$ Pa ;

$\Delta V = 100.5 \times 10^{-3} - 100.0 \times 10^{-3} = 0.5 \times 10^{-3}$ m^3

$$\therefore \quad K_{water} = 100.0\times10^{-3} \times \frac{100.0\times1.013\times10^5}{0.5\times10^{-3}} = \mathbf{2.026 \times 10^9 \text{ Nm}^{-2}}$$

$$\text{Now} \quad \frac{K_{water}}{K_{air}} = \frac{2.026\times10^9}{1.0\times10^{-4}} = \mathbf{2.026 \times 10^{13}}$$

Note that the ratio is very large. It is because large forces are needed to produce even a small change in the volume of solids and liquids *i.e.,* bulk modulus of solids and liquids is very large. On the other hand, gases are easily compressed and, therefore, have very small bulk modulus.

Ans. Here F = 50,000 N ; d = 0.50 mm = 0.50×10^{-3} m

$\therefore$ Pressure at the tip of the anvil is

$$P = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$$

$$= \frac{50,000}{\pi d^2/4} = \frac{4 \times 50,000}{\pi \times (0.50 \times 10^{-3})^2}$$

$$= \mathbf{2.55 \times 10^{11}\ Pa}$$

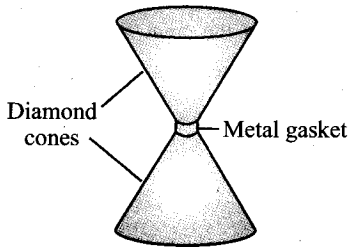

Fig. 9.04

Q.18. A rod of length 1.05 m having negligible mass is supported at its ends by two wires of steel (wire *A*) and aluminium (wire *B*) of equal lengths as shown in Fig. 9.05. The cross-sectional areas of wires *A* and *B* are 1.0 mm^2 and 2.0 mm^2, respectively. At what point along the rod should a mass *m* be suspended in order to produce (*a*) equal stresses and (*b*) equal strains in both steel and aluminium wires? Young's modulus for steel = 2×10^{11} Pa and that for aluminium = 7×10^{10} Pa.

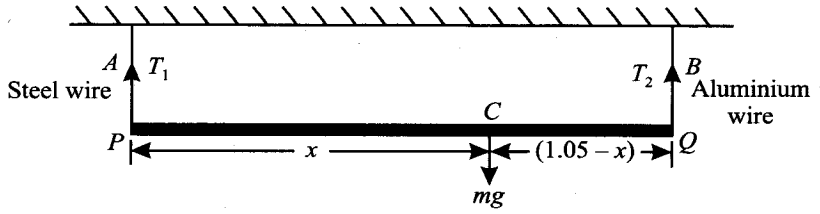

Fig. 9.05

Ans. Fig. 9.05 shows the conditions of the problem. Here mass m is suspended from point C at a distance x from the end P (where wire A is connected) of the rod. Let A_1 and A_2 be the area of X-section of steel wire and aluminium wire respectively.

(*a*) **When the stresses in the two wires are equal.** We are to find the value of x for this condition. Let T_1 and T_2 be the tensions in the wires A and B respectively. For equal stress in the two wires, we have,

$$\frac{T_1}{A_1} = \frac{T_2}{A_2} \quad \text{or} \quad \frac{T_1}{T_2} = \frac{A_1}{A_2} = \frac{1\ \text{mm}^2}{2\ \text{mm}^2} = \frac{1}{2}$$

The suspended system is in equilibrium. Therefore, taking moment of forces about point C, we have,

$$T_1 x = T_2 (1.05 - x) \quad \text{or} \quad \frac{T_1}{T_2} = \frac{1.05 - x}{x}$$

$$\therefore \quad \frac{1}{2} = \frac{1.05 - x}{x} \quad \text{or} \quad x = 0.7\ \text{m} = \mathbf{70\ cm}$$

(*b*) **When the strains in the two wires are equal.** We are to find the value of x for this condition. Let T_1' and T_2' be the tensions in the wires A and B respectively. For equal strain in the two wires, we have,

$$\frac{T_1'}{A_1 Y_1} = \frac{T_2'}{A_2 Y_2} \quad \text{or} \quad \frac{T_1'}{T_2'} = \frac{A_1}{A_2} \times \frac{Y_1}{Y_2} = \frac{1}{2} \times \frac{2 \times 10^{11}}{7 \times 10^{10}} = \frac{10}{7}$$

Again for the equilibrium of the rod, we have,

$$T_1' x = T_2' (1.05 - x) \quad \text{or} \quad \frac{T_1'}{T_2'} = \frac{1.05 - x}{x}$$

$$\therefore \quad \frac{10}{7} = \frac{1.05 - x}{x} \quad \text{or} \quad x = 0.432\ \text{m} = \mathbf{43.2\ cm}$$

Q.19. A mild steel wire of length 1.0 m and cross-sectional area 0.50×10^{-2} cm^2 is stretched, well within its elastic limit, horizontally between two pillars. A mass of 100 g is suspended from the mid-point of the wire. Calculate the depression at the mid-point. Given that Young's modulus for steel = 2×10^{11} Nm^{-2}.

Ans. Fig. 9.06 shows the conditions of the problem. Here $AO = BO = L = 0.5$ m and x is the depression at the point O *i.e.*, $OC = x$.

Increase in the length of the wire is

$$\Delta L = AC + CD - AB = 2AC - AB$$

$$= 2(L^2 + x^2)^{1/2} - 2L = 2L\left(1+\frac{x^2}{L^2}\right)^{1/2} - 2L$$

$$= 2L\left(1+\frac{x^2}{2L^2}\right) - 2L = \frac{x^2}{L}$$

Strain in wire $= \dfrac{\Delta L}{2L} = \dfrac{x^2/L}{2L} = \dfrac{x^2}{2L^2}$

If T is the tension in each segment of the wire, then,

$$2T\cos\theta = mg \quad \text{or} \quad T = \frac{mg}{2\cos\theta}$$

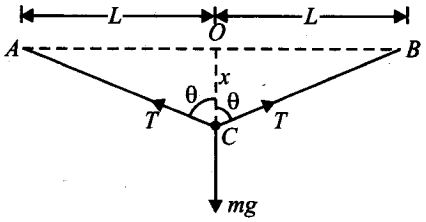

Fig. 9.06

Now $\quad *\cos\theta = \dfrac{x}{L}$ so that $T = \dfrac{mg}{2(x/L)} = \dfrac{mgL}{2x}$

Now $\quad \text{Stress} = \dfrac{T}{A} = \dfrac{mgL}{2Ax}$

Also $\quad Y = \dfrac{\text{Stress}}{\text{Strain}} = \dfrac{mgL}{2Ax} \times \dfrac{2L^2}{x^2} = \dfrac{mgL^3}{Ax^3}$

$$\therefore \quad x = L\left[\frac{mg}{YA}\right]^{1/3} = 0.5\left[\frac{0.1\times10}{2\times10^{11}\times0.5\times10^{-6}}\right]^{1/3} = 1.07 \times 10^{-2} \text{ m} = \mathbf{1.07 \text{ cm}}$$

Q.20. Two strips of metal are riveted together at their ends by four rivets, each of diameter 6.0 mm. What is the maximum tension that can be exerted by the riveted strip if the shearing stress on the rivet is not to exceed 6.9×10^7 Pa? Assume that each rivet is to carry one quarter of the load.

Ans. Radius of rivet, $r = 6/2 = 3$ mm ; Maximum stress $= 6.9 \times 10^7$ Pa

Maximum tension on a rivet = Maximum stress × Area of X-section of rivet

$$= (6.9 \times 10^7) \times \pi \times (3 \times 10^{-3})^2 = 1950 \text{ N}$$

$\therefore$ Maximum tension that can be exerted by riveted strip

$$= 4 \times 1950 = \mathbf{7.8 \times 10^3 \text{ N}}$$

Q.21. The Marina trench is located in the Pacific Ocean, and at one place it is nearly eleven km beneath the surface of water. The water pressure at the bottom of the trench is about 1.1×10^8 Pa. A steel ball of initial volume 0.32 m^3 is dropped into the ocean and falls to the bottom of the trench. What is the change in the volume of the ball when it reaches the bottom? Bulk modulus of steel = 1.6×10^{11} Nm^{-2}.

Ans. Here $\Delta p = 1.1 \times 10^8$ Pa ; $V = 0.32$ m^3 ; $K = 1.6 \times 10^{11}$ Pa

Now $\quad K = V\dfrac{\Delta p}{\Delta V}$ $\quad$ or $\quad 1.6\times10^{11} = 0.32\times\dfrac{1.1\times10^8}{\Delta V}$ $\quad \therefore \quad \Delta V = \mathbf{2.2 \times 10^{-4} \text{ m}^3}$

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* $\cos\theta = \dfrac{OC}{AC} = \dfrac{x}{(L^2+x^2)^{1/2}} = \dfrac{x}{L\left(1+\dfrac{x^2}{L^2}\right)^{1/2}} = \dfrac{x}{L\left(1+\dfrac{1}{2}\dfrac{x^2}{L^2}\right)}$

As $x << L$, therefore, $1+\dfrac{1}{2}\dfrac{x^2}{L^2} \simeq 1$ so that $\cos\theta = \dfrac{x}{L}$.

NCERT CHAPTER 10 : MECHANICAL PROPERTIES OF FLUIDS

Q.1. Explain why

(*a*) The blood pressure in humans is greater at the feet than at the brain.

(*b*) Atmospheric pressure at a height of about 6 km decreases to nearly half of its value at the sea level, though the height of the atmosphere is more than 100 km.

(*c*) Hydrostatic pressure is a scalar quantity even though pressure is force divided by area.

Ans. (*a*) Pressure, $P = h\rho g$. The height (h) of the blood column in the human body is more at the feet than at the brain. Therefore, blood pressure in humans is greater at the feet than at the brain.

(*b*) Atmospheric pressure, $P = h\rho g$ where ρ is the density of air. As we go up, the density of air decreases rapidly and at a height of about 6 km, it decreases to nearly half of its value at the sea level. Due to this reason, atmospheric pressure at a height of about 6 km decreases to nearly half of its value at the sea level.

(*c*) When force is applied on a liquid, hydrostatic pressure is transmitted equally in all directions in the liquid. Therefore, there is no sense to assign direction to hydrostatic pressure and it is considered a scalar quantity.

Q.2. Explain why

(*a*) The angle of contact of mercury with glass is obtuse, while that of water with glass is acute.

(*b*) Water on a clean glass surface tends to spread out while mercury on the same surface tends to form drops. (Put differently, water wets glass while mercury does not).

(*c*) Surface tension of a liquid is independent of the area of the surface.

(*d*) Water with detergent dissolved in it should have small angles of contact.

(*e*) A drop of liquid under no external forces is always spherical in shape.

Ans. (*a*) When a small quantity of liquid (mercury or water) is poured on a solid surface, three interfaces *viz.* liquid-air, solid-air and solid-liquid are formed as shown in Fig. 10.01. Let the surface tensions corresponding to these three layers be T_{LA}, T_{SA} and T_{SL} respectively. Suppose θ is the angle of contact between the liquid and solid. The molecules in the region where the three interfaces meet are in equilibrium *i.e.*, the net force acting on them is zero. Thus for the molecule at O to be in equilibrium, we have,

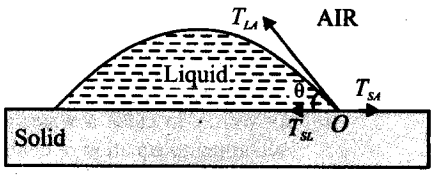


Fig. 10.01

$$T_{SL} + T_{LA}\cos\theta = T_{SA}$$

$$\therefore \quad \cos\theta = \frac{T_{SA} - T_{SL}}{T_{LA}}$$

In case of mercury-glass, $T_{SA} < T_{SL}$ so that $\cos\theta$ is negative *i.e.*, $\theta > 90°$ or the angle of contact θ is obtuse. For the case of water-glass, $T_{SA} > T_{SL}$ so that $\cos\theta$ is positive *i.e.*, $\theta < 90°$ or the angle of contact is acute.

(*b*) For mercury-glass, the angle of contact θ is obtuse (*i.e.*, $> 90°$). In order to achieve an obtuse value of θ, the mercury tends to form a drop. For the case of water-glass, the angle of contact θ is acute (*i.e.*, $< 90°$). To achieve this acute value of θ, the water tends to spread.

(*c*) The surface tension is the force acting per unit length on either side of the imaginary line drawn on the surface of the liquid. This force is independent of the area of the liquid surface.

(*d*) A cloth has small pores in the form of fine capillaries. The rise of a liquid in a capillary is given by ;

$$h = \frac{2T\cos\theta}{r\rho g}$$

If angle of contact for a detergent is small, $\cos\theta$ and hence capillary rise h will be large. Therefore, the detergent will penetrate more in cloth and remove dirt from the cloth.

(*e*) A drop of liquid tends to acquire minimum surface area due to surface tension. For a given volume, a sphere has the least surface area. Therefore, when no external forces (*e.g.,* gravitational force) are acting, a drop of liquid has spherical shape.

Q.3. Fill in the blanks using the word(s) from the list appended with each statement :

(*a*) Surface tension of liquids generally ..... with temperatures (increases/decreases).

(*b*) Viscosity of gases ..... with temperature, whereas viscosity of liquids ..... with temperature (increases/decreases).

(*c*) For solids with elastic modulus of rigidity, the shearing force is proportional to ....., while for fluids it is proportional to ..... (shear strain/rate of shear strain).

(*d*) For a fluid in a steady flow, the increase in flow speed at a constriction follows ..... (conservation of mass/Bernoulli's principle).

(*e*) For the model of a plane in a wind tunnel, turbulence occurs at a ..... speed for turbulence for an actual plane (greater/smaller).

Ans. (*a*) decreases (*b*) increases ; decreases (*c*) shear strain ; rate of shear strain (*d*) conservation of mass and Bernoulli's principle (*e*) greater.

Q.4. Explain why

(*a*) To keep a piece of paper horizontal, you should blow over, not under, it.

(*b*) When we try to close a water tap with our fingers, fast jets of water gush through the openings between our fingers.

(*c*) The size of the needle of a syringe controls flow rate better than the thumb pressure exerted by a doctor while administering an injection.

(*d*) A fluid flowing out of a small hole in a vessel results in a backward thrust on the vessel.

(*e*) A spinning cricket ball in air does not follow a parabolic trajectory.

Ans. (*a*) When we blow over the piece of paper, the velocity of air above the paper increases and pressure of air on it decreases (Bernoulli's principle). The result is that pressure below the paper becomes greater than the pressure above the paper. Therefore, piece of paper remains horizontal.

(*b*) As we try to close a water tap with our fingers, the area of outlet of water jet is reduced. According to equation of continuity (av = constant), the velocity of water increases enormously.

(*c*) According to Bernoulli's principle,

$$P + \rho g h + \frac{1}{2}\rho v^2 = \text{Constant}$$

The size of the needle controls the velocity (v) of flow and thumb controls pressure (P). It is clear from the above equation that total energy of medicine to be injected depends upon the second power of velocity and first power of pressure. Therefore, velocity has more influence in the process. For this reason, the needle has a better control over the flow of medicine.

(*d*) Due to small area of cross-section of the hole, the fluid flows out of the hole with a large velocity and hence possesses large momentum. Since no external force is acting on the system, to conserve momentum, the vessel acquires a backward velocity. As a result, the vessel experiences backward thrust.

(*e*) When a spinning ball is thrown with a large velocity as shown in Fig. 10.02, the velocity of air flow above the ball becomes smaller than that below the ball. It is because the air moving above the ball is in a direction opposite to the direction of the spinning ball. However, below the ball, the two motions help each other. Therefore, according to Bernoulli's principle, the pressure of air above the ball becomes greater that below it. Due to this action, a force acts on the ball in the vertically downward direction. As a result, the ball no longer follows a parabolic path. The effect is called *Magnus effect.*

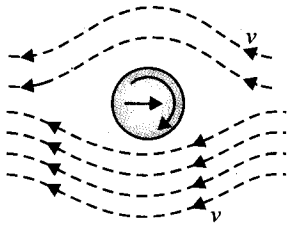


Fig. 10.02

Q.5. A 50 kg girl wearing high heel shoes balances on a single heel. The heel is circular with a diameter 1.0 cm. What is the pressure exerted by the heel on the horizontal floor?

Ans. Mass of girl, m = 50 kg ; Radius of heel, r = 1/2 cm = 1/200 m

$$\therefore \quad \text{Pressure, } P = \frac{\text{Force}}{\text{Area}} = \frac{mg}{\pi r^2} = \frac{50 \times 9.8}{\pi \times (1/200)^2} = 6.24 \times 10^6 \text{ Nm}^{-2}$$

Q.6. Toricelli's barometer used mercury. Pascal duplicated it using French wine of density 984 kg m^{-3}. Determine the height of the wine column for normal atmospheric pressure.

Ans. Normal atmospheric pressure P is

P = Pressure exerted by 0.76 m of mercury column

$= 0.76 \times (13.6 \times 10^3) \times 9.8 \text{ N m}^{-2}$

If h is the height of the wine column for pressure P, then,

$$P = h \rho g \quad \therefore \quad h = \frac{P}{\rho g} = \frac{0.76 \times 13.6 \times 10^3 \times 9.8}{984 \times 9.8} = 10.5 \text{ m}$$

Q.7. A vertical off-shore structure is built to withstand a maximum stress of 10^9 Pa. Is the structure suitable for putting up on top of an oil well in the ocean? Take the depth of the ocean to be roughly 3 km, and ignore ocean currents.

Ans. Pressure, $P = h \rho g$

Here h = 3 km = 3000 m ; ρ = 1000 kg m^{-3} ; g = 9.8 m s^{-2}

$$\therefore \quad P = 3000 \times 1000 \times 9.8 = 2.94 \times 10^7 \text{ Pa}$$

Since the pressure exerted by sea water is less than the maximum stress the structure can withstand, the structure is suitable.

Q.8. A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross-section of the piston carrying the load is 425 cm^2. What maximum pressure would the smaller piston have to bear?

Ans. Area of cross-section of bigger piston (carrying load), A = 425 cm^2 = 425 × 10^{-4} m^2

Max. load bigger piston can bear, F = 3000 × 9.8 N

$$\therefore \text{ Max. pressure on bigger piston, } P = \frac{F}{A} = \frac{3000 \times 9.8}{425 \times 10^{-4}} = 6.92 \times 10^5 \text{ N m}^{-2}$$

Since liquids transmit pressure equally in all directions, the smaller piston can bear maximum pressure = **6.92×10^5 Nm^{-2}**.

Q.9. A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the relative density of spirit?

Ans. Fig. 10.03 shows the conditions of the problem. The pressure at the interfaces marked D and A is the same. It means pressure exerted by water at D and pressure exerted by spirit at A are the same *i.e.*,

$$h_1 \rho_1 g = h_2 \rho_2 g$$

or $$10 \times 1 \times g = 12.5 \times \rho_2 \times g$$

$$\therefore \quad \rho_2 = \frac{10}{12.5} = 0.8 \text{ g cm}^{-3}$$

$\therefore$ Relative density of spirit = **0.8**

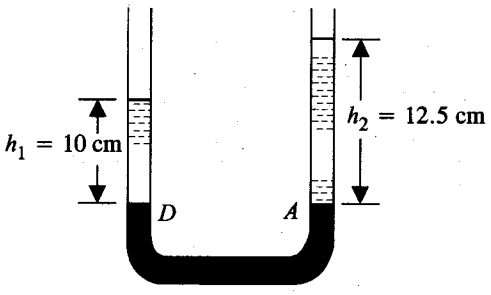


Fig. 10.03

Q.10. In the previous problem, if 15.0 cm of water and spirit each are further poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms? (Relative density of mercury = 13.6)

Ans. When we pour 15 cm of water and spirit in the respective arms, h_1 = 10 + 15 = 25 cm and h_2 = 12.5 + 15 = 27.5 cm.

Pressure at the mercury – water interface is

$$P_1 = h_1 \rho_1 g = 25 \times 1 \times g = 25 \text{ g}$$

Pressure at the mercury – spirit interface is

$$P_2 = h_2 \rho_2 g = 27.5 \times 0.8 \times g = 22 \text{ g}$$

Since pressure at water-mercury interface is more than the pressure at spirit-mercury interface, the mercury will rise in the spirit arm. If h is the difference in the levels of mercury in the two arms, then,

$$P_1 - P_2 = h \rho g$$

or $$25\ g - 22\ g = h \times 13.6 \times g$$

$\therefore$ $$h = \frac{3}{13.6} \text{ cm} = 0.221 \text{ cm}$$

Q.11. Can Bernoulli's equation be used to describe the flow of water through a rapid in a river? Explain.

Ans. No. Bernoulli's equation can be applied only when the flow of liquid is streamline.

Q.12. Does it matter if one uses gauge instead of absolute pressures in applying Bernoulli's equation ? Explain.

Ans. No, provided the atmospheric pressures at the two points where Bernoulli's equation is applied do not differ significantly.

Q.13. Glycerine flows steadily through a horizontal tube of length 1.5 m and radius 1.0 cm. If the amount of glycerine collected per second at one end is 4.0×10^{-3} kg s^{-1}, what is the pressure difference between the two ends of the tube? (Density of glycerine = 1.3×10^3 kg m^{-3} and viscosity of glycerine = 0.83 Pas).

Ans. $Q = \frac{\pi p R^4}{8 \eta l}$. Here l = 1.5 m ; R = 1 cm = 10^{-2} m ; η = 0.83 Nsm^{-2}.

Now $$Q = \frac{\text{Mass collected/sec}}{\text{Density}} = \frac{4 \times 10^{-3}}{1.3 \times 10^3} = \frac{4}{1.3} \times 10^{-6} \text{ m}^3 \text{ s}^{-1}$$

$\therefore$ $$\frac{4}{1.3} = \frac{\pi \times p \times (10^{-2})^4}{8 \times 0.83 \times 1.5}$$

or $$p = \frac{4 \times 10^{-6} \times 8 \times 0.83 \times 1.5}{1.3 \times \pi \times (10^{-2})^4} = 975.52 \text{ Pa}$$

Q.14. In a test experiment on a model aeroplane in a wind tunnel, the flow speeds on the upper and lower surfaces of the wing are 70 m s^{-1} and 63 m s^{-1} respectively. What is the lift on the wing if its area is 2.5 m^2? Take the density of air to be 1.3 kg m^{-3}.

Ans. Let v_1 and v_2 be the speeds on the upper and lower surfaces of the wing of aeroplane and P_1 and P_2 the corresponding pressures. According to Bernoulli's equation for horizontal flow,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

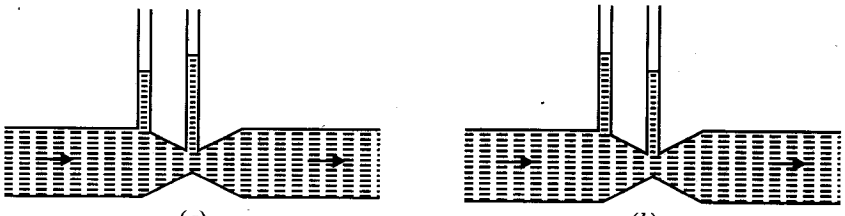
$\therefore$ $$P_2 - P_1 = \frac{1}{2}\rho (v_1^2 - v_2^2)$$

Here ρ = 1.3 kg m^{-3} ; v_1 = 70 ms^{-1} ; v_2 = 63 ms^{-1}

$\therefore$ $$P_2 - P_1 = \frac{1}{2} \times 1.3 \left[(70)^2 - (63)^2\right] = 605.15 \text{ N m}^{-2}$$

$\therefore$ Lifting force = $(P_2 - P_1)$ × Area of wing = 605.15 × 2.5 = 1.51×10^3 N

Q.15. Figures 10.04 (*a*) and (*b*) refer to the steady flow of a (non-viscous) liquid. Which of the two figures is incorrect? Why?



(*a*) (*b*)

Fig. 10.04

Ans. Fig. 10.04 (*a*) is incorrect. According to equation of continuity and Bernoulli's equation for horizontal flow, we have,

$$av = \text{Constant} \quad ; \quad P + \frac{1}{2}\rho v^2 = \text{Constant}$$

According to equation of continuity, the velocity of flow is more at the constriction (where area of X-section is less) and according to Bernoulli's equation where velocity is more, pressure should be less. Fig. 10.04 (*a*) shows that pressure is more (higher level of liquid) at the constriction and hence is incorrect.

Q.16. The cylindrical tube of a spray pump has a cross-section of 8.0 cm^2, one end of which has 40 fine holes each of diameter 1.0 mm. If the liquid flow inside the tube is 1.5 m min^{-1}, what is the speed of ejection of the liquid through the holes?

Ans. According to equation of continuity, $a_1 v_1 = a_2 v_2$.

Area of X–section of tube, $a_1 = 8.0\ cm^2 = 8.0 \times 10^{-4}\ m^2$; speed of liquid in tube, $v_1 = \frac{1.5}{60}\ ms^{-1}$

Area of cross-section of 40 holes, $a_2 = 40 \times \pi r^2 = 40 \times \pi \times (5 \times 10^{-4})^2\ m^2$; v_2 = ?

∴ Speed of ejection of liquid through the holes is

$$v_2 = \frac{a_1 v_1}{a_2} = \frac{(8\times10^{-4})\times 1.5/60}{40\times\pi(5\times10^{-4})^2} = \mathbf{0.637\ m\ s^{-1}}$$

Q.17. A U-shaped wire is dipped in a soap solution, and removed. The thin soap film formed between the wire and the light slider supports a weight of 1.5 × 10^{-2} N (which includes the small weight of the slider). The length of the slider is 30 cm. What is the surface tension of the film?

Ans. The force due to surface tension acts upward on the slider. Since the soap film has the free surfaces, the total length of the film to be supported = $2l$ = 2 × 30 = 60 cm = 0.6 m.

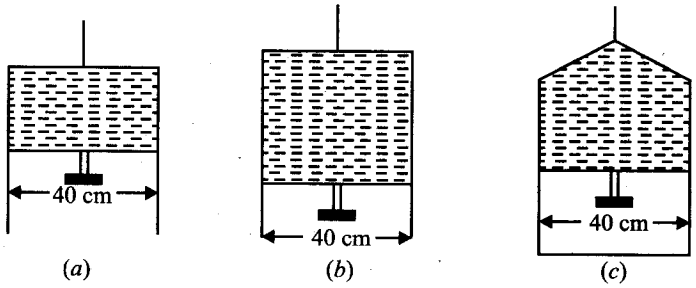
∴ Total upward force on the slider due to surface tension is

$$F = T \times 2l = T \times 0.6 = 0.6\ T \text{ newton}$$

Since the slider is in equilibrium, the total upward force F on the slider must be equal to its weight *i.e.*,

$$F = mg \quad \text{or} \quad 0.6\,T = 1.5 \times 10^{-2} \quad \therefore \quad T = \frac{1.5\times10^{-2}}{0.6} = \mathbf{0.025\ N\ m^{-1}}$$

Q.18. Figure 10.05 (*a*) shows a thin liquid film supporting a small weight = 4.5 × 10^{-2} N. What is the weight supported by a film of the same liquid at the same temperature in Figs. 10.5 (*b*) and (*c*)? Explain your answer physically.



(*a*) (*b*) (*c*)

Fig. 10.05

Ans. The length of the slider is l = 40 cm = 0.4 m. Since the soap film has two free surfaces, the total length of the film to be supported = $2l$ = 2 × 0.4 = 0.8 m. Total upward force on the slider due to surface tension is $F = T \times 2l = T \times 0.8$ and weight supported by the film is 4.5×10^{-2} N.

$$\therefore \quad T \times 0.8 = 4.5 \times 10^{-2} \quad \therefore \quad T = \frac{4.5\times10^{-2}}{0.8} = 5.625 \times 10^{-2} \text{ N m}^{-1}$$

As the liquid and temperature in the three cases are the same, therefore, surface tension for cases (*b*) and (*c*) will also be the same (*i.e.*, 5.625×10^{-2} N m^{-1}). Again the length of the slider is the same in the three cases, the weight supported by the liquid film in cases (*b*) and (*c*) will be the same *i.e.*, 4.5×10^{-2} N.

Q.19. What is the pressure inside the drop of mercury of radius 3.00 mm at room temperature? Surface tension of mercury at that temperature (20°C) is 4.65×10^{-1} N m^{-1}. The atmospheric pressure is 1.01×10^5 Pa. Also give the excess pressure inside the drop.

Ans. The excess pressure inside a liquid drop is given by ;

$$p_2 - p_1 = \frac{2T}{R}$$

Here $p_2 - p_1$ = excess pressure ; $T = 4.65 \times 10^{-1}$ Nm^{-1} ; R = 3.0 mm = 3.0×10^{-3} m

$$\therefore \quad p_2 - p_1 = \frac{2\times(4.65\times10^{-1})}{3.0\times10^{-3}} = 3.1 \times 10^2 \text{ N/m}^2$$

Pressure inside the drop, $p_2 = p_1 + 3.1 \times 10^2 = 1.01 \times 10^5 + 3.1 \times 10^2 = 1.0131 \times 10^5$ Pa

Q.20. What is the excess pressure inside a bubble of soap solution of radius 5.00 mm, given that the surface tension of soap solution at the temperature (20°C) is 2.50×10^{-2} Nm^{-1}? If an air bubble of the same dimensions were formed at depth of 40.0 cm inside a container containing the soap solution (of relative density 1.20), what would be the pressure inside the bubble? (1 atmospheric pressure is 1.01×10^5 Pa).

Ans. The excess pressure inside a soap bubble is given by ;

$$p_2 - p_1 = \frac{4T}{R}$$

Here $T = 2.5 \times 10^{-2}$ N m^{-1} ; R = 5.0 mm = 5.0×10^{-3} m

$$\therefore \quad p_2 - p_1 = \frac{4\times(2.5\times10^{-2})}{5.0\times10^{-3}} = 20 \text{ Pa}$$

Excess pressure inside an air bubble in soap solution

$$= \frac{2T}{R} = \frac{2\times(2.5\times10^{-2})}{5.0\times10^{-3}} = 10 \text{ Pa}$$

Total pressure inside the air bubble

$$= P_a + h\rho g + \frac{2T}{R}$$

$$= 1.01 \times 10^5 + 0.4 \times 1200 \times 9.8 + 10 = 1.06 \times 10^5 \text{ Pa}$$

Q.21. A tank with a square base of area 1.0 m^2 is divided by a vertical partition in the middle. The bottom of the partition has a small-hinged door of area 20 cm^2. The tank is filled with water in one compartment, and an acid (of relative density 1.7) in the other, both to a height of 4.0 m. Compute the force necessary to keep the door closed.

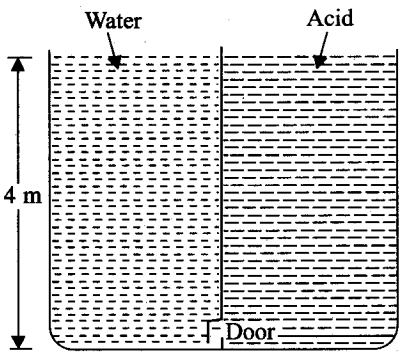


Fig. 10.06

Ans. Fig. 10.06 shows the conditions of the problem.

For compartment containing water.

Pressure on the door due to water is

$P_w = h\ \rho\ g = 4 \times 10^3 \times 9.8 = 39.2 \times 10^3$ N/m^2

For compartment containing acid.

Pressure on the door due to acid is

$$P_{acid} = h\,\rho\,g = 4 \times (1.7 \times 10^3) \times 9.8 = 66.64 \times 10^3 \text{ N/m}^2$$

$$\text{Pressure difference} = P_{acid} - P_w = 66.64 \times 10^3 - 39.2 \times 10^3$$

$$= 27.44 \times 10^3 \text{ N/m}^2$$

$$\text{Area of door, } A = 20 \text{ cm}^2 = 20 \times 10^{-4} \text{ m}^2$$

$$\text{Force on the door} = (P_{acid} - P_w) \times A = 27.44 \times 10^3 \times 20 \times 10^{-4} = \mathbf{54.88\ N}$$

Therefore, in order to keep the door closed, a force of 54.88 N must be applied on it from the water side.

Q.22. A manometer reads the pressure of a gas in an enclosure as shown in Fig. 10.07 (*a*). When a pump removes some of the gas, the manometer reads as in Fig. 10.07 (*b*). The liquid used in the manometers is mercury and the atmospheric pressure is 76 cm of mercury.

(*a*) Give the absolute and gauge pressure of the gas in the enclosure for cases (*a*) and (*b*), in units of cm of mercury.

(*b*) How would the levels change in case (*b*) if 13.6 cm of water (immiscible with mercury) are poured into the right limb of the manometer? (Ignore the small change in the volume of the gas).

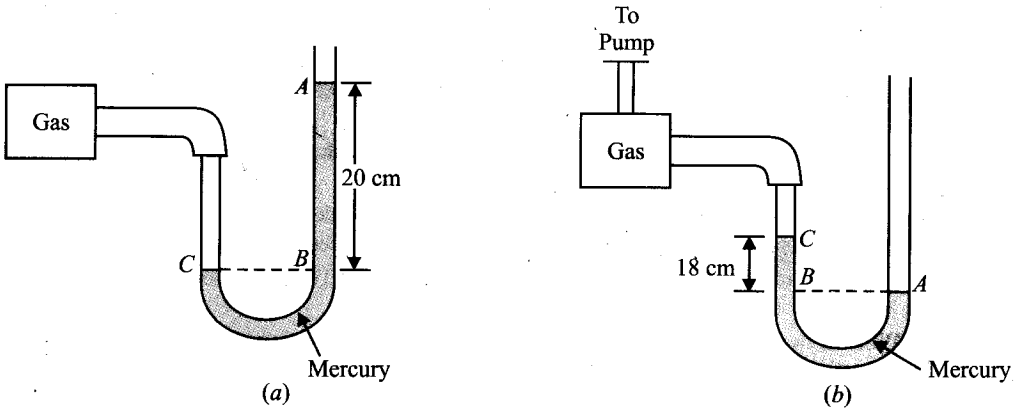


Fig. 10.07

Ans. Atmospheric pressure, P_a = 76 cm of Hg

(*a*) **Fig. 10.07 (*a*).** Referring to Fig. 10.07 (*a*),

Pressure at $A = P_a = 76$ cm of Hg

$\therefore$ Absolute pressure at $B = P_a + h = 76 + 20 = 96$ cm of Hg

Since C is at the same level as B,

$\therefore$ Absolute pressure of gas = Absolute Pressure at B = **96 cm of Hg**

Gauge pressure of gas = h = **20 cm of Hg**

Fig. 10.07 (*b*). Referring to Fig. 10.07 (*b*),

Pressure at $A = 76$ cm of Hg

$\therefore$ Pressure at $B = 76$ cm of Hg

Absolute pressure at $C = 76 - h = 76 - 18 = 58$ cm of Hg

Therefore, absolute pressure of gas is **58 cm of Hg**

Gauge pressure of gas = $-h$ = **−18 cm of Hg**

(*b*) Pouring 13.6 cm of water in the right limb in Fig. 10.07 (*b*) is equivalent to 13.6/13.6 = 1 cm of Hg column = h'.

$\therefore$ Pressure at $A = P_a + h' = 76 + 1 = 77$ cm of Hg

If h_1 is the difference in mercury levels in the two limbs, then, $P_B = 58 + h_1$.

Now $P_A = P_B$ or $77 = 58 + h_1$ $\therefore$ h_1 = **19 cm of Hg**

Q.23. Two vessels have the same base area but different shapes. The first vessel takes twice the volume of water that the second vessel requires to fill upto a particular common height. Is the force exerted by the water on the base of the vessel the same in the two cases? If so, why do the vessels filled with water to that same height give different readings on a weighing scale ?

Ans. Force on the base = Pressure × Area of base = $(h\ \rho\ g) \times A$

Since the two vessels are filled to the same height (h), the pressure at the bases of the two vessels will be equal. Further, base area (A) of each vessel is the same. Therefore, the force exerted by the water on the base of the vessel is the same in the two cases.

Water also exerts force on the sides of the vessels. In case the walls of the vessel are not perpendicular to the base, the force exerted by water on the walls has a net non-zero vertical component. This net vertical component is greater for the first vessel than the second. Hence the vessels weigh different on a weighing machine.

Q.24. During blood transfusion the needle is inserted in a vein where the gauge pressure is 2000 Pa. At what height must the blood container be placed so that blood may just enter the vein? Density of blood = 1.06×10^3 kg/m^3.

Ans. Let h be the height of the blood container.

Now $$P = h\ \rho\ g \quad \text{or} \quad h = \frac{P}{\rho g} = \frac{2000}{1.06 \times 10^3 \times 9.8} = \mathbf{0.193\ m}$$

Q.25. In deriving Bernoulli's equation, we equated the work done on the fluid in the tube to its change in the potential and kinetic energy.

(*a*) How does the pressure change as the fluid moves along the tube if the dissipative forces are present?

(*b*) Do the dissipative forces become more important as the fluid velocity increases. Discuss quantitatively.

Ans. (*a*) If the dissipative forces are present, a part of energy possessed by the fluid is spent in doing work against these forces. Due to this, pressure drop becomes large.

(*b*) As the fluid velocity increases, there is more turbulence. Therefore, dissipative forces become more important as the fluid velocity increases.

Q.26. (*a*) What is the largest average velocity of blood flow in an artery of radius 2×10^{-3} m if the flow must remain laminar? (*b*) What is the corresponding flow rate? (Take viscosity of blood to be 2.084×10^{-3} Pa s and density of blood = 1.06×10^3 kg/m^3).

Ans. (*a*) For laminar flow, the average velocity of blood flow should not exceed the critical velocity (v_c).

$$\text{Critical velocity, } v_c = \frac{N_R\ \eta}{\rho\ D} \qquad \text{... } D \text{ is diameter of artery}$$

Here $\eta = 2.084 \times 10^{-3}$ Pa s ; $\rho = 1.06 \times 10^3$ kg m^{-3} ; For laminar flow, $N_R = 2000$

$$\therefore \quad v_c = \frac{2000 \times 2.084 \times 10^{-3}}{(1.06 \times 10^3) \times (2 \times 2 \times 10^{-3})} = \mathbf{0.983\ ms^{-1}}$$

(*b*) Volume flowing/sec $= \pi\ r^2\ v_c = \pi\ (2 \times 10^{-3})^2 \times 0.983 = \mathbf{1.23 \times 10^{-5}\ m^3\ s^{-1}}$

Q.27. A plane is in level flight at constant speed and each of its two wings has an area of 25 m^2. If the speed of the air is 180 km/h over the lower wing and 234 km/h over the upper wing surface, determine the plane's mass. (Take air density to be 1 kg m^{-3}).

Ans. Let v_1 and v_2 be the speeds on the upper and lower surfaces of the wing of aeroplane and P_1 and P_2 the corresponding pressures. According to Bernoulli's equation for horizontal flow,

$$P_1 + \frac{1}{2}\rho\ v_1^2 = P_2 + \frac{1}{2}\rho\ v_2^2$$

$$\therefore \quad P_2 - P_1 = \frac{1}{2}\rho\ (v_1^2 - v_2^2)$$

Here $v_1 = 234$ km/h $= 65$ ms^{-1} ; $v_2 = 180$ km/h $= 50$ ms^{-1} ; $\rho = 1$ kg/m^3

$$\therefore \quad P_2 - P_1 = \frac{1}{2}\times 1\times\left[(65)^2-(50)^2\right] = 862.5 \text{ N m}^{-2}$$

Upward lift, $F = (P_2 - P_1) \times$ Area of two wings

$= 862.5 \times 2(25)$ N

Let M be the mass of the plane. As the plane is in level flight,

$$\therefore \quad F = Mg \quad \text{or} \quad M = \frac{F}{g} = \frac{862.5\times 2(25)}{9.8} = 4.4 \times 10^3 \text{ kg}$$

Q.28. In Millikan's oil drop experiment, what is the terminal speed of an uncharged drop of radius 2.0×10^{-5} m and density 1.2×10^3 kg m^{-3}. Take the viscosity of air at the temperature of the experiment to be 1.8×10^{-5} Pa s. How much is the viscous force on the drop at that speed? Neglect buoyancy of the drop due to air.

Ans. The terminal speed v_T of the drop is given by ;

$$v_T = \frac{2}{9}\times\frac{r^2(\rho-\sigma)g}{\eta}$$

Since buoyancy in neglected, $\sigma = 0$.

$$\therefore \quad v_T = \frac{2}{9}\times\frac{r^2\rho g}{\eta}$$

Here $r = 2.0 \times 10^{-5}$ m ; $\rho = 1.2 \times 10^3$ kg m^{-3} ; $\eta = 1.8 \times 10^{-5}$ N s m^{-2}

$$\therefore \quad v_T = \frac{2}{9}\times\frac{(2.0\times10^{-5})^2\times1.2\times10^3\times9.8}{1.8\times10^{-5}} = 5.8 \times 10^{-2} \text{ m s}^{-1}$$

$$= 5.8 \text{ cm s}^{-1}$$

The viscous force F at the terminal velocity v_T is

$$F = 6\pi\eta r v_T = 6\pi\times(1.8\times10^{-5})\times(2.0\times10^{-5})\times 5.8\times10^{-2}$$

$$= 3.9 \times 10^{-10} \text{ N}$$

Q.29. Mercury has an angle of contact equal to 140° with soda lime glass. A narrow tube of radius 1.00 mm made of this glass is dipped in a trough containing mercury. By what amount does the mercury dip down in the tube relative to the liquid surface outside? Surface tension of mercury at the temperature of the experiment is 0.465 N m^{-1}. Density of mercury = 13.6×10^3 kg m^{-3}.

Ans.
$$h = \frac{2T\cos\theta}{r\rho g}$$

Here $\theta = 140°$; $r = 1.0$ mm $= 1.0 \times 10^{-3}$ m ; $T = 0.465$ N m^{-1} ; $\rho = 13.6 \times 10^3$ kg m^{-3}

$$\therefore \quad h = \frac{2\times0.465\times\cos140°}{1.0\times10^{-3}\times13.6\times10^3\times9.8} = -5.34 \times 10^{-3} \text{ m} = -5.34 \text{ mm}$$

The negative sign shows that mercury suffers capillary depression.

Q.30. Two narrow bores of diameters 3.0 mm and 6.0 mm are joined together to form a U-tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube? Surface tension of water at the temperature of the experiment is 7.3×10^{-2} N m^{-1}. Take the angle of contact to be zero and density of water to be 1.0×10^3 kg m^{-3} ($g = 9.8$ m s^{-2}).

Ans. Here $T = 7.3 \times 10^{-2}$ N m^{-1} ; $\theta = 0°$; $\rho = 1.0 \times 10^3$ kg m^{-3}.

For tube of diameter 3.0 mm. The radius of the tube is r (= 1.5 mm = 1.5×10^{-3} m). If the water rises to height h in this tube, then,

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2\times7.3\times10^{-2}\times\cos0°}{1.5\times10^{-3}\times1.0\times10^3\times9.8}$$

$$= 9.93 \times 10^{-3} \text{ m} = 9.93 \text{ mm}$$

For tube of diameter 6.0 mm. The radius of the tube is r' (= 3 mm = 3×10^{-3} m). If the water rises to a height h' in this tube, then,

$$h' = \frac{2T\cos\theta}{r'\rho g} = \frac{2\times 7.3\times 10^{-2}\times\cos 0°}{3\times 10^{-3}\times 1.0\times 10^{3}\times 9.8}$$

$$= 4.96 \times 10^{-3} \text{ m} = 4.96 \text{ mm}$$

$\therefore$ Difference of levels of water in the two limbs

$$= h - h' = 9.93 - 4.96 = \mathbf{4.97\ mm}$$

Q.31. (*a*) It is known that density ρ of air decreases with height y as :

$$\rho = \rho_0 e^{-y/y_0}$$

where $\rho_0 = 1.25$ kg m^{-3} is the density at sea level, and y_0 is a constant. This density variation is called the law of atmospheres. Obtain this law assuming that the temperature of atmosphere remains constant (isothermal conditions). Also assume that the value of g remains constant.

(*b*) A large *He* balloon of volume 1425 m^3 is used to lift a payload of 400 kg. Assume that the balloon maintains constant radius as it rises. How high does it rise?

[Take $y_o = 8000$ m and $\rho_{He} = 0.18$ kg m^{-3}].

Ans. (*a*) Suppose ρ is the density of air at a height h above the sea level. The rate of decrease of density of air is directly proportional to ρ *i.e.*,

$$\frac{d\rho}{dy} \propto -\rho \quad \text{or} \quad \frac{d\rho}{dy} = -K\rho$$

where K is a constant of proportionality. Here –ve sign shows that the density of air (ρ) decreases as height (h) increases.

$$\therefore \quad \frac{d\rho}{\rho} = -K\,dy$$

Integrating both sides within proper limits, we have,

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\int_0^y K\,dy$$

or $$[\log_e \rho]_{\rho_0}^{\rho} = -Ky \quad \text{or} \quad \log_e \rho - \log_e \rho_0 = -K y$$

$\therefore$ $$\log_e \frac{\rho}{\rho_0} = -K y \quad \text{or} \quad \frac{\rho}{\rho_0} = e^{-Ky}$$

Choosing the constant K as to be equal to $1/y_0$, we have,

$$\rho = \rho_0 e^{-y/y_0}$$

(*b*) The balloon will rise to a point where its density becomes equal to the density of air at that point.

$$\text{Density of balloon, } \rho = \frac{\text{Total mass}}{\text{Volume}} = \frac{(400 + 1425\times 0.18)\text{ kg}}{1425\text{ m}^3} = 0.46 \text{ kg m}^{-3}$$

Now $$\rho = \rho_0 e^{-y/y_0}$$

Here $y_0 = 8000$ m ; $\rho_0 = 1.25$ kg m^{-3} ; $\rho = 0.46$ kg m^{-3}

$\therefore$ $$0.46 = 1.25\, e^{-y/8000} \quad \text{or} \quad e^{y/8000} = \frac{1.25}{0.46} = 2.72$$

$\therefore$ $$y = 8000 \log_e 2.72 = 8000 \times 1 = 8000 \text{ m} = \mathbf{8\ km}$$

NCERT CHAPTER 11 : THERMAL PROPERTIES OF MATTER

Q.1. The triple points of neon and carbon dioxide are 24.57 K and 216.55 K respectively. Express these temperatures on the Celsius and Fahrenheit scales.

Ans. On Celsius scale. $t\ (°C) = T\ (K) - 273.15$

$\therefore$ Triple point of neon $= 24.57 - 273.15 = -248.58°C$

$\therefore$ Triple point of $CO_2 = 216.55 - 273.15 = -56.6°C$

On Fahrenheit scale. $t\ (°F) = \frac{9}{5}(t°C) + 32$

$\therefore$ Triple point of neon $= \frac{9}{5}(-248.58) + 32 = -415.44°F$

$\therefore$ Triple point of $CO_2 = \frac{9}{5}(-56.6) + 32 = -69.88°F$

Q.2. Two absolute scales *A* and *B* have triple points of water defined to be 200 *A* and 350 *B*. What is the relation between T_A and T_B?

Ans. Triple point of water on absolute scale $A = 200\ A$

Triple point of water on absolute scale $B = 350\ B$

Temperature of triple point of water on Kelvin scale = 273.16 K

Value of 1 degree on absolute scale $A = \frac{273.16}{200}$ K

$\therefore$ Value of temperature T_A on absolute scale $A = \frac{273.16}{200} \times T_A$

Similarly, value of temperature T_B on absolute scale $B = \frac{273.16}{350} \times T_B$

As T_A and T_B represent the same temperature,

$$\therefore \quad \frac{273.16}{200} \times T_A = \frac{273.16}{350} \times T_B$$

or $$T_A = \frac{200}{350} T_B = \frac{4}{7} T_B$$

Q.3. The electrical resistance in ohms of a certain thermometer varies with temperature according to the approximate law :

$$R = R_0[1 + \alpha(T - T_0)]$$

The resistance is 101.6 Ω at the triple-point of water 273.16 K and 165.5 Ω at the normal melting point of lead (600.5 K). What is the temperature when the resistance is 123.4 Ω?

Ans. Here $R_0 = 101.6\ \Omega$; $T_0 = 273.16$ K

For case (i) : $R_1 = 165.5\ \Omega$; $T_1 = 600.5$ K ; *For case (ii)* : $R_2 = 123.4\ \Omega$; $T_2 = ?$

For case (i), $R_1 = R_0[1 + \alpha(T_1 - T_0)]$

or $165.5 = 101.6\ [1 + \alpha(600.5 - 273.16)]$

$\therefore$ $$\alpha = \frac{165.5 - 101.6}{101.6 \times (600.5 - 273.16)} = 1.921 \times 10^{-3}/K$$

For case (ii), $R_2 = R_0[1 + \alpha(T_2 - T_0)]$

or $123.4 = 101.6[1 + 1.921 \times 10^{-3}\ (T_2 - 273.16)]$

or $123.4 = 101.6 + 101.6 \times 1.921 \times 10^{-3}\ (T_2 - 273.16)$

On solving, we have, $T_2 = 384.8$ K

Q.4. Answer the following :

(*a*) The triple-point of water is a standard fixed point in modern thermometry. Why? What is wrong in taking the melting point of ice and the boiling point of water as standard fixed points (as was originally done in the Celsius scale)?

(*b*) There were two fixed points in the original Celsius scale as mentioned above which were assigned the number 0°C and 100°C respectively. On the absolute scale, one of the fixed points is the triple-point of water, which on the Kelvin absolute scale is assigned the number 273.16 K. What is the other fixed point on this (Kelvin) scale?

(*c*) The absolute temperature (Kelvin scale) T is related to the temperature t_c on the Celsius scale by ;

$$t_c = T - 273.15$$

Why do we have 273.15 in this relation and not 273.16?

(*d*) What is the temperature of the triple-point of water on an absolute scale whose unit interval size is equal to that of the Fahrenheit scale?

Ans. (*a*) The triple point of water is unique *i.e.,* it occurs at one particular pressure and temperature. On the other hand, melting point of ice and boiling point of water are not unique. It is because they change due to change in pressure or due to the presence of impurities in water.

(*b*) The other fixed point on the Kelvin scale is absolute zero. It corresponds to the temperature where the volume and pressure of a gas will become zero.

(*c*) The temperature 273.16 K is the triple point of water. In the relation $t_c = T - 273.15$, the temperature 273.15 K is the absolute zero and corresponds to 0°C.

(*d*) The size of degree on Celsius scale and absolute scale are equal and the temperature difference between melting point of ice and steam point is divided into 100°. But this temperature range is divided into 180° on Fahrenheit scale.

$\therefore$ Size of 1 degree on absolute scale = 1.8 × Size of 1 degree on Fahrenheit scale

Therefore, the temperature of triple point of water (*i.e.,* 273.16 K) on the new scale = 1.8 × 273.16 = **491.69.**

Q.5. Two ideal gas thermometers *A* and *B* use oxygen and hydrogen respectively. The following observations are made :

Temperature	*Pressure thermometer A*	*Pressure thermometer B*
Triple-point of water	1.250×10^5 **Pa**	0.200×10^5 **Pa**
Normal melting point of sulphur	1.797×10^5 **Pa**	0.287×10^5 **Pa**

(*a*) What is the absolute temperature of normal melting point of sulphur as read by thermometers *A* and *B*?

(*b*) What do you think is the reason behind the slight difference in answers of thermometers *A* and *B*? (The thermometers are not faulty). What further procedure is needed in the experiment to reduce the discrepancy between the two readings?

Ans. (*a*) We know that triple point of water is T_{tr} = 273.16 K. Let T be the melting point of sulphur.

For thermometer A : $$T = \frac{P}{P_{tr}} \times 273.16 = \frac{1.797 \times 10^5}{1.250 \times 10^5} \times 273.16 = \mathbf{392.69\ K}$$

For thermometer B : $$T = \frac{P}{P_{tr}} \times 273.16 = \frac{0.287 \times 10^5}{0.200 \times 10^5} \times 273.16 = \mathbf{391.98\ K}$$

(*b*) We see that there is a slight difference in the answers of the two thermometers. This is because the gases are not perfectly ideal. In order to reduce this discrepancy, the readings should be taken under very reduced pressure because as the pressure is reduced, the gases approach to ideal gas behaviour.

Q.6. A steel tape 1 m long is correctly calibrated for a temperature of 27.0°C. The length of a steel rod measured by this tape is found to be 63.0 cm on a hot day when the temperature is 45.0°C. What is the actual length of the steel rod on that day? What is the length of the same steel rod on a day when the temperature is 27.0°C? Coefficient of linear expansion of steel = 1.20×10^{-5} °C^{-1}.

Ans. $l' = l(1 + \alpha\, \Delta T)$ where ΔT = Increase in temperature

At 27°C, the size of 1 cm on the steel tape has correct magnitude. Let l' be the length of 1 cm at 45°C.

$\therefore \quad l' = l_{27}\,(1 + \alpha\,\Delta T)$

Here $\alpha = 1.20 \times 10^{-5}/°C$; $\Delta T = 45 - 27 = 18°C$; $l_{27} = 1$ cm

$\therefore \quad l' = 1 \times (1 + 1.20 \times 10^{-5} \times 18) = 1.000216$ cm

The size of the rod measured by the tape at 45°C is 63 cm.

$\therefore$ Actual length of rod at 45°C = $63 \times 1.000216 =$ **63.0136 cm**

Since the tape reads correctly at 27°C, the length of the rod measured at 27°C is **63 cm**.

Q.7. A large steel wheel is to be fitted on to a shaft of the same material. At 27°C, the outer diameter of the shaft is 8.70 cm and the diameter of the central hole in the wheel is 8.69 cm. The shaft is cooled using 'dry ice'. At what temperature of the shaft does the wheel slip on the shaft? Assume coefficient of linear expansion of the steel to be constant over the required temperature range. $\alpha_{steel} = 1.20 \times 10^{-5}\ °C^{-1}$.

Ans. The diameter of the shaft is decreased from 8.70 cm to 8.69 cm using dry ice.

Here $D_1 = 8.70$ cm ; $T_1 = 273 + 27 = 300$ K ; $D_2 = 8.69$ cm ; $T_2 = ?$

Now $\quad D_2 - D_1 = D_1\,\alpha\,(T_2 - T_1)$

or $\quad 8.69 - 8.70 = 8.70 \times 1.2 \times 10^{-5}\,(T_2 - 300)$

$\therefore \quad T_2 - 300 = -\dfrac{0.01}{8.70 \times 1.2 \times 10^{-5}} = -95.8$

or $\quad T_2 = 300 - 95.8 =$ **204.2 K** or **−68.8°C**

Q.8. A hole is drilled in a copper sheet. The diameter of the hole is 4.24 cm at 27.0°C. What is the change in the diameter of the hole when the sheet is heated to 227°C? Coefficient of linear expansion of copper = $1.70 \times 10^{-5}\ °C^{-1}$.

Ans. When the copper sheet is heated, the diameter of its hole increases in the same manner as the length of a rod.

$\therefore$ Increase in the diameter of the hole is

$$\Delta l = l\,\alpha\,\Delta T$$

Here $l = 4.24$ cm ; $\alpha = 1.70 \times 10^{-5}\ °C^{-1}$; $\Delta T = 227 - 27 = 200°C$.

$\therefore \quad \Delta l = (4.24) \times (1.70 \times 10^{-5}) \times 200 = 1.44 \times 10^{-2}$ cm = **0.0144 cm**

Q.9. A brass wire 1.8 m long at 27°C is held taut with little tension between two rigid supports. If the wire is cooled to a temperature of −39°C, what is the tension developed in the wire, if its diameter is 2.0 mm? Co-efficient of linear expansion of brass = $2.0 \times 10^{-5}\ °C^{-1}$; Young's modulus of brass = 0.91×10^{11} Pa.

Ans. Let F be the tension developed in the wire.

Now $\quad \Delta l = \dfrac{Fl}{AY}$. Also $\Delta l = \alpha\,l\,\Delta T$

$\therefore \quad F = \alpha\,Y\,A\,\Delta T$

Here $\alpha = 2.0 \times 10^{-5}/°C$; $Y = 0.91 \times 10^{11}$ Pa ; $A = \pi \times (1.0 \times 10^{-3})^2\ m^2$;

$$\Delta T = -39 - 27 = -66°C$$

$\therefore \quad F = (2.0 \times 10^{-5}) \times (0.91 \times 10^{11}) \times [\pi \times (1.0 \times 10^{-3})^2] \times 66 =$ **3.77×10^2 N (magnitude)**

Q.10. A brass rod of length 50 cm and diameter 3.0 mm is joined to a steel rod of the same length and diameter. What is the change in length of the combined rod at 250°C, if the original lengths are at 40.0°C? Is there a 'thermal stress' developed at the junction? The ends of the rod are free to expand (Co-efficient of linear expansion of brass = $2.0 \times 10^{-5}\ °C^{-1}$, steel = $1.2 \times 10^{-5}\ °C^{-1}$).

Ans. Change in length of the brass rod is

$$\Delta l_b = \alpha_b\,l\,\Delta T$$

Here $\alpha_b = 2.0 \times 10^{-5}/°C$; $l = 50$ cm $= 0.5$ m ; $\Delta T = 250 - 40 = 210°C$

$\therefore$ $\Delta l_b = (2.0 \times 10^{-5}) \times (0.5) \times (210) = 2.10 \times 10^{-3}$ m $= 0.21$ cm

Change in length of the steel rod is

$$\Delta l_s = \alpha_s \, l \, \Delta T$$
$$= (1.2 \times 10^{-5}) \times (0.5) \times (210) = 1.26 \times 10^{-3} \text{ m} = 0.13 \text{ cm}$$

$\therefore$ Change in length of the combined rod

$$= \Delta l_b + \Delta l_s = 0.21 + 0.13 = \mathbf{0.34 \text{ cm}}$$

Since the rods are free to expand, no thermal stress is developed at the junction.

Q.11. The coefficient of volume expansion of glycerine is 49×10^{-5} °C^{-1}. What is the fractional change in its density for a 30°C rise in temperature?

Ans. Let M be the mass of the glycerine and V be its initial volume. If V' is the volume of glycerine after a rise of temperature of 30°C, then,

$$V' = V(1 + \gamma \Delta t) = V(1 + 49 \times 10^{-5} \times 30) = 1.0147 \, V$$

$$\text{Initial density, } \rho = \frac{M}{V}$$

$$\text{Final density, } \rho' = \frac{M}{V'} = \frac{M}{1.0147\,V} = \frac{\rho}{1.0147} = 0.9855 \, \rho$$

$$\therefore \text{ Fractional change in density } = \frac{\rho - \rho'}{\rho} = \frac{\rho - 0.9855\,\rho}{\rho} = \mathbf{0.0145}$$

Q.12. A 10 kW drilling machine is used to drill a bore in a small aluminium block of mass 8.0 kg. How much is the rise in temperature of the block in 2.5 minutes, assuming 50% of power is used up in heating the machine itself or lost to the surroundings? Specific heat of aluminium = 0.91 J g^{-1} °C^{-1}.

Ans. Power of the machine, $P = 10$ kW $= 10 \times 10^3 = 10^4$ W or J/s

Energy obtained from machine in 2.5 minutes

$$= 10^4 \times 2.5 \times 60 \text{ J} = 15 \times 10^5 \text{ J}$$

Energy transferred to the aluminium block is

$$Q = \frac{50}{100} \times 15 \times 10^5 = 7.5 \times 10^5 \text{ J}$$

If m is the mass of aluminium block and ΔT is the rise of its temperature, then,

$$Q = m \, c \, \Delta T$$

or
$$\Delta T = \frac{Q}{mc}$$

Here, $Q = 7.5 \times 10^5$ J ; $m = 8.0$ kg $= 8.0 \times 10^3$ g ; $c = 0.91$ J g^{-1} °C^{-1}

$\therefore$
$$\Delta T = \frac{7.5 \times 10^5}{8.0 \times 10^3 \times 0.91} = \mathbf{103.02°C}$$

Q.13. A copper block of mass 2.5 kg is heated in a furnace to a temperature of 500°C and then placed on a large ice block. What is the maximum amount of ice that can melt? (Specific heat of copper = 0.39 J g^{-1} °C^{-1} ; heat of fusion of water = 335 J g^{-1}).

Ans. When copper block remains on ice, its temperature will decrease from 500°C to 0°C.

Heat lost by copper block = mass × specific heat × fall in temperature

$$= (2.5 \times 10^3) \times 0.39 \times (500 - 0) = 487500 \text{ J}$$

Let m grams be the mass of ice melted.

Heat gained by ice = mass melted × heat of fusion of water

$$= m \times 335 \text{ J}$$

But Heat lost = Heat gained

or $487500 = m \times 335$

$\therefore$
$$m = \frac{487500}{335} = 1455.2 \text{ g} = \mathbf{1.455 \text{ kg}}$$

Q.14. In an experiment on the specific heat of a metal, a 0.20 kg block of the metal at 150°C is dropped in a copper calorimeter (of water equivalent 0.025 kg) containing 150 cm^3 of water at 27°C. The final temperature is 40°C. Compute the specific heat of the metal. If heat losses to the surroundings are not negligible, is your answer greater or smaller than the actual value for specific heat of the metal?

Ans. Mass of metal block, $m_2 = 0.20 \text{ kg} = 0.20 \times 10^3 \text{ g}$

Initial temperature of metal block, $\theta_2 = 150°C$

Mass of water, m_1 = volume × density = $150 \times 1 = 150$ g

Initial temperature of water and calorimeter, $\theta_1 = 27°C$

Water equivalent of calorimeter, $w = 0.025 \text{ kg} = 0.025 \times 10^3 \text{ g}$

Final temperature of mixture, $\theta = 40°C$

Let the specific heat of the metal be c Jg^{-1} $°C^{-1}$.

$$\text{Heat lost by metal} = m_2 c (\theta_2 - \theta) = 0.20 \times 10^3 \times c \times (150 - 40)$$
$$= 22000\, c$$

$$\text{Heat gained by water and calorimeter} = (m_1 + w) \times 4.2 \times (\theta - \theta_1)$$
$$= (150 + 0.025 \times 10^3) \times 4.2 \times (40 - 27)$$
$$= 175 \times 4.2 \times 13$$

Now Heat lost = Heat gained

or $22000\, c = 175 \times 4.2 \times 13$

$$\therefore \quad c = \frac{175 \times 4.2 \times 13}{22000} = \mathbf{0.434\ J\ g^{-1}\ °C^{-1}}$$

If some heat is lost to the surroundings, the value of c obtained will be less than the actual value.

Q.15. Given below are observations on molar specific heats at room temperature of some common gases.

Gas	*Molar specific heat (C_v) (cal mol^{-1} K^{-1})*
Hydrogen	**4.87**
Nitrogen	**4.97**
Oxygen	**5.02**
Nitric oxide	**4.99**
Carbon monoxide	**5.01**
Chlorine	**6.17**

The measured molar specific heats of these gases are markedly different from those for monoatomic gases. Typically, molar specific heat of a monoatomic gas is 2.92 cal/mol K. Explain this difference. What can you infer from the somewhat larger (than the rest) value for chlorine?

Ans. A monatomic gas molecule has only the translational motion. However, diatomic gas molecule has translational, vibrational and rotational motion. Therefore, to raise the temperature of 1 mole of a diatomic gas through 1°C, more heat is required as compared to raise the temperature of 1 mole of monoatomic gas through 1°C. Hence the molar specific heat of a diatomic gas is greater than that of monoatomic gas.

The molar specific heat of a diatomic gas at constant volume is

$$C_v = \frac{5}{2}R = \frac{5}{2} \times 1.985 \simeq 5 \text{ cal mol}^{-1} \text{ K}^{-1}$$

In the given table all the gases are diatomic gases. It is clear from the table that for all the gases (except chlorine) the value of C_v is about 5 cal mol^{-1} K^{-1}. The chlorine molecules, even at room temperature, usually have vibrational motion along with rotational and translational motions. For this reason, chlorine has somewhat larger value of molar specific heat.

Q.16. Answer the following questions based on the *P-T* phase diagram of carbon dioxide shown in Fig. 11.01 :

(*a*) At what temperature and pressure can the solid, liquid and vapour phases of CO_2 co-exist in equilibrium?

(*b*) What is the effect of decrease of pressure on the fusion and boiling point of CO_2?

(*c*) What are the critical temperature and pressure for CO_2? What is their significance?

(*d*) Is CO_2 solid, liquid or gas at (*a*) – 70°C under 1 atm (*b*) –60°C under 10 atm and (*c*) 15°C under 56 atm?

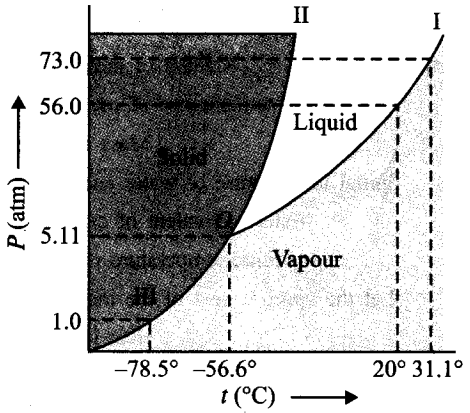


Fig. 11.01

Ans. (*a*) The solid, liquid and vapour phases of CO_2 can exist in equilibrium at its triple point *O i.e., temperature = –56.6°C and pressure = 5.11 atm.*

(*b*) It is clear from the vaporisation curve (I) and fusion curve (II) that both the fusion and boiling points of CO_2 *decrease with the decrease of pressure.*

(*c*) For CO_2, the critical temperature is 31.1°C and critical pressure is 73.0 atm. Above its critical temperature (*i.e.,* 31.1°C), CO_2 cannot be liquefied however large pressure may be applied.

(*d*) (*a*) vapour (*b*) solid (*c*) liquid [See *P-T* phase diagram].

Q.17. Answer the following questions based on the *P-T* phase diagram of CO_2 in Fig. 11.01 :

(*a*) CO_2 at 1 atm pressure and temperature – 60°C is compressed isothermally. Does it go through a liquid phase?

(*b*) What happens when CO_2 at 4 atm pressure is cooled from room temperature at constant pressure?

(*c*) Describe qualitatively the changes in a given mass of solid CO_2 at 10 atm pressure and temperature – 65°C as it is heated up to room temperature at constant pressure.

(*d*) CO_2 is heated to a temperature 70°C and compressed isothermally. What changes in its properties do you expect to observe?

Ans. (*a*) The temperature – 60°C lies in the region of vapour and solid phase [See Fig. 11.01]. If CO_2 is compressed isothermally (*i.e.,* pressure is increased without changing temperature), it will condense directly into solid state without becoming liquid.

(*b*) CO_2 at 4 atm and at room temperature (say 25°C) is a vapour. Now the pressure 4 atm is less than 5.11 atm and, therefore, CO_2 will condense directly into solid without becoming liquid.

(*c*) When solid CO_2 at 10 atm and at – 65°C is heated at constant pressure, it is first converted into liquid. On further heating, it will go into vapour state. The fusion and boiling points of CO_2 can be determined as follows. At P = 10 atm, draw a horizontal line parallel to the temperature axis. Then the points of intersection of this horizontal line with the fusion and vaporisation curves gives the fusion and boiling point of CO_2 at 10 atm.

(*d*) Since 70°C is more than the critical temperature (31.1°C) of CO_2, CO_2 gas cannot be converted into liquid by compressing it isothermally. However, as the pressure is increased, the gas will depart more and more from the ideal gas behaviour.

Q.18. A child running a temperature of 101°F is given an antipyrine (*i.e.,* a medicine that lowers fever) which causes an increase in the rate of evaporation of sweat from his body. If the fever is brought down to 98°F in 20 min, what is the average rate of extra evaporation caused by the drug? Assume the evaporation mechanism to be the only way by which heat is lost. The mass of the child is 30 kg. The specific heat of human body is approximately the same as that of water and latent heat of evaporation of water at that temperature is about 580 cal g^{-1}.

Ans. Mass of child, $m = 30 \text{ kg} = 30 \times 10^3 \text{ g}$

Fall in temperature, $\Delta T = 101 - 98 = 3°\text{F} = 3 \times \frac{5}{9} = \frac{5}{3}°\text{C}$

Specific heat of human body, $c = 1 \text{ cal g}^{-1} °\text{C}^{-1}$

Heat lost by child in 20 min $= m\, c\, \Delta T$

$$= (30 \times 10^3) \times 1 \times \left(\frac{5}{3}\right) = 50000 \text{ cal}$$

Heat lost by child in 1 min $= \frac{50000}{20} = 2500 \text{ cal/min}$

$\therefore$ Average rate of extra evaporation $= \frac{2500}{580} =$ **4.31 g/min**

Q.19. A 'thermocole' icebox is a cheap and efficient method for storing small quantities of cooked food in summer in particular. A cubical thermocole icebox of side 30 cm has a thickness of 5.0 cm. If 4.0 kg of ice is put in the box, estimate the amount of ice remaining after 6 h. The outside temperature is 45°C and co-efficient of thermal conductivity of thermocole is 0.01 J s^{-1} m^{-1} $°C^{-1}$ [Heat of fusion of water = 335 × 10^3 J kg^{-1}].

Ans. The heat entering the box through all its six faces in 6 h is

$$Q = 6 \times \frac{k\, A\, (\theta_1 - \theta_2)\, t}{x}$$

Here $k = 0.01 \text{ J s}^{-1} \text{ m}^{-1} °\text{C}^{-1}$; $A = 30 \times 30 = 0.3 \times 0.3 \text{ m}^2 = 0.09 \text{ m}^2$; $\theta_1 - \theta_2 = 45°\text{C}$;

$t = 6 \times 60 \times 60 = 21.6 \times 10^3$ s ; $x = 5.0 \text{ cm} = 5.0 \times 10^{-2}$ m

$\therefore$ $$Q = 6 \times \frac{0.01 \times 0.09 \times 45 \times 21.6 \times 10^3}{5.0 \times 10^{-2}} = 104976 \text{ J}$$

Suppose m kg of ice melts in 6 hours.

Now $Q = m\, L$ or $m = \frac{Q}{L} = \frac{104976}{3.35 \times 10^5} = 0.3$ kg

$\therefore$ Amount of ice remaining = 4 − 0.3 = **3.7 kg**

Q.20. A brass boiler has a base area of 0.15 m^2 and thickness 1.0 cm. It boils water at the rate of 6.0 kg/min when placed on a gas stove. Estimate the temperature of the part of the flame in contact with the boiler. Thermal conductivity of brass = 109 J s^{-1} m^{-1} $°C^{-1}$; Heat of vaporisation of water = 2256 × 10^3 J kg^{-1}.

Ans. Rate of boiling of water is

$$m = 6.0 \text{ kg min}^{-1} = \frac{6.0 \times 10^3}{60} = 100 \text{ gs}^{-1}$$

$\therefore$ Rate at which heat is supplied by the flame to water is

$$Q = m\, L = 100 \times 2256 = 225600 \text{ Js}^{-1} \text{ ; } \theta_2 = 100°\text{C}$$

But $$Q = \frac{k\, A (\theta_1 - \theta_2)\, t}{x}$$

or $$225600 = \frac{k\, A (\theta_1 - \theta_2)\, t}{x}$$

$\therefore$ $$\theta_1 - \theta_2 = \frac{225600 \times x}{k\, A\, t}$$

Here $x = 1.0 \text{ cm} = 1.0 \times 10^{-2}$ m ; $k = 109 \text{ J s}^{-1} \text{ m}^{-1} °\text{C}^{-1}$; $A = 0.15 \text{ m}^2$; $t = 1$s

$\therefore$ $$\theta_1 - \theta_2 = \frac{225600 \times 1.0 \times 10^{-2}}{109 \times 0.15 \times 1} = 137.98°\text{C}$$

or $$\theta_1 = 137.98 + \theta_2 = 137.98 + 100 = \mathbf{237.98°C}$$

Q.21. Explain why :

(*a*) a body with large reflectivity is a poor emitter.

(*b*) a brass tumbler feels much colder than a wooden tray on a chilly day.

(*c*) an optical pyrometer (for measuring high temperatures) calibrated for an ideal black body radiation gives too low a value for the temperature of a red hot iron piece in the open, but gives a correct value for the temperature when the same piece is in the furnace.

(*d*) the earth without its atmosphere would be inhospitably cold.

(*e*) heating systems based on circulation of steam are more efficient in warming a building than those based on circulation of hot water.

Ans. (*a*) Poor absorbers of heat are also poor emitters. Therefore, a body with large reflectivity is a poor emitter.

(*b*) When we touch a brass tumbler on a chilly day, heat flows quickly from our body to the tumbler because thermal conductivity of brass is very high. As a result, brass tumbler feels colder. On the other hand, the thermal conductivity of wood is very small so that on touching the wooden tray, practically no heat flows from our body to the tray.

(*c*) When red hot iron at temperature T is in a furnace, the heat radiated per second per unit area is

$$E = \sigma T^4$$

However, when the hot iron at temperature T is placed in open (say at temperature T_0), then,

$$E' = \sigma(T - T_0)^4$$

Since $E' < E$, the pyrometer gives too low a value for temperature of red hot iron in the open.

(*d*) During the day, heat radiations received from the sun are reflected back by earth's surface. However, the atmosphere sends them back to earth. Thus heat radiations are kept trapped by the atmosphere. Therefore, earth without atmosphere would be too cold to live.

(*e*) Steam at 100°C possesses more heat than water at 100°C. Thus 1 g of steam at 100°C has 640 calories of heat while 1 g of water at 100°C has only 100 calories of heat. For this reason, heating systems based on circulation of steam are more efficient in warming a building than those based on circulation of hot water.

Q.22. A body cools from 80°C to 50°C in 5 minutes. Calculate the time it takes to cool from 60°C to 30°C. The temperature of the surroundings is 20°C.

Ans. According to Newton's law of cooling,

$$\text{Rate of loss of heat} = k(\theta - \theta_0)$$

or $$mc\frac{\theta_1 - \theta_2}{t} = k(\theta - \theta_0)$$

Here $\theta_1 = 80°C$; $\theta_2 = 50°C$; $\theta = \dfrac{\theta_1 + \theta_2}{2} = \dfrac{80 + 50}{2} = 65°C$; $\theta_0 = 20°C$; $t = 5 \text{ min} = 300 \text{ s}$

$\therefore$ $$mc\frac{80 - 50}{300} = k(65 - 20) \quad ...(i)$$

If the liquid takes t seconds to cool from 60°C to 30°C, then $\theta = (60 + 30)/2 = 45°C$.

$\therefore$ $$mc\frac{60 - 30}{t} = k(45 - 20) \quad ...(ii)$$

Dividing eq. (*i*) by eq. (*ii*), we get,

$$\frac{30}{300} \times \frac{t}{30} = \frac{45}{25}$$

or $$t = \frac{45}{25} \times 300 = 540 \text{ s} = \mathbf{9 \text{ min}}$$

Unit IX

THERMODYNAMICS

- Thermodynamics
- N.C.E.R.T. Textbook Exercises

23

Thermodynamics

INTRODUCTION

The study of heat and its transformation to mechanical energy is called **thermodynamics** (which stems from Greek meaning "movement of heat"). The science of thermodynamics was developed long before the atomic and molecular theory of matter was accepted. It involves the description of matter in terms of *macroscopic properties such as pressure, temperature, volume and heat flow. The laws of thermodynamics are very important for our understanding of heat and energy flow. The first law of thermodynamics is an expression of law of conservation of energy. The second law of thermodynamics sets limits on the efficiency of machines (*e.g.*, air conditioners, heat pumps, refrigerators, electrical generators, internal combustion engines etc.) and processes that change one form of energy into another. In this chapter, we shall discuss the various aspects of thermodynamics.

23.1. THERMODYNAMICS

The branch of physics which deals with the relationship between heat and work is called **thermodynamics**.

Thermodynamics deals with conversion of heat into mechanical work. It is easy to produce heat by doing mechanical work. For example, if you rub your hands briskly, practically the whole of mechanical work done is converted into heat. Thus the conversion from mechanical energy to heat can be almost 100%. However, the opposite process (*i.e.*, continuous conversion of heat into mechanical work) is far less than 100%. In fact, the most efficient heat engine (*i,e.*, a device which converts heat into mechanical work) has an efficiency less than 50%. In other words, not more than 50% of heat can be converted into mechanical work. The reason for this lies in the laws of thermodynamics.

23.2. LAWS OF THERMODYNAMICS AT A GLANCE

The laws of thermodynamics, like Newton's laws of motion, are elegant and compact generalisations of experience. Therefore, they can be treated in a wholly macroscopic way, without reference to microscopic models. Indeed, much of thermodynamics was developed before the molecular theory of matter was accepted.

1. ****Zeroth Law of thermodynamics.** This law may be stated as under :

Two bodies or systems that are in thermal equilibrium with a third body are in thermal equilibrium with each other.

* Macroscopic properties are those that involve the average effects of a very great number of molecules.

** The zeroth law of thermodynamics, which is fundamental to all thermodynamics, is so named because it has priority over the first and second laws of thermodynamics and because it was recognised as a law only after the first and second laws had been stated and assigned their numbers.

Two bodies A and B are in thermal equilibrium if they are at the same temperature. One way to determine if two bodies A and B are in thermal equilibrium would be to make use of a third body C—a thermometer. If the thermometer reads the same temperature for bodies A and B, then bodies A and B are in thermal equilibrium with each other. The reader may feel that the zeroth law is merely the statement of the obvious—may be it is — but the principle it embraces is fundamental to the whole of thermodynamics and therefore needs to be stated formally.

2. First law of thermodynamics. When the law of conservation of energy is applied to thermal systems, we call it the first law of thermodynamics. It may be stated as under :

Whenever heat is added to a system, it transforms to an equal amount of some other form/ forms of energy.

Suppose our thermodynamic system is a quantity of a gas confined in a cylinder fitted with a piston. Now suppose we add heat to this system (*i.e.*, gas). Depending upon the conditions imposed on the system, the added energy does one or both of two things : (*i*) it can raise the temperature of the gas and thus increase its internal energy and (*ii*) it can cause the gas to expand, thus allowing the gas to do work on the piston. The amount of heat added to the system is equal to the increase in internal energy of the system and/or work done by the system. This is what law of conservation of energy states.

3. Second law of thermodynamics. The second law of thermodynamics can be stated in many equivalent ways but the most simple statement is given below :

*Heat flows naturally from a hot body to a cold body; heat will never of *itself flow from a cold body to a hot body.*

The first law of thermodynamics states that energy is conserved. However, we can imagine many processes that conserve energy but are not observed to occur in nature. For example, when a hot object is placed in contact with a cold object, heat flows from the hotter one to the colder one, never the reverse. If the heat were to leave the colder and pass to the hotter, energy would still be conserved *i.e.*, first law of thermodynamics would be obeyed. Yet it does not happen. Although second law of thermodynamics is a simple statement, it has very important applications related to the working of heat engines.

23.3. COMMONLY USED TERMS IN THERMODYNAMICS

The following terms are commonly used in the study of thermodynamics :

(*i*) **Thermodynamic system.** *It is an assembly of an extremely large number of particles (atoms or molecules) having a certain value of pressure, volume and temperature.* A typical thermodynamic system may be the molecules in a gas filled container. Here gas is the thermodynamic system and the rest of the universe is the environment (surroundings) of the system.

A thermodynamic system may be (*a*) **isolated system** or (*b*) **closed system**. An *isolated thermodynamic system* is one in which no heat can enter or leave the system. A *closed thermodynamic system* is one in which there can be heat exchange between the system and the surroundings.

(*ii*) **Thermodynamic state.** When a given sample of gas reaches equilibrium, the gas has a definite pressure (P), volume (V) and temperature (T). These definite values of P, V and T represent the thermodynamic state of the gas (system).

The definite values of pressure (P), volume (V) and temperature (T) of a gas at any time represent the **thermodynamic state** *of the gas.*

For example, if a given sample of gas has pressure P_1, volume V_1 and temperature T_1, then P_1, V_1 and T_1 represent the thermodynamic state (or simply state) of the gas.

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* Heat can be made to flow from a cold body to a hot body only by doing proper amount of external work.

(*iii*) **State variables.** *The variables which determine the thermodynamic behaviour of a system are called* **state variables**. If the system is a gas, then P, V and T (for a given mass) are the state variables. No matter how a gas reaches a particular equilibrium state, the state is described by the same values of these state variables. The equilibrium state of a gas is completely specified by knowing P, V and T (and m).

(*iv*) **Equation of state.** *The general relationship between pressure, volume and temperature for a given mass of the system (e.g., gas) is called* **equation of the state**. Thus for n moles of an ideal gas, the equation of state is

$$PV = n\,R\,T$$

If any two of the three quantities T, P and V are given, the third can be calculated from the above equation of state.

23.4. *P-V* DIAGRAM

For a given mass of a gas, its thermodynamic state (or simply state) is completely specified if we know the values of P, V and T. If any two of the quantities P, V and T are given, the third can be calculated from the gas equation ($PV = n\,R\,T$). The state of a given sample of gas can be represented as a point on pressure versus volume graph called *P-V diagram.* Generally, volume V of the gas is taken along X-axis and pressure P along the Y-axis (See Fig. 23.1).

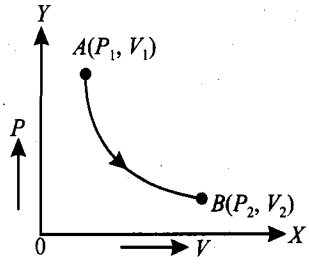


Fig. 23.1

The graphical representation of the state of a given sample of gas with the help of state variables P and V is called the **P-V diagram** *of the gas.*

Thus referring to Fig. 23.1, point $A(P_1, V_1)$ represents one state of the gas while point $B(P_2, V_2)$ represents a different state of the gas. It may be noted that temperature T_1 for the state A of the gas can be found from the gas equation $P_1V_1 = nRT_1$. Similarly, temperature T_2 for state B can be determined from $P_2V_2 = nRT_2$. Thus the value of temperature is associated with each point on the P-V diagram *e.g.,* point $A(P_1, V_1, T_1)$ and point $B(P_2, V_2, T_2)$.

Thermodynamic Process. *The continuous change in the state of a given sample of a gas (system) is called a* **thermodynamic process**.

A thermodynamic process is shown on the P-V diagram as a line or curve going from the initial state to the final state of the gas. Thus in Fig. 23.1, the initial state of the gas is $A(P_1, V_1)$ and its final state is $B(P_2, V_2)$. The curve connecting A and B represents the thermodynamic process by which the state of the gas changes. Clearly, many possible paths or processes can connect A (initial state) and B(final state). Each one would represent a different thermodynamic process.

23.5. TYPES OF THERMODYNAMIC PROCESSES

The following are the important thermodynamic processes :

(*i*) **Isothermal process.** *If the temperature of a system (gas) does not change during a process, the process is called* **isothermal** (See Fig. 23.2).

An isothermal process proceeds so slowly that there is a negligible change in temperature. Slow compression of air in a tyre pump is an example of isothermal process provided that air is not allowed to flow out of the cylinder.

(*ii*) **Isobaric process.** *If the pressure does not change during a process, the process is called* **isobaric** (See Fig. 23.2).

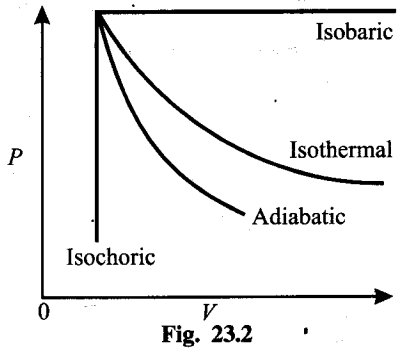


Fig. 23.2

One example of isobaric process is the boiling of water in an open container. Since the container is open, the process occurs at constant atmospheric pressure. At the boiling point, the temperature of water no longer increases with the addition of heat ; instead there is change of phase from water to steam.

(*iii*) **Isochoric process.** *A process in which the volume of the system (gas) does not change is called* **isochoric** (See Fig. 23.2).

For example, when a gas is heated in a tightly sealed container, there is no change in the volume of the gas. Therefore, the process is isochoric.

(*iv*) **Adiabatic process.** *If no heat enters or leaves the system (gas) during the process, the process is called* **adiabatic** (See Fig. 23.2).

Any thing that happens so rapidly that heat does not have time to flow in or out of the system is an adiabatic process. An example is the rapid compression of air in a tyre pump. Thus in an adiabatic process, the system does not exchange heat with the environment.

23.6. WORK DONE DURING EXPANSION

Consider a gas contained in a cylinder fitted with a movable piston (See Fig. 23.3). Suppose the initial state of the gas is A and corresponds to pressure P_1 and volume V_1. Let the gas expand *very slowly* and attain the state B corresponding to pressure P_2 and volume V_2. It is desired to find the work done by the gas during this expansion.

1. Analytical method. Suppose at any instant during expansion, the gas occupies a volume V and exerts a uniform pressure P on the piston. If the cross-sectional area of the piston is A, then force exerted by the gas on the piston is $F = P\,A$. Now suppose the gas expands and moves the piston outward through a small distance dx where dx is so small that P can be considered to be constant. Then small work dW done by the gas on the piston is

$$dW = \vec{F} \cdot \overrightarrow{dx} = (PA)\,dx = P\,(A\,dx)$$

But $A\,dx = dV$ = small increase in the volume of the gas

$$\therefore \quad dW = P\,dV \qquad ...(i)$$

The total work W done by the gas as its volume changes from V_1 to V_2 is given by the integral of eq. (*i*).

$$\therefore \quad W = \int_{V_1}^{V_2} P\,dV \qquad ...(ii)$$

In order to integrate eq. (*ii*), one must know how the pressure varies during the process and this depends upon the type of process. Nevertheless, the eq. (*ii*) holds no matter what the process is.

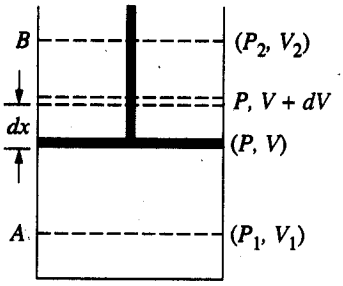


Fig. 23.3

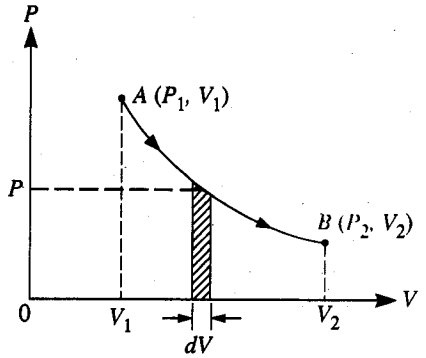


Fig. 23.4

For an expansion or volume increase, dV is positive (*i.e.*, $dV > 0$). Therefore, the force exerted by the gas is in the same direction as the displacement and the work done by the gas is positive

($W > 0$). If the volume of the gas decreases (as in compression), dV is negative (*i.e.*, $dV < 0$), and the force exerted by the gas is opposite to the displacement of the piston. Therefore, work done by the gas is negative ($W < 0$). In other words, *when a gas is compressed, work done by it is negative;* which is equivalent to saying that positive work is done *on* the gas, not by it.

2. By *PV* diagram. The work done by a gas during a volume change can also be obtained graphically from the *PV* diagram of the gas. Suppose a gas expands from the initial state A (P_1, V_1) to the final state B (P_2, V_2) as shown in Fig. 23.4. Suppose the gas undergoes a small expansion ΔV at pressure P. Then work done by gas in such an expansion is

$$\Delta W = P\Delta V$$

The small work done ΔW is equal to the shaded area shown in Fig. 23.4 since the area has height P and width ΔV. To find the total work done (W), we must add all such small works as the volume expands from V_1 to V_2.

$$\therefore \qquad W = \Sigma\Delta W = \Sigma P\Delta V$$

Note that this sum is a sum of rectangular areas that approximates the total area under PV diagram from V_1 to V_2. This statement about area is made exact as $\Delta V \to 0$. Thus, we can replace the sum by an integral *i.e.*,

$$W = \int_{V_1}^{V_2} P\,dV = \text{Area under the curve from } V_1 \text{ to } V_2 \text{ in the } PV \text{ diagram}$$

Thus work done during expansion from the initial state A (P_1, V_1) to the final state B (P_2, V_2) is equal to the area under the curve from V_1 to V_2 in the PV diagram.

Discussion. The following points are worth noting :

(*i*) For expansion, ΔV is positive so that work done by the gas is positive and the area under the curve in the *P-V* diagram is positive. Note that for expansion, the area under the curve (area ABV_2V_1) is traced in the *clockwise direction.*

(*ii*) For compression, ΔV is negative so that work done by the gas is negative (*i.e.*, work is done on the gas) and the area under the curve in the *P-V* diagram is negative. Note that for compression, the area under the curve (BAV_1V_2) will be traced in the *anticlockwise direction.*

(*iii*) *If area under the P-V diagram is traced in the clockwise direction, the work done by the gas will be* **positive** *and if the area is traced in the anticlockwise direction, the work done by the gas will be* **negative.**

(*iv*) *Whenever the volume of a gas changes, the work done by or on the gas changes regardless of the process involved.*

(*v*) *The work done in taking a gas (system) from one state to another depends not only on the initial and final states but also depends upon the type of process (path).*

23.7. NON-CYCLIC AND CYCLIC PROCESS

A thermodynamic process may be (*i*) non-cyclic or (*ii*) cyclic. We shall briefly discuss each.

(*i*) **Non-cyclic process.** A **non-cyclic process** *is one in which the system (gas) does not return to the initial state i.e., the final state of the system is different from the initial state.*

For example, in Fig. 23.4 (refer back), the initial state $A(P_1, V_1)$ of the gas is different from the final state $B(P_2, V_2)$. Therefore, the process is non-cyclic. As shown above, the work done by the gas will be positive if the gas is expanding and work done by the gas will be negative if it is compressed.

(*ii*) **Cyclic process.** *A* **cyclic process** *is one in which the system (gas) returns to its initial state.*

For example, in Fig. 23.5, the initial state of the gas is A and after passing through two processes, it returns to its initial state A. Therefore, the process is cyclic. It will be shown in the next article that net work done on the gas or by the gas is equal to the area enclosed by the loop (shaded area) in the P-V diagram.

23.8. WORK DONE IN A CYCLIC PROCESS

When a system after passing through different states returns to its original state, then it is called a *cyclic process.* Consider a gas enclosed in a cylinder fitted with a movable piston. Suppose the gas expands from the initial state A to the final state B along the path ACB as shown in Fig. 23.5. Since the gas has expanded, the work done (W_1) by the gas is positive and is given by ;

$$W_1 = +\text{ (area } ACBB'A'A)$$

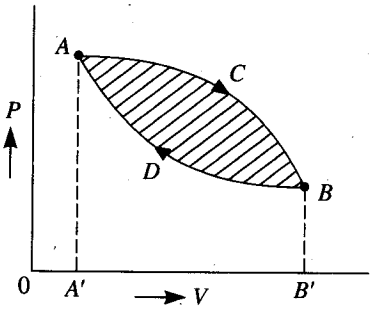


Fig. 23.5

Now suppose the gas is compressed and is brought to the original state A along the path BDA. Since the gas is compressed, the work done (W_2) by the gas is negative and is given by ;

$$W_2 = -\text{ (area } BDAA'B'B)$$

$\therefore$ Net work done in the cyclic process is

$$W = W_1 + W_2$$
$$= (\text{area } ACBB'A'A) + (-\text{ area } BDAA'B'B)$$
$$= +\text{ area } ACBDA$$

Since the net area is positive, the work done is positive. It means that in this cyclic process, the gas has done the net work.

Thus net work done on the system or by the system in any closed cycle is equal to the area enclosed by the closed loop (shaded area) in the PV diagram.

Note that if the expansion curve lies above the compression curve, then net work done is positive. It means the gas (the system) has done work (positive). On the other hand, if the expansion curve lies below the compression curve, it means the gas has done negative work *i.e.*, positive work is done on the gas.

Note. The reader may note that in a non-cyclic process, the work done is equal to the area under the curve on the P–V diagram. It depends not only on the initial and final states, but also on the type of process (or path). On the other hand, in a cyclic process, the work done per cycle is equal to the area enclosed by the closed loop.

Example 23.1. Fig. 23.6 shows the PV graph of the thermodynamic behaviour of an ideal gas. Find out from this graph (*i*) work done in the processes $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow D$ and $D \rightarrow A$ (*ii*) work done in complete cycle $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$.

Solution. The work done in a thermodynamic process is equal to the area under the curve in a PV graph of the gas. Now 1 litre = 10^3 cm^3 = $10^3 \times 10^{-6}$ m^3 = 10^{-3} m^3.

(*i*) Work done by the gas in the process $A \rightarrow B$ is

$$W_1 = +\text{ (Area } AB\,B'A'A) = AB \times A'A$$
$$= [(6 - 1) \times 10^{-3}]\,[12 \times 10^5]$$
$$= 6000 \text{ N m} = \mathbf{6000\ J}$$

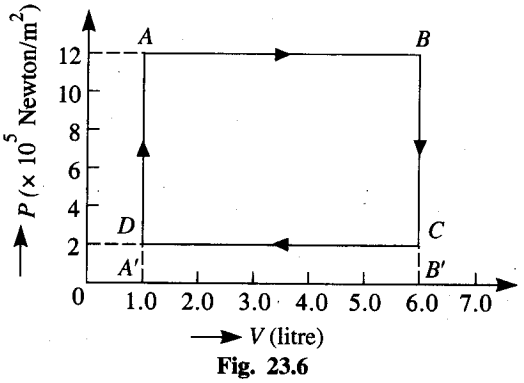


Fig. 23.6

Work done during the process $B \rightarrow C$ is **zero** since volume remains constant.

Work done by the gas in the process $C \rightarrow D$ is

$$W_2 = -(\text{Area } CDA'B') = -(CD \times DA')$$
$$= -[(5 \times 10^{-3}) \times (2 \times 10^5)] = \mathbf{-1000\ J}$$

Work done during the process $D \rightarrow A$ is **zero** since volume remains constant.

(*ii*) The net work done by the gas in the complete cycle is

$$W = W_1 + W_2 = (6000) + (-1000) = \mathbf{5000\ J}$$

Example 23.2. Fig. 23.7 shows the *PV* graph of the thermodynamic behaviour of a gas. Find the work done by the gas in going from the situtation represented by point *A*, through point *B*, to point *C*.

Solution. Work done by the gas in the process $A \rightarrow B$ is

$$W_1 = -(\text{area under curve } AB)$$
$$= -[(5 \times 10^5) \times (800 - 500) \times 10^{-6}]$$
$$= -150\ \text{J}$$

Work done by the gas in the process $B \rightarrow C$ is

$$W_2 = -[\text{area under the curve } BC]$$
$$= -\left[(2 \times 10^5)(200 \times 10^{-6}) + \frac{1}{2}(3 \times 10^5)(200 \times 10^{-6})\right] = -70\ \text{J}$$

$\therefore$ Net work done by the gas in the whole process is

$$W = W_1 + W_2 = (-150) + (-70)$$
$$= \mathbf{-220\ J}$$

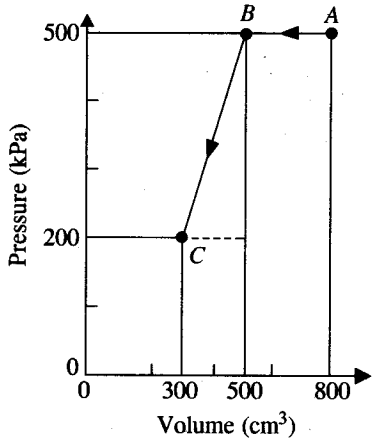


Fig. 23.7

Example 23.3. The *P-V* diagram for a cyclic process is a triangle *ABC* drawn in order. The co-ordinates of *A*, *B*, *C* are (4, 1), (2, 4) and (2, 1). The co-ordinates are in the order (*P*, *V*). Pressure is in Nm^{-2} and volume is in litre. Calculate work done during the process from *A* to *B*, *B* to *C* and *C* to *A*. Also calculate work done in the complete cycle.

Solution. Fig. 23.8 shows the conditions of the problem. The cyclic process is the triangle *ABC* represented by points *A*(4, 1), *B*(2, 4) and *C*(2, 1) on the *P-V* diagram.

Work done during the process *A* to *B*. The work done is positive because the system (*e.g.*, gas) expands along *AB*.

$\therefore$ W_{AB} = + Area of trapezium *ABDEA*

$$= \frac{AE + BD}{2} \times DE$$

Here $AE = 4\ \text{N/m}^2$; $BD = 2\text{N/m}^2$; $DE = 4 - 1 = 3$ litres $= 3 \times 10^{-3}\ \text{m}^3$

$$\therefore \quad W_{AB} = \frac{4+2}{2} \times 3 \times 10^{-3} = \mathbf{9 \times 10^{-3}\ J}$$

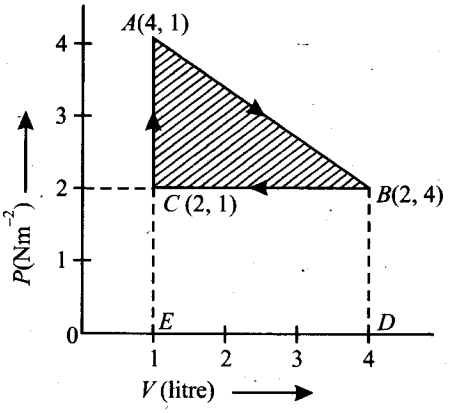


Fig. 23.8

Work done during the process *B* to *C*. The work done is negative because the system is compressed along *BC*.

$\therefore$ W_{BC} = − Area of rectangle *BCED* = − (*DE* × *CE*)

Here $DE = 3 \times 10^{-3}\ \text{m}^3$; $CE = 2\ \text{N/m}^2$

$\therefore$ $W_{BC} = -(3 \times 10^{-3} \times 2) = \mathbf{-6 \times 10^{-3}\ J}$

Work done during the process C to A. In going from C to A, there is no change in volume *i.e.*, $\Delta V = 0$.

$$\therefore \quad W_{CA} = P \times \Delta V = P \times 0 = \mathbf{0}$$

Work done in complete cycle.

$$W = W_{AB} + W_{BC} + W_{CA} = (9 \times 10^{-3}) + (-6 \times 10^{-3}) + (0)$$
$$= \mathbf{3 \times 10^{-3} J}$$

Alternatively.

$$W = +\text{Area of } \Delta ABC = +\left(\frac{1}{2} \times BC \times CA\right)$$

Here $BC = 3 \times 10^{-3}$ m^3 ; $CA = 2$ N/m^2

$$\therefore \quad W = \frac{1}{2} \times 3 \times 10^{-3} \times 2 = \mathbf{3 \times 10^{-3}\ J}$$

PROBLEMS FOR PRACTICE

1. In Fig. 23.9, the PV diagram of a gas is shown. Find out from the graph (*i*) work done in taking the gas from the state A to B (*ii*) work done in taking the gas from state B to C (*iii*) work done in the complete cycle. **[(*i*) 60 × 10^{-3} J (*ii*) zero (*iii*) 36 × 10^{-3}J]**

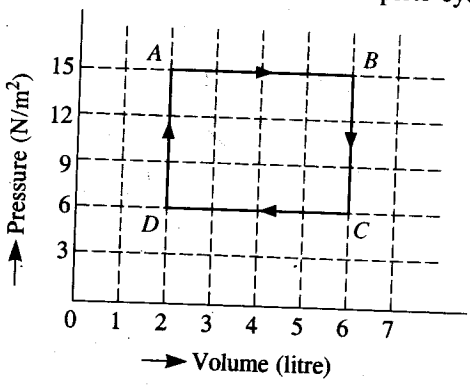


Fig. 23.9

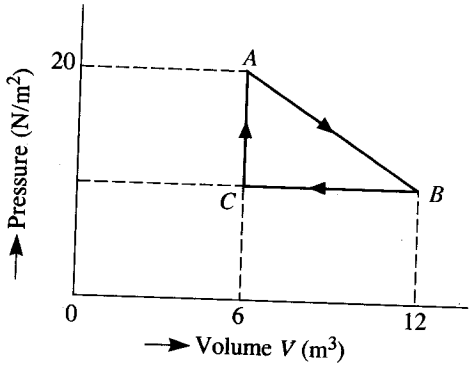


Fig. 23.10

2. In Fig. 23.10, the PV graph for the thermodynamic behaviour of a system is shown. Determine from the graph (*i*) work done by the system in the processes $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow A$ (*ii*) Net work done by the system in the complete cycle $ABCA$.

[(*i*) 90 J ; – 60 J ; zero (*ii*) 30 J]

3. A gas expands from 75 litres to 125 litres at a constant pressure of 4 atmosphere. If one atmosphere pressure is 1.0×10^5 Nm^{-2}, calculate the work done by the gas during expansion.

[2.0 × 10^4 J]

23.9. INTERNAL ENERGY OF A GAS

The sum of the kinetic and potential energies of the molecules of a gas is called its **internal energy.**

According to kinetic theory of gases, the molecules of an ideal gas do not exert intermolecular forces. Therefore, the molecules of an ideal gas do not possess potential energy. As a result, the internal energy of an ideal gas is wholly kinetic in nature and is a function of temperature *only.*

In practice, no gas is ideal. The molecules of a real gas exert forces of attraction on each other and so they possess potential energy; in addition to kinetic energy. Since work is required to increase the separation of two molecules, the potential energy associated with them increases with separation. If the volume of a gas increases at constant temperature, the kinetic energy remains

constant (since it depends on temperature only) but the potential energy increases. Therefore, the total internal energy of a gas depends on the volume of the gas as well as on the temperature.

Experiments show that potential energy associated with the molecules of a real gas does depend to some extent on pressure or volume. However, the contribution of this energy to the internal energy will be small if the pressure of the gas is small because then average separation of any two molecules is very great. In the limit of very low pressures, the potential energy will be negligible. In that case, the internal energy is totally kinetic and depends *only* on temperature. The internal energy of an ideal gas is entirely due to the kinetic energy of the molecules.

23.10. FIRST LAW OF THERMODYNAMICS

The first law of thermodynamics is based on the idea that energy is neither created nor destroyed in any thermodynamic system. Suppose our system is a gas contained in a cylinder fitted with a movable piston. The heat Q added to the system can cause two changes : (*i*) it can raise the temperature of the gas and thus increase its internal energy (ΔU) and (*ii*) it can cause the gas to expand, allowing the gas to do work.

$$\begin{pmatrix}\text{Heat added}\\ \text{to system}\end{pmatrix} = \begin{pmatrix}\text{Increase in internal energy}\\ \text{of the system}\end{pmatrix} + \begin{pmatrix}\text{External work done by}\\ \text{the system}\end{pmatrix}$$

or $$Q = \Delta U + W \qquad ...(i)$$

Eq. (*i*) is the mathematical statement of the **first law of thermodynamics.** In words, it may be stated as under :

The heat energy (Q) supplied to a system is equal to the increase in the internal energy (ΔU) of the system plus the work done (W) by the system.

Note that the first law of thermodynamics is a statement of the law of conservation of energy applied to a thermodynamic system.

Signs. In using first law of thermodynamics, we must be careful about signs.

(*i*) If heat is *added* to a system, Q is *positive* and if heat is *lost* by the system, Q is *negative.*

(*ii*) If work is done *by* the system (*e.g.*, gas), W *is positive* and if work is done *on* the system (gas), W is *negative.*

(*iii*) $\Delta U = Q - W$; *increase* in ΔU is taken *positive* while *decrease* is taken *negative.*

If the work is done *by* the system (*e.g.*, expansion of a gas), then W is positive and internal energy ΔU of the system will decrease. If work is done *on* the system (*e.g.*, compressing a gas), then W is negative and internal energy ΔU of the system will increase as shown below :

$$\Delta U = Q - W = Q - (-W) = Q + W$$

Explanation. Let us explain first law of thermodynamics considering particular situations.

(*i*) Suppose our system is a gas contained in a cylinder fitted with a movable piston. Suppose heat energy Q flows into the gas. If the piston is locked in place, the volume of the gas is unchanged *i.e.*, $\Delta V = 0$. Therefore, work done by the gas is $W = P\Delta V = P \times 0 = 0$. According to first law of thermodynamics,

$$Q = \Delta U + W$$

or $$Q = \Delta U \qquad (\because W = 0)$$

Therefore, all the heat (Q) added goes into increasing the internal energy of the gas.

(*ii*) If the piston is free to move, the gas will expand as it receives heat. Therefore, the added heat Q does two things: it increases the internal energy of the gas and does external work.

$\therefore$ $$Q = \Delta U + W$$

If you examine any other thermodynamic system, then a similar situation is always found. The heat can be accounted for in terms of ΔU and W.

Notes. (*i*) $$Q = \Delta U + W$$

The values of Q and W depend on the path the system is following in going from initial to final state. However, ΔU does not depend on path; it depends only on the initial and final states of the system.

(*ii*) In an isolated system, there is no heat flow and the work done is zero.

$$\therefore \quad 0 = \Delta U + 0 \quad \text{or} \quad \Delta U = 0$$

We conclude that *the internal energy of an isolated system remains constant.*

(*iii*) In a cyclic process, the change in internal energy is *zero.*

$$\therefore \quad Q = 0 + W \quad \text{or} \quad Q = W$$

We conclude that *the heat added in the cyclic process is equal to the work done during the cycle.*

23.11. APPLICATIONS OF FIRST LAW OF THERMODYNAMICS

Let us apply the first law of thermodynamics to some simple processes.

(*i*) **Relation between C_p and C_v.** Consider n moles of an ideal gas. Suppose the gas is heated at *constant volume* so that its temperature increases by ΔT. From the definition of C_v, the amount of heat Q supplied is given by ;

$$Q = n\, C_v\, \Delta T$$

Since there is no change in volume, work done by the gas is zero ($W = P\Delta V$; $\Delta V = 0$). According to first law of thermodynamics,

$$Q = \Delta U + W$$

or $$Q = \Delta U \qquad (\because W = 0)$$

$$\therefore \quad nC_v \Delta T = \Delta U$$

or $$C_v = \frac{\Delta U}{n\Delta T} \qquad ...(i)$$

Suppose now that n moles of the same gas are heated at *constant pressure* so that its temperature increases by the same amount ΔT. From the definition of C_p, the heat Q' supplied is given by ;

$$Q' = nC_p \Delta T$$

or $$C_p = \frac{Q'}{n\Delta T}$$

According to first law of thermodynamics,

$$Q' = \Delta U + W$$

or $$Q' = \Delta U + P\Delta V \qquad (\because \text{Work done, } W = P\Delta V)$$

When P is constant, perfect gas equation tells us that :

$$P\Delta V = n\, R\Delta T$$

$$\therefore \quad Q' = \Delta U + n\, R\Delta T \qquad \text{(Constant } P\text{)}$$

$$\therefore \quad C_p = \frac{Q'}{n\Delta T} = \frac{\Delta U + nR\Delta T}{n\Delta T} = \frac{\Delta U}{n\Delta T} + \frac{nR\Delta T}{n\Delta T}$$

or $$C_p = C_v + R \quad \left(\because C_v = \frac{\Delta U}{n\Delta T}\right)$$

or $$C_p - C_v = R$$

Note that C_p and C_v are measured in the same units as for R.

(ii) **Boiling process.** Suppose a liquid of mass m vaporises at a constant pressure P. Its volume in the liquid state is V_l and its volume in the vapour state is V_v. Let us find the work done in the expansion and the change in internal energy of the system.

Since expansion takes place at constant pressure, the work done by the system is

$$W = \int_{V_l}^{V_v} PdV = P\int_{V_l}^{V_v} dV = P(V_v - V_l)$$

The amount of heat (Q) required to convert the whole of liquid in the vapour state is

$$Q = m L_v \text{ where } L_v = \text{latent heat of vaporisation of the liquid}$$

According to first law of thermodynamics,

$$Q = \Delta U + W$$

Here $Q = mL_v$ and $W = P\ (V_v - V_l)$

$$\therefore \quad mL_v = \Delta U + P(V_v - V_l)$$

$\therefore$ Increase in internal energy of the system is

$$\Delta U = m\ L_v - P\ (V_v - V_l)$$

(iii) **Melting process.** When a solid changes into liquid state (melting), its internal energy increases. This can be calculated from first law of thermodynamics.

Let m = mass of the solid

L = specific latent heat of fusion

$\therefore$ Heat absorbed during the melting process is

$$Q = m\ L$$

Since during melting process, the change in volume (ΔV) is very small and can be considered zero.

$$\therefore \quad W = P \times \Delta V = P \times 0 = 0$$

Now $$Q = \Delta U + W \quad \text{or} \quad mL = \Delta U$$

Therefore, during melting process, internal energy increases by mL. Since temperature remains constant during melting, the kinetic energy remains the same. Therefore, the increase in internal energy is due to increase in potential energy.

(iv) **Isobaric process.** A process that occurs at constant pressure is called an *isobaric process.* Suppose a gas expands at constant pressure.

Now $$Q = \Delta U + W$$

In this case, the heat added increases the internal energy of the gas as well as the gas does external work.

(v) **Isochoric process.** A process that occurs at constant volume is called *isochoric process.* In such a process, work done is zero ($W = P\Delta V = P \times 0 = 0$).

$$\therefore \quad Q = \Delta U \quad (\because\ W = 0\ )$$

We conclude that *if heat is added to a system (e.g. gas) at constant volume, all the heat goes into increasing the internal energy of the system.*

Example 23.4. One gram of water (1 cm^3) becomes 1671 cm^3 of steam at a pressure of 1 atm (= 1.013 × 10^5 Pa). The latent heat of vaporisation at this pressure is 2256 J g^{-1}. Calculate the external work and the increase in internal energy.

Solution. External work, $W = P\ (V_v - V_l)$

Here $P = 1.013 \times 10^5$ Pa; $V_v = 1671 \times 10^{-6}$ m^3 ; $V_l = 1 \times 10^{-6}$ m^3

$\therefore \quad W = 1.013 \times 10^5\ (1671 \times 10^{-6} - 1 \times 10^{-6}) =$ **169 J**

The heat (Q) required to convert 1 g of water into vapour state is

$$Q = mL_v = 1 \times 2256 = 2256 \text{ J}$$

According to first law of thermodynamics,

$$Q = \Delta U + W$$

$$\therefore \quad \Delta U = Q - W = 2256 - 169 = \mathbf{2087\ J}$$

Thus when 1 g of water vaporises, 2256 J of heat are added. Of the added energy, 2087 J remain in the system as an increase in the internal energy and the remaining 169 J leaves the system as it does work in expanding liquid to vapour state.

Example 23.5. 1.0 m^3 of water is converted into 1671 m^3 of steam at atmospheric pressure and 100°C temperature. The latent heat of vaporisation of water is 2.3×10^6 J/kg. If 2.0 kg of water is converted into steam at atmospheric pressure and 100°C temperature, then how much will be the increase in its internal energy? Density of water = 1.0×10^3 kg m^{-3} and atmospheric pressure = 1.01×10^5 N m^{-2}.

Solution. Heat supplied to 2.0 kg of water to convert it into steam at 100°C is

$$Q = m L_v = 2.0 \times 2.3 \times 10^6 = 4.6 \times 10^6 \text{ J}$$

$$\text{Volume of 2 kg of water, } V_l = \frac{\text{Mass}}{\text{Density}} = \frac{2.0}{1.0 \times 10^3} = 2.0 \times 10^{-3} \text{ m}^3$$

$$\text{Volume of steam formed, } V_v = 1671 \times 2.0 \times 10^{-3} = 3342 \times 10^{-3} \text{ m}^3$$

$$\text{Work done, } W = P(V_v - V_l)$$

Here, $P = 1.01 \times 10^5$ Nm^{-2} ; $V_v = 3342 \times 10^{-3}$ m^3 ; $V_l = 2.0 \times 10^{-3}$ m^3

$$\therefore \quad W = 1.01 \times 10^5 (3342 \times 10^{-3} - 2.0 \times 10^{-3}) = 0.337 \times 10^6 \text{ J}$$

According to first law of thermodynamics,

$$Q = \Delta U + W$$

$\therefore$ Increase in internal energy is

$$\Delta U = Q - W = 4.6 \times 10^6 - 0.337 \times 10^6 = \mathbf{4.263 \times 10^6 J}$$

Example 23.6. At 0°C and normal atmospheric pressure, the volume of 1 g of water increases from 1 cm^3 to 1.091 cm^3 on freezing. What will be the change in its internal energy? Normal atmospheric pressure is 1.013×10^5 N m^{-2} and the latent heat of melting of ice is 80 cal/g.

Solution. Heat given out by 1 g of water in freezing is

$$Q = -(mL_f) = -(1 \times 80) = -80 \text{ cal}$$

Note that negative sign is assigned to Q because it is given out by water.

During freezing, the water expands against atmospheric pressure. Hence external work done (W) by water is

$$W = P \times \Delta V$$

Here $P = 1.013 \times 10^5$ Nm^{-2} ; $\Delta V = 1.091 - 1 = 0.091$ $cm^3 = 0.091 \times 10^{-6}$ m^3

$$\therefore \quad W = (1.013 \times 10^5) \times (0.091 \times 10^{-6}) = 0.0092 \text{ J}$$

Since 1 cal = 4.2 J, $W = 0.0092/4.2 = 0.0022$ cal

Since the work has been done by ice, it will be taken positive.

According to first law of thermodynamics,

$$Q = \Delta U + W$$

or $$\Delta U = Q - W = (-80) - (0.0022) = \mathbf{-80.0022\ cal}$$

Note that negative ΔU means that internal energy of water decreases on freezing. This decrease is more than the heat given up by the water.

PROBLEMS FOR PRACTICE

1. What amount of work will have to be done in converting 1 kg of water into steam at 100°C and normal atmospheric pressure? The volume of 1 kg of water at 100°C is 10^{-3} m^3 and that of 1 kg of steam is 1.671 m^3 and the normal pressure is 10^5 Nm^{-2}. **[1.670 × 10^5 J]**
2. 1 g of water at 373 K is converted into steam at the same temperature. The volume of 1 cm^3 of water becomes 1671 cm^3 on boiling. Calculate the change in internal energy of the system if heat of vaporisation is 540 cal g^{-1}. Given that standard atmospheric pressure = 1.013 × 10^5 Nm^{-2}. **[499.5 cal]**
3. The volume of 1.0 kg of water at 100°C is 1 × 10^{-3} m^3 and the volume of 1.0 kg of steam at normal pressure is 1.671 m^3. The latent heat of steam is 2.3 × 10^6 J/kg and the normal pressure is 1.0 × 10^5 Nm^{-2}. How much work will be done in converting 5.0 kg of water at 100°C into steam at the same temperature at normal pressure? What will be the increase in internal energy of water in this process? **[8.35 × 10^5 J ; 1.066 × 10^7 J]**
4. Calculate the change in internal energy of a block of copper of mass 200 g when it is heated from 25°C to 75°C. Assume the change in volume is negligible. Specific heat of copper is 0.1 cal $g^{-1}K^{-1}$. **[4200 J]**
5. At normal pressure and 0°C temperature, the volume of 1 kg of ice is reduced by 91 cm^3 on melting. Latent heat of melting of ice is 3.4 × 10^5 J/kg. Compute the change in the internal energy when 2 kg of ice melts at normal pressure and 0°C. Normal pressure = 1.01 × 10^5 Nm^{-2}. **[+ (6.8 × 10^5 + 18.4) J]**

23.12. ISOTHERMAL PROCESS

A thermodynamic process which takes place at constant temperature is called an **isothermal process** *i.e., only P and V change and not T.*

Consider an ideal gas contained in a cylinder of heat-conducting bottom and fitted with a piston. To carry out isothermal process (expansion or compression), we place the heat-conducting bottom of the cylinder in good thermal contact with a second system of very large heat capacity and at the same temperature as the gas [See Fig.23.11]. This second system is known as *heat reservoir.*

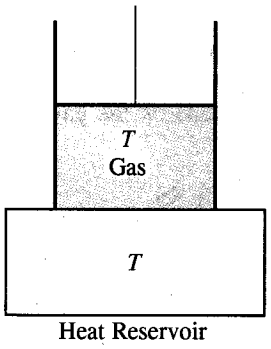


Fig. 23.11

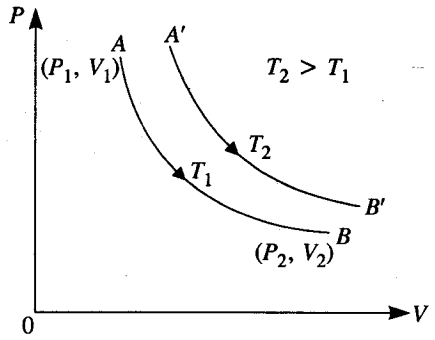


Fig. 23.12

Since the heat capacity of a heat reservoir is very large and the volume of the gas is changed very slowly in the process, the temperature of the gas remains constant throughout the process. Therefore, for isothermal process to take place :

(*a*) *the cylinder should have conducting walls.*

(*b*) *the gas should be compressed or allowed to expand very slowly.*

Equation of isothermal process. We know that pressure P and volume V of a mole of an ideal gas are related by the equation :

$$PV = RT$$

where T is the absolute temperature of the gas and R is the universal gas constant. In an isothermal process, temperature (T) is constant so that :

$$PV = \text{Constant} \qquad \text{(For ideal gas)} \quad ...(i)$$

Eq. *(i)* is called the equation of the isothermal process for an ideal gas. It gives us the path of an isothermal process (called an *isotherm)* on the PV diagram as shown in Fig. 23.12. The higher the temperature of an isotherm, the farther it lies from the coordinate axes. Thus in Fig. 23.12, the isotherm $A'B'$ represents a higher temperature than the isotherm AB. Note that in an isothermal process, pressure and volume change in such a manner that product PV is always constant. Thus, if the gas expands isothermally from the initial state A (P_1, V_1) to the final state B (P_2, V_2), then,

$$P_1 V_1 = P_2 V_2$$

Application of first law to isothermal process. Let us apply first law of thermodynamics to isothermal process. According to this law :

$$Q = \Delta U + W$$

The internal energy of an ideal gas depends only on temperature. Since temperature does not change in an isothermal process, it means there will be no change in the internal energy of the gas *i.e.*, $\Delta U = 0$.

$$\therefore \qquad Q = W = P\Delta V$$

(i) **Isothermal expansion.** When a gas expands isothermally, ΔV is positive and so W and Q will also be positive. This means that *for isothermal expansion of the gas, an amount of heat equal to the work done by the gas will have to be supplied from the external source in order to maintain constant temperature of the gas.* This heat comes from the heat reservoir.

(ii) **Isothermal compression.** When a gas is compressed isothermally, ΔV is negative and so W and Q will also be negative. This means that *for isothermal compression of the gas, an amount of heat equal to the work done on the gas must flow out of the gas if its temperature is not to change.* This heat is given to the heat reservoir.

We conclude that during isothermal expansion or compression of an ideal gas, there is no change in the internal energy of the gas. During isothermal expansion, all heat added to the gas does external work. During isothermal compression, all the work done on the gas is lost as heat to the surroundings.

23.13. WORK DONE DURING ISOTHERMAL PROCESS

Suppose an ideal gas is allowed to expand very slowly at constant temperature as described by the PV diagram in Fig. 23.13. The curve is a hyperbola that obeys the equation PV = constant. Let us calculate the work done by the gas in expanding from state A (P_1, V_1) to state B (P_2, V_2).

The work done by the gas in expanding from state A to state B is given by ;

$$W = \int_{V_1}^{V_2} P\,dV$$

For an ideal gas, $PV = nRT$ where n = number of moles.

$$\therefore \qquad P = \frac{nRT}{V}$$

$$\therefore \quad W = \int_{V_1}^{V_2} \frac{nRT}{V} dV$$

Since the gas expands isothermally, T is constant. Therefore, T can be taken out of the integral sign.

$$\therefore \quad W_{iso} = nRT \int_{V_1}^{V_2} \frac{dV}{V} = nRT \left| \log_e V \right|_{V_1}^{V_2}$$

or $$W_{iso} = nRT \log_e \frac{V_2}{V_1}$$

or $$W_{iso} = 2.303 \times nRT \log_{10} \frac{V_2}{V_1} \quad \ldots (i)$$

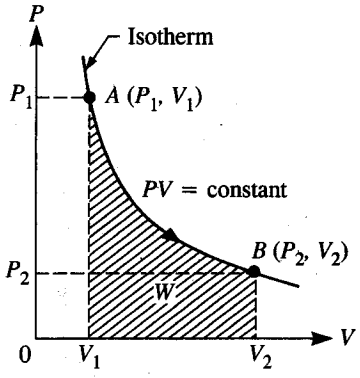


Fig. 23.13

Eq. (*i*) gives the expression for work done during isothermal expansion for n moles of an ideal gas.

If M is the molecular mass of the gas, then for 1 gram of ideal gas,

$$W_{iso} = 2.303 \frac{RT}{M} \log_{10} \frac{V_2}{V_1}$$

or $$W_{iso} = 2.303\ rT\ \log_{10} \frac{V_2}{V_1} \quad \ldots (ii)$$

where r is the gas constant for 1 gram of an ideal gas.

For an isothermal process, $P_1V_1 = P_2V_2$ so that $V_2 / V_1 = P_1 / P_2$. Therefore, eqs. (*i*) and (*ii*) can be expressed as :

$$W_{iso} = 2.303\ nRT \log_{10} \frac{P_1}{P_2} \quad \ldots (iii)$$

$$W_{iso} = 2.303\ r\ T \log_{10} \frac{P_1}{P_2} \quad \ldots (iv)$$

Discussion. The following points may be noted :

(*i*) For isothermal expansion, $V_2 > V_1$ (or $P_1 > P_2$) and we see from above equations that work done by the gas is positive as we would expect.

(*ii*) If the gas is compressed isothermally, $V_2 < V_1$ (or $P_1 < P_2$) and the work done by the gas is negative. The above equations clearly indicate it because $\log_{10}$ of a number less than 1 is negative. Note that negative work done by the gas means that positive work must be done *on* the gas by some external agent to compress it.

(*iii*) In an isothermal process, temperature remains constant. We know that the internal energy of an ideal gas depends only on temperature. Therefore, for an ideal gas undergoing isothermal process (expansion or compression), there will be no change in the internal energy of the gas *i.e.*, $\Delta U = 0$. According to first law of thermodynamics,

$$Q = \Delta U + W = 0 + W = W$$

$$\therefore \quad Q = W = 2.303 nRT \log_{10} \frac{V_2}{V_1}$$

In case of isothermal expansion, work done by the gas is equal to the heat energy (Q) which flowed into the gas in order to replace the energy lost by the gas as it did work. In case of isothermal compression, the energy furnished to the gas by the external agent must be lost as an outward heat flow from the gas.

23.14. ADIABATIC PROCESS

An **adiabatic process** *is one which takes place in such a way that no heat enters or leaves the system during the process i.e., P, V and T change in the process.*

An adiabatic process can by achieved either by thermally insulating a system from its surroundings or by performing the process rapidly so that heat has little time to leave or enter the system. Two examples of approximate adiabatic processes are : (*i*) the rapid escape of air from a burst tyre. (*ii*) the rapid expansions and contractions of air through which a sound wave is passing.

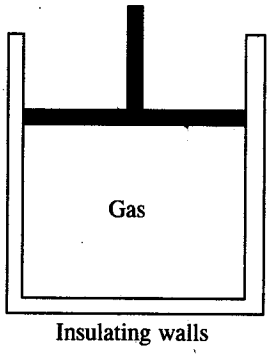


Fig. 23.14

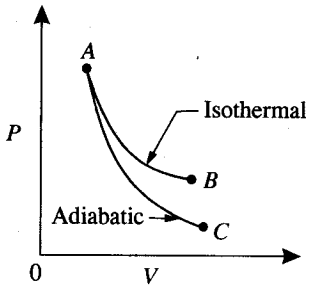


Fig. 23.15

Consider an ideal gas contained in a thick-walled cylinder having insulating walls; the cylinder being fitted with a frictionless movable piston [See Fig. 23.14]. If expansion and contraction of the gas take place rapidly, the process is nearly adiabatic. It is because the cylinder has insulating walls as well as the speed of the process is high so that no heat can enter or leave the system (the gas). Therefore, for adiabatic process to take place :

(*a*) *the cylinder should have insulating walls.*

(*b*) *the gas should be compressed or allowed to expand very quickly.*

Application of first law to adiabatic process. Let us apply first law of thermodynamics to adiabatic process. According to this law :

$$Q = \Delta U + W$$

In an adiabatic process, no heat enters or leaves the system so that $Q = 0$.

$$\therefore \quad 0 = \Delta U + W$$

or $$\Delta U = - W = - P\Delta V \qquad (\because \text{ Work done } = P\Delta V)$$

(*i*) **Adiabatic expansion.** When a gas expands adiabatically, ΔV is positive so that work done by the gas is positive.

$\therefore$ Change in internal energy, $\Delta U = - W$

The negative change in internal energy means that internal energy of the gas decreases and temperature of the gas falls.

Thus when a gas expands adiabatically, it loses internal energy equal to the work done by the gas and the temperature of the gas falls.

This is not surprising because if no heat is added, then any energy transferred out as work must come from the system's internal energy. Consequently, the temperature of the system decreases.

(*ii*) **Adiabatic compression.** When a gas is compressed adiabatically, ΔV is negative so that work done *by the gas is negative.

* In the equation $Q = \Delta U + W$, Q is heat transferred to the gas and W is the work done by the gas.

$\therefore$ Change in internal energy, $\Delta U = -(-W) = +W$

The positive change in internal energy means that internal energy of the gas increases and temperature of the gas rises.

Thus when a gas is compressed adiabatically, it gains internal energy equal to the work done on the gas and the temperature of the gas rises.

This makes sense. It is because if no heat leaves, then work done on the gas must be added to the internal energy of the gas. As a result, the internal energy of the gas increases and its temperature rises.

Equations of an adiabatic process. We have noted above that in an adiabatic process, unlike the isothermal case, T will vary as well as P and V. Therefore, adiabatic process for an ideal gas is not described in terms of $PV=nRT$ alone because all three state variables (P, V and T) change during the process. To find how these changes are related, we need a second equation. Indeed, we can find out the relation which involves only P and V without T .If an ideal gas undergoes an adiabatic change from P_1, V_1, T_1 to P_2, V_2 , T_2, then it can be shown that :

$$P_1V_1^{\gamma} = P_2V_2^{\gamma}$$

or $$PV^{\gamma} = \text{Constant} \qquad \text{...Adiabatic process, Ideal gas}$$

where $\gamma = C_p/C_v$ for the gas. We can write this adiabatic relation as :

$$P = \frac{\text{Constant}}{V^{\gamma}}$$

Since $\gamma > 1$ for all gases, P decreases faster as V increases in an adiabatic process than it does in the isothermal relation P = constant / V . This behaviour is shown in Fig. 23.15.

Note. It might seem that equation PV^{γ} = constant contradicts the ideal gas equation $PV = nRT$. But the ideal gas equation does not imply PV = constant except in an isothermal process where T is constant. Since a process cannot be both adiabatic (no heat addition) and isothermal (no temperature change), there is no contradiction. In fact, we can find the temperature change by using ideal gas equation to eliminate pressure from the adiabatic equation. We may write the adiabatic equation in the form :

$$PV \times V^{\gamma-1} = \text{Constant}$$

But the ideal gas equation gives $PV = nRT$ so that :

$$nRT\,V^{\gamma-1} = \text{Constant}$$

or $$TV^{\gamma-1} = \text{Constant} \qquad \text{...(i)}$$

Eq. (*i*) clearly shows that an adiabatic process is not isothermal, for the temperature must drop in an adiabatic expansion and rise in an adiabatic compression. We can also write adiabatic equation in a third way, eliminating V in favour of T and P. In that case, we have,

$$\frac{P^{\gamma-1}}{T^{\gamma}} = \text{Constant}$$

23.15. WORK DONE DURING ADIABATIC PROCESS

Consider one mole of a perfect gas contained in a cylinder having insulating walls. Suppose the gas expands adiabatically from the initial state A (P_1, V_1) to the final state B (P_2, V_2). Then work done by the gas during this adiabatic expansion is

$$W = \int_{V_1}^{V_2} P\,dV$$

For adiabatic change, $PV^{\gamma} = K$ (constant) so that $P = KV^{-\gamma}$.

$\therefore$ $$W_{adi} = \int_{V_1}^{V_2} KV^{-\gamma}dV = K\int_{V_1}^{V_2} V^{-\gamma}dV$$

$$= K\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_1}^{V_2} = \frac{K}{1-\gamma}\left[V_2^{1-\gamma} - V_1^{1-\gamma}\right]$$

$$= \frac{1}{\gamma-1}\left[K V_1^{1-\gamma} - K V_2^{1-\gamma}\right]$$

But $K = P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$

$$\therefore \quad W_{adi} = \frac{1}{\gamma-1}\left[P_1 V_1^{\gamma} V_1^{1-\gamma} - P_2 V_2^{\gamma} V_2^{1-\gamma}\right]$$

or $$W_{adi} = \frac{1}{\gamma-1}\left[P_1 V_1 - P_2 V_2\right]$$

If the temperature of the initial state is T_1 and that of the final state is T_2, then,

$$P_1 V_1 = R T_1 \quad \text{and} \quad P_2 V_2 = R T_2$$

$$\therefore \quad W_{adi} = \frac{1}{\gamma-1}\left[RT_1 - RT_2\right]$$

or $$W_{adi} = \frac{R(T_1 - T_2)}{\gamma-1} \qquad ...(i)$$

Eq. (*i*) gives the work done for one mole of perfect gas during adiabatic process. For 1 g of the gas, the work done during adiabatic process is

$$W_{adi} = \frac{R(T_1 - T_2)}{(\gamma-1)M} = \frac{r(T_1 - T_2)}{\gamma-1}$$

where r is the gas constant for one gram of the perfect gas and M is the molecular mass of the gas.

Alternate method. Suppose one mole of a perfect gas expands adiabatically from the initial state A (P_1, V_1) to the final state B (P_2, V_2). Suppose the initial state temperature of the gas is T_1 and the final state temperature is T_2. Let W be the work done by the gas during this adiabatic expansion. For an adiabatic process, there is no heat transfer (*i.e.*, $Q = 0$) so that $\Delta U = - W$. We know that $\Delta U = C_v\ \Delta T = C_v\ (T_2 - T_1)$ for any process. Therefore, in an adiabatic process,

$$W = -\Delta U = C_v(T_1 - T_2)$$

From ideal gas equation, $P V = R T$, $T_1 = \frac{P_1 V_1}{R}$ and $T_2 = \frac{P_2 V_2}{R}$

$$\therefore \quad W_{adi} = C_v\left(\frac{P_1 V_1 - P_2 V_2}{R}\right) = \frac{C_v}{R}(P_1 V_1 - P_2 V_2)$$

But $C_p - C_v = R$ so that :

$$\frac{C_v}{R} = \frac{C_v}{C_p - C_v} = \frac{1}{(C_p / C_v) - 1} = \frac{1}{\gamma - 1}$$

$$\therefore \quad W_{adi} = \frac{P_1 V_1 - P_2 V_2}{\gamma-1}$$

But $P_1 V_1 = R T_1$ and $P_2 V_2 = R T_2$

$$\therefore \quad W_{adi} = \frac{RT_1 - RT_2}{\gamma-1}$$

or $$W_{adi} = \frac{R(T_1 - T_2)}{\gamma-1}$$

Note that this is the same expression we derived earlier for the work done for one mole of a perfect gas during the adiabatic process.

23.16. ISOTHERMAL AND ADIABATIC PROCESSES COMPARED

Fig. 23.16 (*i*) illustrates isothermal and adiabatic expansions of an ideal gas which is initially at a pressure P_1 and volume V_1. In an adiabatic expansion, the internal energy of the gas decreases and the temperature of the gas falls. This results in a lower final pressure ($PV = nRT$) than that produced by the isothermal expansion. Note that area under the isothermal is greater than that under the adiabatic *i.e.*, more work is done by isothermal expansion than by adiabatic expansion. Also note that the adiabatic through any point is steeper than the isothermal through that point.

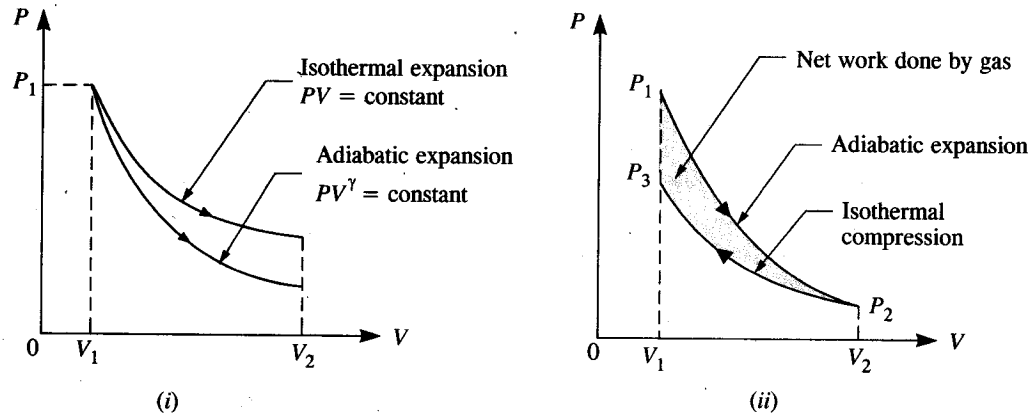


Fig. 23.16

Fig. 23.16 (*ii*) shows the PV diagram of a gas which is expanded adiabatically from volume V_1 to volume V_2 and is then compressed isothermally to its original volume.

Work done by gas in expanding adiabatically = Area $V_1 P_1 P_2 V_2 V_1$

Work done on gas in contracting isothermally = Area $V_2 P_2 P_3 V_1 V_2$

Net work done by gas = Area $P_1 P_2 P_3 P_1$

The salient differences between isothermal process and adiabatic process are given below :

S.No.	Isothermal Process	Adiabatic Process
1.	Temperature remains constant *i.e.* $\Delta T = 0$.	No heat is added or removed *i.e.* $Q = 0$.
2.	The changes occur slowly.	The changes occur suddenly.
3.	The system is thermally conducting to the surroundings.	The system is thermally insulated from the surroundings.
4.	No change in the internal energy of the system.	There is change in internal energy of the system.
5.	Equation of the process is PV = constant.	Equation of the process is PV^γ = constant.
6.	Specific heat becomes infinite.	Specific heat becomes zero.
7.	At any point on isothermal curve, the slope is less than the slope of adiabatic curve at that point.	At any point on the adiabatic curve, the slope is more than the slope of isothermal curve at that point.

Example 23.7. An ideal gas at 17°C has a pressure of 760 mm of Hg and is compressed (*i*) isothermally (*ii*) adiabatically until its volume is halved. Calculate in each case the final pressure and temperature of the gas (γ = 1.4).

Solution. (*i*) Isothermal compression. For isothermal process, $P\ V$ = Constant.

$\therefore \quad P_1V_1 = P_2V_2$

or $\quad P_2 = \dfrac{P_1V_1}{V_2}$

Here $V_1 = V$ (say) ; $V_2 = V/2$; $P_1 = 760$ mm of Hg

$\therefore \quad P_2 = \dfrac{760 \times V}{V/2} = \mathbf{1520\ mm\ of\ Hg}$

The final temperature will remain **17°C** because in isothermal change, temperature remains constant.

(ii) **Adiabatic compression.** For adiabatic process, PV^γ = Constant.

$\therefore \quad P_1V_1^\gamma = P_2V_2^\gamma$

Here $V_1 = V$ (say) ; $V_2 = V/2$; $P_1 = 760$ mm of Hg ; $\gamma = 1.4$

$\therefore \quad 760 \times V^{1.4} = P_2\left(\dfrac{V}{2}\right)^{1.4}$

or $\quad P_2 = 760 \times 2^{1.4} = \mathbf{2006\ mm\ of\ Hg}$

We know that for adiabatic change, $TV^{\gamma-1}$ = Constant.

$\therefore \quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

Here $T_1 = 273 + 17 = 290$ K ; $V_1 = V$ (say) ; $V_2 = V/2$; $\gamma = 1.4$

$\therefore \quad 290 \times V^{0.4} = T_2 \times \left(\dfrac{V}{2}\right)^{0.4}$

or $\quad T_2 = 290 \times 2^{0.4} = 383$ K

$\therefore \quad$ Temperature = 383 − 273 = **110°C**

Example 23.8. Calculate the fall in temperature of helium initially at 15°C when it is suddenly expanded to 8 times its original volume ($\gamma = 5/3$).

Solution. Here $T_1 = 273 + 15 = 288$ K ; $V_2 = 8\ V_1$; $\gamma = 5/3$; $T_2 = ?$

Since helium is suddenly expanded, it is adiabatic expansion.

$\therefore \quad T_2V_2^{\gamma-1} = T_1V_1^{\gamma-1}$

or $\quad T_2 = T_1\left(\dfrac{V_1}{V_2}\right)^{\gamma-1} = 288\left(\dfrac{V_1}{8V_1}\right)^{5/3-1}$

$= 288\left(\dfrac{1}{8}\right)^{2/3} = 71.99$ K $= -201.01$°C

Fall in temperature = Final temp. − Initial temp.

= (−201.01°C) − (15°C) = **−216.01°C**

Example 23.9. A motor car tyre has a pressure of four atmospheres at a room temperature of 27°C. If the tyre suddenly bursts, calculate the temperature of escaping air. Value of γ for air is 1.4.

Solution. Since the tyre suddenly bursts, the change taking place is adiabatic. For adiabatic change,

$$\frac{P_1^{\gamma-1}}{T_1^\gamma} = \frac{P_2^{\gamma-1}}{T_2^\gamma}$$

or $\quad T_2^\gamma = T_1^\gamma \times \left(\dfrac{P_2}{P_1}\right)^{\gamma-1}$

Here $T_1 = 273 + 27 = 300$ K ; $P_1/P_2 = 4$; $\gamma = 1.4$

$$\therefore \quad T_2^{1.4} = (300)^{1.4} \times \left(\frac{1}{4}\right)^{1.4-1}$$

or $$T_2^{1.4} = (300)^{1.4} \times \left(\frac{1}{4}\right)^{0.4}$$

or $$T_2 = 300 \times \left(\frac{1}{4}\right)^{\frac{0.4}{1.4}} = 201.8 \text{ K}$$

$\therefore$ Temperature = 201.8 – 273 = – 71.2°C

Example 23.10. A gram-molecule of a gas at 127°C expands isothermally until its volume is doubled. Find the amount of work done and heat absorbed.

Solution. $V_1 = V$ (say) ; $V_2 = 2V$; $T = 273 + 127 = 400$ K

Now $$R = \frac{PV_m}{T}$$

Here $P = h\rho g = 0.76 \times 13600 \times 9.8 = 1.013 \times 10^5$ Nm^{-2};

$V_m = 22.4$ litres $= 22.4 \times 10^{-3}$ m^3 ; $T = 273$ K

$$\therefore \quad R = \frac{(1.013 \times 10^5) \times (22.4 \times 10^{-3})}{273} = 8.31 \text{ Jmol}^{-1}\text{K}^{-1}$$

For one mole of a gas, the work done during isothermal change is

$$W = 2.303 RT \log_{10}\left(\frac{V_2}{V_1}\right)$$

$$= 2.303 \times 8.31 \times 400 \log_{10} 2 = 2304.4 \text{ J}$$

$$\text{Heat absorbed, } Q = \frac{W}{J} = \frac{2304.4}{4.2} = 548.7 \text{ cal}$$

Example 23.11. A cylinder containing one gram molecule of the gas was compressed adiabatically until its temperature rose from 27°C to 97°C. Calculate the work done and heat produced in the gas ($\gamma = 1.5$).

Solution. When a gas is compressed adiabatically, work done W is given by ;

$$W = \frac{R}{\gamma - 1}(T_1 - T_2)$$

Here T_1 = Initial temperature in $K = 273 + 27 = 300$ K

T_2 = Final temperature in $K = 273 + 97 = 370$ K

$$\therefore \quad W = \frac{8.31}{1.5 - 1}(300 - 370) = -11.63 \times 10^2 \text{ J}$$

Negative sign shows that work is done on the gas.

$$\text{Heat produced, } Q = \frac{W}{J} = \frac{11.63 \times 10^2}{4.2} = 277 \text{ cal}$$

Example 23.12. A quantity of oxygen is compressed isothermally until its pressure is doubled. It is then allowed to expand adiabatically until its original volume is restored. Find the final pressure in terms of the initial pressure. Take $\gamma = 1.4$.

Solution. Let P_1 and V_1 be the original pressure and volume of the oxygen respectively.

Isothermal compression. For isothermal change, $P\,V$ = Constant.

$\therefore \quad P_1\,V_1 = P_2\,V_2$

Here $V_2 = V_1/2$ so that $P_2 = 2\,P_1$.

Adiabatic expansion. For adiabatic change, PV^{γ} = Constant.

$\therefore \quad P_2V_2^{\gamma} = P_3V_3^{\gamma}$

Here $P_2 = 2\,P_1$; $V_2 = V_1/2$; $V_3 = V_1$; $\gamma = 1.4$

$\therefore \quad 2P_1\times\left(\frac{V_1}{2}\right)^{1.4} = P_3\times(V_1)^{1.4}$

$\therefore \quad P_3 = 2\,P_1 \times \left(\frac{1}{2}\right)^{1.4} = 0.76\,P_1$

Example 23.13. **A quantity of air (γ = 1.4) at 27°C is compressed (*i*) slowly and (*ii*) suddenly to one third of its volume. Find the change in temperature in each case.**

Solution. (*i*) **Isothermal compression.** When the air is compressed slowly, the heat produced during compression is given to the surroundings so that the temperature remains constant i.e., the change is isothermal. For an isothermal change, $P\,V$ = constant.

$\therefore \quad$ Temperature = 27°C

(*ii*) **Adiabatic compression.** When the air is compressed suddenly, the heat produced during compression will not be able to escape from the gas. The work done on air increases its internal energy and hence temperature. Clearly, it is a case of adiabatic compression. For an adiabatic change, $TV^{\gamma-1}$ = constant.

$\therefore \quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

or $\quad T_2 = T_1\times\left(\frac{V_1}{V_2}\right)^{\gamma-1} = (273+27)\times 3^{0.4} = 465.5\ \text{K}$

$\therefore \quad$ Temperature = 465.5 − 273 = 192.5°C

Change in temperature = 192.5 − 27 = **165.5°C**

PROBLEMS FOR PRACTICE

1. Gas in a cylinder, initially at a temperature of 17°C and a pressure of 1.01×10^5 Nm^{-2}, is to be compressed to one-eighth of its volume. What would be the difference between pressures if the compression were done (*a*) isothermally (*b*) adiabatically? What would be the final temperature in the latter case? Take γ = 1.4. **[10.48×10^5 Nm^{-2} ; 393°C]**
2. A tyre pumped to a pressure of 6 atmosphere suddenly bursts. The room temperature is 15°C. Calculate the temperature of escaping air. Take γ = 1.4. **[−94.4°C]**
3. A litre of air, initially at 20°C and at 760 mm of mercury pressure, is heated at constant pressure until its volume is doubled. Find (*i*) the final temperature (*ii*) the external work done by the air in expanding (*iii*) the quantity of heat supplied. Assume that density of air at N.T.P. is 1.293 kg m^{-3} and C_v = 714 J kg^{-1} K^{-1}. **[(*i*) 313°C (*ii*) 101.2 J (*iii*) 355 J]**
4. A gas is suddenly compressed to one-half of its volume. Calculate the rise in temperature, the original temperature being 27°C. Take γ = 1.5. **[124.2°C]**
5. A certain volume of dry air at N.T.P. is allowed to expand four times its original volume under (*i*) isothermal conditions (*ii*) adiabatic conditions. Calculate the final pressure and temperature in each case (γ = 1.4):

[(*i*) 19 cm of Hg ; 0°C (*ii*) 10.91 cm of Hg ; − 116.2°C]

6. A quantity of perfect gas at 15°C is compressed adiabatically to one-fourth of its volume. Calculate the final temperature. Take $\gamma = 1.4$. **[228°C]**

7. Ten moles of hydrogen gas at NTP are compressed adiabatically so that its temperature becomes 400°C. How much work is done by the gas? Also find the increase in internal energy of the gas. Given $R = 8.4$ J mol^{-1} K^{-1} and $\gamma = 1.4$.

[– 8.4× 10^4 J ; + 8.4 ×10^4J]

8. Calculate the work done when one mole of a perfect gas is compressed adiabatically. The initial pressure and volume of the gas are 10^5 Nm^{-2} and 6 litres respectively. The final volume of the gas is 2 litres. Molar specific heat of the gas at constant volume is $3R/2$. **[972 J]**

9. A cylinder contains 1 mole of oxygen at a temperature of 27°C. The cylinder is provided with a frictionless piston which maintains a constant pressure of 1 atm on the gas. The gas is heated until its temperature rises to 127°C. (*i*) How much work is done by the gas in the process? (*ii*) What is the increase in internal energy of the gas? (*iii*) How much heat was supplied to the gas? $C_p = 7.03$ cal mol^{-1} $°C^{-1}$; $R = 1.99$ cal mol^{-1} $°C^{-1}$.

[(*i*) 835.6 J (*ii*) 2117 J (*iii*) 2952.6 J]

10. A volume of gas at atmospheric pressure is compressed adiabatically to half its original volume. Calculate the resulting pressure ($\gamma = 1.4$). **[200.5 cm of Hg]**

23.17. REVERSIBLE AND IRREVERSIBLE PROCESSES

1. Reversible process. *Any process which can be made to proceed in the reverse direction so that it passes through exactly the same states in all respects as in the direct process is called a* **reversible process.**

In a reversible process, the system is very nearly in equilibrium all the time. A reversal of controlling factors causes the system to exactly retrace its path in the opposite direction, back to the initial state. Fig. 23.17 shows an isothermal reversible processes from initial state i (P_i, V_i, T) to final state f (P_f, V_f, T) and back. The direct process is from i to f and reverse process is represented by the same curve from f to i. In the direct process, the system has passed through certain equilibrium states in a sequence. When the process is reversed, the system passes through the same states in the reverse sequence. Thus, if some work is done *by* the system in the direct process, the same work will be done *on* the system in the reverse process. Similarly, if some heat is absorbed from the surroundings in the direct process, the same amount of heat is returned to the surroundings in the reverse process.

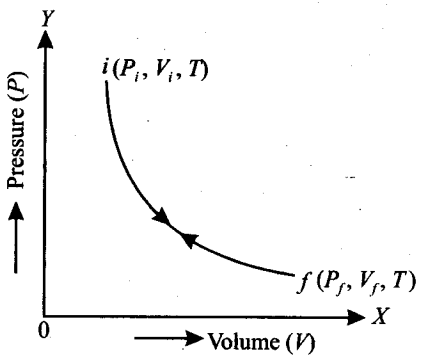


Fig. 23.17

Conditions for reversible process. A process can be reversible if it satisfies the following conditions :

(*i*) *The process should take place at extremely slow rate.* This will ensure that at any instant during the process, the system is in thermal equilibrium.

(*ii*) The system should be free from friction and any other form of energy loss.

In practice, completely reversible process is not possible because above conditions cannot be realised.

Examples. Some examples of approximate reversible processes are :

(*i*) An ideal gas allowed to expand slowly and then compressed slowly so as to reach its initial state.

(*ii*) All mechanical processes taking place under the action of conservative forces.

(*iii*) Slow compression and expansion of a spring.

2. Irreversible process. *A process that cannot be made to proceed in the reverse direction is called an* **irreversible process.**

In an irreversible process, a part of energy of the system is lost and cannot be recovered back.

Examples. There are many examples of irreversible processes viz.

(*i*) Dissolving of sugar or salt in water is an irreversible process.

(*ii*) The decay of organic matter is an irreversible process.

(*iii*) Rusting of iron is an irreversible process.

(*iv*) The sudden expansion or compression of a gas is an irreversible process.

23.18. LIMITATIONS OF FIRST LAW OF THERMODYNAMICS

The first law of thermodynamics is a statement of the principle of conservation of energy. Any process which would violate it cannot occur. However, there are many processes that would not violate first law but are never observed to occur in nature. The first law of thermodynamics fails to answer the following questions :

(*i*) *The first law does not indicate the direction of transfer of heat.*

We know that heat always flows from a hot body to a cold body, never the reverse. If the heat were to leave the cold body and pass to the hot body, energy would still be conserved. Yet it does not happen. Thus first law does not explain why heat cannot flow from a cold body to a hot body.

(*ii*) *The first law does not tell anything about the conditions under which heat can be converted into work.*

Consider what happens when you drop a stone and it hits the ground. The initial potential energy of the stone changes to kinetic energy as the stone falls. When the stone hits the ground, the kinetic energy is transformed into internal energy of the stone and the ground in the vicinity of the impact; the molecules move faster and the temperature rises slightly. But have you seen the reverse happen—a stone at rest on the ground suddenly rise up in the air because the thermal energy of the molecules is transformed into kinetic energy of the stone as a whole? Energy would be conserved in the process, yet we never see it happen. The first law does not explain why it canot happen.

(*iii*) *The first law does not tell why the whole of heat energy cannot be converted into mechanical work continuously.*

The conversion of mechanical energy into heat can be almost 100%. For example, if you rub your hands briskly, practically the whole of mechanical work done against friction is transformed into heat. However, the opposite process, conversion of heat into mechanical work, is never 100% efficient.

There are many other examples of processes that occur in nature but whose reverse does not. The first law of thermodynamics would not be violated if any of these processes occurred in reverse. To explain this lack of reversibility, scientists in the later half of the nineteenth century came to formulate a new principle known as the **second law of thermodynamics.** This law is a statement about which processes occur in nature and which do not.

23.19. SECOND LAW OF THERMODYNAMICS

The second law of thermodynamics is a statement about which processes occur in nature and which do not. It can be stated in a variety of ways, all of which are equivalent.

(*i*) **Heat flows naturally from a hot body to a cold body; heat will never of itself flow from a cold body to a hot body.**

Heat flows one way, downhill from hot to cold. Heat can be made to flow the other way, but only by doing appropriate amount of external work. There is an enormous quantity of internal energy in the ocean, but without external effort all this energy cannot be used to light a single flashlight lamp. Gas expands adiabatically and cools. Once cooled, it will never spontaneously (*i.e.*, by itself) contract to its original volume and warm up to its original temperature. Outside work would be necessary. We could squeeze it back, but it would not do so of its own. To do so would violate the second law.

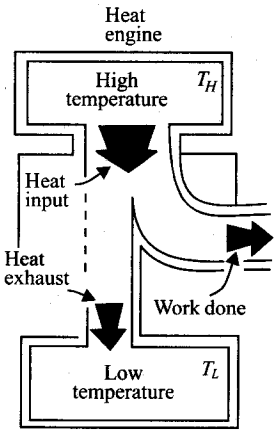


Fig. 23.18

(*ii*) The internal energy converted to useful work in heat engines flows from a hot region to a cold region. Every heat engine will (*a*) absorb energy from a reservoir of higher temperature (T_H) (*b*) convert some of this energy into mechanical work and (*c*) discard the remaining energy to some lower temperature (T_L) reservoir, usually called a *sink* (See Fig. 23.18). For example, in a gasoline engine, the burning fuel in the combustion chamber is the high-temperature reservoir, mechanical work is done on the piston and the discarded energy goes out as exhaust. The second law of thermodynamics tells us that no heat engine can convert all the heat energy into mechanical work. Some heat is always exhausted in the process. Applied to heat engines, **second law of thermodynamics** may be stated as under :

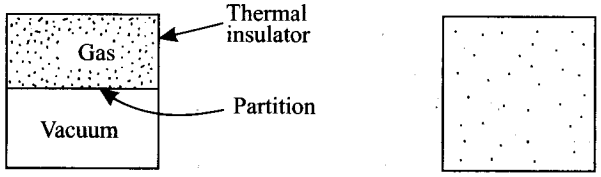
When work is done by a heat engine, running between two temperatures, T_H (hot) and T_L (cold), only some of the input energy at T_H can be converted to work ; the rest is rejected as heat at T_L.

Comments. The above two statements of second law of thermodynamics are equivalent as one can be explained from the other. It appears that nature has a preferred direction for the course of spontaneous events. We shall try to explain second law of thermodynamics on the basis of entropy.

23.20. ENTROPY

Entropy *is a measure of the disorder of a system.*

Whenever a physical system is allowed to distribute its energy freely, it always does so in a manner that disorder increases *i.e.* entropy increases. By disorder, we mean random. For example, suppose you have a box with a partition dividing it in two, with a gas on one side of the partition and other side evacuated [See Fig. 23.19 (*i*)]. With all the molecules of gas in one side, gas has a highly ordered situation. However, if the partition is removed [See Fig. 23.19 (*ii*)], the gas molecules will distribute themselves throughout the box and be moving in random directions — a less ordered situation. We say that entropy of the gas has increased because the amount of disorder has increased.



(*i*) **Fig. 23.19** (*ii*)

As the disorder of a system increases, its entropy increases. In order to change the disordered system to the ordered one, some work will have to be done. For example, water in a refrigerator freezes and becomes more ordered because work is put into the refrigeration cycle.

23.21. ENTROPY AND SECOND LAW OF THERMODYNAMICS

When the entropy of a system increases, there is decrease in the energy available to do useful work. For example, consider the natural process of heat conduction from a hot body to a cold one. In this process, entropy increases and the order goes to disorder. The fact that disorder has increased as a result of such a process can be explained as follows. Hot and cold objects could serve as high and low temperature regions for a heat engine and thus could be used to obtain useful work. But after the two objects are put in contact and reach the same uniform temperature, no useful work can be obtained from them. With regard to being able to do useful work, order has gone to disorder in the process. Therefore, second law of thermodynamics can also be stated in another useful way.

In any natural (irreversible) process, the entropy increases i.e. some energy becomes unavailable to do useful work.

Entropy is a measure of how much energy or heat is unavailable for conversion into useful work. When a system at kelvin temperature T undergoes a reversible process by absorbing an amount of heat Q, its increase in entropy ΔS is

$$\Delta S = \frac{Q}{T}$$

Note that we are defining entropy for a reversible process which does not occur in nature. However, we can use this definition for processes which are approximately reversible.

23.22. PRINCIPLE OF A HEAT ENGINE

It is easy to produce thermal energy by doing work. For example, if you rub your hands briskly, practically the whole of mechanical work done against friction is transformed into heat. Thus the conversion from mechanical energy to heat can be almost 100%.

However, the opposite process, the continuous conversion of heat into work is never 100% efficient. *A device which converts heat into mechanical work is called a* **heat engine** *e.g.*, internal combustion engines and steam engines. In order that the heat engine works continuosuly , it *must work in a cycle* .To illustrate this point, consider a simple heat engine consisting of an ideal gas enclosed in a cylinder by a piston. If heat is supplied to the gas, it expands and pushes back the pistion. Thus heat has been converted to work. However, the process stops as soon as the pressure of the gas becomes equal to the pressure outside the cylinder. Before there can be any further conversion of heat into work, the gas has to be returned to its initial state. It follows that if there is to be a continuous conversion of heat into work, the gas has to undergo a cycle. Furthermore, the gas can be returned to its initial state only if some of the heat it initially absorbed is given up to a low-temperature reservoir (usually called a *sink*) which is at lower temperature than the source which provided the heat in the first place. We conclude that all heat engines operate by taking some working substance around a cycle. Thus every heat engine will

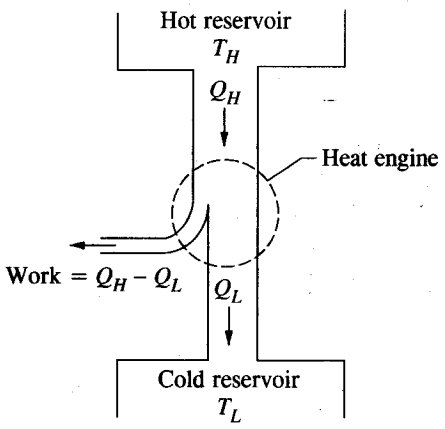


Fig. 23.20

(*i*) absorb heat from a reservoir of higher temperature,

(*ii*) convert some of this heat energy into mechanical work and

(*iii*) discard the remaining energy to some low-temperature reservoir (called sink).

The principle of a heat engine is illustrated in Fig. 23.20. Since the engine rejects some of the heat it initially takes in, it has converted only a part of it into mechanical work.

23.23. HEAT AND MECHANICAL WORK IN ENGINES

We have seen that a heat engine continuously converts heat into mechanical work by taking some working substance around a cycle. Let us illustrate it further. A simple heat engine consists of a cylinder containing a gas; the cylinder being fitted with a frictionless movable piston. We shall see that if the engine operates in a cycle so that it returns to its original state after a series of operations, then it can be made to do work continuously *i.e.*, there will be continuous conversion of heat into mechanical work. Suppose the gas is taken through a cyclic process as shown in the *PV* diagram in Fig. 23.21. Starting from the state represented by *A*, the gas expands from *A* to *B* and then it expands from *B* to *C*. Here the mechanical work is done *by* the gas, which is the integral of *PdV* for the two stages in the cycle, is represented by the area *ABCXYA* on the *PV* diagram. The gas takes a quantity of heat Q_H (from a high-temperature reservoir) along *AB* and *BC* while doing mechanical work. To restore the gas to its initial state *A*, the gas is now compressed along *CD* and *DA*. In this case, work done by the gas is negative (*i.e.*, work is done on the gas by external agent) and is represented by the area *CDAYXC*. The gas now gives up a quantity of heat Q_L (to a low-temperature reservoir). The difference between the two areas is the area enclosed by the cyclic curve *ABCDA* and represents the net work done by the gas in a cycle.

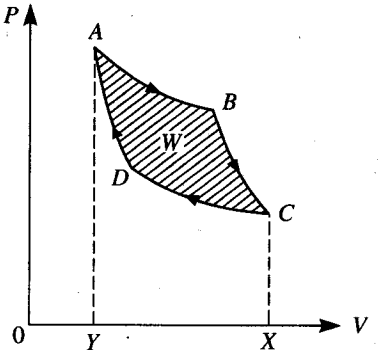


Fig. 23.21

Thus the net work done by a gas in any closed cycle is equal to the area enclosed by the closed loop (shaded area) in the PV diagram.

Since the gas returns to its initial state *A*, there is no change in the internal energy of the gas at the end of the cycle. Then from the first law of thermodynamics, net work done in a cycle = heat gained by gas = $Q_H - Q_L$.

$\therefore$ Net work done by the gas in a cycle is

$$W = Q_H - Q_L$$

where Q_H = heat taken in at the higher temperature

Q_L = heat rejected at the lower temperature

Efficiency of heat engine. The efficiency η of any heat engine can be *defined as the ratio of the work (W) it does to the heat input (Q_H) at the higher temperature *i.e.*,

$$\eta = \frac{\text{Work done}}{\text{Energy supplied}} = \frac{W}{Q_H}$$

As shown above, $W = Q_H - Q_L$

$$\therefore \quad \eta = \frac{Q_H - Q_L}{Q_H}$$

or

$$\eta = 1 - \frac{Q_L}{Q_H} \qquad ...(i)$$

---

* This is a very practical definition since W is the output (what you get from the engine) whereas Q_H is what you put in and pay for in burned fuel. Since energy is conserved, $Q_H = W + Q_L$.

It is clear from eq. (*i*) that efficiency of a heat engine will be greater if Q_L can be made small. However, from experience with a wide variety of systems, it has not been found possible to reduce Q_L to zero. It follows that a heat engine can never be 100% efficient *i.e.*, all the heat supplied can never be converted into mechanical energy during a complete cycle. **The fact that 100% efficient heat engine is not possible is another way of expressing second law of thermodynamics.** It can be stated as under :

No cyclic process is possible whose only result is to transfer an amount of heat Q from a source at a single temperature totally into work W (so W = Q).

This is known as the **Kelvin-Planck statement of the second law of thermodynamics.** If the second law were not true, so that a perfect engine (100% efficient) could be built, some rather remarkable things could happen. For example, if the engine of a ship did not need a low-temperature reservoir to exhaust heat into, the ship could sail across the ocean using the vast resources of the internal energy of the ocean water. Indeed, we would have no fuel problems at all !

It has not been found possible to construct a perfect heat engine. It has also not been found possible to construct its reverse—which would be a device to take heat from a low-temperature region to a high-temperature region without requiring any work ($W = 0$; $Q_L = Q_H$). This can be stated as under :

No cyclic process is possible whose only result is a flow of heat out of a system at one temperature and an equal magnitude flow of heat into a second system at a higher temperature.

This is the **Clausius statement of the second law of thermodynamics.** It is more formal statement of the one given in Art. 23.19 that heat does not flow spontaneously (*i.e.*, by itself) from a cold object to a hot object. To accomplish such a task, external work must be done. The Clausius statement can also be stated as *there can be no perfect refrigerator.*

23.24. THE CARNOT CYCLE

One particular type of cycle is of great importance from both a theoretical and practical standpoint. It was first discussed theoretically by Sadi Carnot in 1824 and is called the *Carnot cycle.* He recognised that work could be done only when heat flowed from a higher temperature to a lower one. So Carnot proposed an ideal engine that operates cyclically and reversibly between two temperatures. Carnot showed that his proposed cycle is the most efficient cycle possible *i.e.*, work done by a gas when following this cycle is the greatest possible for a given amount of heat energy added to the gas. As we shall see, a *Carnot cycle consists of two adiabatic and two isothermal processes, all reversible.*

A **Carnot cycle** *consists of two reversible adiabatic and two reversible isothermal processes operating between temperatures T_H and T_L.*

Working. To describe the Carnot cycle, we shall assume that the working substance is an ideal gas contained in a cylinder fitted with a movable piston. The cylinder walls and the piston are thermally non-conducting. We use an ideal gas for mathematical simplicity. However, the results would be the same for any working substance in the Carnot cycle. Fig. 23.22 shows the four stages of the Carnot cycle while Fig. 23.23 shows the *PV* diagram of the cycle.

Stage 1 (Process $A \rightarrow B$). We start the cycle with the cylinder in contact with the heat reservoir where the working substance (gas) takes in an amount of heat Q_H at a high temperature T_H [See Fig. 23.22(*i*)]. Therefore, the gas acquires the temperature T_H of the heat source. The gas is allowed to expand at constant temperature T_H (*isothermal expansion*). This isothermal expansion is represented in the *P-V* diagram by the isothermal curve *AB*. We know that in an isothermal process, the internal energy of the system (the gas) does not change. Therefore, according to first law of thermodynamics, the work done by the gas (the system) is equal to the heat added (Q_H).

Stage 2 (Process $B \rightarrow C$). The cylinder is now removed from the source and is placed in contact with an insulating stand [See Fig. 23.22(*ii*)]. The gas is completely thermally isolated from

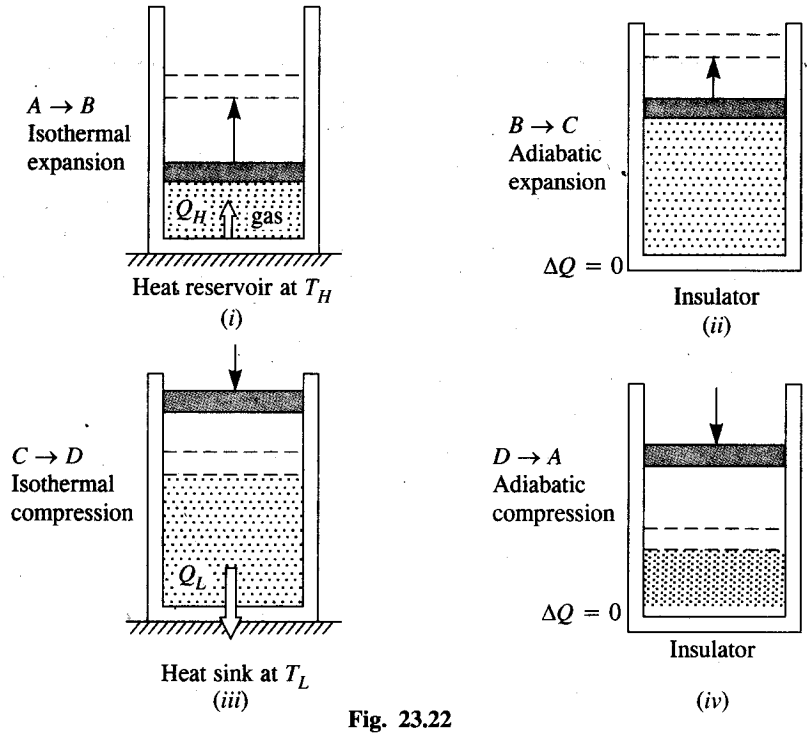


Fig. 23.22

the surroundings. Therefore, as the gas is allowed to expand further, the expansion will be adiabatic. This *adiabatic expansion* is represented in the *PV* diagram by the adiabatic curve *BC*. We know that in an adiabatic expansion, the internal energy and hence temperature of the gas changes. Therefore, the adiabatic expansion is accompanied by a decrease in temperature along the curve *BC* [See Fig. 23.23] until the cylinder reaches the temperature of the heat sink (T_L). Note that during adiabatic expansion, the work is done by the gas; the work done being equal to the decrease in the internal energy of the gas.

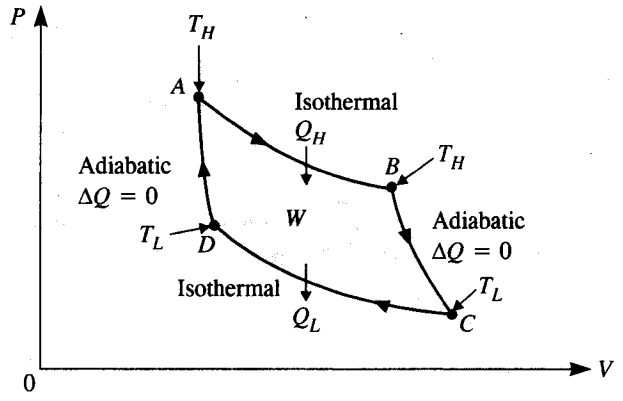


Fig. 23.23

Stage 3 (Process $C \rightarrow D$). Now the cylinder is moved to the heat sink (temperature T_L) as shown in Fig. 23.22 (*iii*). Here the gas undergoes an *isothermal contraction* (curve *CD* in *PV*

diagram) in which an amount of heat Q_L is expelled to the heat sink at temperature T_L. Since external work is done on the gas, heat (Q_L) must flow out of the gas in order for the temperature to remain constant at T_L.

Stage 4 (Process $D \rightarrow A$). This is the final stage of the Carnot cycle. Here the cylinder is moved back to the insulating stand and the gas is *compressed adiabatically.* Again the heat exchange is zero and because the volume is decreasing (work being done on the gas), the internal energy and temperature of the gas increase. When the temperature of the gas reaches that of the heat reservoir (T_H), the cylinder is transferred to the heat reservoir and the cycle starts again.

During the full cycle, the gas did an amount of work W given by the area of the enclosed curve. Since the gas returns to its initial state A, there is no change in the internal energy of the gas at the end of the cycle. Then from the first law of thermodynamics, the net work done by the gas in one cycle = heat gained by gas = $Q_H - Q_L$.

$\therefore$ Net work done by the gas in one cycle is

$$W = Q_H - Q_L$$

23.25. EFFICIENCY OF CARNOT ENGINE

Figure 23.24 shows the PV diagram for the Carnot cycle. The Carrot engine receives thermal energy Q_H isothermally from the heat reservoir maintained at a constant temperature T_H. It rejects thermal energy Q_L isothermally to a heat sink maintained at a constant lower temperature T_L. As shown above, the net work (W) done by the gas in one cycle is

$$W = Q_H - Q_L$$

$$\therefore \text{ Efficiency, } \eta = \frac{\text{Work Output}}{\text{Energy Input}} = \frac{Q_H - Q_L}{Q_H}$$

or
$$\eta = 1 - \frac{Q_L}{Q_H} \qquad ...(i)$$

It is illuminating to express eq. (i) in terms of temperatures rather than quantities of heat. To do this, we seek expressions for Q_L and Q_H in terms of temperatures. We know that work done W_{AB} in an isothermal expansion from V_A to V_B is

$$W_{AB} = nRT_H \log_e \frac{V_B}{V_A}$$

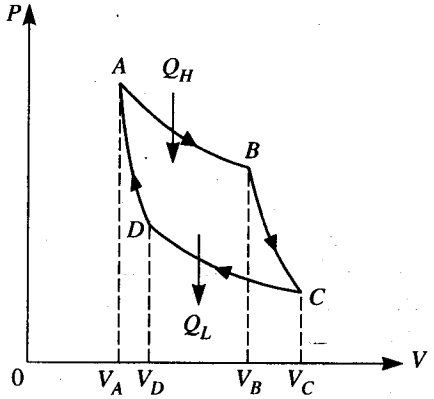


Fig. 23.24

where T_H is the temperature at the time of expansion. Since the expansion is isothermal, W_{AB} is equal to the heat energy Q_H supplied to the gas during the expansion AB in the Carnot cycle of Fig. 23.24.

$$\therefore \qquad Q_H = W_{AB} = nRT_H \log_e \frac{V_B}{V_A}$$

During the portion CD of the cycle, an amount of heat Q_L is expelled from the gas. Since curve CD is isothermal compression at temperature T_L, heat rejected is

$$Q_L = -^{*} nRT_L \log_e \left(\frac{V_D}{V_C}\right) = nRT_L \log_e \left(\frac{V_C}{V_D}\right)$$

Dividing Q_L by Q_H, we have,

---

* We put the minus sign because in our statement of the first law of thermodynamics, Q is the heat absorbed. Here Q_L is the heat rejected.

$$\frac{Q_L}{Q_H} = \frac{T_L \log_e (V_C / V_D)}{T_H \log_e (V_B / V_A)} \quad ...(ii)$$

This expression can be simplified because for adiabatic change, $TV^{\gamma-1}$ = constant. Applying this fact to the adiabatic processes *BC* and *DA* in the Carnot cycle, we have,

$$T_H V_B^{\gamma-1} = T_L V_C^{\gamma-1} \text{ and } T_H V_A^{\gamma-1} = T_L V_D^{\gamma-1}$$

Dividing these two equations, we get,

$$\left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{V_C}{V_D}\right)^{\gamma-1} \text{ or } \frac{V_B}{V_A} = \frac{V_C}{V_D}$$

Therefore, eq. (*ii*) becomes :

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H}$$

The efficiency of Carnot engine given by eq. (*i*), therefore, becomes :

$$\eta = 1 - \frac{T_L}{T_H} \quad ...(iii)$$

where T_L and T_H are the *absolute temperatures. Eq. (*iii*) tells us that efficiency of a Carnot engine depends only on the highest and lowest temperatures of the working fluid. For a practical engine, the low temperature (T_L) is usually the ambient temperature. Then to maximise the efficiency, we must make high temperature (T_H) as high as possible. Most real engines represent a compromise between efficiency and the ability of materials to withstand high temperatures and pressures. For example, a steam engine cannot easily operate at a temperature much higher than 100°C, we should have T_H = 100 + 273 = 373 K. During the cool portion of the cycle, the gas cannot be at a temperature lower than that of the atmosphere (say 27°C). Therefore, T_L = 27 + 273 = 300 K. Our conclusion is that the best steam engine will have an efficiency of only about :

$$\eta = 1 - \frac{T_L}{T_H} = 1 - \frac{300}{373} = 0.2 \text{ or } 20\%$$

Important Points. $\eta = 1 - \frac{T_L}{T_H}$

(*i*) Note that if $T_L = T_H$, $\eta = 0$ as one would expect. It is clear that efficiency will be 100% if T_L = 0 K. This is, of course, impossible. Therefore, the maximum efficiency is always less than 100%. This result is a direct consequence of the second law of thermodynamics and is often quoted as an alternative statement of second law *viz. A device that converts 100% of input heat into mechanical work is a physical impossibility.*

(*ii*) All reversible engines operating in cycles between the same temperature T_H (high) and T_L (low) have the same efficiency regardless of the working substance.

(*iii*) The efficiency is increased by decreasing T_L/T_H *i.e., the efficiency is increased by taking in heat at as high temperature as possible and rejecting heat at as low temperature as possible.*

(*iv*) The efficiency of a real heat engine is less than that given by the above efficiency equation. It is because of losses due to frictional effects etc.

(*v*) $\eta = 1 - \frac{Q_L}{Q_H} = 1 - \frac{T_L}{T_H} \quad \therefore \quad \frac{Q_L}{Q_H} = \frac{T_L}{T_H}$

* Whenever ratios of temperatures are involved, the absolute temperatures must be used.

23.26. CARNOT'S THEOREM

The Carnot cycle is the most efficient cycle of all reversible cycles operating between two fixed temperature extremes. This is known as Carnot's theorem and may be state as under :

No engine working between two temperatures can be more efficient than a Carnot's reversible engine working between the same temperatures.

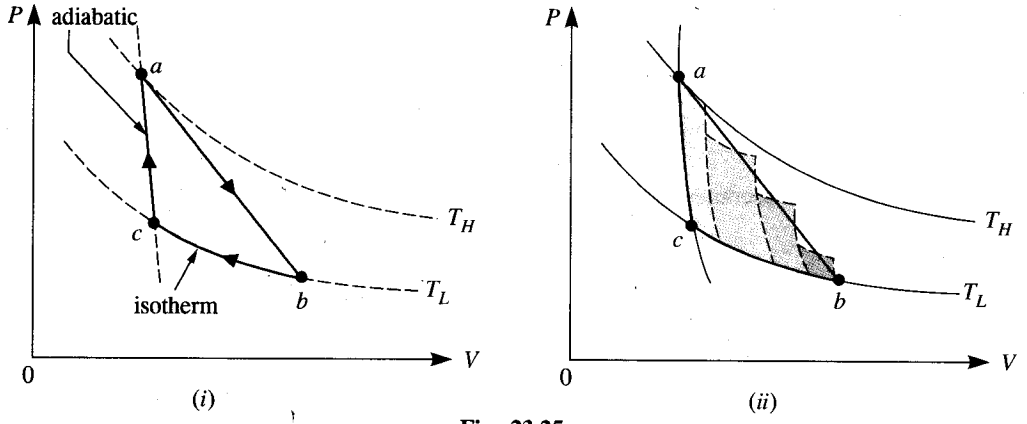


Fig. 23.25

Now we show that the Carnot cycle is the most efficient of all reversible cycles operating between two fixed temperatures. Consider a cycle shown in Fig. 23.25 (*i*). From point *a*, the gas expands along the line *ab* (neither an adiabatic nor an isothermal path) as temperature decreases from T_H to T_L. This is followed by an isothermal compression to point *c* and then adiabatic compression which returns the system to starting point *a*. How does this reversible cycle compare in efficiency with a Carnot cycle between the same temperature extremes? Fig. 23.25 (*ii*) shows how expansion *ab* can be approximated as closely as we wish by a series of small isothermals and adiabatic steps. We then replace the reversible cycle of Fig. 23.25 (*i*) by the small adjacent Carnot cycles shown in Fig. 23.25 (*ii*). The efficiency of any one of these small Carnot cycles depends upon its upper and lower temperatures. The upper and lower temperature of any small Carnot cycle is less than that of the actual Carnot cycle.

$\therefore$ For any non-Carnot reversible engine, $\eta < 1 - \frac{T_L}{T_H}$.

23.27. TYPES OF HEAT ENGINES

Heat engines can be divided into two classes *viz*

(*i*) External combustion engine

(*ii*) Internal combustion engine

(*i*) **External combustion Engine.** In an external combustion engine, the working substance (*i.e.* steam) is produced by burning the fuel outside the cylinder. The steam engine is an external combustion engine. The efficiency of a steam engine is about 12% to 16% .

(*ii*) **Internal combustion Engine.** In an internal combustion engine, fuel is burnt inside the cylinder. Petrol engine and diesel engine are internal combustion engines. The efficiency of a petrol engine is about 25% while that of diesel engine is about 40%.

Petrol Engine. A petrol engine is an internal combustion engine and is used in most cars. In this engine, each cycle consists of four processes called *strokes*. The four strokes are :

(*a*) Induction stroke (*b*) Compression stroke

(*c*) Power stroke (*d*) Exhaust stroke

In the induction stroke, a mixture of air and petrol at atmospheric pressure is drawn into (*i.e.*, inducted) the cylinder. In the compression stroke, the petrol-air is compressed adiabatically from volume V_2 to V_1. The ratio V_2/V_1 is called *adiabatic compression ratio.* In the power stroke, the gas expands adiabatically from V_1 to V_2. In the exhaust stroke, the burnt gases are pushed out.

Efficiency of petrol engine. It can be proved that the efficiency (η) of petrol engine is given by ;

$$\eta = 1 - \frac{1}{(V_2/V_1)^{\gamma-1}} \qquad ...(i)$$

where V_2/V_1 = *adiabatic compression ratio i.e.*, the ratio of maximum volume of the gas to its minimum value

$\gamma = C_p/C_v$ for the gas

It is clear from eq. (*i*) that efficiency increases with increasing compression ratios. For a compression ratio of 8 and $\gamma = 1.4$, we have,

$$\eta = 1 - \frac{1}{(V_2/V_1)^{\gamma-1}} = 1 - \frac{1}{(8)^{1.4-1}} = 1 - \frac{1}{8^{0.4}} = 0.56 \text{ or } 56\%$$

The efficiency of a real engine will be less than this value if we take into account such effects as friction, heat loss to cylinder walls and incomplete combustion.

23.28. REFRIGERATOR

A refrigerator is a heat engine running in reverse *i.e.*, it transfers heat from a colder body (the contents of the refrigerator) to a hotter body (the room). Fig. 23.26 shows the principle of a refrigerator. It absorbs heat Q_L from the cold reservoir and expels heat Q_H to the hot reservoir. This can be achieved only if work is done *on* the refrigerator. From the first law of thermodynamics, we see that the heat given up to the hot reservoir must be equal to the sum of the work done and the heat absorbed from the cold reservoir *i.e.*,

$$Q_L + W = Q_H$$

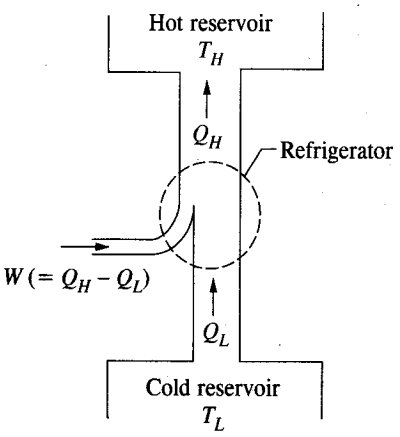


Fig. 23.26

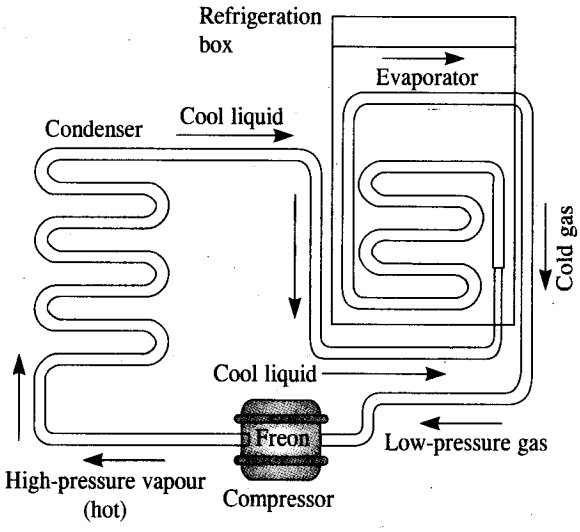


Fig. 23.27

Working. Fig. 23.27 shows the schematic diagram of a refrigerator. The **compressor** at the bottom compresses a refrigerant gas (freon) to a pressure high enough that it will liquefy when cooled somewhat. During this nearly adiabatic compression, the work (W) done on the gas causes

it to become quite hot. The hot freon passes through the coils of the **condenser** (external to or on the walls of the refrigerator) where it *loses some of its heat to the surrounding air. The cooling causes the freon to become liquid, giving up its heat of vaporisation to the surrounding air. The heat given up by cooling plus the heat of vaporisation add up to Q_H. The near-room temperature liquid freon under high pressure is then sent to a throttling valve where it evaporates into low pressure region called the **evaporator**. The coiled tubing of the evaporator is inside the refrigerator and as now very cold gaseous freon flows through these coils, heat (Q_L) flows from the warmer contents of the refrigerator into the freon, cooling the interior. Eventually the (now warmer) freon leaves the evaporator coils and returns to the compressor. There it is compressed again and both the heat it absorbed in the cooling coils and the heat generated during its compression are removed in the condenser coils and the cycle repeats.

Coefficient of performance (*COP*). The effectiveness of a refrigerator is measured by its coefficient of performance.

The **coefficient of performance** *(COP) of a refrigerator is defined as the ratio of heat extracted per cycle at low temperature to the work input required i.e.,*

$$COP = \frac{Q_L}{W}$$

Now $Q_H = Q_L + W$ so that $W = Q_H - Q_L$.

$$\therefore \quad COP = \frac{Q_L}{Q_H - Q_L}$$

In terms of temperatures, the coefficient of performance becomes :

$$COP = \frac{T_L}{T_H - T_L}$$

Another expression for *COP*. We can also express *COP* in terms of efficiency η of Carnot cycle.

$$COP = \frac{T_L}{T_H - T_L} = \frac{T_L/T_H}{1 - T_L/T_H}$$

Now $$\eta = 1 - \frac{T_L}{T_H} \quad \text{or} \quad \frac{T_L}{T_H} = 1 - \eta$$

$$\therefore \quad COP = \frac{1-\eta}{1-(1-\eta)} = \frac{1-\eta}{\eta}$$

Important points. The following points may be noted about *COP* :

(*i*) The smaller is the temperature difference ($T_H - T_L$), the greater is *COP*. This makes sense because it takes less work to force heat flow to a reservoir that is at a slightly higher temperature than to one at a much higher temperature.

(*ii*) The *COP* indicates the efficiency of the refrigerator. The greater the value of *COP*, more efficient is the refrigerator.

(*iii*) The practical value of *COP* is less than calculated from the above formulae.

Example 23.14. The source temperature of a Carnot engine is 127°C. It takes 500 cal of heat from the source and rejects 400 cal to the sink during each cycle. What is the temperature of the sink?

Solution. $$\frac{Q_H}{Q_L} = \frac{T_H}{T_L} \quad \therefore \quad T_L = T_H \times \frac{Q_L}{Q_H}$$

* When you reach behind a refrigerator, you can feel that heated air is in the vicinity of the coils of the condenser.

Here T_H = 127 + 273 = 400 K ; Q_L = 400 cal ; Q_H = 500 cal

$$\therefore \quad T_L = 400 \times \frac{400}{500} = 320 \text{ K}$$

$$\therefore \quad \text{Temperature} = 320 - 273 = 47^\circ\text{C}$$

Example 23.15. **A Carnot engine absorbs 1000 J of heat from a reservoir at 127°C and rejects 600 J of heat during each cycle. Calculate (*i*) the efficiency of engine (*ii*) the temperature of the sink and (*iii*) the amount of the useful work done during each cycle**

Solution. Here Q_H = 1000 J ; Q_L = 600 J ; T_H = 273 + 127 = 400 K

(*i*) Engine efficiency, $\eta = 1 - \frac{Q_L}{Q_H} = 1 - \frac{600}{1000} = 1 - 0.6 = 0.4$ or 40%

(*ii*) For Carnot's engine, $\frac{T_L}{T_H} = \frac{Q_L}{Q_H}$

$$\therefore \quad T_L = T_H \times \frac{Q_L}{Q_H} = 400 \times \frac{600}{1000} = 240 \text{ K} = -33^\circ\text{C}$$

(*iii*) The amount of useful work done (W) during each cycle is

Useful work done/cycle, $W = Q_H - Q_L = 1000 - 600 = 400$ J

Example 23.16. **The efficiency of a Carnot cycle is 1/6. If on reducing the temperature of the sink by 65°C, the efficiency becomes 1/3, find the initial and final temperatures between which the cycle is working.**

Solution. Suppose the initial temperature (source temperature) is T_H and final temperature (sink temperature) is T_L. Then,

$$\eta = 1 - \frac{T_L}{T_H}$$

For the first case, $\frac{1}{6} = 1 - \frac{T_L}{T_H}$ ...(*i*)

For the second case, $\frac{1}{3} = 1 - \frac{T_L - 65}{T_H}$ ...(*ii*)

Solving eqs. (*i*) and (*ii*), we get,

T_H = 390 K = 117°C; T_L = 325 K = 52°C

Example 23.17. **A Carnot engine develops 100 h.p. and operates between 27°C and 227°C. Find (*i*) thermal efficiency (*ii*) heat supplied (*iii*) heat rejected.**

Solution. W = 100 h.p. = 100 × 746 W = (100 × 746)/4.2 cal/s ; T_H = 227°C = 500 K ; T_L = 27°C = 300 K.

(*i*) Thermal efficiency, $\eta = 1 - \frac{T_L}{T_H} = 1 - \frac{300}{500} = 0.4$ or 40%

(*ii*) The heat supplied Q_H is given by ;

$$Q_H = \frac{W}{\eta} = \frac{100 \times 746}{4.2 \times 0.4} = 4.44 \times 10^4 \text{ cal/s}$$

(*iii*) The heat rejected Q_L is given by ;

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H}$$

or $$Q_L = Q_H \times \frac{T_L}{T_H} = 4.44 \times 10^4 \times \frac{300}{500} = 2.66 \times 10^4 \text{ cal/s}$$

Example 23.18. **A reversible engine converts one fifth of heat which is absorbed from source into work. When the temperature of the sink is reduced by 70°C, the efficiency is doubled. Calculate the temperature of the source and sink.**

Solution. $W = \frac{1}{5}Q_H \quad \therefore Q_L = Q_H - W = Q_H - \frac{1}{5}Q_H = \frac{4}{5}Q_H$

$\therefore \quad \frac{T_L}{T_H} = \frac{Q_L}{Q_H} = \frac{4}{5}$

Now $\eta = 1 - \frac{T_L}{T_H} = 1 - \frac{4}{5} = \frac{1}{5}$

On reducing the temperature of the sink by 70°C, the efficiency is doubled *i.e.*, it becomes 2η or 2/5.

$\therefore \quad \frac{2}{5} = 1 - \frac{T_L - 70}{T_H} = 1 - \frac{T_L}{T_H} + \frac{70}{T_H}$

$= 1 - \frac{4}{5} + \frac{70}{T_H} \qquad \left(\because \frac{T_L}{T_H} = \frac{4}{5}\right)$

or $\frac{70}{T_H} = \frac{1}{5} \quad \therefore \quad T_H = \mathbf{350\ K}$

Now $T_L = T_H \times \frac{4}{5} = 350 \times \frac{4}{5} = \mathbf{280\ K}$

Example 23.19. **A 25 h.p. petrol engine of a car consumes 10 kg of petrol per hour. The compression ratio is 5 and 1 g of petrol on complete combustion yields 50 kJ of energy. Given γ for air is 1.4. Find the ratio of actual to theoretical efficiency of the engine.**

Solution. Output power = 25 h.p. = 25 × 746 W = 18650 Js^{-1}

Input petrol = 10 kg per hour = $\frac{10 \times 1000}{3600} g/s = \frac{25}{9} g/s$

$\therefore \quad$ Input power = $\frac{25}{9} \times 50 \times 10^3 = \frac{125}{9} \times 10^4\ Js^{-1}$

$\therefore \quad$ Actual efficiency, $\eta = \frac{\text{Output power}}{\text{Input power}} = \frac{18650 \times 9}{125 \times 10^4} = 0.1343$

Theoretical efficiency, $\eta' = 1 - \frac{1}{(V_2/V_1)^{\gamma-1}} = 1 - \frac{1}{(5)^{1.4-1}} = 0.4747$

$\therefore \quad \frac{\eta}{\eta'} = \frac{0.1343}{0.4747} = \mathbf{0.283}$

Example 23.20. **How much heat energy is required to do 10 kWh of work by a petrol engine (Otto cycle) whose adiabatic compression ratio is 10? Given γ = 1.4.**

Solution. Output energy = 10 kWh= 36 × 10^6 J [$\because$ 1 kWh= 36 × 10^5 J]

Compression ratio = V_2/V_1 = 10

Efficiency = $\frac{\text{Output energy}}{\text{Input energy}}$ or $1 - \frac{1}{(V_2/V_1)^{\gamma-1}} = \frac{36 \times 10^6}{\text{Input energy}}$

or $1 - \frac{1}{(10)^{1.4-1}} = \frac{36 \times 10^6}{\text{Input energy}}$ or $0.602 = \frac{36 \times 10^6}{\text{Input energy}}$

$\therefore \quad$ Input energy = $\frac{36 \times 10^6}{0.602} = 5.98 \times 10^7\ J = \mathbf{1.42 \times 10^7\ cal}$

Example 23.21. A heat engine operates between two reservoirs at 300 K and 500 K. During each cycle, it absorbs 200 cal from the high-temperature reservoir. Find (*i*) the maximum efficiency of the engine (*ii*) the maximum work the engine can perform in each cycle of operation.

Solution. (*i*) The maximum efficiency is equal to the efficiency of a Carnot engine operating between these temperatures.

$$\therefore \quad \text{Maximum efficiency, } \eta_{max} = 1-\frac{T_L}{T_H} = 1-\frac{300}{500} = 0.4 \text{ or } 40\%$$

(*ii*) $W_{max} = \eta_{max} \times Q_H = 0.4 \times 200 = 80$ cal

Any real engine operating between these two temperatures will have an efficiency less than 40% and will do less than 80 cal of work in each cycle.

Example 23.22. A household refrigerator has a coefficient of performance of 6.0. If the room temperature outside the refrigerator is 30°C, what is the lowest temperature that can be obtained inside the refrigerator?

Solution. $$COP = \frac{T_L}{T_H - T_L}$$

Here $COP = 6.0$; $T_L = ?$; $T_H = 30°C = 303$ K

$$\therefore \quad 6 = \frac{T_L}{303 - T_L} \quad \text{or} \quad T_L = 260 \text{ K} = -13°C$$

Example 23.23. In a refrigerator, heat from inside at 277 K is transferred to a room at 300 K. How many joules of heat shall be delivered to the room for each joule of electrical energy consumed ideally?

Solution. $$COP = \frac{Q_L}{W} = \frac{T_L}{T_H - T_L}$$

$$\therefore \quad Q_L = W \times \frac{T_L}{T_H - T_L}$$

Here $W = 1$ J ; $T_L = 277$ K ; $T_H = 300$ K

$$\therefore \quad Q_L = 1 \times \frac{277}{300 - 277} = 12 \text{ J}$$

$$\therefore \quad Q_H = W + Q_L = 1 + 12 = 13 \text{ J}$$

Example 23.24. Refrigerator *A* works between –10°C and 27°C while refrigerator *B* works between –27°C and 17°C, both removing heat equal to 2000 J from the freezer. Which of the two is the better refrigerator?

Solution. That refrigerator is better whose *C.O.P.* is more.

For refrigerator *A* : $T_H = 273 + 27 = 300$ K ; $T_L = 273 - 10 = 263$ K

$$\therefore \quad \text{C.O.P.} = \frac{T_L}{T_H - T_L} = \frac{263}{300 - 263} = \frac{263}{37} = 7.1$$

For refrigerator *B* : $T_H = 273 + 17 = 290$ K ; $T_L = 273 - 27 = 246$ K

$$\therefore \quad \text{C.O.P.} = \frac{T_L}{T_H - T_L} = \frac{246}{290 - 246} = \frac{246}{44} = 5.6$$

Hence refrigerator *A* is better.

Example 23.25. A Carnot engine has the same efficiency when operated (*i*) between 100 K and 500 K (*ii*) between *T K* and 900 K. Calculate the temperature *T K* of the sink.

Solution. (*i*) $T_H = 500\text{K}$; $T_L = 100\text{ K}$

$$\therefore \quad \eta = 1-\frac{T_L}{T_H}=1-\frac{100}{500}=0.8$$

(*ii*) $\eta = 0.8$; $T_H = 900$ K ; $T_L = T$ K

Now $$\eta = 1-\frac{T_L}{T_H} \quad \text{or} \quad 0.8=1-\frac{T}{900} \quad \therefore \quad T = \mathbf{180\ K}$$

Example 23.26. How much energy in watthour may be required to convert 2 kg of water into ice at 0°C, assuming that the refrigerator is ideal? Given temperature of freezer is –15°C, room temperature is 25°C and initial temperature of water is 25°C.

Solution. $T_H = 25 + 273 = 298$ K ; $T_L = -15 + 273 = 258$ K

Specific heat of water, $c = 4.2 \times 10^3$ J kg^{-1} °C^{-1}

Latent heat of ice, $L = 3.36 \times 10^5$ Jkg^{-1}

Mass of water, $m = 2$ kg

Amount of heat to be removed from 2 kg of water at 25°C to change it into ice at 0°C is

$$Q_L = mc\,\Delta T + mL = 2 \times 4.2 \times 10^3(25–0) + 2 \times 3.36\times 10^5 = 8.82 \times 10^5\text{J}$$

If Q_H is the heat rejected to the surroundings, then,

$$Q_H = Q_L \times \frac{T_H}{T_L} = 8.82\times10^5 \times \frac{298}{258} = 10.15\times10^5\text{J}$$

Energy supplied to convert water into ice is

$$W = Q_H - Q_L = (10.15 - 8.82)\times 10^5 = 1.33 \times 10^5\text{ J}$$

$$= \frac{1.33\times10^5}{3600} = \mathbf{36.96\ Wh} \qquad [\because 1\text{Wh} = 3600\text{ J}]$$

Example 23.27. Five moles of an ideal gas are taken in a Carnot engine working between 100°C and 30°C. The useful work done in one cycle is 420 J. Calculate the ratio of the volume of the gas at the end and beginning of the isothermal expansion. Take R = 8.4 J mol^{-1}K^{-1}.

Solution. $T_H = 100 + 273 = 373$ K; $T_L = 30 + 273 = 303$ K

Amount of gas, $n = 5$ moles

Useful work done per cycle, $W = Q_H - Q_L = 420$ J ...(*i*)

Now $$\frac{Q_H}{Q_L} = \frac{T_H}{T_L} = \frac{373}{303} \quad \therefore \quad Q_H = \frac{373}{303}Q_L$$

Putting $Q_H = \frac{373}{303}Q_L$ in eq. (*i*), we get, $\frac{373}{303}Q_L - Q_L = 420$ J $\quad \therefore \quad Q_L = 1818\text{J}$

$\therefore \quad Q_H - 1818 = 420$ or $Q_H = 420 + 1818 = 2238$ J

When the gas is carried through Carnot cycle, the heat absorbed Q_H during isothermal expansion is equal to the work done by the gas. If V_1 and V_2 are the volumes of the gas at the beginning and at the end of the isothermal expansion, then,

$$Q_H = 2.303nRT_H \log_{10}\frac{V_2}{V_1}$$

or $$2238 = 2.303 \times 5 \times 8.4 \times 373 \log_{10}\frac{V_2}{V_1}$$

or $$\log_{10}\frac{V_2}{V_1} = \frac{2238}{2.303\times5\times8.4\times373} = 0.0620 \quad \therefore \quad \frac{V_2}{V_1} = \mathbf{1.153}$$

PROBLEMS FOR PRACTICE

1. A Carnot engine has the same efficiency between (*i*) 100 K and 500 K and (*ii*) between T K and 900 K. Calculate the temperature of the sink. **[180 K]**
2. At the end of a cycle, 17 J are extracted from a Carnot engine operating between 375°C and 150°C. How many joules were taken in during each cycle? How much work was done ? **[26 J ; 9 J]**
3. A Carnot engine absorbs 1000 J of heat from a reservoir at 127°C and rejects 600 J of heat during each cycle. Calculate (*i*) the efficiency of the engine (*ii*) temperature of the sink and (*iii*) useful work done during each cycle. **[(*i*) 40% (*ii*) – 33°C (*iii*) 400 J]**
4. What is the maximum efficiency of an engine for which the ratio of the energy exhausted to the work done is 0.25? **[80%]**
5. A Carnot engine converts one-sixth of heat input into work. When the temperature of the sink is reduced by 62°C, its efficiency is doubled. Find the temperature of the source and the sink. **[372 K ; 310 K]**
6. An engine does 2000 J of work and exhausts 2.25 kcal in each cycle while operating between 700°C and 400°C. What is its actual efficiency? What is its maximum theoretical efficiency? **[17.5%, 30.8%]**
7. Five moles of an ideal gas are taken in a Carnot engine working between 100°C and 30°C. The useful work done in one cycle is 420 J. Calculate the ratio of the volumes of the gas at the end and beginning of the isothermal expansion. R = 8.4 J mol^{-1} K^{-1}. **[1.1535]**
8. What is the efficiency of an ideal engine operating between 275°C and 0°C? What is the efficiency if the engine operates over the same temperature difference but with input and output temperatures both raised by 100°C? **[50.2% ; 42.4%]**
9. What is the minimum theoretical temperature that can be maintained in a refrigerator with a coefficient of performance of 5 when operated in a room where the temperature is 16°C? **[– 32°C]**
10. How much energy in watthours may be required to convert 2 kg of water into ice at 0°C, assuming that the refrigerator is ideal? Given temperature of freezer is – 15°C, room temperature is 25°C and the initial temperature of water is 25°C. **[37.98 Wh]**

CONCEPTUAL QUESTIONS

Q.1. What is the foundation of thermodynamics?

Ans. The foundation of thermodynamics is the law of conservation of energy and the fact that heat flows from a hot body to a cold body and not the other way around.

Q.2. 100 J of energy is added to a system that does 40 J of external work. By how much the internal energy of the system is raised?

Ans. According to first law of thermodynamics,

$$Q = \Delta U + W$$

or $$\Delta U = Q - W = 100 - 40 = 60 \text{ J}$$

Q.3. What would be the ideal efficiency of an engine if its hot reservoir and exhaust were at the same temperature–say 400K?

Ans. $$\eta = 1 - \frac{T_L}{T_H} = 1 - \frac{400}{400} = 1 - 1 = 0$$

Therefore, no work output is possible for any heat engine unless a temperature difference exists between the hot reservoir and heat sink.

Q.4. Give a common example of adiabatic process.

Ans. A common example of adiabatic process is the compression and expansion of gases in the cylinders of an automobile engine. Compression and expansion occur very rapidly so that energy cannot leave the combustion chamber.

Q.5. Milk is poured into a cup of tea and is mixed with a spoon. Is this an example of reversible process?

Ans. When milk is mixed in tea, certain amount of work is done on the system which appears in the form of heat. The milk cannot be separated from tea with the recovery of same work from heat. Hence this process is not reversible.

Q.6. Is it possible that there is a change in the temperature of a body without giving/ taking heat to/from it?

Ans. Yes in an adiabatic process. During adiabatic change, no heat is given or taken from the system but the temperature changes. In adiabatic compression, the temperature increases while in adiabatic expansion, the temperature falls.

Q.7. How is Carnot cycle operated?

Ans. A Carnot cycle operates as follows :

(*i*) Receive thermal energy isothermally from some hot reservoir maintained at a constant high temperature T_H.

(*ii*) Reject thermal energy isothermally to a constant low-temperature reservoir (T_L).

(*iii*) Change temperature in reversible adiabatic processes.

Such a cycle, which consists of two isothermal processes bounded by two adiabatic processes, is called Carnot cycle.

Q.8. The internal energy of a compressed gas is less than that of the rarified gas at the same temperature. Why?

Ans. In a compressed (real) gas, the mutual attraction between the molecules increases as the molecules come close. Therefore, potential energy is added to the internal energy. Since the potential energy is negative, the total internal energy of the gas decreases. The kinetic energy of the molecules is the same as both are at the same temperature.

Q.9. What happens to the air temperature in a valley when cold air blowing across the mountain tops descends into the valley?

Ans. As the cold air blowing across the mountain tops descends into the valley, it is adiabatically compressed. Consequently, the temperature in the valley is increased.

Q.10. A box with rigid insulating walls is divided into two parts by a partition. An ideal gas occupies half the box and the other half is completely evacuated. The partition is suddenly removed. Will there be any temperature change in the gas?

Ans. The temperature of the gas remains unchanged. It can be easily explained. According to the first law of thermodynamics :

$$Q = \Delta U + W = \Delta U + PdV$$

Here $Q = 0$ (adiabatic process) and $P = 0$ because the gas expands in vacuum. Since the gas is ideal, there is no internal potential energy. Therefore, internal kinetic energy of the gas is unchanged and so is the temperature.

Q.11. Absolute zero is not the temperature of zero energy. Explain.

Ans. The total energy of a gas is the sum of kinetic energy and potential energy of its molecules. Only the kinetic energy is a function of the temperature of the gas. Hence at absolute zero, the kinetic energy of the molecules ceases but potential energy is not zero. For this reason, absolute zero temperature is not the temperature of zero energy.

Q.12. What happens to the internal energy of a gas during (i) isothermal expansion (ii) adiabatic expansion?

Ans. According to first law of thermodynamics,

$$Q = \Delta U + W$$

(*i*) In an isothermal process, temperature remains constant so that $\Delta U = 0$. Hence in isothermal expansion, internal energy remains constant and heat supplied to the gas is equal to the work done by the gas.

(*ii*) In an adiabatic process, no heat enters or leaves the system so that $Q = 0$. Therefore, $\Delta U = -W$. As the gas expands adiabatically, work is done by the gas and its internal energy as well as temperature decreases. The decrease in internal energy of the gas is equal to the work done by the gas.

Q.13. A heat engine with 100% efficiency is only a theoretical possibility. Explain.

Ans. The efficiency of an ideal engine (Carnot engine) is given by ;

$$\eta = 1 - \frac{T_L}{T_H}$$

For 100% efficiency, $\eta = 1$ so that $T_L = 0$ K. Since it is not possible to have $T_L = 0$K (temperature of sink), an ideal engine has less than 100% efficiency. A practical heat engine associated with various losses will be even far less efficient.

Q.14. If the door of a refrigerator is kept open in a room, will it make the room warm or cool?

Ans. A refrigerator is a heat engine that operates in the reverse direction *i.e.*, it extracts heat from a cold body and transfers it to a hot body. No refrigerator is 100% efficient. Thus it exhausts more heat into the room than it extracts from it. Therefore, the net effect is an increase in temperature of the room.

Q.15. Heat cannot be completely converted into mechanical work. Why?

Ans. The thermal energy possessed by a body is the energy of the random motions of its molecules. This is quite distinct from, say, the kinetic energy the body has when it is moving. The kinetic energy of a moving body represents the ordered motion which its molecules have superimposed on their random motion. When we try to convert heat into work, we are trying to change the random molecular motion into ordered motion. The reason that we cannot accomplish this with 100% efficiency is that we cannot control the individual motions of the molecules.

Q.16. Are melting and boiling isothermal changes?

Ans. Yes. Melting process is an isothermal change because temperature of the substance remains the same during this process. For the same reason, boiling is an isothermal change.

Q.17. What is the specific heat in an isothermal change?

Ans. In an isothermal change, the specific heat is infinite.

Now, $c = \frac{Q}{m\Delta T}$. Since in an isothermal change, temperature is constant so $\Delta T = 0$.

$$\therefore \quad c = \frac{Q}{m \times 0} = \infty$$

Q.18. What is the specific heat in an adiabatic change?

Ans. In an adiabatic change, specific heat is zero. Now, $c = \frac{Q}{m\Delta T}$. Since in an adiabatic change, no heat can enter or leave the system, $Q = 0$.

$$\therefore \quad c = \frac{0}{m\Delta T} = 0$$

Q.19. What is the slope of an isothermal curve?

Ans. The slope of P-V curve is dP/dV. For an isothermal change,

$$PV = \text{constant} = K$$

Differentiating both sides, we get,

$$P.dV + V.dP = 0$$

$$\therefore \quad \text{Slope of curve, } \frac{dP}{dV} = \frac{-P}{V}$$

Q.20. What is the slope of an adiabatic curve?

Ans. For an adiabatic change, $PV^{\gamma} = \text{constant} = K$

Differentiating both sides, we get,

$$\gamma P V^{\gamma-1} dV + V^{\gamma} dP = 0$$

or
$$V^{\gamma} dP = -\gamma P V^{\gamma-1} dV$$

$$\therefore \quad \text{Slope of curve, } \frac{dP}{dV} = -\frac{\gamma P V^{\gamma-1}}{V^{\gamma}} = -\gamma\left(\frac{P}{V}\right)$$

Since $\gamma > 1$, the adiabatic curve at any point is steeper than the isothermal curve at that point. For this reason, isothermal curve lies above the adiabatic curve during expansion. Reverse is the case for compression.

Q.21. Give two practical examples of second law of thermodynamics.

Ans. The first law of thermodynamics is simply law of conservation of energy applied to thermodynamic processes. The second law tells us whether a given thermodynamic process can take place or not.

(*i*) **Transfer of heat.** When two bodies at different temperatures are put in thermal contact, heat flows from a body at higher temperature to a body at lower temperature till they attain a common intermediate temperature. According to second law of thermodynamics, the entropy of the system increases in the process. Therefore, heat cannot by itself flow from a colder body to a hotter body.

(*ii*) **Expansion of a gas.** Suppose you have a box with a partition dividing it in two parts with a gas on one side of the partition and other side evacuated. With all the molecules of gas on one side, the gas molecules have highly ordered situation. However, if the partition is removed, the gas molecules will distribute themselves throughout the box and be moving in random directions — a less ordered situation. In other words, the entropy of the gas has increased. The molecules of the gas, on their own, cannot go back into one half of the box. Hence the process is irreversible.

Q.22. Where do we use petrol and diesel engines?

Ans. Since petrol engine is light in weight and small in size, it is used in cars, aeroplanes, scooters etc. On the other hand, diesel engine is heavy and, therefore, it is used in heavy vehicles *e.g.* buses, trucks, tractors etc.

Q.23. Why is defrosting necessary for refrigerators?

Ans. $$COP = \frac{T_L}{T_H - T_L}$$

As the refrigerator operates, T_L goes on decreasing due to the formation of ice. However, T_H is almost constant. As a result, COP decreases. When the refrigerator is defrosted, T_L increases and hence COP increases.

VERY SHORT ANSWER QUESTIONS

Q.1. What is thermodynamics?

Ans. The study of heat and its transformation to mechanical energy is called thermodynamics.

Q.2. What do you mean by internal energy of a gas?

Ans. The sum of kinetic and potential energies of the molecules of a gas is called its internal energy.

Q.3. What is the nature of internal energy of an ideal gas?

Ans. Since there is no molecular attraction in an ideal gas, the potential energy of the molecules is zero. Therefore, the internal energy of an ideal gas is wholly kinetic in nature.

Q.4. An ideal gas is compressed at a constant temperature. Will its internal energy change?

Ans. No. It is because the internal energy of an ideal gas depends only on temperature.

Q.5. Does the internal energy of an ideal gas change in an (*i*) isothermal process (*ii*) adiabatic process?

Ans. (*i*) In an isothermal process, temperature does not change. Therefore, the internal energy of an ideal gas does not change in an isothermal process.

(*ii*) In an adiabatic process, temperature changes so that the internal energy of the ideal gas changes.

Q.6. All the heat supplied to the system goes into increasing in its internal energy. What is the process involved?

Ans. It happens during an isochoric process (*i.e.,* process at constant volume). According to first law of thermodynamics,

$$Q = \Delta U + W = \Delta U + P\Delta V$$

In isochoric process, $\Delta V = 0$ so that $Q = \Delta U$.

Q.7. A given system undergoes a change in which work done by the system equals decrease in its internal energy. What change system has undergone?

Ans. It is given that $W = -\Delta U$. Therefore $\Delta U + W = 0$ or $Q = 0$. Since no heat enters or leaves the system, the change is adiabatic.

Q.8. What is (*i*) non-cyclic (*ii*) cyclic thermodynamic process?

Ans. (*i*) A non-cyclic process is one in which the system does not return to its initial state *i.e.,* the final state of the system is different from the initial state.

(*ii*) A cyclic process is that in which the system returns to its initial state.

Q.9. What is the change in the internal energy of a system over one cycle in a cyclic process?

Ans. Zero. It is because in one complete cycle, the system returns to its initial state so that $\Delta U = 0$.

Q.10. What is the amount of work done in a cyclic process?

Ans. It is equal to the area enclosed by the closed loop in the PV diagram.

Q.11. Is it possible to increase the temperature of a gas without giving it heat?

Ans. Yes. For example, when a gas is subjected to adiabatic compression, its temperature increases.

Q.12. Can heat be added to a system without increasing its temperature?

Ans. Yes. Examples are melting and boiling processes.

Q.13. Give two examples of isothermal change.

Ans. Two examples of isothermal change are (*i*) melting process (*ii*) boiling process.

Q.14. Give two examples of adiabatic change.

Ans. (*i*) A gas contained in a non-conducting cylinder is allowed to expand quickly. (*ii*) Air escaping from the cycle tube on removing the valve.

Q.15. Can two isothermal curves intersect each other?

Ans. No. It is because intersection of two isothermal curves means that at two different temperatures, the volume and pressure of the gas are the same which is not possible.

Q.16. Represent equation of an adiabatic process in terms of (*i*) T and V (*ii*) P and T.

Ans. (*i*) $T V^{\gamma-1}$ = Constant (*ii*) $\dfrac{P^{\gamma-1}}{T^{\gamma}}$ = Constant.

Q.17. **What is the relation between heat energy, work done and change in internal energy?**

Ans. The relation is : $Q = \Delta U + W$.

Q.18. **Out of the parameters—pressure, volume, temperature and work—which parameter is not thermodynamic coordinate?**

Ans. The physical quantities which are used to specify the thermodynamic state of a system are pressure (P), volume (V) and temperature (T). Therefore, work is not a thermodynamic coordinate.

Q.19. **A gas expands from a volume of 2 m^3 to 6 m^3 at a constant pressure of 10 N/m^2. What is the work done by the gas?**

Ans. The work done W by gas at constant pressure P (= 10 N/m^2) is

$$W = P\,\Delta V = 10(6 - 2) = 40 \text{ J}$$

Q.20. **Which thermodynamic variable is defined by (*i*) zeroth law (*ii*) first law of thermodynamics ?**

Ans. (*i*) Temperature (*ii*) Internal energy.

Q.21. **What is a heat engine?**

Ans. A device which converts heat into work is called heat engine *e.g.,* petrol engine, diesel engine etc.

Q.22. **During adiabatic expansion of 2 moles of a gas, the internal energy of the gas is found to decrease by 2 J. What is the work done by the gas during the process?**

Ans. According to first law of thermodynamics, $Q = \Delta U + W$. For adiabatic process, $Q = 0$. Therefore, $0 = -2 + W$ or $W = 2$ J.

Q.23. **Why does a gas get heated on compression?**

Ans. When a gas is compressed, work is done on it. Therefore, the internal energy and hence temperature of the gas increases.

Q.24. **A piece of metal is hammered. Does its internal energy increase?**

Ans. Yes. It is because work done during hammering is converted into heat and internal energy of metal increases.

Q.25. **Can the efficiency of a heat engine be 100%?**

Ans. No. It is because $\eta = 1 - (T_L/T_H)$ and it is impossible to make $T_L = 0$ K or $T_H = \infty$ K.

SHORT ANSWER QUESTIONS

Q.1. **Is the internal energy of a gas a function of the pressure? Explain.**

Ans. The internal energy of an ideal gas depends *only* on the temperature of the gas. However, the internal energy of a real gas depends upon temperature and volume (and hence pressure).

Q.2. **Give three important differences between isothermal process and adiabatic process.**

Ans. (*i*) In an isothermal process, temperature remains constant (*i.e.,* $\Delta T = 0$) while in an adiabatic process, the heat added or removed is zero (*i.e.,* $Q = 0$).

(*ii*) In an isothermal process, there is no change in the internal energy of the system while in an adiabatic process, the internal energy of the system changes.

(*iii*) In an isothermal process, changes in pressure and volume take place slowly while in an adiabatic process, these changes take place quickly.

Q.3. **Why does the temperature of a gas decrease when it is allowed to expand adiabatically?**

Ans. According to first law of thermodynamics, $Q = \Delta U + W = \Delta U + P\,dV$.

For an adiabatic change, $Q = 0$.

$\therefore \quad 0 = \Delta U + PdV$ or $PdV = -\Delta U$

During expansion dV is positive so that PdV is positive. It means that ΔU is negative *i.e.,* internal energy of the gas decreases or temperature of the gas decreases.

Q.4. **A gas is compressed adiabatically till its temperature is doubled. Show that the ratio of final to initial volume is less than 1/2.**

Ans. For an adiabatic process (expansion or compression), $TV^{\gamma-1}$ = Constant.

$$\therefore \quad T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \quad \text{or} \quad \frac{V_2}{V_1} = \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} = \left(\frac{1}{2}\right)^{\frac{1}{\gamma-1}}$$

Since $\gamma > 1$, V_2/V_1 is less than 1/2.

Q.5. An ideal gas undergoes an adiabatic change. Show that $TV^{\gamma-1}$ = constant.

Ans. For an adiabatic change, PV^{γ} = constant. It can be written as :

$$PV \times V^{\gamma-1} = \text{constant}$$

For an ideal gas, $PV = n\,R\,T$ so that :

$$(n\,RT) \times V^{\gamma-1} = \text{constant}$$

or $$TV^{\gamma-1} = \text{constant}$$

Q.6. Show that in an adiabatic expansion, temperature must fall.

Ans. For an adiabatic process (expansion or compression), $TV^{\gamma-1}$ = constant. Therefore if expansion takes place (*i.e.,* V increases), the temperature must decrease. On the other hand, in case of adiabatic compression, the temperature must increase.

Q.7. A Carnot engine takes steam at 200°C and after doing work exhausts it to a sink at 100°C. Calculate the %age of heat which is utilised for doing useful work.

Ans. Here $T_H = 200 + 273 = 473$ K ; $T_L = 100 + 273 = 373$ K

$$\therefore \quad \eta = 1-\frac{T_L}{T_H} = 1-\frac{373}{473} = \frac{100}{473} = 0.2114 \text{ or } 21.14\%$$

Therefore, 21.14% heat is utilised for doing useful work.

Q.8. A Carnot engine absorbs 1000 J of heat energy from a reservoir at 127°C and rejects 600 J of heat energy during each cycle. Calculate (*i*) temperature of sink (*ii*) useful work done per cycle.

Ans. Here $Q_H = 1000$ J ; $T_H = 127 + 273 = 400$ K ; $Q_L = 600$ J

(*i*) $$\frac{Q_L}{Q_H} = \frac{T_L}{T_H} \text{ or } \frac{600}{1000} = \frac{T_L}{400} \quad \therefore T_L = 240 \text{ K} = -33°\text{C}$$

(*ii*) $$\text{W.D./cycle} = Q_H - Q_L = 1000 - 600 = 400 \text{ J}$$

Q.9. What is the shape of *P-V* diagram for (*i*) isobaric process (*ii*) isochoric process?

Ans. (*i*) The *P-V* diagram for an isobaric process is a straight line parallel to volume axis.

(*ii*) The *P-V* diagram for isochoric process is a straight line parallel to pressure axis.

Q.10. Can we make heat to flow from a body at lower temperature to a body at higher temperature ? Explain.

Ans. Yes. It can be done by supplying energy from an external source. For example, in a refrigerator, the working substance extracts heat from a cold body and delivers it to a hot body when certain work is done on it by the compressor. Here cold body is the food in the refrigerator and the hot body is the surroundings.

Q.11. What is a heat pump?

Ans. A device that makes it possible to cause heat to flow from a lower temperature to a higher temperature is called a *heat pump.* Examples of heat pump are : refrigerator and air conditioner.

Q.12. Why is whole of heat cannot be converted into mechanical work?

Ans. According to second law of thermodynamics, it is not possible to convert heat into mechanical work unless there is a temperature difference. Due to the temperature difference, a large part of heat is lost in the process. Hence whole of the heat cannot be converted into mechanical work.

Q.13. The temperature of the surface of the sun is about 6000 K. If we focus the sun rays by a big lens, can we produce a temperature of 8000 K?

Ans. No. This would mean that heat will flow from lower temperature (6000 K) to higher temperature (8000 K) without doing any work. This is against second law of thermodynamics. Therefore, the maximum temperature that we can produce by this method is 6000 K.

Q.14. Why is the conversion of heat into work not possible without a sink at lower temperature?

Ans. For a reversible engine, $\eta = 1-\frac{T_L}{T_H}$.

If $T_L = T_H$, when $\eta = 0$ *i.e.,* the engine shall not work. Therefore, T_L should not be equal to T_H *i.e.,* it is not possible to convert heat into work unless sink temperature T_L is less than source temperature T_H.

Q.15. What will be the efficiency of a Carnot engine operating with reservoir and sink temperatures of 100°C and –23°C respectively?

Ans. $T_H = 100 + 273 = 373$ K ; $T_L = -23 + 273 = 250$ K

$$\therefore \eta = 1 - \frac{T_L}{T_H} = 1 - \frac{250}{373} = 0.3297 \text{ or } 32.97\%$$

LONG ANSWER QUESTIONS

1. Explain (*i*) thermodynamic system (*ii*) thermodynamic state and (*iii*) state variables. **[Refer to Art. 23.3]**
2. Write a short note on the internal energy of a gas. **[Refer to Art. 23.9]**
3. State and explain first law of thermodynamics. **[Refer to Art. 23.10]**
4. State zeroth law of thermodynamics. How does it lead to the definition of temperature? **[Refer to Art. 23.2]**
5. Using first law of thermodynamics, derive the relation between C_p and C_v. **[Refer to Art. 23.11]**
6. Explain isothermal process and adiabatic process and give two examples of each. **[Refer to Arts. 23.12 and 23.14]**
7. Derive an expression for the work done by a gas in an isothermal expansion. **[Refer to Art. 23.13]**
8. Derive an expression for the work done by a gas in an adiabatic expansion. **[Refer to Art. 23.15]**
9. Compare isothermal and adiabatic processes. **[Refer to Art. 23.16]**
10. Discuss three applications of first law of thermodynamics. **[Refer to Art. 23.11]**
11. State and explain second law of thermodynamics. **[Refer to Art. 23.19]**
12. How is heat converted into mechanical work by an engine? Explain your answer with a *PV* diagram. **[Refer to Art. 23.23]**
13. Describe Carnot cycle in detail. **[Refer to Art. 23.24]**
14. Derive an expression for the efficiency of Carnot engine. **[Refer to Art. 23.25]**
15. With a neat diagram, explain the working of a refrigerator. **[Refer to Art. 23.28]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The study of heat and its transformation to mechanical energy is called thermodynamics.
2. The variables which determine the thermodynamic behaviour of a system are called state variables. If the system is a gas, then P, V and T (for a given mass) are the state variables.
3. The work done by (or on) a gas to change its volume by dV is $dW = PdV$ where P is the pressure.
4. Work and heat are not the functions of the state of a system (as are P, V, T, n and U) but depend on the type of process that takes a system from one state to another.
5. The sum of the kinetic and potential energies of the molecules of a gas is called its internal energy.
6. The internal energy of an *ideal gas* is due entirely to the kinetic energy of the molecules.
7. The first law of thermodynamics states that *the heat energy (Q) supplied to a system (e.g. gas) is equal to the increase in the internal energy (ΔU) of the system plus the work done (W) by the system i.e.*

$$Q = \Delta U + W$$

8. Two important thermodynamic processes are isothermal process and adiabatic process.
9. A process, usually of expansion or compression, of a system (e.g a gas), carried out at constant temperature is called an *isothermal process.*
10. In an isothermal process, T = constant.

$$\therefore \quad \Delta T = 0 \text{ and } \Delta U = 0$$
$$\therefore \quad Q = W = PdV$$

11. A process, usually of expansion or compression, of a system (e.g a gas), wherein no heat enters or leaves the system is called an *adiabatic process.*
12. In an adiabatic process, $Q = 0$. $\therefore \quad W = -\Delta U$
13. In an isochoric process, volume is constant *i.e.* $dV = 0$.

$$\therefore \quad W = P\,dV = P \times 0 = 0 \text{ and } Q = \Delta U$$

14. The work done during expansion of a system (e.g. a gas) from the initial state A (P_1, V_1) to the final state B (P_2, V_2) is equal to the area under the curve from V_1 to V_2 in the PV diagram.
15. The work done in taking a system (*e.g.* a gas) from one state to another depends not only on the initial and final states but also on the type of process (or path).
16. When a system (*e.g.* a gas) after passing through different states returns to its original state, then it is called a *cyclic process.*
17. When the gas (the system) does work, the work done is positive. When the work is done on the gas, the work done is negative.
18. When a gas expands, the work done by the gas (the system) is *positive. When a gas is compressed, the work done by the gas is negative.
19. When a cyclic process is represented *clockwise* on PV diagram, the work done by the system (*e.g.* a gas) is positive and vice-versa.

---

* $W = FS\cos\theta$. During expansion of gas $\theta = 0°$ and during compression $\theta = 180°$.

20. The change in internal energy of a system (*e.g.* a gas)) is independent of the path on the *PV* diagram because it is only a function of temperature.

21. In an *isothermal process* and *cyclic process,* there is no change in the internal energy of the system (*e.g.* a gas).

22. The second law of thermodynamics states that :

(*i*) no heat engine that works in a cycle completely converts heat into work.

(*ii*) when a cold body and a hot body are brought into contact with each other, heat *always* flows from the hot body to the cold body – never from the cold body to the hot body.

23. A device which converts heat into work is called a *heat engine* (petrol engines, diesel engines and steam engines are the examples).

24. All heat engines operate by taking some working substance (*e.g.* air, steam *etc)* around a cycle and :

(*i*) take in heat at a high temperature,

(*ii*) do work,

(*iii*) reject some of the heat at a lower temperature.

25. The efficiency η of all reversible heat engines is given by ;

$$\eta = \frac{T_H - T_L}{T_H} = 1 - \frac{T_L}{T_H}$$

where T_H and T_L are the kelvin temperatures at which the heat is respectively absorbed and rejected. This expression is valid for all heat engines regardless of the particular cycle and particular working substance.

26. The efficiency of a heat engine can be increased by decreasing T_L/T_H.

27. The efficiency of a heat engine is always less than 100% because it is impossible to make $T_L = 0$ K.

28. Entropy is a measure of the disorder of a system (*e.g.* a gas). When energy freely transforms from one form to another, the disorder of the system and hence its entropy increases.

29. Increase in entropy, $\Delta S = \frac{Q}{T}$

where Q is heat absorbed by the system at constant temperature T measured in kelvin.

30. For an adiabatic change, Q = 0, so $\Delta S = 0$.

31. When a substance is taken through a complete cycle, the net change in entropy is zero.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A scientist says that the efficiency of his heat engine which works at source temperature of 127°C and sink temperature 27°C is 26%. **[CBSE PMT 2001]**

(*a*) It is impossible

(*b*) It is possible but less probable

(*c*) It is quite possible

(*d*) Data is incomplete

2. If heat of 110 J is added to a gaseous system, the increase in internal energy is 40 J. Then the amount of external work done is **[AFMC Pune 1999]**

(*a*) 150 J (*b*) 110 J

(*c*) 70 J (*d*) 40 J

3. Which of the following is not a thermodynamic coordinate? **[AIIMS 2001]**

(*a*) *P* (*b*) *V*

(*c*) *T* (*d*) *R*

4. The *W*/*Q* of a Carnot engine is 1/6. Now the temperature of the sink is reduced by 62°C,

then this ratio becomes twice. The initial temperature of the sink and source are respectively **[CBSE – PMT 1999]**

(*a*) 33°C, 67°C (*b*) 37°C , 99°C

(*c*) 67°C, 33°C (*d*) 9K, 37 K

5. A sample of gas expands from volume V_1 to V_2. The amount of work done by the gas is greatest when the expansion is **[AFMC-Pune 1998]**

(*a*) isothermal (*b*) isobaric

(*c*) adiabatic (*d*) equal in all cases

6. Air in a cylinder is suddenly compressed by a piston which is then maintained at the same position. After some time, the **[AIIMS 2000]**

(*a*) pressure will increase

(*b*) pressure will decrease

(*c*) pressure will remain the same

(*d*) pressure becomes zero

7. Initial pressure and volume of a gas are P and V respectively. First the gas is expanded isothermally to $4V$ and then its volume is made V by adiabatic process. Its final pressure ($\gamma = 1.5$) is **[CBSE PMT 1999]**

(*a*) 8 P (*b*) 4 P

(*c*) P (*d*) 2 P

8. An ideal gas is compressed adiabatically to 8/27 times of its present value at 27°C. The temperature of the gas becomes ($\gamma = 5/3$) **[CBSE PMT1999]**

(*a*) 475°C (*b*) 275°C

(*c*) 402°C (*d*) 175°C

9. In an adiabatic process, the quantity which remains constant is **[AIIMS 1999]**

(*a*) volume

(*b*) pressure

(*c*) temperature

(*d*) total heat of the system

10. We consider a thermodynamic system. If ΔU represents the increase in its internal energy and W the work done by the system, which of the following statement is true? **[CBSE PMT 1998]**

(*a*) $\Delta U = -W$ is an isothermal process

(*b*) $\Delta U = W$ is an isothermal process

(*c*) $\Delta U = -W$ is an adiabatic process

(*d*) $\Delta U = W$ is an adiabatic process

11. The efficiency of a Carnot engine operating with reservoir temperatures of 100°C and –23°C will be **[CBSE PMT 1997]**

(*a*) $\frac{100+23}{100}$ (*b*) $\frac{100-23}{100}$

(*c*) $\frac{373+250}{373}$ (*d*) $\frac{373-250}{373}$

12. A diatomic gas initially at 18°C is compressed adiabatically to one-eighth of its original volume. The temperature after compression will be **[CBSE PMT 1996]**

(*a*) 18°C (*b*) 887.4°C

(*c*) 395.4°C (*d*) 144 °C

13. An ideal gas undergoing adiabatic change has which of the following pressure-temperature relation: **[CBSE PMT 1996]**

(*a*) $P^{\gamma-1}T^{\gamma}$ = constant (*b*) $P^{\gamma}T^{\gamma+1}$ = constant

(*c*) $P^{\gamma}T^{\gamma-1}$ = constant (*d*) $P^{1-\gamma}T^{\gamma}$ = constant

14. A gas expands from a volume of 2 m^3 to 6 m^3 at constant pressure of 10 N/m^2 and then at constant volume, the pressure is changed from 10 N/m^2 to 20 N/m^2. The total work done by the gas is **[MP PMT 1998]**

(*a*) 40 J (*b*) 100 J

(*c*) 60 J (*d*) 240 J

15. An ideal Carnot engine whose efficiency is 40% receives heat at 500 K. If its efficiency is 50%, then the intake temperature for the same exhaust temperature is **[CBSE PMT 1995]**

(*a*) 600 K (*b*) 700 K

(*c*) 800 K (*d*) 900 K

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*a*)	**2.** (*c*)	**3.** (*d*)	**4.** (*b*)	**5.** (*b*)
6. (*b*)	**7.** (*d*)	**8.** (*c*)	**9.** (*d*)	**10.** (*c*)
11. (*d*)	**12.** (*c*)	**13.** (*d*)	**14.** (*a*)	**15.** (*a*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. $T_H = 127 + 273 = 400$ K ; $T_L = 27 + 273 = 300$ K

Efficiency of heat engine, $\eta = \left(1 - \frac{T_L}{T_H}\right) \times 100 = \left(1 - \frac{300}{400}\right) \times 100 = 25\%$

The mathematical value is less than that claimed by the scientist. Therefore, it is **impossible.**

2. Heat added to system, $Q = 110$ J

Increase in internal energy, $\Delta U = 40$ J

According to first law of thermodynamics, we have,

$$Q = \Delta U + W \text{ or } 110 = 40 + W \quad \therefore W = 110 - 40 = \mathbf{70\ J}$$

3. The physical quantities which are used to specify the state of a thermal system are called thermodynamic coordinates *e.g.* pressure (P), volume (V) and temperature (T). Therefore, R (molar gas constant) is not a thermodynamic coordinate.

4. Efficiency of Carnot engine, $\eta = \frac{W}{Q} = 1 - \frac{T_L}{T_H}$

For the first case, $1/6 = 1 - \frac{T_L}{T_H}$

For the second case, $2/6 = 1 - \frac{T_L - 62}{T_H} = 1 - \frac{T_L}{T_H} + \frac{62}{T_H}$

or $$2/6 = \frac{1}{6} + \frac{62}{T_H} \quad \left(\because 1 - \frac{T_L}{T_H} = 1/6\right)$$

$\therefore$ $$T_H = 372 \text{ K} = 99^\circ\text{C} \text{ and } T_L = 310\ K = 37^\circ\text{C}$$

5. Fig. 23.28 shows the PV diagram for the three processes. It is clear that work done by the gas is maximum for isobaric expansion.

6. The sudden compression causes heating and rise in temperature. Later the temperature falls and pressure decreases (P/T = constant).

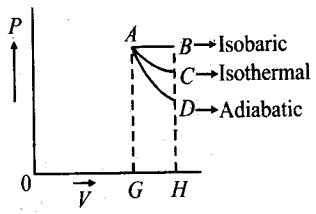


Fig. 23.28

7. When the gas is expanded isothermally, we have,

$P_1V_1 = P_2V_2$ or $P \times V = P_2 \times 4V \therefore P_2 = P/4$

When the volume of the gas is made V adiabatically, we have,

$$P_2V_2^\gamma = P_3V_3^\gamma \text{ or } \frac{P}{4} \times (4V)^\gamma = P_3 \times (V)^\gamma$$

$\therefore$ $$P_3 = \frac{P}{4} \times \frac{4^\gamma \times V^\gamma}{V^\gamma} = \frac{P}{4} \times 4^{1.5} = \mathbf{2P}$$

8. In adiabatic expansion, $P_1 V_1^\gamma = P_2 V_2^\gamma$.

According to ideal gas equation, $PV = RT$.

$\therefore$ $$\frac{RT_1}{V_1}V_1^\gamma = \frac{RT_2}{V_2}V_2^\gamma \text{ or } T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$$

$$\therefore \quad \frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = \left(\frac{V_1}{8/27\,V_1}\right)^{\frac{5}{3}-1} = \left(\frac{27}{8}\right)^{2/3} = \frac{9}{4}$$

or $$T_2 = T_1 \times \frac{9}{4} = (273 + 27) \times \frac{9}{4} = 675K = 402°C$$

9. In an adiabatic process, no heat enters or leaves the system. Therefore, the total heat of the system remains constant.

10. According to first law of thermodynamics, $Q = \Delta U + W$

For an adiabatic process, $Q = 0 \therefore 0 = \Delta U + W$ or $\Delta U = -W$

11. Temperature of source, $T_H = 100 + 273 = 373$ K

Temperature of sink, $T_L = -23 + 273 = 250$ K

Efficiency of Carnot engine, $\eta = \frac{T_H - T_L}{T_H} = \frac{373 - 250}{373}$

12. For adiabatic compression, $TV^{\gamma-1} =$ Constant.

Now $T_1 = 18 + 273 = 291$ K ; $V_1 = V$; $V_2 = V/8$; $T_2 = ?$

$$\therefore \quad T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

or $$T_2 = T_1 \times \left(\frac{V_1}{V_2}\right)^{\gamma-1} = 291 \times (8)^{0.4} = 668.4 \text{ K} = 395.4 °C$$

13. For adiabatic change, $PV^{\gamma} =$ constant. According to perfect gas equation, $PV = RT$ or $V = RT/P$.

$\therefore P \times \left(\frac{RT}{P}\right)^{\gamma} =$ constant or $P^{1-\gamma}T^{\gamma} =$ constant.

14. Work done by gas at constant pressure P (= 10N/m^2) is

$W_1 = P(V_2 - V_1) = 10\,(6 - 2) = 40$ J

Work done by gas at constant volume, $W_2 = 0$

$\therefore$ Total work done by gas, $W = W_1 + W_2 = 40 + 0 = 40$ J

15. Efficiency of Carnot engine, $\eta = 1 - \frac{T_L}{T_H}$

For the first case, $0.4 = 1 - \frac{T_L}{500} \quad \therefore \quad T_L = (1 - 0.4) \times 500 = 300$ K

For the second case, $0.5 = 1 - \frac{300}{T_H'} \quad \therefore \quad T_H' = \frac{300}{1 - 0.5} = 600$ K

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. An ideal gas is taken around ABC as shown in the PV diagram in Fig. 23.29. The work done during the cycle is **[Karnataka CET 2001]**

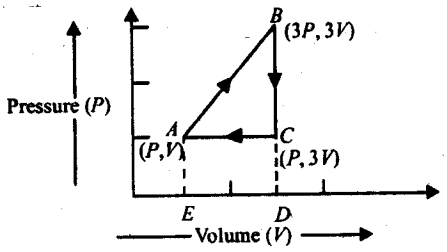


Fig. 23.29

(a) $2PV$ (b) PV
(c) $PV/2$ (d) zero

2. The internal energy of a gas during isothermal expansion **[Karnataka CET 2001]**
(a) increases (b) decreases
(c) becomes zero (d) remains constant

3. A given system undergoes a change in which work done by the system equals decrease in its internal energy. The system must have undergone an **[CEE 1996]**
(a) isothermal change (b) adiabatic change
(c) isobaric change (d) isocheric change

4. A gas is compressed adiabatically till its temperature is doubled. The ratio of its final volume to initial volume is **[BHU 1997]**
(a) 1/2 (b) more than 1/2
(c) less than 1/2 (d) between 1 and 2

5. Which of the following possesses the largest internal energy? **[SCRA 1998]**
(a) 2 moles of helium occupying $1 m^3$ at 300 K
(b) 56 g of nitrogen at 107 N/m^2 at 300 K
(c) 8 g of oxygen at 8 atm at 300 K
(d) 6×10^{26} molecules of argon occupying 40 m^3 at 900 K

6. The ratio of slopes of P-V graphs of adiabatic and isothermal processes is **[Karnataka CET 1998]**
(a) γ (b) $1/\gamma$
(c) $1 + \gamma$ (d) $1 - \gamma$

7. The specific heat of a gas in an isothermal process is **[MP PET 1998]**
(a) infinite (b) zero
(c) negative (d) remains constant

8. Air 0.5 m^3 in volume is expanded three times at two atmospheric pressure. What will be the external work done? (1 atmospheric pressure = 10^5 Nm^{-2}) **[Haryana CET 2001]**
(a) 2×10^{-5} J (b) 5000 J
(c) 500 J (d) 2×10^5 J

9. When a gas expands adiabatically : **[Haryana CET 2000]**
(a) law of conservation of energy does not hold.
(b) internal energy of the gas is used in doing work.
(c) no energy is required for expansion.
(d) energy is required and comes from the walls of the container of the gas.

10. During adiabatic expansion of 2 moles of a gas, the internal energy of the gas is found to decrease by 2 J. The work done by the gas during the process is **[Haryana CET 1996]**
(a) – 2 J (b) – 1 J
(c) 2 J (d) 1 J

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (a)	**2.** (d)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (a)	**8.** (d)	**9.** (b)	**10.** (c)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Work done during the cycle = Area $ABCA = \frac{1}{2} \times AC \times BC$

$$= \frac{1}{2} \times 2V \times 2P = 2\ PV$$

2. Internal energy of a gas is a function of temperature. Since in isothermal process (expansion or compression), temperature remains constant, the internal energy of the gas during isothermal expansion remains constant.

3. As per question, $W = -\Delta U$ or $\Delta U + W = 0$ *i.e.* $Q = 0$. Since no heat enters or leaves the system, the change is adiabatic
4. For an adiabatic process (expansion or compression), $TV^{\gamma-1}$ = constant.

$$\therefore \quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \text{ or } \frac{V_2}{V_1} = \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} = \left(\frac{1}{2}\right)^{\frac{1}{\gamma-1}}$$

Since $\gamma > 1$, V_2 / V_1 is less than 1/2.

5. Internal energy, U = No. of moles × no. of degrees of freedom × $\frac{1}{2}RT$. Out of the given options, the product of no. of moles (2000), degrees of freedom (3) and T (= 900 K) is largest for argon gas.
6. The slope of P–V graph is dP/dV.

For an isothermal process (PV = constant), $\frac{dP}{dV} = \frac{P}{V}$

For an adiabatic process (PV^{γ} = constant), $\frac{dP}{dV} = \gamma\frac{P}{V}$

Therefore, the ratio of adiabatic slope to isothermal slope is γ.

7. Specific heat, $c = \frac{Q}{m\Delta T}$. In an isothermal process, temperature remains constant (*i.e.* ΔT = 0). Therefore, c is infinite.
8. Work done = $P\ (V_2 - V_1) = 2 \times 10^5\ (3 \times 0.5 - 0.5) = 2 \times 10^5$ J
9. According to first law of thermodynamics, $Q = \Delta U + W$. For an adiabatic process, $Q = 0$. Therefore, $0 = \Delta U + W$ or $W = -\Delta U = U_1 - U_2$.
10. $Q = \Delta U + W$. For an adiabatic process, $Q = 0$. Therefore, $0 = -2 + W$ or $W = 2$ J.

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **Consider the cyclic process *ABCA* on a sample of 2.0 mol of an ideal gas as shown in Figure 23.30. The temperature of the gas at *A* and *B* are 300 K and 500 K respectively. A total of 1200 J heat is withdrawn from the sample. Find the work done by the gas in part *BC*. Take *R* = 8.3J/mol. K.**

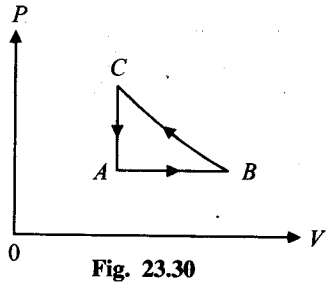


Fig. 23.30

Hint. The change in internal energy during the cyclic process is zero. Therefore, heat supplied to the gas is equal to work done by it.

$$\therefore \quad W_{AB} + W_{BC} + W_{CA} = -1200 \text{ J}^*$$

The work done during the process AB is

$$W_{AB} = P_A\ (V_B - V_A) = nR\ (T_B - T_A)$$
$$= 2 \times 8.3\ (500 - 300) = 3320 \text{ J}$$

Since in this process, the volume increases, the work done by the gas is positive.

Now $W_{CA} = 0$ ($\because$ volume of gas remains constant)

$$\therefore \quad 3320 + W_{BC} + 0 = -1200$$

or $$W_{BC} = -1200 - 3320 = -4520 \text{ J}$$

2. **A cylinder contains 3 moles of oxygen at a temperature of 27° C. The cylinder is provided with a frictionless piston which maintains a constant pressure of 1 atm on the gas. The gas is heated unless its temperature rises to 127°C.**

* Since the cyclic process is traced anticlockwise, the net work done by the system is negative.

(*i*) How much heat is supplied to the gas?

(*ii*) What is the change in internal energy of the gas?

(*iii*) How much work is done by the gas in the process?

For oxygen, C_p= 7.03 cal mol^{-1} °C^{-1}

Hint. (*i*) Heat supplied to the gas is given by :

$$Q = nC_p \Delta T$$

Here n = 3; C_p= 7.03 cal mol^{-1} °C $^{-1}$; ΔT = 127 – 27 = 100°C

$$\therefore \quad Q = 3 \times 7.03 \times 100 = 2109 \text{ cal}$$

(*ii*) Increase in internal energy of the gas is given by ;

$$\Delta U = nC_v \Delta T$$

Here n = 3; $C_v = C_p/\gamma$ = 7.03/1.4 cal mol^{-1} °C^{-1}; ΔT = 100°C

$$\therefore \quad \Delta U = 3 \times \frac{7.03}{1.4} \times 100 = 1506 \text{ cal.}$$

(*iii*) According to first law of thermodynamics,

$$Q = \Delta U + W$$

$\therefore$ Work done by gas, $W = Q - \Delta U$= 2109 – 1506= 603 cal

$$= 603 \times 4.186\text{J} = 2524 \text{ J}$$

3. **What amount of heat is to be transferred to nitrogen in an isobaric heating so that the gas may perform 2 J of work?**

Hint. Internal energy of gas is given by ;

$$U = \frac{m}{M} C_v T = \frac{m}{M} \frac{R}{\gamma - 1} \times T \qquad \left[\because C_v = \frac{R}{\gamma - 1}\right]$$

$$\therefore \quad U = \frac{PV}{\gamma - 1}$$

or $$\Delta U = \frac{P \times \Delta V}{\gamma - 1} \quad \text{at constant pressure}$$

or $$\Delta U = \frac{W}{\gamma - 1} \qquad (\because P \times \Delta V = W)$$

According to first law of thermodynamics,

$$Q = \Delta U + W = \frac{W}{\gamma - 1} + W = \frac{\gamma W}{\gamma - 1}$$

Here W = 2 J ; γ = 1.4

$$\therefore \quad Q = \frac{1.4 \times 2}{1.4 - 1} = \frac{2.8}{0.4} = 7 \text{ J}$$

4. **As a result of isobaric heating by ΔT = 72 K, one mole of a certain ideal gas obtains an amount of heat Q = 1.60 kJ. Find (*i*) work done by the gas (*ii*) the increment in its internal energy and (*iii*) the value of γ.**

Hint. (*i*) According to first law of thermodynamics,

$$Q = \Delta U + W$$

In an isobaric process, $Q = C_p \Delta T$. Also $\Delta U = C_v \Delta T$.

$$\therefore \quad C_p \Delta T = C_v \Delta T + W$$

or $$W = (C_p - C_v)\Delta T = R \times T \qquad (\because C_p - C_v = R)$$

$$= 8.3 \times 72 = 597.6 \text{ J}$$

(*ii*) $\Delta Q = C_p \Delta T$

or $C_p = \dfrac{\Delta Q}{\Delta T} = \dfrac{1.6\times1000}{72} = 22.2\,\text{J mol}^{-1}\text{K}^{-1}$

Now $\Delta U = Q - W = 1.6 \times 1000 - 597.6 = \mathbf{1002.4\ J}$

(*iii*) $\Delta U = C_v \Delta T$

$\therefore \quad C_v = \dfrac{\Delta U}{\Delta T} = \dfrac{1002.4}{72} = 13.9\,\text{J mol}^{-1}\text{K}^{-1}$

$\therefore \quad \gamma = \dfrac{C_p}{C_v} = \dfrac{22.2}{13.9} = \mathbf{1.60}$

5. **Figure 23.31 shows a process *ABCA* performed on an ideal gas. Find the net heat given to the system during the process.**

Hint. Since the process is cyclic, the change in internal energy is zero. Therefore, the heat given to the system is equal to the work done by it. The net work done by the gas in the process *ABCA* is

$$W = W_{AB} + W_{BC} + W_{CA} \qquad ...(i)$$

Now $W_{AB} = 0$ ($\because$ volume remains constant)

During the part *BC*, temperature remains constant. This means path *BC* represents an isothermal process so that work done by the gas during this part is

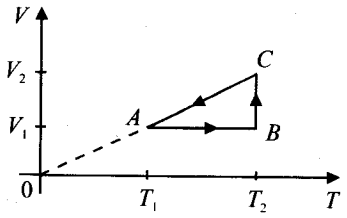


Fig. 23.31

$$W_{BC} = nRT_2 \log_e \frac{V_2}{V_1}$$

During the part *CA*, $V \propto T$ so that V/T is constant.

$\therefore \quad P = \dfrac{nRT}{V} = \text{Constant}$

$\therefore$ Work done by the gas during the part *CA* is

$$W_{CA} = P(V_1 - V_2) = nRT_1 - nRT_2$$
$$= -nR\,(T_2 - T_1)$$

$\therefore \quad W = 0 + nRT_2 \log_e \dfrac{V_2}{V_1} - nR\,(T_2 - T_1)$

or $\quad W = nR\left[T_2 \log_e \dfrac{V_2}{V_1} - (T_2 - T_1)\right]$

This is the amount of heat given to the gas.

6. **When 1000 calories of heat is added to a system, it performs 1675 joules of external work and the increase in its internal energy is 2515 joules. Find the value of J.**

Hints. According to first law of thermodynamics,

$$Q = \Delta U + W$$

Here Q= 1000 cal = 1000 × J joules ; ΔU= 2515 joules ; W = 1675 joules

$\therefore \quad 1000 \times J = 2515 + 1675$

or $\quad J = \dfrac{2515+1675}{1000} = \dfrac{4190}{1000} = \mathbf{4.19\ Joule/cal}$

7. **In a thermodynamic process, the pressure of a fixed mass of gas is changed in such a manner that the gas releases 20 J of heat and 8 J of work is done on the gas. If the initial energy of the gas was 30 J, what will be its final internal energy?**

Hint. According to first law of thermodynamics,

$$Q = \Delta U + W$$

Here $Q = -20$ J; $W = -8$ J

$\therefore$ $\Delta U = Q - W = (-20) - (-8) = -12$ J

Now $\Delta U = U_f - U_i$

$\therefore$ Final internal energy is given by ;

$$U_f = \Delta U + U_i = (-12) + (30) = \mathbf{18\ J}$$

8. **The temperature of 3 kg of nitrogen is raised from 10°C to 100°C. Calculate the heat added, the work done and the change in internal energy if the heating is (*i*) at constant volume (*ii*) at constant pressure. For nitrogen C_p = 1040 J kg^{-1}K^{-1} and C_v = 740 J kg^{-1} K^{-1}.**

Hint. (*i*) **Heating at constant volume.** At constant volume, work done is zero, *i.e.*, $W = 0$

Change in internal energy of the gas is

$$\Delta U = mc_v \Delta T = 3 \times 740 \times (100 - 10) = \mathbf{199800\ J}$$

According to first law of thermodynamics,

$$Q = \Delta U + W = 199800 + 0 = \mathbf{199800\ J}$$

(*ii*) **Heating at constant pressure**

$$Q = m\, c_p \Delta T = 3 \times 1040 \times (100 - 10) = \mathbf{280800\ J}$$

Change in internal energy is

$$\Delta U = m\, c_v\, \Delta T = \mathbf{199800\ J}$$

According to first law of thermodynamics,

$$Q = \Delta\, U + W$$

or $W = Q - \Delta\, U = 280800 - 199800 = \mathbf{81000\ J}$

9. **A cyclic process *ABCA* shown in the *V-T* diagram [See Fig. 23.32 (*i*)] is performed with a constant mass of an ideal gas. Show the same process on a *P-V* diagram. In Fig. 23.32 (*i*), *CA* is parallel to the *V*-axis and *BC* is parallel to the *T*-axis.**

Hint. Figure 23.32 (*ii*) shows the *P-V* diagram corresponding to the *V-T* diagram shown in Fig. 23.32 (*i*). During the part *AB* in *V-T* diagram, $V \propto T$ so that V/T = constant. Now $PV/T = nR$. This means that pressure P is constant during this process. This is represented by the part $A'B'$ in the *P-V* diagram.

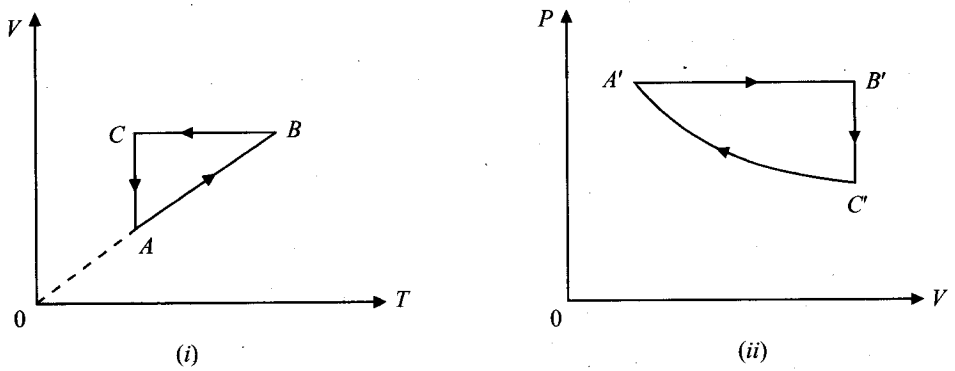


Fig. 23.32

During the part BC in the V-T diagram, the volume is constant. From equation $PV/T = nR$, this means that P/T= constant,*i.e.*, as the temperature decreases, the pressure also decreases and *vice-versa*. This is represented by the part $B'C'$ in the P-V diagram. During the part CA in the V-T diagram, the temperature remains constant so that PV= constant. In other words, P is inversely proportional to V. This is represented by $C'A'$ in the P-V diagram.

10. Let the temperatures T_1 and T_2 of the two heat reservoirs in the ideal Carnot engine be 1500°C and 500°C respectively. Which of these, increasing T_1 by 100°C or decreasing T_2 by 100°C, would result in a greater improvement in the efficiency of the engine? [Roorkee]

Hint. The efficiency of Carnot engine is

$$\eta = \frac{W}{Q_H} = 1 - \frac{T_2}{T_1}$$

Here T_1 = 1500°C = 1500 + 273 = 1773K ; T_2= 500°C = 773 K

When the temperature of source is increased by 100°C. When the temperature of source is increased by 100°C, then,

$$T_1 = 1600°C = 1873 \text{ K and } T_2 = 773 \text{ K}$$

$$\therefore \text{ Efficiency, } \eta_1 = 1 - \frac{T_2}{T_1} = 1 - \frac{773}{1873} = \frac{1100}{1873} = 0.59$$

When the temperature of sink is decreased by 100°C. When the temperature of sink is decreased by 100°C, then,

$$T_1 = 1773 \text{ K}; \; T_2 = 400°C = 673 \text{ K}$$

$$\therefore \quad \text{Efficiency, } \eta_2 = 1 - \frac{T_2}{T_1} = 1 - \frac{673}{1773} = \frac{1100}{1773} = 0.62.$$

Therefore, decreasing the temperature of sink by 100°C is more effective than increasing the temperature of source by 100°C for the improvement of the efficiency of heat engine.

11. Find the efficiency of the Carnot's engine working between the steam point and ice-point.

Hint. $$\text{Efficiency, } \eta = 1 - \frac{T_L}{T_H}$$

Here T_L= 273 + 0 = 273 K ; T_H = 273 + 100 = 373 K

$$\therefore \quad \eta = 1 - \frac{273}{373} = \frac{100}{373} = 0.2681 = \mathbf{26.81\%}$$

12. A Carnot engine, whose low temperature reservoir is at 7°C, has an efficiency of 50%. It is desired to increase the efficiency to 70%. By how many degrees, the temperature of the high-temperature reservoir be increased?

Hint. In the first case; $\eta = 0.5$; $T_L = 273 + 7 = 280$K ; T_H = ?

Now $$\eta = 1 - \frac{T_L}{T_H}$$

or $$0.5 = 1 - \frac{280}{T_H}$$

$\therefore$ $T_H = 560$ K

In the second case ; $\eta' = 0.7$; $T_L = 280$ K ; $T_H' = ?$

Now $\eta' = 1 - \frac{T_L}{T_H'}$

or $0.7 = 1 - \frac{280}{T_H'}$

$\therefore$ $T_H' = 933.3$ K

$\therefore$ Temperature of hot reservoir to be increased

$= 933.3 - 560 = 373.3$ K

13. A refrigerator is driven by a 1000 W electric motor having an efficiency of 60%. The refrigerator can be treated as a reversible heat engine operating between 0°C and 30°C (room temperature). Calculate the time required by it to freeze 32.5 kg of water which is at 0°C. Heat losses may be neglected .Given L for ice = 80,000 cal/kg and J = 4.2 J/cal.

Hint. Power of electric motor = 1000 W

$\therefore$ Net power (energy per second) available for useful work is

$$P = 1000 \times \frac{60}{100} = 600\,\text{W} = 600\,\text{J}s^{-1}$$

Heat to be drawn from cold reservoir is

$Q_L = 32.5 \times 80{,}000 = 26 \times 10^5$ cal $= 26 \times 10^5 \times 4.2$ J $= 1092 \times 10^4$ J

Now $T_H = 30 + 273 = 303$ K ; $T_L = 0 + 273 = 273$ K

Now $\frac{Q_H}{Q_L} = \frac{T_H}{T_L}$

or $Q_H = \frac{T_H}{T_L} \times Q_L = \frac{303}{273} \times 1092 \times 10^4 = 1212 \times 10^4$ J

Mechanical work to be done by compressor is

$W = Q_H - Q_L = (1212 - 1092) \times 10^4 = 120 \times 10^4$ J

Let t seconds be the required time. Then,

$P \times t = 120 \times 10^4$

$\therefore$ $t = \frac{120 \times 10^4}{P} = \frac{120 \times 10^4}{600} = 2000$ s $= 33$ min. 20 s

14. A refrigerator placed in a room at 300 K has inside temperature 264 K. How many calories of heat shall be delivered to the room for each 1 kcal of energy consumed by the refrigerator ideally?

Hint. $T_H = 300$ K ; $T_L = 264$ K ; $W = 1$ kcal

Coefficient of performance is given by ;

$$COP = \frac{T_L}{T_H - T_L} = \frac{264}{300 - 264} = \frac{22}{3}$$

Also $COP = \frac{Q_L}{W}$ $\therefore$ $Q_L = COP \times W = \frac{22}{3} \times 1 = \frac{22}{3}$ kcal

The mechanical work done by the compressor of the refrigerator is

$$W = Q_H - Q_L \quad \text{or } Q_H = Q_L + W = \frac{22}{3} + 1 = \frac{25}{3}\ \text{kcal} = 8.33\ \text{kcal}$$

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 12 : THERMODYNAMICS

Q.1. A geyser heats water flowing at the rate of 3.0 litres per minute from 27°C to 77°C. If the geyser operates on a gas burner, what is the rate of consumption of the fuel if its heat of combustion is 4.0×10^4 J/g?

Ans. Mass of 3 litres of water = 3 kg

$\therefore$ Mass of water flowing per minute, m = 3 kg min^{-1} = 3000 g min^{-1}

Rise of temperature, $\Delta\theta = 77 - 27 = 50°C$

Heat absorbed by water/minute $= m\, c\, \Delta\theta = 3000 \times 1 \times 50$ cal

$= 3000 \times 1 \times 50 \times 4.2$ J min^{-1}

$= 630000$ J min^{-1}

$\therefore$ Heat supplied by gas burner = 630000 J min^{-1}

Heat of combustion of fuel = 4.0×10^4 Jg^{-1}

$\therefore$ Rate of combustion of fuel $= \dfrac{630000}{4.0 \times 10^4} = 15.75$ g min^{-1}

Q.2. What amount of heat must be supplied to 2.0×10^{-2} kg of nitrogen (at room temperature) to raise its temperature by 45°C at constant pressure? [Molecular mass of N_2 = 28 ; R = 8.3 J mol^{-1} K^{-1}; C_p (diatomic gas) = 7/2 R]

Ans. Heat required at constant pressure is

$$Q = n\, C_p\, \Delta T$$

Here $n = \dfrac{m}{M} = \dfrac{2.0 \times 10^{-2} \times 10^3}{28} = \dfrac{20}{28}$; $C_p = \dfrac{7}{2} \times R$; $\Delta T = 45°C$

$\therefore$ $$Q = \frac{20}{28} \times \left(\frac{7}{2} \times 8.3\right) \times 45 = 934 \text{ J}$$

Q.3. Explain why

(a) Two bodies at different temperatures T_1 and T_2 if brought in thermal contact do not necessarily settle to the mean temperature $(T_1 + T_2)/2$.

(b) The coolant in a chemical or a nuclear plant (*i.e.*, the liquid used to prevent the different parts of a plant from getting too hot) should have high specific heat.

(c) Air pressure in a car tyre increases during driving.

(d) The climate of a harbour town is more temperature than that of a town in a desert at the same latitude.

Ans. (a) When two bodies at temperatures T_1 and T_2 are brought in thermal contact, heat flows from the body at higher temperature to the body at lower temperature till they reach a common temperature. The common temperature will be equal to mean temperature *i.e.*, $(T_1 + T_2)/2$ only if the two bodies have the same thermal capacities (*i.e.*, mc is the same for the two). Since two bodies may have different thermal capacities, they do not settle to the mean temperature.

(b) The heat absorbed ($Q = m\, c\, \Delta T$) by a body is directly proportional to its specific heat. Therefore, a coolant that has a high specific heat will absorb greater amount of heat from the plant. Consequently, different parts of the plant are prevented from getting too hot.

(c) During driving, the temperature of tyre and hence air inside it increases due to friction between the tyre and the road. Since the volume of the tyre practically remains constant, $P \propto T$ (pressure law). Therefore, pressure inside the tyre increases.

(d) The relative humidity in a harbour town is more than that in a desert town. For this reason, the climate of a harbour town is more temperate (*i.e.*, without extremes of hot and cold) than that of a town in a desert.

Q.4. A cylinder with a movable piston contains 3 moles of hydrogen at standard temperature and pressure. The walls of the cylinder are made of a heat insulator and the piston is insulated by having a pile of sand on it. By what factor does the pressure of the gas increase if the gas is compressed to half its original volume? Given : $\gamma = 1.4$.

Ans. Since the walls of the cylinder and the piston are insulated, it is an adiabatic process. For adiabatic change, $P V^{\gamma}$ = constant.

$$\therefore \quad P_1 V_1^{\gamma} = P_2 V_2^{\gamma} \quad \text{or} \quad \frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{\gamma}$$

Here $V_1/V_2 = 2$; $\gamma = 1.4$

$$\therefore \quad \frac{P_2}{P_1} = (2)^{1.4} = 2.64$$

Q.5. In changing the state of a gas adiabatically from an equilibrium state *A* to another equilibrium state *B*, an amount of work equal to 22.3 J is done on the system. If the gas is taken from state *A* to *B* via a process in which the net heat absorbed by the system is 9.35 cal, how much is the net work done by the system in the latter case? (Take 1 cal = 4.19 J)

Ans. For first case, the change is adiabatic so that $Q = 0$. Also as per problem, $W = -22.3$ J ($\because$ work is done on the system).

Now $\quad Q = \Delta U + W \quad \text{or} \quad 0 = \Delta U + (-22.3) \quad \therefore \quad \Delta U = 22.3$ J

For the second case, $Q = 9.35$ cal $= 9.35 \times 4.19 = 39.18$ J

Now $\quad Q = \Delta U + W \quad \text{or} \quad 39.18 = 22.3 + W \quad \therefore \quad W = 16.88$ J

Q.6. Two cylinders *A* and *B* of equal capacity are connected to each other via a stopcock. *A* contains a gas at standard temperature and pressure. *B* is completely evacuated. The entire system is thermally insulated. The stopcock is suddenly opened. Answer the following :

(*a*) What is the final pressure of the gas in *A* and *B*?

(*b*) What is the change in internal energy of the gas?

(*c*) What is the change in the temperature of the gas?

(*d*) Do the intermediate states of the system (before settling to the final equilibrium state) lie on its *P-V-T* surface?

Ans. (*a*) When the stopcock is suddenly opened, the volume available to the gas at 1 atmosphere pressure will become two times. Therefore, pressure of the gas in *A* and *B* will become one-half *i.e.*, 0.5 atmosphere.

(*b*) There will be no change in the internal energy of the gas. It is because no work is done on/by the gas.

(*c*) There will be no change in the temperature of the gas as its internal energy remains the same.

(*d*) No, since the process (rapid expansion) is rapid and cannot be controlled. Moreover, the intermediate states are non-equilibrium states and do not satisfy the gas equation. In due course, the gas does return to an equilibrium state.

Q.7. A steam engine delivers 5.4×10^8 J of work per minute and services 3.6×10^9 J of heat per minute from its boiler. What is the efficiency of the engine? How much heat is wasted per minute?

Ans. Output $= 5.4 \times 10^8$ J min^{-1} ; Input $= 3.6 \times 10^9$ J min^{-1}

$$\therefore \quad \text{Efficiency of engine, } \eta = \frac{\text{Output}}{\text{Input}} = \frac{5.4 \times 10^8}{3.6 \times 10^9} = 0.15 \text{ or } 15\%$$

Heat wasted per minute $= 3.6 \times 10^9 - 5.4 \times 10^8 = 3.06 \times 10^9$ J

Q.8. An electric heater supplies heat to a system at a rate of 100 W. If system performs work at a rate of 75 joules per second, at what rate is the internal energy increasing?

Ans. Heat supplied, $Q = 100$ W $= 100$ J s^{-1} ; Work done, $W = 75$ J s^{-1}

According to first law of thermodynamics, the increase in internal energy (ΔU) per second is

$$\Delta U = Q - W = 100 - 75 = 25 \text{ J s}^{-1}$$

Q.9. **A thermodynamic system is taken from an original state *D* to an intermediate state *E* by the linear process shown in Fig. 12.01**

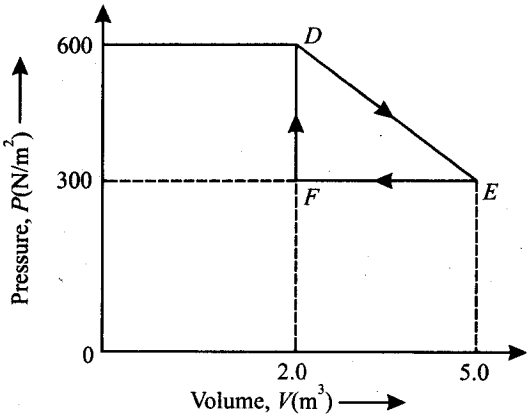


Fig. 12.01

Its volume is then reduced to the original value from *E* to *F* by an isobaric process. Calculate the total work done by the gas from *D* to *E* to *F*.

Ans. Referring to Fig. 12.01, the total work done (W) by the gas from D to E to F is

$$W = \text{Area of } \Delta DEF = \frac{1}{2} \times EF \times DF$$

Here EF = Change in volume = $\Delta V = (5.0 - 2.0)\ m^3 = 3.0\ m^3$

DF = Change in pressure = $\Delta P = (600 - 300)\ N/m^2 = 300\ N/m^2$

$$\therefore \quad W = \frac{1}{2} \times 3.0 \times 300 = 450\ J$$

Q.10. **A refrigerator is to maintain eatables kept inside at 9°C. If room temperature is 36°C, calculate the coefficient of performance.**

Ans. $T_H = 36 + 273 = 309\ K$; $T_L = 9 + 273 = 282\ K$

$\therefore$ Coefficient of performance (COP) of the refrigerator is

$$COP = \frac{T_L}{T_H - T_L} = \frac{282}{309 - 282} = \frac{282}{27} = 10.44$$

Unit X

BEHAVIOUR OF PERFECT GAS AND KINETIC THEORY

- Behaviour of Perfect Gas and Kinetic Theory
- N.C.E.R.T. Textbook Exercises

24

Behaviour of Perfect Gas and Kinetic Theory

INTRODUCTION

The concept that matter is made up of atoms or molecules which are in continuous random motion is called the **kinetic theory**. Since the theory assumes that matter consists of particles in motion, it is usually called kinetic theory. In order to keep the mathematics relatively simple, we shall consider only the molecular behaviour of gases where the interactions between molecules are much weaker than in liquids or solids. Gas laws tell us that all gases obey a simple relation among pressure, temperature and volume but they say nothing about why gases act this way. The answer to this question lies in the **kinetic theory of gases**. In the kinetic theory of gases, we explain the behaviour of gases by considering the motion of their molecules. Thus we suppose that the pressure of a gas is due to the molecules bombarding the walls of its container. This theory also provides us with a physical basis upon which the concept of temperature can be understood. Before considering the behaviour of gases on the basis of kinetic theory of gases, we briefly review the gas laws as they were determined over the years by observations and experiments.

24.1. IDEAL GAS OR PERFECT GAS

An ideal or perfect gas is that which obeys the gas laws (Boyle's law, Charles' law etc.) at all values of temperature and pressure.

In practice, the gases do not obey the gas laws at all values of pressure and temperature. It is because of the presence of intermolecular forces between the molecules of a gas. The following are the important characteristics of an ideal gas :

(*i*) The molecules of an ideal gas are point masses.

(*ii*) There are no intermolecular forces between the molecules of an ideal gas.

The above conditions cannot be realised in the real gases (*e.g.*, hydrogen, oxygen, nitrogen etc.). However, real gases *nearly* behave as perfect gases so long as the pressure and density of the gas are not too high and the gas is not too close to condensation.

24.2. GAS LAWS

The gas laws tell us what happens to a gas under various conditions. The condition of a gas is described by its volume, pressure and temperature. Therefore, gas laws reveal the relation between any of these two quantities when the third is kept constant. Accordingly, there are following three gas laws :

(*i*) **Boyle's law.** It gives the relation between pressure and volume of a given mass of a gas at constant temperature.

(*ii*) **Charles' law.** It gives the relation between volume and temperature of a given mass of a gas at constant pressure.

(*iii*) **Gay Lussac's law.** It gives the relation between pressure and temperature of a given mass of a gas at constant volume.

Since the real gases are not perfect gases, they do not obey the gas laws exactly under all conditions of pressure and temperature. However, at very low pressure and high temperature, the real gases obey the gas laws to a reasonable accuracy.

24.3. BOYLE'S LAW

This law was discovered by Robert Boyle in 1662 and is stated below :

Temperature remaining constant, the volume of a given mass of a gas is inversely proportional to its pressure i.e.,

$$V \propto \frac{1}{P} \quad \text{... at constant temperature}$$

For example, if the pressure of a given mass of a gas is doubled, its volume is reduced to one-half of the original value when the temperature is kept constant. Boyle's law can also be written as :

$$PV = \text{Constant} \quad \text{...at constant temperature}$$

Thus, Boyle's law can also be stated as under :

For a given mass of a gas at constant temperature, the product of its pressure and volume is constant.

Thus for a given mass of a gas, if pressure changes from P_1 to P_2 and the corresponding volume changes from V_1 to V_2 at constant temperature, then according to Boyle's law,

$$P_1V_1 = P_2V_2$$

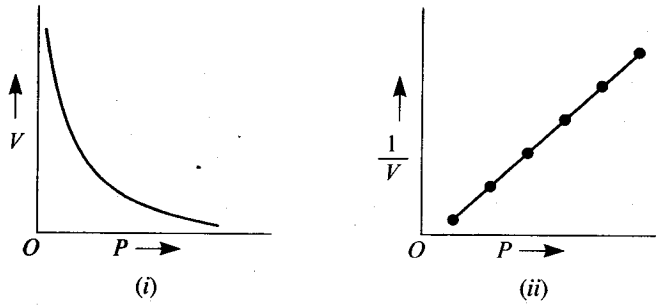


Fig. 24.1

Fig. 24.1(*i*) shows Boyle's law graphically. It is the graph between pressure (P) and volume (V) of a given mass of a gas at constant temperature. The graph is a hyperbola and shows that with the increase in pressure, there is a decrease in the volume of the gas at constant temperature. The same fact is illustrated in the graph between P and $1/V$ in Fig. 24.1(*ii*). Note that Boyle's law is perfectly obeyed at low pressure and high temperature.

24.4. CHARLES' LAW

A.C. Charles studied the effect of temperature on the volume of a given mass of a gas at constant pressure. He found that *if pressure is kept constant, the volume of a given mass of a gas increases with temperature at a nearly constant rate.* Thus if the pressure of a given mass of a gas is kept constant, and we draw the graph between volume and temperature (°C), it will be a straight line as shown in Fig. 24.2(*i*). If we extend this graph backward (dashed line), it meets the

temperature axis at –273.15°C indicating *zero volume. This means that –273.15°C is the lowest possible temperature. This is called the **absolute zero** of temperature. The temperature scale which begins at absolute zero is called **kelvin scale** of temperature. The intervals are the same as for Celsius scale but zero on this scale is –**273.15°C or 0 K. We can relabel the temperature axis so that it shows the absolute temperature as shown in Fig. 24.2(*ii*). We see from the graph in Fig. 24.2(*ii*) that the graph between volume (V) and absolute temperature (T) at constant pressure (P) is a straight line. Therefore, Charles' law can be stated as under :

Pressure remaining constant, the volume of a given mass of a gas is directly proportional to its absolute temperature i.e.,

$$V \propto T \qquad \text{...at constant } P$$

or

$$\frac{V}{T} = \text{Constant} \qquad \text{...at constant } P$$

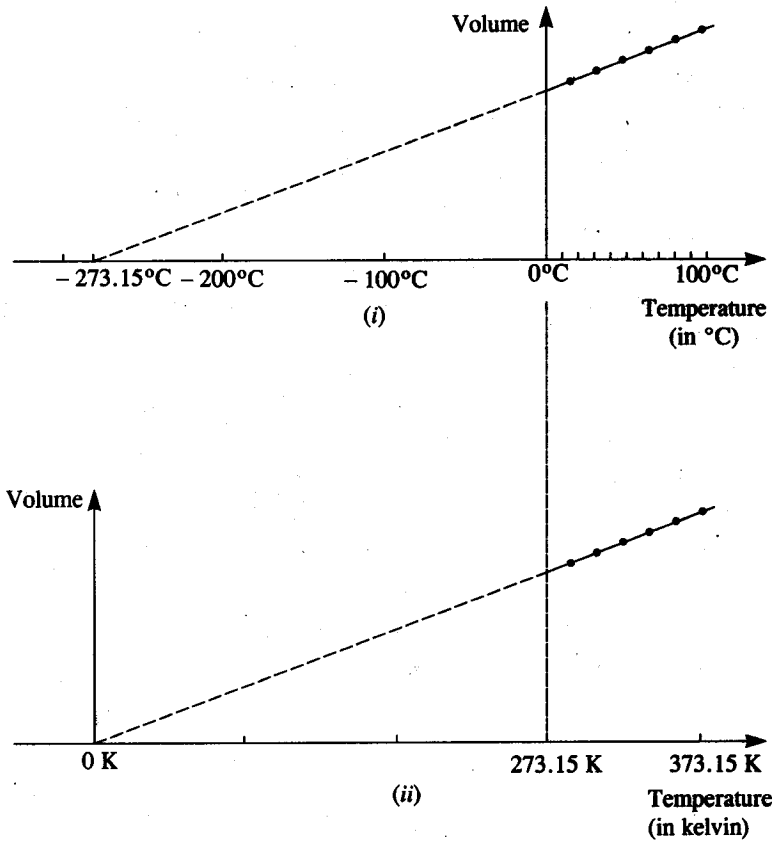


Fig. 24.2

Thus if a given mass of a gas has a volume V_1 at θ_1°C and is heated at constant pressure to θ_2°C, its new volume V_2 is given by ;

$$\frac{V_2}{V_1} = \frac{273.15 + \theta_2}{273.15 + \theta_1} = \frac{T_2}{T_1}$$

---

* This seems to imply that if a gas could be cooled to –273.15°C, it would have zero volume. Although a gas would liquefy before cooling to this temperature (–273.15°C), the graph suggests that there is a limit to how cold objects can be.

** This means 0°C becomes 273.15 kelvin (written as T = 273.15 K) and 100°C = 373.15 K.

Another statement of Charles' law. The equation of the graph in Fig. 24.2(*i*) is

$$y = mx + c$$

Here c = Intercept on Y-axis = Volume at 0°C = V_0

m = Slope of graph = $\dfrac{V_0}{273.15}$

x = Temperature θ_1°C

y = Volume V_1 of gas at temperature θ_1°C

$$\therefore \quad V_1 = \frac{V_0}{273.15}\theta_1 + V_0$$

or $$V_1 = V_0 + \left(\frac{V_0}{273.15}\right)\theta_1$$

Hence Charles' law can also be stated as under :

Pressure remaining constant, the volume of a given mass of a gas increases or decreases by $\dfrac{1}{273.15}$ *of its volume at 0°C for each 1°C rise or fall in temperature.*

24.5. GAY LUSSAC'S LAW (PRESSURE LAW)

Joseph Gay Lussac studied the effect of temperature on the pressure of a given mass of a gas at constant volume. He arrived at the following result :

Volume remaining constant, the pressure of a given mass of a gas is directly proportional to its absolute temperature i.e.,

$$P \propto T \quad \text{... at constant } V$$

or $$\frac{P}{T} = \text{Constant} \quad \text{...at constant } V$$

Thus if a given mass of gas has a pressure P_1 at θ_1°C and is heated at constant volume to θ_2°C, its new pressure P_2 is given by ;

$$\frac{P_2}{P_1} = \frac{273.15 + \theta_2}{273.15 + \theta_1} = \frac{T_2}{T_1}$$

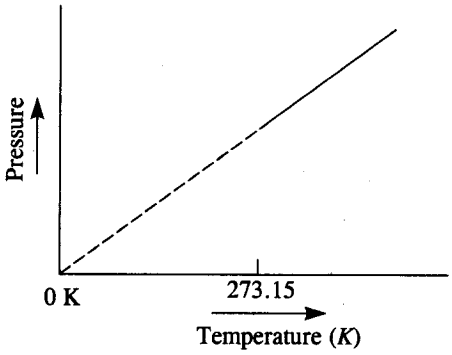


Fig. 24.3

Fig. 24.3 shows the graph between pressure (P) and absolute temperature (T) of a given mass of a gas at constant volume (V).

Another statement of Gay Lussac's Law. Referring to Fig. 24.3,

Pressure of gas at 0°C (= 273.15 K) = P_0

Slope of the graph, $m = \dfrac{P_0}{273.15}$

∴ Pressure at θ_1°C is given by ;

$$P_1 = P_0 + m\theta_1 = P_0 + \left(\frac{P_0}{273.15}\right)\theta_1$$

$$\therefore \quad P_1 = P_0 + \left(\frac{P_0}{273.15}\right)\theta_1$$

Hence Gay Lussac's law can also be stated as under :

Volume remaining constant, the pressure of a given mass of a gas increases or decreases by $\dfrac{1}{273.15}$ *of its pressure at 0°C for each 1°C rise or fall in temperature.*

24.6. PERFECT GAS EQUATION

The laws of Charles and Boyle can be combined to obtain the general relationship between pressure, volume and temperature of a given mass of gas. *This relationship is called* **perfect gas equation.**

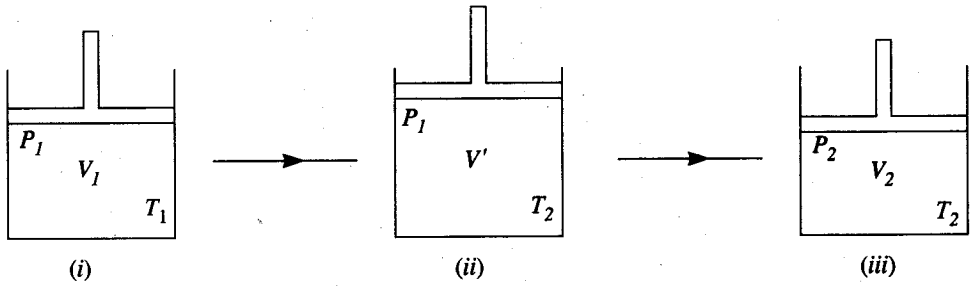


Fig. 24.4

In Fig. 24.4 (*i*), we have the gas occupying a volume V_1 at a pressure P_1 and an absolute temperature T_1. It is desired to calculate its volume V_2 at an absolute temperature T_2 and pressure P_2 as at Fig. 24.4 (*iii*). We proceed via Fig. 24.4 (*ii*), raising the temperature to T_2 while keeping the pressure constant at P_1. If V' is the volume of the gas in Fig. 24.4 (*ii*), then by Charles' law,

$$\frac{V'}{V_1} = \frac{T_2}{T_1} \qquad ...(i)$$

We proceed now to Fig. 24.4 (*iii*), by increasing the pressure to P_2 while keeping the temperature constant at T_2. By Boyle's law,

$$\frac{V_2}{V'} = \frac{P_1}{P_2} \qquad ...(ii)$$

Eliminating V' between eqs. (*i*) and (*ii*), we have,

$$\frac{V_2}{V_1} = \frac{T_2 P_1}{T_1 P_2}$$

or
$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

or
$$\frac{PV}{T} = \text{Constant}$$

or
$$PV = \text{Constant} \times T$$

or
$$PV_m = RT \quad \text{or} \quad PV = nRT \qquad ...(iii)$$

where V_m is the volume of one mole of a gas and V is the volume occupied by n moles. The constant R is called *universal gas constant* and has the same value for all gases. The experimentally measured value of R is

$$R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1}$$

The amount of gas in a sample is measured by the number of moles n given by ;

$$n = \frac{m}{M}$$

where m = mass of the gas sample

M = atomic or molecular mass of the gas

(*i*) Eq. (*iii*) is called *perfect (or ideal) gas equation.*

(*ii*) An ideal gas obeys the gas laws and $PV = n\,R\,T$ exactly. No such gas exists.

(*iii*) We use the term ideal or perfect because the real gases do not follow eq. (*iii*). However, perfect gas equation gives reasonable accuracy for many real gases of low density at temperatures well above the temperature at which the gas condenses to a liquid.

24.7. UNIVERSAL GAS CONSTANT

For n moles of a gas, $PV = n\,R\,T$

or $$R = \frac{PV}{nT} = \frac{\text{pressure} \times \text{volume}}{\text{number of moles} \times \text{temperature}}$$

$\therefore$ $$\text{Units of } R = \frac{\text{Nm}^{-2} \times \text{m}^3}{\text{Number of moles} \times K} = \frac{\text{Nm}}{\text{Number of moles} \times \text{K}}$$

$$= \text{J mol}^{-1}\,\text{K}^{-1} \qquad (\because \text{N m} = \text{J})$$

Numerical value of *R*. The fact that constant R has the same value for all gases is a remarkable reflection of the simplicity in nature. From Avogadro's hypothesis, it follows that one mole of all gases, at the same temperature and pressure, occupy equal volumes; at S.T.P. 1 mole of any gas occupies 22.4 litres. Consequently, if V_m is the volume of 1 mole, then the ratio PV_m/T is the same for all gases. So the universal gas constant, which is equal to this ratio, is the *same for all gases*.

At S.T.P., $V_m = 22.4 \text{ litres} = 22.4 \times 10^{-3}\ \text{m}^3$

$P = 760 \text{ mm of Hg} = h\,\rho\,g = (760 \times 10^{-3}) \times (13600) \times 9.8$

$= 1.013 \times 10^5\ \text{N m}^{-2}$

$T = 273.15\ \text{K}$

$\therefore$ $$R = \frac{P\,V_m}{T} = \frac{1.013 \times 10^5 \times 22.4 \times 10^{-3}}{273.15} = 8.31\ \text{J mol}^{-1}\text{K}^{-1}$$

Notes. (*i*) $$R = \frac{PV}{nT} = \frac{\text{Pressure} \times \text{Volume}}{\text{Number of moles} \times \text{Temperature}}$$

$$= \frac{\text{Work Done}}{\text{Number of moles} \times \text{Temperature}}$$

Therefore, gas constant indicates the work done by (or on) a gas per mole per kelvin.

(*ii*) If M is the molecular weight of the gas and v is the volume of 1 gram of a gas at pressure P and temperature T, then perfect gas equation for 1 gram of gas is

$$P\,v = \frac{R}{M}T$$

or $$P\,v = r\,T$$

where $r = \dfrac{R}{M}$ is gas constant for 1 gram of the gas.

Since the value of M is different for different gases, the value of r (= gas constant for 1 gram of gas) is different for different gases.

(*iii*) An ideal or perfect gas has the following properties :

(*a*) An ideal gas obeys the gas laws and $P\,V = n\,R\,T$ exactly. No such gas exists.

(*b*) The internal energy of an ideal gas is entirely translational kinetic energy and depends only on its temperature. It is because the molecules of an ideal gas are considered point masses.

Example 24.1. A flexible container of oxygen (O_2) has a volume of 10 m^3. What is the mass of the gas enclosed?

Solution. Since 1 mole occupies a volume of 22.4×10^{-3} m^3, the number of moles of oxygen in 10 m^3.

$$= \frac{10}{22.4 \times 10^{-3}} = 446 \text{ mol}$$

Since one mol of oxygen (O_2) has a mass of 0.032 kg, the mass of 446 mol of oxygen

$$= 446 \times 0.032 = \mathbf{14.3 \text{ kg}}$$

Example 24.2. An automobile tyre is filled to a gauge pressure of 200 k Pa at 10°C. After driving 100 km, the temperature within the tyre rises to 40°C. What is the pressure within the tyre now?

Solution. $$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

Since the volume remains essentially the same, $V_1 = V_2$.

$$\therefore \quad \frac{P_1}{T_1} = \frac{P_2}{T_2}$$

Since pressure given is the gauge pressure, we must add atmospheric pressure (= 101 kPa) to get absolute pressure P_1, *i.e.*, $P_1 = 200 + 101 = 301$ kPa $= 3.01 \times 10^5$ Pa.

Now $$P_2 = P_1 \times \frac{T_2}{T_1}$$

Here, $P_1 = 3.01 \times 10^5$ Pa ; $T_1 = 10 + 273 = 283$ K ; $T_2 = 40 + 273 = 313$ K

$$\therefore \quad P_2 = 3.01 \times 10^5 \times \frac{313}{283} = 3.33 \times 10^5 \text{ Pa} = 333 \text{ k Pa}$$

$$\therefore \quad \text{Gauge pressure} = P_2 - 101 = 333 - 101 = \mathbf{232 \text{ Pa}}$$

Example 24.3. **How many molecules you breathe in 1 litre breath of air?**

Solution. One mol corresponds to 22.4 litres. Therefore, number of mol in 1 litre

$$= \frac{1}{22.4} \times 1 = 0.045 \text{ mol}$$

$\therefore$ Number of air molecules in 0.045 mol

$$= (0.045) \times (6.02 \times 10^{23}) = \mathbf{2.7 \times 10^{22} \text{ molecules}}$$

PROBLEMS FOR PRACTICE

1. Air is filled at 60°C in a vessel of open mouth. Upto what temperature should the vessel be heated so that one-fourth of the air may escape? **[171°C]**
2. Molecular weight of oxygen is 32. At S.T.P., volume of 11 g of oxygen is 700 cm^3. Find the value of gas constant R. **[8.31 J mol^{-1} K^{-1}]**

24.8. KINETIC THEORY OF GASES

The *kinetic theory of gases* states that gases consist of molecules (or atoms) which are in continuous random motion. The molecules in a gas are, on the average, far apart from one another *i.e.*, their average separation is much greater than the diameter of each molecule. Therefore, gas molecules can be considered to be moving freely with respect to each other. The gas molecules collide with each other and the walls of the container. The pressure caused by these molecules is a result of collisions of the molecules with the walls of the container. The kinetic theory of gases and Newton's laws of motion are used to derive an expression for the pressure caused by a gas against the walls of its container. To complete this derivation, we will make several assumptions

that simplify our calculations without affecting the general nature of the results. These assumptions are needed because of the huge number of molecules in a gas and because of the complicated motion of the individual molecules. The **assumptions** which represent the basic postulates of the kinetic theory are :

(*i*) *A sample of gas consists of a large number of identical molecules for statistics to be applied*

(*ii*) *The molecules are very far apart in comparison to their sizes i.e., the total volume of the molecules is negligible when compared with the size of their container.*

(*iii*) *The molecules are treated as if they were hard spheres.*

(*iv*) *The molecules make perfectly elastic collisions with each other and with the walls of the container.*

(*v*) *The molecules exert no force on each other except during collisions (which are assumed to have negligible duration) and the effect of gravity is ignored so that :*

(*a*) *between collisions, the molecules move in straight lines at constant speed.*

(*b*) *the motion is random.*

(*vi*) *The molecules obey Newton's laws of motion.*

In fact, these assumptions mean that the **gas is ideal one**.

24.9. PRESSURE EXERTED BY A GAS

The pressure caused by a gas is the result of collisions of gas molecules with the walls of the container. When molecules collide with the walls, they transfer momentum to the walls. The total change in momentum per second is the force exerted on the walls by the gas. Then pressure on the walls is force divided by area of the wall. We neglect collisions the molecules make with each other.

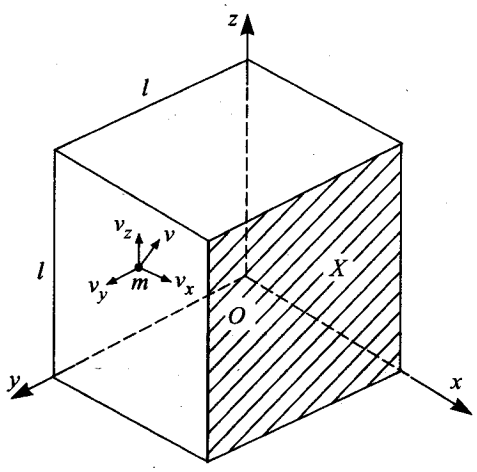


Fig. 24.5

Since their collisions with each other are assumed perfectly elastic, both kinetic energy and momentum are conserved in them. The average momentum with which all the molecules strike the walls is then not changed by their collisions with one another; what one loses, the other gains.

Expression for Pressure. Consider a cubical container of side l containing N molecules of gas, each of mass m (See Fig. 24.5). Suppose the edges of the cubical container are along OX, OY and OZ axes. Consider a single molecule that has velocity v at any instant. If v_x , v_y and v_z are the components of the velocity along OX, OY and OZ axes respectively, then,

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

Consider the force exerted on the face X of the cube due to the component v_x. Just before impact, the momentum of the molecule due to v_x is $m\,v_x$. The molecule will reverse the direction of its momentum by colliding with wall X. Since the collision will be elastic [assumption (*iv*)], it will rebound with the same speed so that its momentum will now be $-\,m\,v_x$.

$\therefore$ Change of momentum of the molecule

$$= (-\,m\,v_x) - (m\,v_x) = -\,2\,m\,v_x$$

$\therefore$ Momentum transferred to the face X

$$= +\,2\,m\,v_x$$

The time taken by the molecule to move across the cube to the opposite face and back to face X

$$= \frac{2\,l}{v_x}$$

This is the time to make one impact. Therefore, the number of impacts per second n' on face X is

$$n' = 1 \div \frac{2\,l}{v_x} = \frac{v_x}{2\,l}$$

Momentum change per second on face X

$$= n' \times \text{one momentum change}$$

$$= \frac{v_x}{2\,l} \times 2mv_x = \frac{m\,v_x^2}{l}$$

$\therefore$ Force on face $X = \dfrac{m\,v_x^2}{l}$ ...(*i*)

Note that expression (*i*) gives the average force on face X due to one molecule that has an x-component v_x.

$\therefore$ Pressure on face X $= \dfrac{\text{Force}}{\text{Area}} = \dfrac{\frac{mv_x^2}{l}}{l^2} = \dfrac{mv_x^2}{l^3}$ ...(*ii*)

We now take account of N molecules in the cubical container. Each has a different velocity and hence a component of different magnitude in the direction OX. If these are represented by v_{x1}, v_{x2}, v_{x3} .....v_{xN}, it follows from exp. (*ii*) that total pressure P on face X is given by ;

$$P = \frac{mv_{x1}^2}{l^3} + \frac{mv_{x2}^2}{l^3} + \frac{mv_{x3}^2}{l^3} + \ldots\ldots\ldots\ldots + \frac{mv_{xN}^2}{l^3}$$

$$= \frac{m}{l^3}(v_{x1}^2 + v_{x2}^2 + v_{x3}^2 + \ldots\ldots\ldots\ldots + v_{xN}^2) \quad \ldots(iii)$$

Let the symbol $\overline{v_x^2}$ represent the average or mean value of all the squares of the components in the OX direction *i.e.*,

$$\overline{v_x^2} = \frac{v_{x1}^2 + v_{x2}^2 + v_{x3}^2 + \ldots\ldots\ldots\ldots + v_{xN}^2}{N}$$

or $$\overline{v_x^2}\,N = v_{x1}^2 + v_{x2}^2 + v_{x3}^2 + \ldots\ldots\ldots\ldots + v_{xN}^2$$

Therefore, eq. (*iii*) becomes :

$$P = \frac{N m \overline{v_x^2}}{l^3} \quad \ldots(iv)$$

The square of the velocity for any molecule is given by ;

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

Since there is no preferred direction for the molecules, the average values of $\overline{v_x^2}$, $\overline{v_y^2}$ and $\overline{v_z^2}$ are equal.

$$\therefore \quad \overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2}$$

Therefore, equation (*iv*) becomes :

$$P = \frac{1}{3}\frac{N m \overline{v^2}}{l^3}$$

But $l^3 = V$ = Volume of the container.

$$\therefore \quad P = \frac{1}{3}\frac{N m \overline{v^2}}{V} \qquad ...(v)$$

Eq. (*v*) gives the expression for the pressure exerted by the gas. We can write eq. (*v*) as :

$$P V = \frac{1}{3} N m \overline{v^2} \qquad ...(vi)$$

Note the importance of this equation. The quantities on the left-hand side of eq. (*vi*) are macroscopic (large-scale) quantities. The quantities on the right-hand side of this equation are microscopic (molecular level) quantities. Thus, we have derived an expression linking the microscopic properties of molecules such as their speed and mass (generally unobservable) with easily observed large-scale properties such as pressure and volume.

24.10. EXPRESSION FOR PRESSURE IN ALTERNATE FORMS

The expression for pressure exerted by a gas can be given in several alternate forms.

(*i*) $$P = \frac{1}{3}\frac{N m \overline{v^2}}{V} \qquad ...(i)$$

Here N is the total number of molecules each of mass m and V is the volume of the gas.

Here $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$ = mean square velocity of gas molecules

(*ii*) Now $m N$ is the total mass of the gas so that $m N/V$ is the density ρ of the gas.

$$\therefore \quad P = \frac{1}{3}\rho \overline{v^2} \qquad ...(ii)$$

(*iii*) $$P = \frac{1}{3}\frac{N m \overline{v^2}}{V}$$

If we multiply and divide the right-hand side of this equation by 2, we get,

$$P = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2} m \overline{v^2}\right)$$

Now $\frac{1}{2} m \overline{v^2}$ = average kinetic energy per molecule = $\overline{\text{K.E.}}$

$$\therefore \quad P = \frac{2}{3}\frac{N}{V}\left(\overline{\text{K.E.}}\right) \qquad ...(iii)$$

Eq. (*iii*) tells us that the pressure exerted by a gas is proportional to the number of molecules per unit volume (N/V) and to the average kinetic energy of the molecule.

(*iv*) The product $N\ \overline{\text{K.E}}$. is the total average kinetic energy of the gas. Therefore, $N\ \overline{\text{K.E}}./V$ is the average kinetic energy of the gas per unit volume. This permits us to write eq. (*iii*) as :

$$P = \frac{2}{3} \text{ (average kinetic energy of the gas per unit volume)}$$

Note. For an ideal gas, K.E. of molecules means translational kinetic energy.

24.11. AVERAGE KINETIC ENERGY OF A GAS MOLECULE

Consider one mole of a gas. Let P, V and T be the pressure, volume and temperature of the gas respectively. Clearly, the gas will contain N_A molecules. According to perfect gas equation,

$$PV = RT \qquad ...(i)$$

Also $$P = \frac{2}{3}\frac{N_A}{V}\left(\frac{1}{2}m\overline{v^2}\right)$$

or $$PV = \frac{2}{3}N_A\left(\frac{1}{2}m\overline{v^2}\right) \qquad ...(ii)$$

From eq. (*i*) and eq. (*ii*), we have,

$$\frac{2}{3}N_A\left(\frac{1}{2}m\overline{v^2}\right) = RT$$

or $$\frac{1}{2}m\overline{v^2} = \frac{2}{3}\frac{R}{N_A}T \qquad ...(iii)$$

Both R and N_A are universal constants. Therefore, R/N_A is a constant called **Boltzmann's constant** k ($= 1.38 \times 10^{-23}$ J K^{-1}) and is the gas constant per molecule. The left-hand term is the average kinetic energy of a single molecule (*i.e.*, $\overline{\text{K.E.}}$).

$\therefore$ Average kinetic energy of a gas molecule is

$$\overline{\text{K.E.}} = \frac{3}{2}kT \qquad ...(iv)$$

Thus the average kinetic energy of a molecule in an ideal gas is directly proportional to the absolute temperature T of the gas.

Note that the gas constant R refers to 1 mole of a substance, as does Avogadro's number N_A. The Boltzmann's constant k ($= R/N_A$) is therefore independent of the concept of the mole and refers to the molecule. We can think of the Boltzmann's constant k as the gas constant *per molecule.*

Average K.E. of one mole of a gas. Consider one mole of a gas. Let P, V and T be the pressure, volume and temperature of the gas respectively. Clearly, the gas will contain N_A (Avogadro's number) molecules. If M is the molecular mass of the gas (*i.e.*, mass of 1 mole of the gas), then,

$$\text{Average K.E. of gas} = \frac{1}{2}M\overline{v^2} \qquad ...(v)$$

Since there are N_A molecules in one mole of gas,

$\therefore$ Average K.E. of gas = $N_A \times$ Average K.E. of one molecule

$$= N_A\left[\left(\frac{3}{2}\right)kT\right]$$

$$= N_A\left(\frac{3}{2}\right)\frac{R}{N_A}T \qquad \left(\because\ k = \frac{R}{N_A}\right)$$

$$= \frac{3}{2}RT \qquad ...(vi)$$

From eqs. (*v*) and (*vi*), we have,

$$\frac{1}{2}M\overline{v^2} = \frac{3}{2}RT \qquad ...(vii)$$

Eq. (*vii*) gives the average kinetic energy of one mole of the gas.

Perfect gas equation. We can write perfect gas equation in terms of the number of gas molecules present. Suppose N is the total number of molecules in a gas. Let P, V and T be the pressure, volume and temperature of the gas respectively. According to perfect gas equation :

$$P\,V = n\,R\,T$$

Here n is the number of moles of the gas so that $N = n\,N_A$ where N_A is Avogadro's number.

$$\therefore \qquad PV = \left(\frac{N}{N_A}\right)RT = N\left(\frac{R}{N_A}\right)T$$

or $$P\,V = N\,kT \qquad (\because\ k = R/N_A) \quad ...(viii)$$

Eq. (*viii*) gives the perfect gas equation in terms of the number of molecules present in the gas. Note that kinetic theory of gases helps us to understand the gas laws in a logical way.

24.12. KINETIC-THEORY INTERPRETATION OF TEMPERATURE

We can obtain some insight into the meaning of temperature by writing the formula for the average kinetic energy of a gas molecule *viz.*

$$\overline{\text{K.E.}} = \frac{3}{2}kT$$

or $$\frac{1}{2}m\,\overline{v^2} = \frac{3}{2}kT \qquad ...(i)$$

or $$\frac{1}{2}m\,\overline{v^2} \propto T$$

Thus the kinetic energy of a gas molecule is directly proportional to the absolute temperature (T) of the gas. Indeed, eq. (i) is one of the triumphs of the kinetic theory of gases.

Note that the average kinetic energy of a molecule in a gas depends *only* on the temperature, not on the pressure or volume or type of gas. The connection between temperature and the average kinetic energy of a molecule is an important result. It tells us what is happening on a molecular scale. An increase in temperature corresponds to an increase in the speed of the molecules ; a decrease in temperature corresponds to a slowing down of the molecules.

According to kinetic theory of gases, if $T = 0$, $\overline{\text{K.E.}} = 0$. In other words, at 0 K (*i.e.*, – 273.15°C), the kinetic energy of a gas molecule is zero. No gas molecules can have kinetic energy less than zero, so no temperature can be lowerer than – 273.15°C. *For this reason, – 273.15°C is known as* **absolute zero**. This seems quite logical. If temperature is to be a direct measure of the kinetic energy of a molecule, it must be read on a scale at which the temperature is zero when the molecular kinetic energy is zero *i.e.*, it must start at absolute zero (*i.e.*, – 273.15°C). *Such a temperature scale is called* **Kelvin scale of temperature**. It has the same size of degree as the Celsius scale but starts with its zero at –273°C (or more accurately – 273.15°C). Since the degrees are of the same size on both the scales, one need only add 273 to Celsius reading to convert temperature from °C to K. Thus on the Kelvin scale (named for a prominent physicist), the freezing point of water is 273 K and the boiling point at one atmosphere is 373 K.

Note. According to kinetic theory of gases, the average kinetic energy of one molecule of a gas at absolute temperature $T = (3/2)\,kT$. If a container holds N molecules at absolute temperature T, then average kinetic energy of all the molecules

$$= N\left(\frac{3}{2}\right)kT$$

24.13. MEAN VELOCITY OF GAS MOLECULES

It is defined as the arithmetic mean of the velocities of the gas molecules. If v_1, v_2, v_3... v_N are the velocities of N gas molecules, then mean velocity of gas molecules is

Mean velocity, $\bar{v} = \dfrac{v_1 + v_2 + v_3 + \ldots\ldots\ldots + v_N}{N}$

Let us suppose that the velocities of six gas molecules are 1, 2, 3, 4, 5 and 6 units. Their mean velocity $\bar{v}$ is

$$\bar{v} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5 \text{ units}$$

Note. It can be shown that mean velocity is given by ;

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$$

where m = mass of each molecule

k = Boltzmann's constant

T = temperature of the gas in kelvin.

24.14. ROOT MEAN SQUARE VELOCITY OF GAS MOLECULES

Suppose $v_1, v_2, v_3 \ldots\ldots v_N$ are the velocities of N molecules. Then their mean square velocity $\overline{v^2}$ is given by ;

$$\overline{v^2} = \frac{v_1^2 + v_2^2 + v_3^2 + \ldots\ldots\ldots + v_N^2}{N}$$

The square root of $\overline{v^2}$ is called the *root mean square velocity* $v_{r.m.s.}$ (since we are taking the square *root* of the *mean* of the *square* of velocities).

$$\therefore \quad v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{\frac{v_1^2 + v_2^2 + v_3^2 + \ldots\ldots\ldots + v_N^2}{N}}$$

Hence root mean square velocity of a gas molecule is defined as the square root of the mean of the squares of the velocities of the gas molecules.

It should be noted that pressure P of a gas depends upon the 'mean square' of the velocity $(\overline{v^2})$. Now the mean square velocity is *not* equal to the square of the average velocity. As an example,let us suppose that the velocities of six molecules are 1, 2, 3, 4, 5 and 6 units. Their mean velocity $\bar{v}$ is given by ;

$$\bar{v} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5 \text{ units}$$

and its square is $(\bar{v})^2 = (3.5)^2 = 12.25$

Now the mean square velocity is

$$\overline{v^2} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{91}{6} = 15.2$$

$$\therefore \quad v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{15.2} = 3.9 \text{ units}$$

Note that mean velocity is 3.5 units, whereas the root mean square velocity is 3.9 units.

Thus the difference between the two velocities in this simple case is about 12%.

Different expressions for $v_{r.m.s.}$. The root mean square velocities of gas molecules can be given in several alternate forms.

(*i*) $$P = \frac{1}{3}\rho \overline{v^2}$$

$$\therefore \quad v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{\frac{3P}{\rho}} \qquad \ldots(i)$$

(*ii*) $$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$$

$$\therefore \quad v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{m}} \qquad ...(ii)$$

This is an interesting result. This shows that *the root mean square velocity of gas molecules is directly proportional to the square root of the absolute temperature of the gas and inversely proportional to the square root of the mass of each molecule.* Thus for a given temperature, lighter molecules move faster, on the average, than heavier molecules. For example, hydrogen with a molecular weight of 2 g/mol moves four times as fast as oxygen whose molecular weight is 32 g/mol.

(*iii*) $$\frac{1}{2}M\overline{v^2} = \frac{3}{2}RT$$

Here M is the molecular mass of the gas (*i.e.*, mass of one mole of gas).

$$\therefore \quad v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{\frac{3RT}{M}} \qquad ...(iii)$$

Therefore, root mean square velocity of gas molecules can be expressed as :

$$v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3\,RT}{M}}$$

(*iv*) At $T = 0$ K ; $v_{r.m.s.} = 0$.

Therefore, absolute zero (i.e., 0 K) is that temperature at which the root mean square velocity of the gas molecules reduces to zero. This means that molecular motion ceases at absolute zero.

Note. The root mean square velocities of gas molecules are very high. For example, the root mean square velocity of a hydrogen molecule at 27°C is about 1900 m s^{-1}. For this reason, the effect of gravity on root mean square velocities is very small.

24.15. MOST PROBABLE VELOCITY OF GAS MOLECULES

The velocity which is possessed by maximum number of molecules in a gas sample is called **most probable velocity.**

It can be shown that : Most probable velocity, $v_{mp} = \sqrt{\frac{2\,kT}{m}}$

where m = mass of each molecule

T = temperature of the gas in kelvin

k = Boltzmann's constant

It is clear that most probable velocity increases with the increase in temperature.

Now $$v_{mp} : \bar{v} : v_{r.m.s.} = \sqrt{\frac{2\,kT}{m}} : \sqrt{\frac{8\,kT}{\pi\, m}} : \sqrt{\frac{3\,kT}{m}}$$

$$= 1 : 1.3 : 1.23$$

where v_{mp} = most probable velocity of gas molecules

$\bar{v}$ = average velocity of gas molecules

$v_{r.m.s.}$ = r.m.s. velocity of gas molecules

Example 24.4. **Nine particles have speeds of 5, 8, 12, 12, 12, 14, 14, 17 and 20 ms^{-1}. Find (*i*) the average speed (*ii*) the r.m.s. speed (*iii*) the most probable speed of the particles.**

Solution. (*i*) The average speed is the sum of speeds divided by the total number of particles.

$$\therefore \text{ Average speed, } \bar{v} = \frac{5+8+12+12+12+14+14+17+20}{9} = 12.7 \text{ ms}^{-1}$$

(*ii*) The average value of the squares of speeds is given by ;

$$\overline{v^2} = \frac{5^2+8^2+12^2+12^2+12^2+14^2+14^2+17^2+20^2}{9} = 178 \text{ m}^2\text{s}^{-2}$$

$\therefore$ R.M.S. speed, $v_{r.m.s.} = \sqrt{\overline{v^2}} = \sqrt{178}$ = **13.3 ms^{-1}**

(*iii*) Three of the particles have a speed of 12 ms^{-1}, two have a speed of 14 ms^{-1} and the remaining have different speeds. Therefore, we see that the most probable speed v_{mp} = **12 ms^{-1}**.

Example 24.5. What is the average kinetic energy of a molecule of a gas at a temperature of 27°C? Given k = 1.38 × 10^{-23} J K^{-1}.

Solution. Here k = 1.38 × 10^{-23} J K^{-1} ; T = 273 + 27 = 300 K

$\therefore$ Average kinetic energy of the molecule of the gas is

$$\overline{\text{K.E.}} = \frac{3}{2}kT = \frac{3}{2} \times (1.38 \times 10^{-23}) \times 300 = \mathbf{6.21 \times 10^{-21}\ J}$$

Example 24.6. What is the root mean square speed of a hydrogen molecule at 27°C? Given k = 1.38 × 10 $^{-23}$ JK^{-1}.

Solution. $$v_{r.m.s.} = \sqrt{\frac{3kT}{m}}$$

Here k = 1.38 × 10^{-23} J K^{-1} ; T = 273 + 27 = 300 K ; m = mass of H_2 = 2 × 1.674 × 10^{-27} kg

$\therefore$ $$v_{r.m.s.} = \sqrt{\frac{3\times(1.38\times10^{-23})\times300}{2\times1.674\times10^{-27}}} = \mathbf{1926\ m\ s^{-1}}$$

Alternatively, $$v_{r.m.s.} = \sqrt{\frac{3RT}{M}}$$

Here R = 8.31 J mol^{-1}K^{-1} ; T = 300 K ; M = 2 × 1.008 × 10^{-3} kg mol^{-1}

$\therefore$ $$v_{r.m.s.} = \sqrt{\frac{3\times8.31\times300}{2\times1.008\times10^{-3}}} = \mathbf{1926\ m\ s^{-1}}$$

Example 24.7. Find the number of molecules in one cubic metre of air at atmospheric pressure and 0°C.

Solution. Let N be the total number of air molecules in 1 m^3 of air. Then according to perfect gas equation,

$$PV = NkT$$

Here P = 1.013 × 10^5 Pa; V = 1 m^3; k = 1.38 × 10^{-23} J K^{-1}; T = 273 + 0 = 273 K

$\therefore$ $$N = \frac{PV}{kT} = \frac{1.013\times10^5\times(1)}{(1.38\times10^{-23})\times273} = \mathbf{2.69 \times 10^{25}}$$

Example 24.8. At what temperature will the r.m.s. velocity of hydrogen molecules be doubled of its value at N.T.P.?

Hint. The root mean square velocity of molecules is directly proportional to the square root of the absolute temperature of the gas *i.e.*, $v_{r.m.s.} \propto \sqrt{T}$.

$\therefore$ $$\frac{v'_{r.m.s.}}{v_{r.m.s.}} = \sqrt{\frac{T_2}{T_1}}$$

Here $\frac{v'_{r.m.s.}}{v_{r.m.s.}} = 2$; T_1 = 0°C=273 K

$\therefore$ $2 = \sqrt{\frac{T_2}{T_1}}$ or $4 = \frac{T_2}{273}$ $\therefore$ T_2=1092 K

$\therefore$ Required temperature = 1092 − 273 = **819°C**

Example 24.9. Calculate the total random kinetic energy of 1 g of nitrogen at 300 K.

Solution. Total random energy for gram molecule of nitrogen = $\frac{3}{2}RT$

Total random energy for 1 g of nitrogen is $E = \frac{3}{2}\frac{RT}{M}$

where M = molecular mass of N_2 in grams = 28 grams

$$\therefore \quad E = \frac{3}{2}\times\frac{8.3\times10^7\times300}{28} = 133.4\times10^7 \text{ ergs} = 133.4 \text{ J}$$

Example 24.10. Calculate the root mean square speed of an air molecule at a temperature of 27°C. One mole of air has a mass of 29 g. How does $v_{r.m.s.}$ of air molecules compare to the speed of sound in air (340 ms^{-1})?

Solution. Since one mole (6.023×10^{23} molecules) of air has a mass of 29 g, the average mass m of an air molecule is

$$m = \frac{29\times10^{-3}}{6.023\times10^{23}} = 4.8\times10^{-26} \text{ kg}$$

Now $$v_{r.m.s.} = \sqrt{\frac{3kT}{m}}$$

Here $k = 1.38 \times 10^{-23}$ J K^{-1} ; $T = 273 + 27 = 300$ K ; $m = 4.8 \times 10^{-26}$ kg

$$\therefore \quad v_{r.m.s.} = \sqrt{\frac{3\times(1.38\times10^{-23})\times300}{4.8\times10^{-26}}} = 509 \text{ m s}^{-1}$$

Sound waves are caused by air molecules oscillating in and out of regions of varying air density. Since the molecules cannot move into or out of these regions faster than their random speed, the speed with which sound waves are formed and travel is limited by this random speed. The speed of sound in a gas is about two-third the root mean square speed of its molecules.

Example 24.11. Helium gas occupies a volume of 0.04 m^3 at a pressure of 2×10^5 Pa and temperature 300 K. Calculate (*i*) the mass of helium (*ii*) the r.m.s. speed of its molecules (*iii*) the r.m.s. speed at 432 K when the gas is heated at constant pressure to this temperature (*iv*) r.m.s. speed of hydrogen molecules at 432 K. Relative molecular masses of helium and hydrogen are 4 and 2 respectively and molar gas constant = 8.3 J mol^{-1} K^{-1}.

Solution. (*i*) Mass. For n moles, $PV = nRT$

$$\therefore \quad n = \frac{PV}{RT} = \frac{(2\times10^5)\times0.04}{8.3\times300} = 3.2$$

$\therefore$ Mass of helium = $n \times 4 = 3.2 \times 4 = 12.8$ g

(*ii*) R.M.S. speed. Pressure, $P = 2 \times 10^5$ Pa

$$\text{Now density, } \rho = \frac{\text{mass}}{\text{volume}} = \frac{12.8\times10^{-3}}{0.04} = 0.32 \text{ kgm}^{-3}$$

$$\therefore \quad \text{R.M.S. speed, } v_{r.m.s.} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3\times2\times10^5}{0.32}} = 1369 \text{ m s}^{-1}$$

(*iii*) Temperature 432 K. Since r.m.s. speed $\propto \sqrt{T}$,

$$\therefore \quad \frac{v'_{r.m.s.}}{v_{r.m.s.}} = \sqrt{\frac{432}{300}} = \sqrt{1.44} = 1.2$$

or $$v'_{r.m.s.} = 1.2\times v_{r.m.s.} = 1.2\times1369 = 1643 \text{ m s}^{-1}$$

(*iv*) **Hydrogen.** One mole of hydrogen has a mass of 2 g and one mole of helium has a mass of 4 g. So ratio of molar masses = 2 : 4 = 1 : 2. But r.m.s. speed at a given temperature $\propto 1/\sqrt{M}$ where M is the molar mass. So at 432K, r.m.s. speed of hydrogen molecules $= \sqrt{2} \times 1643 = \mathbf{2324\ ms^{-1}}$.

Example 24.12. The kinetic energy of a molecule of oxygen at 0°C is 5.64×10^{-21} J. Calculate Avogadro's number. Given $R = 8.31$ J mol^{-1} k^{-1}.

Solution. $T = 0 + 273 = 273$ K ; $\overline{K.E.} = 5.64 \times 10^{-21}$ J

Let N_A be the Avogadro's number.

Now, $$\overline{\text{K.E.}} = \frac{3}{2}kT = \frac{3}{2}\frac{R}{N_A}T$$

$\therefore$ Avogadro's number, $$N_A = \frac{3}{2} \times \frac{RT}{\overline{K.E.}} = \frac{3}{2} \times \frac{8.31 \times 273}{5.64 \times 10^{-21}} = \mathbf{6.035 \times 10^{23}}$$

Example 24.13. Calculate (*i*) the r.m.s. speed and (*ii*) the mean kinetic energy of one gram molecule of hydrogen at S.T.P. Given density of hydrogen at S.T.P. is 0.09 kg/m^3 and $R = 8.31$ J mol^{-1} K^{-1}.

Solution. At S.T.P., $\rho = 0.09$ kg/m^3 ; $P = 1.013 \times 10^5$ N/m^2

(*i*) Now $$v_{r.m.s.} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3 \times 1.013 \times 10^5}{0.09}} = \mathbf{1837.5\ ms^{-1}}$$

(*ii*) Mass of 1 gram molecule of hydrogen is

$$M = 2g = 2 \times 10^{-3} \text{ kg}$$

$\therefore$ Mean kinetic energy of 1 gram molecule of hydrogen at S.T.P. is

$$\overline{\text{K.E.}} = \frac{1}{2}M v_{r.m.s.}^2 = \frac{1}{2} \times (2 \times 10^{-3}) \times (1837.5)^2 = \mathbf{3376.4\ J}$$

PROBLEMS FOR PRACTICE

1. Estimate the total number of air molecules in a room of capacity 25.0 m^3 at a temperature of 27°C and 1 atm pressure. Given $k = 1.38 \times 10^{-23}$ JK^{-1}. **[6.1×10^{26}]**
2. Calculate the kinetic energy of one gm-molecule of a gas at normal temperature and pressure. What will be its value at 273°C? $R = 8.31$ J mol^{-1}K^{-1}.

   [3.40×10^3 J; 6.80×10^3 J]
3. What is the mean kinetic energy of a molecule of a gas at 300 K temperature? What will be the root-mean-square-speed of a molecule of H_2 gas at this temperature? Given that $R = 8.31$ J mol^{-1} K^{-1} ; $N_A = 6.02 \times 10^{23}$ per mole; mass of hydrogen molecule = $2 \times 1.67 \times 10^{-27}$ kg. **[6.21×10^{-21} J ; 1.93×10^3 m s^{-1}]**
4. A vessel is filled with a gas at a pressure of 76 cm of mercury at a certain temperature. The mass of the gas is increased by 50% by introducing more gas in the vessel at the same temperature. Find out the resultant pressure of the gas. **[114 cm of Hg]**
5. Calculate the r.m.s. velocity of oxygen molecules at S.T.P., the molecular weight of oxygen being 32. **[461.2 m s^{-1}]**
6. The temperature of an ideal gas is T K and the mean kinetic energy $\overline{K.E.}$ of its molecules is given by ;

   $$\overline{K.E.} = 2.07 \times 10^{-23} \text{ J /molecule}$$

   Calculate the number of molecules in one litre of the gas at N.T.P. What will be the average distance between the molecules? 1 atmospheric pressure = 1.01×10^5 N/m^2.

   [2.68×10^{22} ; 3.34×10^{-9} m]

7. Calculate the temperature at which the r.m.s. velocity of O_2 gas molecules will be 1 km s^{-1}. Given R = 8.311 J $mol^{-1}K^{-1}$. **[1010.4° C]**

8. At what temperature will the average velocity of oxygen molecules be sufficient so as to escape from the earth? Escape velocity from earth is 11.0 km s^{-1} and mass of one molecule of oxygen is 5.34×10^{-26} kg. Boltzmann's constant = 1.38×10^{-23} J K^{-1}.

[1.56×10^5 K]

9. (*a*) Calculate : (*i*) the root mean square speed and (*ii*) the mean kinetic energy of one gram molecule of hydrogen at S.T.P. Given that the density of hydrogen at S.T.P. is 0.09 kg m^{-3} and R = 8.31 J mol^{-1} K^{-1}.

(*b*) Given that the mass of a molecule of hydrogen is 3.34×10^{-27} kg, calculate Avogadro's number.

(*c*) Calculate Boltzmann's constant.

[(*a*) 1.8375 km s^{-1}; 3376.4 J (*b*) 5.99×10^{23} (*c*) 1.87×10^{-23} JK^{-1}]

10. The density of carbon dioxide gas at 0°C and at a pressure of 1.0×10^5 N/m^2 is 1.98 kg m^{-3}. Find the root-mean-square velocity of its molecules at 0°C and 30°C. Pressure is constant. **[389 m s^{-1} ; 410 m s^{-1}]**

24.16. DERIVATION OF GAS LAWS FROM KINETIC THEORY

The gas laws describe what happens to a gas under various conditions but they say nothing why gases act this way. The kinetic theory of gases successfully explains the gas laws and other related aspects.

(*i*) **Boyle's law.** *It states that temperature remaining constant, the volume V of a given mass of a gas is inversely proportional to its pressure P i.e., P V = constant.*

According to kinetic theory of gases, the pressure exerted by a gas is given by ;

$$P = \frac{1}{3}\frac{m N \overline{v^2}}{V}$$

or $$P V = \frac{1}{3} m N \overline{v^2}$$

But $m N$ = total mass of the gas = M.

$\therefore$ $$P V = \frac{1}{3} M \overline{v^2}$$

According to kinetic theory of gases, $\overline{v^2} \propto T$ .Therefore, if temperature of the gas is kept constant, then for a given mass M of the gas,

$$P V = \text{Constant} \quad \text{or} \quad P \propto \frac{1}{V}$$

Thus Boyle's law stands proved on the basis of kinetic theory of gases.

(*ii*) **Charles' law.** *It states that if the pressure of a given mass of a gas is kept constant, its volume is directly proportional to the absolute temperature of the gas i.e.*

$$V \propto T$$ ...at constant pressure

According to kinetic theory of gases,

$$P = \frac{1}{3}\frac{M \overline{v^2}}{V}$$

or $$P V = \frac{1}{3} M \overline{v^2}$$

According to kinetic definition of temperature, $\overline{v^2} \propto T$.

$\therefore \quad PV \propto T$

If P is constant, $V \propto T$

Thus kinetic theory of gases successfully explains Charles' law.

(*iii*) **Gay Lussac's law or Pressure law.** *It states that if the volume of a given mass of a gas is kept constant, its pressure is directly proportional to the absolute temperature of the gas i.e.,*

$P \propto T$ ...at constant volume

According to kinetic theory of gases,

$$PV = \frac{1}{3}M\overline{v^2}$$

According to kinetic definition of temperature, $\overline{v^2} \propto T$.

$\therefore \quad PV \propto T$

If V is constant, $P \propto T$

Thus pressure law is proved on the basis of kinetic theory of gases.

(*iv*) **Perfect gas equation.** According to perfect gas equation,

$$PV = RT$$

where R is the gas constant for one mole of the gas.

According to kinetic theory of gases,

$$PV = \frac{1}{3}M\overline{v^2}$$

But $\overline{v^2} \propto T$

$\therefore \quad PV \propto T$

or $PV = RT$

Thus kinetic theory of gases successfully explains the perfect gas equation.

(*v*) **Avogadro's law.** *It states that equal volumes of all gases under similar conditions of temperature and pressure contain equal number of molecules.*

This law was announced by Avogadro in 1811 and was well-established before the kinetic theory was developed. However, we shall see how easily the kinetic theory of gases proves this fact. Consider two gases distinguished by the subscripts 1 and 2.

Let m_1 and m_2 = masses of molecules of 1 and 2 respectively

N_1 and N_2 = number of molecules of 1 and 2 respectively

P_1 and P_2 = pressures of 1 and 2 respectively

V_1 and V_2 = volumes of 1 and 2 respectively

T_1 and T_2 = absolute temperatures of 1 and 2 respectively

$\overline{v_1^2}$ and $\overline{v_2^2}$ = mean square velocities of molecules of 1 and 2 respectively

According to kinetic theory of gases,

$$P_1V_1 = \frac{1}{3}m_1N_1\overline{v_1^2} \quad \text{and} \quad P_2V_2 = \frac{1}{3}m_2N_2\overline{v_2^2}$$

For equal volumes at the same pressure, $P_1V_1 = P_2V_2$.

$\therefore \quad \frac{1}{3}m_1N_1\overline{v_1^2} = \frac{1}{3}m_2N_2\overline{v_2^2}$ ...(*i*)

If the gases are at the same temperature, the average kinetic energies of their moelcules are equal *i.e.*,

$$\frac{1}{2}m_1\overline{v_1^2} = \frac{1}{2}m_2\overline{v_2^2}$$

or $$m_1\overline{v_1^2} = m_2\overline{v_2^2}$$

Therefore, eq. (*i*) becomes :

$$N_1 = N_2$$

Hence equal volumes of all gases under similar conditions of temperature and pressure have the same number of molecules. Note how easily kinetic theory of gases proves Avogadro's law. However, in Avogadro's time, there was no established kinetic theory to which he could turn for proof.

(*vi*) **Graham's law of diffusion.** *It states that the rates of diffusion of two gases are inversely proportional to the square roots of their densities.*

Consider two gases A and B diffusing into each other at pressure P* . Let ρ_A and ρ_B be their densities. The root mean square velocities of the molecules of gases A and B will be :

$$v_{r.m.s.} = \sqrt{\frac{3P}{\rho_A}} \quad \text{and} \quad v'_{r.m.s.} = \sqrt{\frac{3P}{\rho_B}}$$

$$\therefore \quad \frac{v_{r.m.s.}}{v'_{r.m.s.}} = \sqrt{\frac{3P}{\rho_A}} \times \sqrt{\frac{\rho_B}{3P}} = \sqrt{\frac{\rho_B}{\rho_A}}$$

But the rate of diffusion of a gas is directly proportional to r.m.s. velocity of its molecules. If r_A and r_B are the rates of diffusion of gases A and B respectively, then,

$$\frac{r_A}{r_B} = \frac{v_{r.m.s.}}{v'_{r.m.s.}} = \sqrt{\frac{\rho_B}{\rho_A}}$$

$$\therefore \quad \frac{r_A}{r_B} = \sqrt{\frac{\rho_B}{\rho_A}}$$

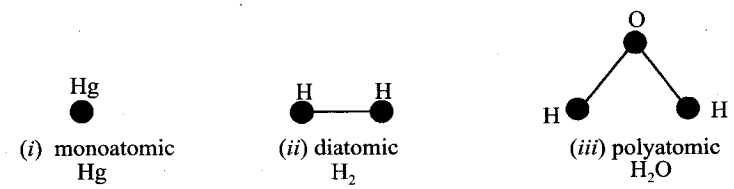
This is Graham's law.

24.17. MONOATOMIC, DIATOMIC AND POLYATOMIC GASES

The internal energy of a gas depends on the number of atoms in its molecule.

(*i*) A gas whose molecules consist of single atoms is called a *monoatomic gas e.g.* chemically inert gases and metallic vapours, Hg, Na, He, Ne etc. A gas whose molecules contain two atoms is called a *diatomic gas e.g.* O_2, H_2, N_2, Cl_2, CO etc. A gas whose molecules consist of more than two atoms is called a *polyatomic gas e.g.* H_2O, O_3, H_2S, CO_2, CH_4 etc.

(*ii*) The molecules of a monoatomic gas may be regarded as points but those, of a diatomic gas we must regard as 'dumb-bells'. The molecules of a polyatomic gas have more complicated structure. This is illustrated in Fig. 24.6. A molecule which extends appreciably in space— a



(*i*) monoatomic Hg (*ii*) diatomic H_2 (*iii*) polyatomic H_2O

Fig. 24.6

* Initially, the pressures of gases A and B are P_A and P_B respectively. When the steady state of diffusion is reached, pressure of the mixture is same, say P.

diatomic or polyatomic molecule—has an appreciable moment of inertia. However, a monoatomic molecule must have much smaller moment of inertia than a diatomic or polyatomic molecule.

(*iii*) The internal energy of a gas is due to *translational kinetic energy and rotational kinetic energy. Since the moment of inertia of the molecules of a monoatomic gas is negligible, their kinetic energy of rotation can be neglected. In other words, the internal energy of a monoatomic gas is solely due to the translational kinetic energy of its molecules. However, the molecules of a diatomic or polyatomic gas have appreciable amount of moment of inertia because their structures appreciably extend in space. Therefore, the molecules of a diatomic or polyatomic gas have translational as well as rotational kinetic energy.

24.18. DEGREES OF FREEDOM

The degrees of freedom mean the number of independent ways the molecules can possess energy.

(*i*) The molecules of a monotomic gas have only translational kinetic energy. Since the molecule of a monoatomic gas can have velocity along *X*-axis, *Y*-axis and *Z*-axis, we say that a monoatomic gas has *three degrees of freedom* (translational).

(*ii*) The molecules of a diatomic gas have translational as well as rotational kinetic energy. A diatomic molecule can have rotational kinetic energy about two axes at right angles to its **own axis as shown in Fig. 24.7 (*i*). Therefore, a diatomic molecule has two degrees of rotational freedom. In addition, it has the same three degrees of freedom associated with translational kinetic energy. We conclude that a diatomic gas has *five degrees of freedom* — two degrees of rotational freedom plus three degrees of translational freedom.

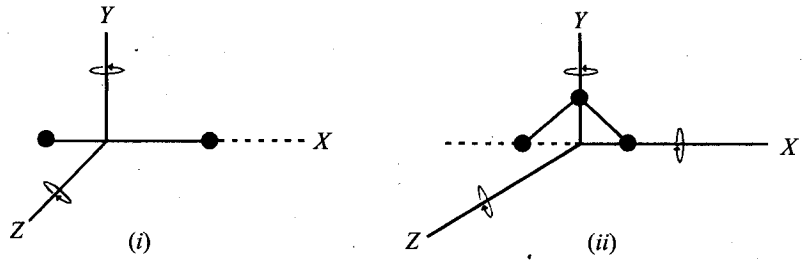


Fig. 24.7

(*iii* A polyatomic molecule, unless it happens to consist of molecules all in a straight line, has no axis about which its moment of inertia is negligible. It can, therefore, have kinetic energy of rotation about three mutually perpendicular axes as shown in Fig. 24.7 (*ii*). It has three degrees of rotational freedom. In addition, it has three degrees of translational freedom. Therefore, a polyatomic molecule has *six degrees of freedom.*

Note 1. The degrees of freedom of the system is given by ;

$$f = 3N - k$$

where f = degrees of freedom

N = number of particles in the system

k = independent relations among the particles

(*i*) For a monoatomic gas, $N = 1$ and $k = 0$.

$$\therefore \quad f = 3 \times 1 - 0 = 3$$

Therefore, a monoatomic gas molecule has 3 degrees of freedom.

(*ii*) For a diatomic gas, $N = 2$, $k = 1$.

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* This kinetic energy is associated with the motion of centre of mass of the molecules.

** The moment of inertia about its own axis is zero.

$\therefore \quad f = 3 \times 2 - 1 = 5$

Therefore, a diatomic gas molecule has 5 degrees of freedom.

(*iii*) For a triatomic gas, $N = 3$, $k = 3$.

$\therefore \quad f = 3 \times 3 - 3 = 6$

Therefore, a triatomic gas molecule has 6 degrees of freedom.

Note 2. At very high temperatures (above 5000 K), a gas molecule possesses vibratory motion also in addition to translatory and rotatory motion. However, at ordinary temperatures, a gas molecule possesses translational and rotational motion only.

24.19. LAW OF EQUIPARTITION OF ENERGY

We know that average kinetic energy of a monoatomic molecule for temperature T is

Average K.E. of monoatomic molecule = $\frac{3}{2}kT$

Since a monoatomic gas has 3 degrees of freedom,

$\therefore$ Average K.E. in each degree of freedom = $\frac{1}{2}kT$

Maxwell assumed that the average kinetic energy of a molecule in each degree of freedom (translational as well as rotational) is the same and is equal to $\frac{1}{2}kT$. This assumption is called *law of equipartition of energy* and may be stated as under :

The average kinetic energy of a molecule in each degree of freedom (translational as well as rotational) is the same and is equal to $\frac{1}{2}kT$.

Experiments show that this is true at room temperature and above. At very low temperatures, when the gas is near liquefication, it fails. At ordinary temperatures, then, we have,

$$\textit{Average K.E. of monoatomic molecule} = 3 \times \frac{1}{2} kT(\textit{trans.}) = \frac{3}{2} kT$$

$$\textit{Average K.E. of diatomic molecule} = 3 \times \frac{1}{2} kT(\textit{trans.}) + 2 \times \frac{1}{2} kT(\textit{rot.}) = \frac{5}{2} kT$$

$$\textit{Average K.E. of polyatomic molecule} = 3 \times \frac{1}{2} kT(\textit{trans.}) + 3 \times \frac{1}{2} kT(\textit{rot.}) = \frac{6}{2} kT$$

24.20. MOLAR SPECIFIC HEATS OF AN IDEAL MONOATOMIC GAS

We now calculate the molar specific heats of an ideal monoatomic gas at constant volume and constant pressure. A monoatomic gas molecule has one atom. Suppose one mole of an ideal monoatomic gas is heated through ΔT at constant volume. Since no work is done, all the heat supplied (Q) goes into raising the internal energy and thus the temperature of the gas.

$$\therefore \quad Q = \Delta U = C_v \Delta T \qquad \ldots(i)$$

$$\text{For one mole of gas, } \Delta U = \frac{3}{2} R \Delta T \qquad \ldots(ii)$$

From eq. (*i*) and eq. (*ii*), we have,

$$C_v \Delta T = \frac{3}{2} R \Delta T$$

or $$C_v = \frac{3}{2} R$$

Also $$C_p = C_v + R = \frac{3}{2} R + R = \frac{5}{2} R$$

Since R = 8.31 J mol^{-1} K^{-1}, kinetic theory predicts that $C_v = 3R/2 = 3 \times 8.31/2 = 12.46$ J mol^{-1} K^{-1}. This is very close to the experimental values for monoatomic gases such as helium and neon (See Table below). The value of $C_p = 5\,R/2 = 5 \times 8.31/2 = 20.8$ J mol^{-1} K^{-1} for monoatomic gases is also in agreement with experiment.

Observed molar specific heats of some real gases

Gas	*Type*	*Temperature* (°C)	C_p	C_v	$C_p - C_v$	$\gamma = C_p/C_v$
			($J\ mol^{-1}\ K^{-1}$)			
Argon (Ar)	Monoatomic	15°C	20.95	12.56	8.39	1.67
Neon (Ne)	Monoatomic	15°C	20.78	12.67	8.11	1.64
Nitrogen (N_2)	Diatomic	15°C	29.04	20.68	8.36	1.40
Oxygen (O_2)	Diatomic	15°C	29.17	20.82	8.35	1.40
Carbon dioxide (CO_2)	Triatomic	15°C	36.64	27.88	8.76	1.30
Steam (H_2O)	Triatomic	100°C	36.35	27.45	8.90	1.32
Ethene (C_2H_2)	Polyatomic	15°C	48.60	39.84	8.76	1.20

The molar specific heats for more complex gases such as diatomic (two atoms) and triatomic (three atoms) gases increases with the increased number of atoms per molecule. This is apparently due to the fact that the internal energy of such complex molecules includes not only translational kinetic energy but other forms of energy as well. It can be shown that for a diatomic gas, $C_p = 7R/2$ and $C_v = 5R/2$ so that $C_p - C_v = R$.

24.21. CALCULATION OF C_p/C_v FOR GASES

We can find C_p and C_v (and hence γ) for monoatomic and diatomic gases by using the law of equipartition of energy and degrees of freedom.

(*i*) **Monatomic gases.** We know that in a monoatomic gas (*e.g.* argon, helium etc), a molecule has *three degrees of freedom.* Since energy per molecule per degree of freedom is $kT/2$,

$$\therefore \text{ Total energy of one molecule } = 3\times\frac{kT}{2}=\frac{3}{2}kT$$

$$\text{Total energy of 1 mole of gas, } U=\frac{3}{2}N_A kT=\frac{3}{2}RT \qquad \left(\because k=\frac{R}{N_A}\right)$$

Here N_A = Avogadro's number ; k = Boltzmann's constant

$$\text{Now} \qquad C_v=\frac{dU}{dT}=\frac{d}{dT}\left(\frac{3}{2}RT\right)=\frac{3}{2}R$$

Now R = 1.985 cal mol^{-1} K^{-1}, $\quad \therefore\ C_v=\frac{3}{2}\times 1.985 = 2.98$ cal mol^{-1} K^{-1}

If C_p is the specific heat of 1 mole of the gas at constant pressure,then,

$$C_p - C_v = R$$

$$\text{or } C_p = R + C_v = R + \frac{3}{2}R = \frac{5}{2}R = 5/2 \times 1.985 = 4.96 \text{ cal mol}^{-1}\ \text{K}^{-1}$$

$$\therefore \qquad \gamma=\frac{C_p}{C_v}=\frac{4.96}{2.98}=1.67$$

(*ii*) **Diatomic gases.** We know that in a diatomic gas (*e.g.* oxygen, nitrogen, hydrogen etc), a molecule has *five degrees of freedom.*

$\therefore$ Total energy per molecule $= 5 \times \frac{1}{2} kT = \frac{5}{2} kT$

Total energy of 1 mole of gas, $U = \frac{5}{2} N_A kT = \frac{5}{2} RT$

Now $$C_v = \frac{dU}{dT} = \frac{d}{dT}\left(\frac{5}{2}RT\right) = \frac{5}{2}R$$

$$= \frac{5}{2} \times 1.985 = 4.96 \text{ cal mol}^{-1} \text{ K}^{-1}$$

Now $C_p = C_v + R = \frac{5}{2} R + R = \frac{7R}{2} = \frac{7 \times 1.985}{2} = 6.95 \text{ cal mol}^{-1} \text{ K}^{-1}$

$\therefore$ $$\gamma = \frac{C_p}{C_v} = \frac{6.95}{4.95} = 1.4$$

Note. For a polyatomic gas, $\gamma = 4/3 = 1.33$, unless it happens to consist of molecules all in a straight line.

24.22. RELATION BETWEEN γ AND DEGREES OF FREEDOM

Suppose a polyatomic gas molecule has f degrees of freedom.

$\therefore$ Total energy associated with 1 mole of the gas is

$$U = f \times \frac{1}{2} kT \times N_A = \frac{f}{2} RT \qquad \left(\because k = \frac{R}{N_A}\right)$$

Now $$C_v = \frac{dU}{dT} = \frac{d}{dT}\left(\frac{f}{2}RT\right) = \frac{f}{2}R$$

Also $$C_p = C_v + R = \frac{f}{2} R + R = \left(\frac{f}{2}+1\right)R$$

We know that $$\gamma = \frac{C_p}{C_v} = \frac{\left(\frac{f}{2}+1\right)R}{\frac{f}{2}R} = \frac{2}{f}\left(\frac{f}{2}+1\right) = \left(1+\frac{2}{f}\right)$$

$\therefore$ $$\gamma = \left(1+\frac{2}{f}\right) \qquad ...(i)$$

Eq. (i) gives the relation between γ and degrees of freedom. Clearly, we can find the value of γ for a polyatomic gas from the degrees of freedom of its molecules.

Example 24.14. Calculate the total number of degrees of freedom possessed by the molecules of 1 cm^3 of H_2 gas at N.T.P.

Solution. Number of H_2 molecules in 22.4 litres (= 22400 cm^3) at N.T.P.

$$= 6.02 \times 10^{23}$$

Number of H_2 molecules in 1 cm^3 at N.T.P.

$$= \frac{6.02 \times 10^{23}}{22400} = 2.6875 \times 10^{19}$$

Number of degrees of freedom associated with each H_2 (diatomic) molecule = 5.

$\therefore$ Total number of degrees of freedom associated with 1 cm^3 of gas

$$= 2.6875 \times 10^{19} \times 5 = 1.34375 \times 10^{20}$$

Example 24.15. How many degrees of freedom are associated with 2 g of He at N.T.P.? Calculate the amount of heat energy required to raise the temperature of this amount from

27°C to 127°C. Given, Boltzmann's constant = 1.38×10^{-23} J molecule^{-1} K^{-1} and Avogadro's number = 6.02×10^{23}.

Solution. Molecular weight of He = 4

$\therefore$ Number of molecules in 4 g of He = 6.02×10^{23}

Number of molecules in 2 g of He = $\frac{1}{2} \times 6.02 \times 10^{23} = 3.01 \times 10^{23}$

Since He is a monoatomic gas, degrees of freedom associated with each of its molecule are 3.

$\therefore$ Total degrees of freedom of 3.01×10^{23} molecules

$$= 3 \times 3.01 \times 10^{23} = \mathbf{9.03 \times 10^{23}}$$

Energy associated with 1 degree of freedom per molecule

$$= \frac{1}{2} kT$$

$\therefore$ Energy associated with 2g of He at 300 K (27 + 273 = 300 K) is

$$E_1 = 9.03 \times 10^{23} \times \frac{1}{2} \times 1.38 \times 10^{-23} \times 300 = 1869.2 \text{ J}$$

Energy associated with 2g of He at 400 K (127 + 273 = 400 K) is

$$E_2 = 9.03 \times 10^{23} \times \frac{1}{2} \times 1.38 \times 10^{-23} \times 400 = 2492.3 \text{ J}$$

Heat energy required to raise the temperature from 27°C to 127°C

$$= E_2 - E_1 = 2492.3 - 1869.2 = \mathbf{623.1 \text{ J}}$$

24.23. MEAN FREE PATH

Although gas molecules move with high speeds (several hundred metres per second), it still takes some time for the strong odour of a gas (*e.g* ammonia) to diffuse through a room. This is due to several collisions which a molecule makes with other molecules. Therefore, molecules do not travel from one side of the room to the other in a straight line. Between collisions, the molecules move with constant speed in straight lines.

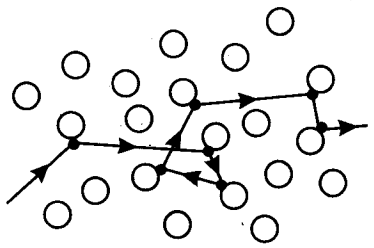

Fig. 24.8

It we were to follow the path of a particular molecule (dark one), we would expect it to follow a zig-zag path as shown in Fig. 24.8. Between each collision, the *molecule would move in a straight line path (Not quite true if we take account of the small intermolecular forces that act between collisions). An important parameter for a given situation is the mean free path. *The* **mean free path** *is defined as the average distance a molecule travels between collisions.* We would expect that the greater the gas density and larger the molecules, the shorter the mean free path would be.

Expression for mean free path. We shall now find the expression for the mean free path for a gas molecule. As an approximation, suppose the gas is made up of molecules which are hard spheres of radius r. A collision will occur whenever the centres of two molecules come within a distance $2r$ of one another. Let us assume that molecule C (See Fig. 24.9) moves a

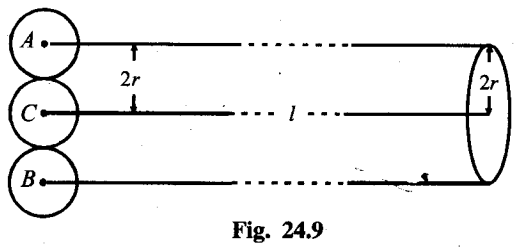

Fig. 24.9

* The size of the considered molecule has been shown smaller for clarity of the figure.

distance l while other molecules remain stationary. Then molecules such as A and B, situated at a distance $2r$ from C, are those with which C *just* makes a collision. Consequently, molecule C makes a collision with all those molecules whose centres lie inside a volume $\pi(2r)^2 \times l = 4\pi r^2 l$

Suppose n is the number of molecules per unit volume in the gas. Then,

$$\text{Number of collisions} = \text{Number of molecules in volume } 4\pi r^2 l$$
$$= n \times 4\pi r^2 l$$

$$\therefore \quad \text{Mean free path, } \lambda = \frac{\text{Distance moved}}{\text{Number of collisions}} = \frac{l}{n \times 4\pi r^2 l}$$

$$\therefore \quad \lambda = \frac{1}{4\pi r^2 n} \qquad ...(i)$$

Thus we find that λ is inversely proportional to the cross–sectional area ($= \pi r^2$) of the molecules and to their concentration n (number of molecules per unit volume).

Collision frequency. *The number of collisions per second is called collision frequency f.* Referring to Fig. 24.9, if v is the average speed of gas molecules and t seconds is the time taken to travel distance l, then, $l = v\,t$.

$$\text{Number of collisions in time } t = n \times 4\pi r^2 l$$
$$= n \times 4\pi r^2 v t \qquad (\because\ l = v\,t)$$

$$\therefore \quad \text{Collision frequency, } f = \frac{n \times 4\pi r^2 v t}{t} = 4\pi n r^2 v \qquad ...(ii)$$

Eqs. (*i*) and (*ii*) are not fully correct since we assumed the other molecules are all at rest. Since the molecules are moving about in all directions, the chance of a collision by a molecule is greater. Taking this into account, the correct results are :

$$\lambda = \frac{1}{4\pi\sqrt{2}\, r^2 n} \quad \ldots\ldots\ldots\ldots \sqrt{2} \text{ times less than eq. } (i)$$

$$f = 4\pi\sqrt{2}\, n r^2 v \quad \ldots\ldots\ldots\ldots \sqrt{2} \text{ times more than eq. } (ii)$$

Note. At extremely low densities (*i.e* very small n), the mean free path becomes comparable with the dimensions of the container and the concept of mean free path then loses its meaning. It is because in this situation collisions with the container may occur more frequently than collisions with other molecules.

Example 24.16. Estimate the mean free path of air molecules at S.T.P. The diameter of O_2 and N_2 molecules is about 3×10^{-10} m.

Solution. We know that 1 mole of an ideal gas at S.T.P. occupies a volume of 22.4×10^{-3} m^3.

$$\therefore \quad n = \frac{6.02 \times 10^{23} \text{ molecules}}{22.4 \times 10^{-3}\ \text{m}^3} = 2.69 \times 10^{25} \text{ molecules/m}^3$$

$$\text{Mean free path, } \lambda = \frac{1}{4\pi\sqrt{2}\, r^2 n}$$
$$= \frac{1}{4\pi\sqrt{2}\left(1.5 \times 10^{-10}\right)^2 \times \left(2.69 \times 10^{25}\right)} \simeq 9 \times 10^{-8} \text{ m}$$

Example 24.17. Find (*i*) the mean free path and (*ii*) collision frequency for nitrogen molecules at a temperature of 20°C and a pressure of 1 atm. Assume a molecular diameter of 2×10^{-10} m. Given that the average speed of a nitrogen molecule at 20°C is 511 m/s.

Solution. $P\,V = N\,k\,T$

Now $$n = \frac{N}{V} = \frac{P}{kT}$$

Here $P = 1.01 \times 10^5$ N/m^2 ; $k = 1.38 \times 10^{-23}$ J/K ; $T = 273 + 20 = 293$ K

$$\therefore \quad n = \frac{1.01\times10^5}{(1.38\times10^{-23})\times(293)} = 2.5\times10^{25} \text{ molecules/m}^3$$

(*i*)Mean free path, $\lambda = \frac{1}{4\pi\sqrt{2}\, r^2 n} = \frac{1}{4\pi\sqrt{2}\,(1\times10^{-10})^2 \times 2.5\times10^{25}} = 2.25 \times 10^{-7}$ m

(*ii*)Collision frequency, $f = 4\pi\sqrt{2}\, r^2 n v$

$$= \frac{v}{\lambda} \quad \left(\because \frac{1}{\lambda} = 4\pi\sqrt{2}\, r^2 n\right)$$

$$= \frac{511}{2.25\times10^{-7}} = 2.27 \times 10^9 \text{ collisions/second}$$

24.24. AVOGADRO'S NUMBER

According to Avogadro's hypothesis, gram atomic masses of all elements contain the same number of atoms and gram molecular masses of all substances contain an equal number of molecules. He had little idea of the value of this number (now called Avogadro's number N_A) except that it must be very large. Now, we know that value of this number is $N_A = 6.02 \times 10^{23}$. The gram atomic mass of oxygen is 16 g. It means that 16 g of oxygen contains 6.02×10^{23} atoms. Again, the gram molecular mass of oxygen is 32g. It means that 32g of molecular oxygen contains 6.02×10^{23} molecules.

There are several methods to determine the value of N_A. We discuss here one simple method. In an electrolytic cell that deposits silver, it takes 96500 C to deposit one gram atomic mass of silver (*i.e.*, 107.88 g of silver) or Avogadro's number of silver atoms (N_A). The 96500 C that pass through the cell are picked up from the cathode, one electron at a time, by Ag^+ ions that are being reduced to silver atoms. Since each silver ion accepts one electron, the question becomes: How many electron charges are there in 96500 C? We know that charge on one electron = 1.602×10^{-19}C.

$$\therefore \quad N_A = \frac{96500}{1.602\times10^{-19}} = 6.02\times10^{23}$$

Thus there are 6.02×10^{23} atoms in 107.88 g of silver. In other words, one gram atomic mass of silver contains 6.02×10^{23} atoms. Similarly, in 22.99 g of Na there are 6.02×10^{23} atoms ; 18.008 g of water contains 6.02×10^{23} molecules. Again 22.4 litres of any gas at N.T.P. contains 6.02×10^{23} molecules.

Now the gram atomic mass of carbon –12 is 12 g. It means that there are 6.02×10^{23} atoms in 12 g of carbon. Now 1 a.m.u. is equal to 1/12th of the mass of carbon atom ($_6C^{12}$).

$$\therefore \quad 1 \text{ a.m.u.} = \frac{1}{12} \times \left(\frac{12\times10^{-3}}{6.02\times10^{23}}\right) = 1.66\times10^{-27} \text{ kg}$$

or $\quad 1$ a.m.u.$= 1.66 \times 10^{-27}$kg

24.25. THE MOLE

The atomic masses give no direct information about the actual masses of the atoms. However, in chemical experiments as well as in industrial chemistry, it is important to know the actual amounts of masses of the ingredients and products. For this purpose, we use the unit *mole*. The mole is the unit of quantity called 'amount of mass'.

One mole *of any substance (solid, liquid or gas) is the amount of substance equal to the atomic mass or molecular mass in grams and contains* N_A (6.02×10^{23}) *atoms or molecules.*

It is abbreviated as mol. Thus 16 g oxygen means one mole of oxygen atoms and it contains 6.02×10^{23} atoms of oxygen. Similarly, 32 g of oxygen means one mole of oxygen molecules and it contains 6.02×10^{23} molecules.

24.26. SIGNIFICANCE OF AVOGADRO'S NUMBER

The importance of Avogadro's number (mole concept) is clear from the following applications :

(*i*) **It helps to calculate the actual mass of one atom of an element.**

$$\text{Mass of one atom of an element} = \frac{\text{Gram atomic mass}}{\text{Avogadro's number}}$$

For example, 16 g of oxygen contains 6.02×10^{23} atoms.

$$\therefore \text{ Mass of one oxygen atom} = \frac{16}{6.02\times10^{23}} = 2.66\times10^{-23} \text{ g} = 2.66\times10^{-26} \text{ kg}$$

(*ii*) **It helps to calculate the mass of one molecule of a substance.**

$$\text{Mass of one molecule of substance} = \frac{\text{Gram molecular mass}}{\text{Avogadro's number}}$$

For example, there are 6.02×10^{23} molecules in 18 g of water.

$$\therefore \text{ Mass of one molecule of water} = \frac{18}{6.02\times10^{23}} = 2.99\times10^{-23} \text{ g} = 2.99\times10^{-26} \text{ kg}$$

(*iii*) **It helps to calculate the number of atoms in a given mass of an element.** Suppose an element has a mass of x gms. Then number of atoms in x gms of element

$$= \frac{6.02\times10^{23}}{\text{Gram atomic mass}} \times x$$

(*iv*) **It helps to calculate the number of molecules in a given mass of a substance.** Suppose a substance has a mass of x gms. Then the number of molecules in x gms of the substance

$$= \frac{6.02\times10^{23}}{\text{Gram molecular mass}} \times x$$

(*v*) **It helps to calculate the number of molecules present in a given volume of a gas at N.T.P.** The volume of 1 mol of any gas at N.T.P. is 22.4 litres.

$\therefore$ Number of molecules in x litres of a gas at N.T.P.

$$= \frac{6.02\times10^{23}}{22.4} \times x$$

Example 24.18. How many copper atoms are there in a solid cube of copper which is 1 cm on each side? Copper has an atomic mass of 63.6 g/mol and a density of 8.96 g/cm^3.

Solution. The volume of cube = $l^3 = (1 \text{ cm})^3 = 1 \text{ cm}^3$

Mass of copper cube = Density × Volume = $8.96 \times 1 = 8.96$ g

Number of moles present in the mass of copper mass

$$= \frac{\text{mass}}{\text{gram atomic mass}} = \frac{8.96}{63.6} = 0.141 \text{ mol}$$

$\therefore$ Number of atoms = Number of moles × N_A

$= 0.141 \times 6.023 \times 10^{23} =$ **8.49×10^{22} atoms**

This example reveals an interesting fact. To find the volume occupied by one atom, we divide the total volume (1 cm^3) by the number of atoms. This gives an average volume per atom of 1.18×10^{-23} cm^3 or 1.18×10^{-29} m^3. If, for simplicity, we assign each atom a small cubic volume

of this size, then the cube edge has a length equal to the cube root of the volume or 2.28×10^{-10} m. Without making any assumption about the shape of atoms or how they are arranged, we may conclude that the linear dimensions of an atom are of the order of 10^{-10} m *i.e.,* size of atom $\simeq 10^{-10}$ m.

CONCEPTUAL QUESTIONS

Q.1. What is meant by Boltzmann's constant?

Ans. Boltzmann's constant, $k = \dfrac{R}{N_A}$

where R = universal gas constant ; N_A = Avogadro's number.

Note that the gas constant R refers to 1 mole of a substance, as does Avogadro's number N_A. The Boltzmann's constant k (= R/N_A) is therefore independent of the concept of the mole and refers to the molecules. We can think of Boltzmann's constant k as the gas constant per molecule.

Q.2. On which factor/factors does the average kinetic energy of gas molecules depend?

Ans. It depends *only* upon the absolute temperature and is directly proportional to it.

Q.3. On reducing the volume of a gas at constant temperature, the pressure of a gas increases. Why?

Ans. On reducing the volume, the number of molecules per unit volume increases. Therefore, more number of molecules collide with the walls per second and hence a larger momentum is transferred to the walls per second. If the volume is halved, then the number of molecules per m^3 will be doubled. Therefore, according to kinetic theory of gases, the pressure will be doubled. This is the explanation of Boyle's law on the basis of kinetic theory.

Q.4. Two vessels of the same size are at the same temperature. One of them holds 1 kg of H_2 gas and the other holds 1 kg of N_2 gas. Which of the vessels contains more number of molecules?

Ans. Now 2 g of H_2 contains N_A molecules where N_A is the Avogadro's number. Therefore, 1 kg (= 1000 g) of hydrogen will contain number of molecules = $(N_A/2) \times 1000 = 500\ N_A$. In case of N_2 gas, the number of molecules in 1 kg = $(N_A/28) \times 1000 \simeq 36\ N_A$. Clearly, the vessel containing hydrogen will have more number of molecules.

Q.5. Under what conditions do the real gases obey more strictly the gas equation $PV = RT$? Explain.

Ans. The essential properties of the molecules of an ideal gas are zero volume of the molecules and no mutual force between them. At low pressure, the volume of a gas is large so that the volumes of the molecules is negligible as compared to the volume of the gas. At high temperature, the kinetic energy of molecules is quite large so that effect of intermolecular forces on the motion of the molecules is negligible. Hence at low pressure and high temperature (*i.e.,* lower density), a real gas is approximately ideal. Consequently, the gas equation gives more accurate results.

Q.6. Suppose that our system consists of a mixture of several ideal gases in thermal equilibrium in a container of volume V. What is the resultant pressure exerted by the mixture of gases?

Ans. It is assumed that each component behaves as an ideal gas, and moreover, that the constituent molecules of different gases do not react with each other—for example, no chemical combinations are formed. Therefore, the molecules of each gas move about in the container at their normal thermal speeds, striking the walls and exerting a pressure.

The total pressure P exerted by all the molecules of the mixture is then,

$$P = P_1 + P_2 + \ldots\ldots\ldots \quad \ldots(i)$$

where P_1, P_2 .....are the *partial pressures* of the constituent gases and are given by ;

$$P_1 = \frac{1}{3}\frac{m_1 N_1 \overline{v_1^2}}{V} \; ; \; P_2 = \frac{1}{3}\frac{m_2 N_2 \overline{v_2^2}}{V} \text{ and so on.}$$

Here N_1, N_2 ..... are the number of molecules of gas 1, gas 2 etc., confined to a volume V.

Eq. (*i*) is known as Dalton's law of partial pressures and may be stated as under : *The resultant pressure exerted by a mixture of ideal gases occupying a given volume is equal to the sum of the pressures exerted separately by its various constituent gases.* Ordinary air is principally a mixture of 80% N_2 and 20% O_2. Since the total atmospheric pressure is 1 atm at sea level, the pressures exerted by N_2 and O_2 separately are 0.8 atm and 0.2 atm respectively.

Q.7. A certain mass of gas is heated first in a small vessel of volume V_1 and then in a large vessel of volume V_2. Draw the P-T graph for the two cases.

Ans. From perfect gas equation,

$$P = \frac{RT}{V}$$

For a given temperature, $P \propto 1/V$. Therefore, when the gas is heated in a small vessel (volume V_1), the pressure will increase more rapidly than when heated in a large vessel (volume V_2). As a result, the slope of $P - T$ graph will be more in case of a small vessel than that of the large vessel (See Fig. 24.10).

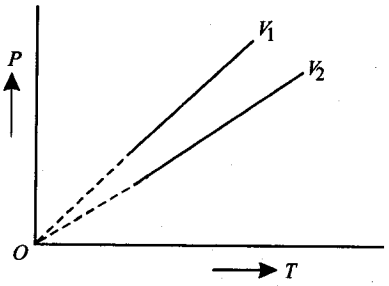

Fig. 24.10

Q.8. The average translational kinetic energy of O_2 molecules (molar mass 32) at a particular temperature is 0.48 eV. What will be the translational kinetic energy of N_2 molecules (molar mass 28) in eV at the same temperature?

Ans. Average translational K.E. of molecules $= \frac{3}{2} kT$

Since the average translational K.E. of molecules depends only on temperature, the average translational K.E. of N_2 molecules will also be 0.48 eV.

VERY SHORT ANSWER QUESTIONS

Q.1. What is an ideal gas?

Ans. An ideal gas or perfect gas is that gas which strictly obeys the gas laws (*e.g.*, Boyle's law, Charles' law etc.).

Q.2. Under which conditions Boyle's law is perfectly obeyed?

Ans. Boyle's law is perfectly obeyed at high temperature and low pressure of a gas.

Q.3. When does Charles' law fail?

Ans. Charles' law fails at low temperatures. However, it is reasonably applicable at high and normal temperatures.

Q.4. What is the value of universal gas constant R?

Ans. $R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1} = 1.98 \text{ cal mol}^{-1} \text{ K}^{-1}$.

Q.5. What is Boltzmann's constant and what is its value in SI units?

Ans. Boltzmann's constant $k = R/N_A$ where R is the universal gas constant and N_A ($= 6.023 \times 10^{23}$) is the Avogadro's number.

$$k = \frac{R}{N_A} = \frac{8.31 \text{ J mol}^{-1} \text{ K}^{-1}}{6.023 \times 10^{23} \text{ mol}^{-1}} = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

Q.6. What is the dimensional formula of R?

Ans. $$PV = RT \quad \therefore \quad R = \frac{PV}{T}$$

$$\therefore \quad [R] = \frac{[M\,LT^{-2}][L^3]}{[L^2][K]} = [M\,L^2\,T^{-2}\,K^{-1}]$$

Q.7. 500 cm^3 of hydrogen and 500 cm^3 of oxygen are given at N.T.P. Which will have larger number of molecules?

Ans. Both gases will have equal number of molecules. It is because according to Avogadro's law, equal volumes of all gases at the same temperature and pressure contain equal number of molecules.

Q.8. Two different gases have the same temperature. Does it mean that their molecules have the same *r.m.s.* speeds?

Ans. $v_{r.m.s.} = \sqrt{\frac{3kT}{m}}$ or $v_{r.m.s.} \propto \frac{1}{\sqrt{m}}$ ... for same T

Thus for a given temperature, the r.m.s. speed of a molecule depends upon its mass (m). Since the molecules of different gases have different masses, the r.m.s. speeds of their molecules will be different.

Q.9. Absolute temperature of a gas is increased four times. What will be the increase in the r.m.s. velocity of its molecules?

Ans. $v_{r.m.s.} \propto \sqrt{T}$. Therefore, if absolute temperature is increased four times, r.m.s. velocity of molecules will be doubled.

Q.10. Why does the pressure of a gas increase on heating?

Ans. Pressure, $P = \frac{2}{3}$ (Average K.E. of gas per unit volume)

When gas is heated, its temperature increases and average K.E. of its molecules increases. Therefore, pressure increases on heating the gas.

Q.11. All molecular motion ceases at 0 K. Explain.

Ans. The average kinetic energy $(\overline{\text{K.E.}})$ of a gas molecule is given by ;

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT \quad \text{or} \quad \frac{1}{2}m\overline{v^2} \propto T$$

If $T = 0$, then $\frac{1}{2}m\overline{v^2} = 0$.

Therefore, at $T = 0$ K, the molecular motion stops.

Q.12. Why temperature less than absolute zero (0 K) is not possible?

Ans. At 0 K, the kinetic energy of the gas molecules becomes zero. Therefore, more decrease in the kinetic energy is not possible. For this reason, the temperature less than 0 K is not possible.

Q.13. Can the temperature of a gas be increased keeping its pressure and volume fixed?

Ans. $PV = RT$. Therefore, we cannot change the temperature of a gas without changing P or V.

Q.14. The pressure of a gas can be increased by decreasing its volume. Why?

Ans. $P = \frac{1}{3}\frac{Nm\overline{v^2}}{V}$. Here V is the volume of the gas.

It is clear that pressure of the gas will increase when its volume is decreased.

Q.15. The r.m.s. velocity of a gas molecule is $v_{r.m.s.}$ at pressure P. If the pressure is increased two times, what will be the r.m.s. velocity?

Ans. $v_{r.m.s.} = \sqrt{\frac{3kT}{m}}$. Therefore, so long as the temperature of the gas remains constant, there is no effect of change of pressure on the r.m.s. velocity of the molecule.

Q.16. A given amount of gas at 20°C has a pressure P. What will be the temperature at $2P$ (volume remains constant)?

Ans. $\frac{P}{T}=\frac{P_1}{T_1}$ or $T_1 = P_1 \times \frac{T}{P} = 2P \times \frac{(273+20)}{P}$ = 586 K = 313°C

Q.17. Two identical cylinders at same temperature contain helium at 2.5 atm. and argon at 1 atm. respectively. What will be the pressure if both gases are filled in one of the cylinders?

Ans. According to Dalton's law of partial pressures, total pressure, $P = P_1 + P_2$ = 2.5 atm. + 1 atm. = 3.5 atm.

Q.18. What are the different ways of increasing the number of molecular collisions per unit time in a gas?

Ans. By (*i*) increasing temperature (T) of the gas (*ii*) increasing the number of molecules (N) of the gas (*iii*) decreasing volume (V) of the gas.

Q.19. In thermal equilibrium, how average velocity of a gas depends on temperature?

Ans. Average velocity ($\bar{v}$) of gas molecule = $\sqrt{\frac{8\,kT}{\pi m}}$. Therefore, $\bar{v} \propto \sqrt{T}$.

Q.20. How does the r.m.s. velocity of a gas molecule change with (*i*) temperature (*ii*) mass of molecule?

Ans. $v_{r.m.s.} = \sqrt{\frac{3\,kT}{m}}$. Therefore, r.m.s. velocity of a gas molecule is (*i*) directly proportional to the square root of absolute temperature (T) of the gas (*ii*) inversely proportional to the square root of the mass (m) of the molecule.

SHORT ANSWER QUESTIONS

Q.1. In terms of kinetic theory of gases, explain why the pressure of a gas in a container increases when a gas is heated?

Ans. When a gas is heated, its temperature increases. The increase in temperature increases the pressure of the gas due to following reasons :

(*i*) With the increase in temperature of the gas, the velocity of gas molecules increases. Therefore, the number of collisions per second against the walls of the container increases and so does the gas pressure.

(*ii*) Due to increased velocity, the transfer of momentum per second to the walls increases and so does the pressure.

Q.2. The volume of a sample of a gas is increased. In terms of kinetic theory of gases, why does the gas pressure decrease?

Ans. When the volume of a gas is increased, its pressure decreases due to the following reasons :

(*i*) The molecules of the gas in the container travel longer distances between collisions on the walls of the container.

(*ii*) The collisions are distributed over a large area.

Q.3. A vessel containing 10 litres of air at atmospheric pressure (760 mm of Hg) is connected to an evacuated vessel of capacity 9 litres. What is the resultant air pressure?

Ans. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$. Since temperature is constant, $P_1V_1 = P_2V_2$.

$\therefore P_2 = \frac{P_1V_1}{V_2} = \frac{760 \times 10}{(10+9)}$ = 400 mm of Hg

Q.4. What is the r.m.s. velocity of a nitrogen molecule at N.T.P.?

Ans. The r.m.s. velocity of nitrogen molecule is $v_{r.m.s.} = \sqrt{\frac{3RT}{M}}$

Here R = 8.31 J $mol^{-1}K^{-1}$; T = 273 K ; M = 28 g = 28 × 10^{-3} kg

$\therefore$ $v_{r.m.s.} = \sqrt{\frac{3 \times 8.31 \times 273}{28 \times 10^{-3}}}$ = 493 ms^{-1}

Q.5. A gas at certain volume and temperature has a pressure equal to 75 cm of Hg. If the mass of the gas is doubled at the same volume and temperature, what will be its new pressure?

Ans. $P = \frac{1}{3} m N \frac{\overline{v^2}}{V}$. Mass of gas, $M = mN$.

$\therefore \quad P = \frac{1}{3} \frac{M \overline{v^2}}{V}$

Therefore, at constant volume and temperature, $P \propto M$. Hence if mass of the gas is doubled, the pressure will also be doubled *i.e.,* 150 cm of Hg.

Q.6. An electric fan is switched on in a closed room. Will the air in the room be heated or cooled ?

Ans. The mechanical energy of the moving blades is converted into heat which increases the K.E. of the air molecules. Therefore, the temperature of the room will increase.

Q.7. There are N molecules of a gas in a container. If the number of molecules is increased to 2N, how does it affect (*i*) gas pressure (*ii*) K.E. of molecules and (*iii*) r.m.s. speed of molecules?

Ans. (*i*) $P = \frac{1}{3} \frac{N m \overline{v^2}}{V}$. Therefore, if N is increased to $2N$, the gas pressure P is doubled.

(*ii*) Average K.E. of gas $= \frac{1}{2} mN \overline{v^2}$. Therefore, if N is increased to $2N$, the K.E. of the gas is doubled.

(*iii*) $v_{r.m.s.} \propto \sqrt{T}$. Since r.m.s. velocity of gas molecules is independent of number of molecules (N), the r.m.s. velocity remains the same.

Q.8. How is average kinetic energy of gas molecules affected by (*i*) nature of gas (*ii*) absolute temperature of gas (*iii*) volume of gas?

Ans. Average kinetic energy of a gas molecule, $\overline{\text{K.E.}} = \frac{3}{2} kT$

(*i*) The average K.E. of gas molecules is independent of the nature of the gas.

(*ii*) The average K.E. of gas molecules depends upon the absolute temperature (T) of the gas.

(*iii*) The average K.E. of gas molecules is independent of the volume of the gas.

Q.9. The speeds of 5 molecules of a gas (in arbitrary units) are 2, 3, 4, 5, 6. What is the root mean square speed of these molecules?

Ans. $v_{r.m.s.} = \sqrt{\frac{2^2+3^2+4^2+5^2+6^2}{5}} = 4.24$

Q.10. When the temperature of a gas filled in a closed vessel is increased by 1°C, its pressure increases by 0.4%. What was the initial temperature of the gas?

Ans. $T_1 = (T + 1)K \; ; \; P_1 = P + \frac{0.4\,P}{100} = 1.004\,P$

Since the volume remains the same, according to pressure law,

$$\frac{T}{T_1} = \frac{P}{P_1} \quad \text{or} \quad \frac{T}{T+1} = \frac{P}{1.004\,P} \quad \therefore \quad T = 250 \text{ K}$$

Q.11. A cylinder contains 10 kg of gas at a pressure of 10^7 N/m^2. Some quantity of gas is taken out from the cylinder and its pressure decreases to 2.5×10^6 N/m^2. What is the quantity of gas (temperature of gas is constant) taken out?

Ans. $P = \frac{1}{3} \frac{N m \overline{v^2}}{V}$. Here Nm (= number of molecules × mass of 1 molecule) is the mass M of the gas.

Therefore, $P \propto M$.

$\therefore \quad \frac{P_2}{P_1} = \frac{M_2}{M_1}$ or $\frac{2.5 \times 10^6}{10^7} = \frac{M_2}{10} \quad \therefore \quad M_2 = 2.5$ kg

Therefore, quantity of gas taken out from the cylinder $= M_1 - M_2 = 10 - 2.5 = 7.5$ kg

LONG ANSWER QUESTIONS

1. State and explain (*i*) Boyle's law (*ii*) Charles' law. **[Refer to Arts. 24.3 and 24.4]**
2. Derive perfect gas equation. **[Refer to Art. 24.6]**
3. What are the basic assumptions of kinetic theory of gases? Explain pressure exerted by an ideal gas. **[Refer to Arts. 24.8 and 24.9]**
4. Derive an expression for the pressure exerted by an ideal gas. **[Refer to Art. 24.9]**
5. Give expressions for pressure exerted by an ideal gas in four alternate forms. **[Refer to Art. 24.10]**
6. From kinetic theory of gases, explain kinetic interpretation of temperature and absolute zero. **[Refer to Art. 24.12]**
7. Explain mean velocity, r.m.s. velocity and most probable velocity of gas molecules. **[Refer to Arts. 24.13, 24.14 and 24.15]**
8. Explain the various gas laws from the kinetic theory of gases. **[Refer to Art. 24.16]**
9. What is meant by degrees of freedom? State the law of equipartition of energy. **[Refer to Arts. 24.18 and 24.19]**
10. Explain the term mean free path. **[Refer to Art. 24.23]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. The experimental relationships between the pressures, volumes and temperatures of gases are called *gas laws.*
2. There are three gas laws.
   (*i*) At constant T, $P\ V$ = constant .....Boyle's law
   (*ii*) At constant P, $\frac{V}{T}$ = constant .....Charles' law
   (*iii*) At constant V, $\frac{P}{T}$ = constant ..... Pressure law
   Note that T is the temperature measured in kelvin.
3. An ideal (or perfect) gas is that which obeys the gas laws exactly. No gas is perfect.
4. The three gas laws can be combined to give the ideal gas equation :
   $$P\ V = n\ R\ T$$
   where n = number of moles ; R = universal gas constant (= 8.31 J $mol^{-1}K^{-1}$)
   For 1 mole of gas (n =1), $PV=RT$
5. The value of gas constant R for one mole of a gas is independent of the atomicity of the gas.
6. The ideal gas equation for 1 gram of the gas is
   $$PV = \frac{RT}{M} = rT$$
   where M is the molecular mass of the gas in grams.
   For m grams of gas, we have,
   $$PV = \frac{mRT}{M} = mr^{*}T$$
7. For a fixed mass of gas whose conditions are changed from P_1, V_1, T_1 to P_2, V_2, T_2, the equation of the state can be written as :
   $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ .... for a fixed mass of gas
   Note that temperature must always be measured in kelvin.
8. Avogadro's law can be stated as under :
   Equal volumes of all gases at the same temperature and pressure contain equal number of molecules.
   Avogadro's law can be easily proved. Consider two gases 1 and 2. The gas equation for each gas is
   For gas 1 : $P_1\ V_1 = n_1 R\ T_1$, For gas 2 : $P_2\ V_2 = n_2\ R\ T_2$
   If $P_1 = P_2$, $V_1 = V_2$ and $T_1 = T_2$, then $n_1 = n_2$. Thus Avogadro's law stands proved.
9. S.T.P. means *standard temperature and pressure i.e.* 273K and 76cm of Hg (= 1.013×10^5 N/m^2).
10. The volume of 1 mole of any gas at S.T.P. is 22.4 litres (= 22.4×10^{-3} m^3).
    Now $PV = nRT$ or $V = \frac{nRT}{P}$

* Note that although R is same for all gases, the value of r is different for different gases depending upon their molecular masses.

Therefore, for 1 mole of any gas at S.T.P., we have,

$$V = \frac{1\times8.31\times273}{1.013\times10^5} = 22.4\times10^{-3}\ \text{m}^3$$

11. In the kinetic theory of gases, we explain the experimentally observed properties of gases by considering the motion of their molecules (or atoms).

12. The pressure of a gas is due to its molecules bombarding the walls of the container.

13. The pressure exerted by a gas can be written in the following ways :

(*i*) $$P = \frac{1}{3}\frac{mN\overline{v^2}}{V}$$

Here N = total number of molecules ; m = mass of each molecule ; $\overline{v^2}$ = mean square velocity of molecules ; V = volume of the gas

(*ii*) Now $m\,N$ is the total mass of the gas so that $m\,N/V$ is the density of the gas.

$$\therefore\ P = \frac{1}{3}\rho\overline{v^2}$$

(*iii*) $$P = \frac{2}{3}\frac{N}{V}\left(\overline{\text{K.E.}}\right)$$

$\overline{\text{K.E}}$ = average kinetic energy of one molecule

(*iv*) $P = \frac{2}{3}$ (Average kinetic energy of gas per unit volume)

14. In an ideal gas, there would be no forces between its molecules. *Therefore, the internal energy of an ideal gas is entirely kinetic and depends only on temperature.*

15. The average speed of a gas molecule of mass m is

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$$

Here k = Boltzmann's constant (= 1.38×10^{-23} JK^{-1})

16. If v_1 and v_2 are r.m.s. velocities of the molecules of two gases of molecular weights M_1 and M_2, then at a given temperature,

$$\frac{v_1}{v_2} = \sqrt{\frac{M_2}{M_1}}$$

17. Average translational K.E. of a molecule = $\frac{3}{2}kT = \frac{3}{2}\frac{R}{N_A}T$.

18. At the same kelvin temperature, the average kinetic energy of the molecules of different gases is the same.

19. One mole of any form of matter contains a mass in grams equal to the atomic weight or molecular weight of that form of matter. For example, 1 mole of atomic hydrogen (H_2) has a mass of lg and one mole of molecular hydrogen (H_2) has a mass of 2 g.

20. One mole of a substance (solid, liquid or gas) contains 6.023×10^{23} atoms or molecules. For example, one mole (lg) of atomic hydrogen contains 6.023×10^{23} atoms and one mole (2 g) of molecular hydrogen contains 6.023×10^{23} molecules. It is interesting to note that kinetic theory of gases has linked the macroscopic world with microscopic world.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. The kinetic energy of one gm - molecule of a gas at normal temperature and pressure is ($R = 8.31$ J mol^{-1} K^{-1}) **[AFMC- Pune 1998]**
   (*a*) 0.56×10^4J (*b*) 1.3×10^2J
   (*c*) 2.7×10^3J (*d*) 3.4×10^3J

2. If an ideal gas has volume V at 27°C and it is heated at a constant pressure so that its volume becomes $1.5V$, then the value of final temperature is **[Pb PMT 2001]**
   (*a*) 327 K (*b*) 873 K
   (*c*) 177°C (*d*) 600°C

3. If an oxygen atom and hydrogen atom are having the same temperature, then ratio of their average kinetic energy is **[Pb PMT 2000]**
   (*a*) 1 : 1 (*b*) 2 :1
   (*c*) 4 : 1 (*d*) 1 : 4

4. R.M.S. velocity of a molecule is C at pressure P. If pressure is increased two times, then r.m.s. velocity becomes **[AFMC - Pune 1996]**
   (*a*) 0.5 C (*b*) C
   (*c*) 2 C (*d*) 3 C

5. A cylinder contains 10 kg of gas at a pressure of 10^7 N/m^2. The quantity taken out of the cylinder if the final pressure is 2.5×10^6N/m^2 (temperature of gas is constant) is **[Pb PMT 1999]**
   (*a*) 7.5 kg (*b*) zero
   (*c*) 9.5 kg (*d*) 14.2 kg

6. When the temperature of a gas filled in a closed vessel is increased by 1°C, its pressure increases by 0.4%. The initial temperature of the gas was **[MP PMT 1999]**
   (*a*) 25°C (*b*) 250 K
   (*c*) 250°C (*d*) 25 K

7. The average kinetic energy of a gas molecule at 27°C is 6.21×10^{-21} J. Its average kinetic energy at 227°C will be **[AIIMS 1999]**
   (*a*) 24.8×10^{-21} J (*b*) 6.7×10^{-21} J
   (*c*) 10.35×10^{-21} J (*d*) 11.35×10^{-21} J

8. The kinetic energy of 1 gram of hydrogen gas at 27°C will be **[AIIMS 1999]**
   (*a*) 1.87×10^3J (*b*) 1.57×10^3J
   (*c*) 1.81×10^3 J (*d*) 1.73×10^3J

9. The speeds of 5 molecules of a gas (in arbitrary units) are 2, 3, 4, 5, 6. The root mean square speed for these molecules is **[MP PMT 2000]**
   (*a*) 2.91 (*b*) 3.52
   (*c*) 4.00 (*d*) 4.24

10. A given amount of gas at 20°C has a pressure P. The temperature at which the pressure will be $2P$ (at constant volume) is approximately **[AMU PMT 2000]**
    (*a*) 113°C (*b*) 40° C
    (*c*) 213°C (*d*) 313°C

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*d*)	**2.** (*c*)	**3.** (*a*)	**4.** (*b*)	**5.** (*a*)
6. (*b*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*d*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. According to kinetic theory of gases,

   K.E. of 1 mol of gas = $\frac{3}{2}RT$. Here $T = 273$ K ; $R = 8.31$ J mol^{-1}K^{-1}

   $\therefore$ K.E. of 1 mol of gas = $\frac{3}{2} \times 8.31 \times 273 = 3.4 \times 10^3$ J

2. $V_1 = V$; $T_1 = 27°C = 300$ K ; $V_2 = 1.5V$; $T_2 = ?$

   According to Charles' law, at constant pressure,

   $$\frac{V_1}{T_1} = \frac{V_2}{T_2} \quad \therefore \quad T_2 = \frac{T_1}{V_1} \times V_2 = \frac{300}{V} \times 1.5V = 450\text{K} = 177°\text{C}$$

3. K.E. of 1 mole of gas = $\frac{3}{2}RT$

Since K.E. depends on temperature only and is independent of nature of the gas, the K.E. of the two atoms will be the same.

4. So long as temperature remains constant, there is no effect of change of pressure on the r.m.s. velocity of the molecule.

5. Initial pressure, $P_1 = 10^7$ N/m^2 ; Initial mass, $M_1 = 10$ kg

Final pressure, $P_2 = 2.5 \times 10^6$N/m^2

According to kinetic theory of gases, $P = \frac{1}{3}\frac{Nm\overline{v^2}}{V}$

Now Nm (number of molecules × mass of 1 molecule) is the mass M of the gas. Therefore, $P \propto M$.

$$\therefore \frac{P_2}{P_1} = \frac{M_2}{M_1}$$

$$\text{or } M_2 = \frac{P_2}{P_1} \times M_1 = \frac{2.5\times10^6}{10^7} \times 10 = 2.5 \text{ kg}$$

$\therefore$ Quantity of gas taken out from the cylinder = $M_1 - M_2 = 10 - 2.5 = \mathbf{7.5\ kg}$

6. $T_1 = (T + 1)$K ; $P_1 = P + \frac{0.4\,P}{100} = 1.004\,P$

Since the volume remains constant, according to pressure law,

$\frac{T}{T_1} = \frac{P}{P_1}$ or $\frac{T}{T+1} = \frac{P}{1.004\,P}$. On solving ; $T = \mathbf{250\ K}$

7. According to Kinetic theory of gases, average K.E. of a molecule is

$$E = \frac{3}{2}kT \text{ or } E \propto T$$

$$\therefore \frac{E_{227}}{E_{27}} = \frac{227+273}{27+273} \text{ or } E_{227} = E_{27} \times \frac{500}{300}$$

$$\therefore E_{227} = 6.21\times10^{-21} \times \frac{500}{300} = \mathbf{10.35 \times 10^{-21}\ J}$$

8. K.E.of lgm of gas $= \frac{3}{2} \times \frac{RT}{M} = \frac{3}{2} \times \frac{8.31\times(273+27)}{2} = \mathbf{1.87 \times 10^3\ J}$

9. $v_{r.m.s.} = \sqrt{\frac{2^2+3^2+4^2+5^2+6^2}{5}} = \mathbf{4.24}$

10. According to pressure law, at constant volume,

$$\frac{P}{T} = \frac{P_1}{T_1} \text{ or } T_1 = P_1 \times \frac{T}{P} = 2P \times \frac{(273+20)}{P} = 586K = \mathbf{313°C}$$

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A gas at a temperature of 250 K is contained in a closed vessel. If the gas is heated through 1 K, then percentage increase in its pressure is **[CET Karnataka 2001]**

(*a*) 0.1% (*b*) 0.2 %

(*c*) 0.3 % (*d*) 0.4%

2. In thermal equilibrium, the average velocity of a gas molecule is **[CET Haryana 2001]**

(*a*) proportional to T^3

(*b*) proportional to T^2

(*c*) proportional to $\sqrt{T}$

(*d*) zero

3. The temperature of an ideal gas is increased from 120 K to 480 K. If at 120 K, the root mean square velocity of gas molecules is v, at 480 K it becomes **[IIT 1996]**

(*a*) 4 v (*b*) 2 v

(*c*) $v/2$ (*d*) $v/4$

4. Air is filled at 60°C in a vessel with open mouth. The vessel is heated to a temperature T so that 1/4th part of air escapes. Assuming the volume of vessel remains constant, the value of T is **[MP PET 1996]**
   (a) 80°C (b) 143°C
   (c) 333°C (d) 416°C

5. The number of degrees of freedom for molecules of a monoatomic gas is **[CET Karnataka 1998]**
   (a) 3 (b) 4
   (c) 5 (d) 6

6. What is the absolute mass of an atom of oxygen? Atomic mass of oxygen is 16 and Avogadro's number = 6.023×10^{23}. **[CET Haryana 2001]**
   (a) 2.657×10^{-15} kg (b) 2.657×10^{-26} kg
   (c) 2.657×10^{-20} kg (d) 2.657×10^{-18} kg

7. The r.m.s. velocity of nitrogen molecules at N.T.P. is **[B.H.U. 1995]**
   (a) 33 m/s (b) 493 m/s
   (c) 517 m/s (d) 546 m/s

8. A vessel contains 1 mole of O_2 gas (molar mass 32) at temperature T. The pressure of the gas is P. An identical vessel containing one mole of He (molar mass 4) at a temperature $2T$ has a pressure of **[IIT 1997]**
   (a) $P/8$ (b) P
   (c) $8P$ (d) $2P$

9. A container has N molecules at absolute temperature T. If the number of molecules is doubled but kinetic energy in the box remains the same, the absolute temperature of the gas is **[CET Karnataka 1998]**
   (a) T (b) $2T$
   (c) $T/2$ (d) zero

10. At what temperature is the r.m.s. speed of molecules of hydrogen twice that at S.T.P.? **[PET Rajasthan 1997]**
   (a) 273 K (b) 546 K
   (c) 819 K (d) 1092 K

11. An electric fan is switched on in a closed room. The air in the room **[MP PET 1996]**
   (a) is cooled (b) is heated
   (c) maintains its temperature
   (d) is heated or cooled depending upon atmospheric pressure

12. The average kinetic energy per molecule of helium gas at temperature T is E and molar gas constant is R. The Avogadro's number is **[CET Haryana 1996]**
   (a) $\frac{3RT}{E}$ (b) $\frac{3RT}{2E}$
   (c) $\frac{E}{2RT}$ (d) $\frac{RT}{2E}$

13. At constant pressure, V_1 and V_2 are the volumes of a given mass of a gas at temperatures 27°C and 54°C respectively. The ratio V_1/V_2 is **[CET Karnataka 1997]**
   (a) 54/27 (b) 27/54
   (c) 100/109 (d) $\sqrt{100/109}$

14. A vessel containing 10 litres of air at atmospheric pressure (760 mm of Hg) is connected with an evacuated vessel of capacity 9 litres. The resultant air pressure will be **[MP PET 1999]**
   (a) 180 mm (b) 760 mm
   (c) 400 mm (d) 40 mm

15. A gas at certain volume and temperature has a pressure equal to 75 cm of Hg. If the mass of the gas is doubled at the same volume and temperature, its new pressure will be **[MP PET 1998]**
   (a) 75 cm (b) 200 cm
   (c) 150 cm (d) 37.5 cm

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (d)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (a)
6. (b)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (d)
11. (b)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (c)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Since volume remains constant , according to pressure law,

$$\frac{P_1}{T_1} = \frac{P_2}{T_2} \quad \text{or} \quad \frac{P_2}{P_1} = \frac{T_2}{T_1} = \frac{251}{250} = 1.004$$

or $\frac{P_2}{P_1}-1=1.004-1$ or $\frac{P_2-P_1}{P_1}=0.004$

$\therefore$ %age increase in pressure = 0.004 × 100 = **0.4%**

2. Average velocity of gas molecule $=\sqrt{\frac{8RT}{\pi M}}$

3. R.M.S.velocity of gas molecule, $v_{r.m.s.}=\sqrt{\frac{3RT}{M}}$

$\therefore \quad v_{r.m.s.} \propto \sqrt{T} \quad \therefore \quad \frac{v_{480}}{v_{120}}=\sqrt{\frac{480}{120}}=2$ or $v_{480}=2\times v_{120}$ = **2** v

4. Suppose V is the initial volume of the air. When the air is heated to temperature T, $V/4$ air escapes. Therefore, final volume = $V+ V/4 = 5V/4 = 1.25\ V$. Since pressure remains constant, according to Charles' law,

$\frac{\text{Volume}}{\text{Asbolute temp.}}=\text{Constant}$ or $\frac{V}{273+60}=\frac{1.25V}{T}$ or $T=(273+60)\times 1.25=416K$ = **143°C.**

5. The number of degrees of freedom for the molecules of a monoatomic gas is **3.**

6. 0.016 kg of oxygen contains 6.023×10^{23} atoms.

$\therefore$ Mass of oxygen atom = $\frac{0.016}{6.023\times10^{23}}$ = **2.657×10^{-26} kg**

7. The r.m.s. velocity of nitrogen molecule is

$$v_{r.m.s.}=\sqrt{\frac{3RT}{M}}$$

Here R = 8.31 J mol^{-1} K^{-1}; T = 273 K ; M = 28 g = 28×10^{-3} kg

$\therefore \quad v_{r.m.s.}=\sqrt{\frac{3\times8.31\times273}{28\times10^{-3}}}$ = **493 m/s**

8. $\frac{P_1V_1}{T_1}=\frac{P_2V_2}{T_2}$ or $P_2=\frac{P_1V_1}{T_1}\times\frac{T_2}{V_2}=\frac{PV}{T}\times\frac{2T}{V}=$ **2** P $\quad (\because V_1=V_2=V)$

9. K.E. of gas $=\frac{1}{2}mN\overline{v^2} \quad \therefore \quad \frac{1}{2}mN\overline{v_1^2}=\frac{1}{2}m(2N)\overline{v_2^2}$ or $\frac{\overline{v_1^2}}{\overline{v_2^2}}=2$

or $\frac{T}{T_2}=2 \quad \therefore \quad T_2 =$ **$T/2$**

10. R.M.S. speed of gas molecule $\propto \sqrt{T}$

$\therefore \quad \frac{2v_{r.m.s.}}{v_{r.m.s.}}=\sqrt{\frac{T}{T_0}}$ or $2=\sqrt{\frac{T}{273}} \quad \therefore \quad T = 4 \times 273 =$ **1092 K**

11. When the electric fan is switched on, first electrical energy is converted into mechanical energy and then mechanical energy is converted into heat. The heat energy will increase the K.E. of air molecules. Therefore, the temperature of the room will increase.

12. Average K.E. of gas molecule $=\frac{3}{2}\frac{RT}{N_A}$ or $E=\frac{3}{2}\frac{RT}{N_A}$

$\therefore$ Avogadro's number, $N_A = \mathbf{\frac{3RT}{2E}}$

13. According to Charles' law, at constant pressure,

$$\frac{V_1}{V_2}=\frac{T_1}{T_2}=\frac{273+27}{273+54} \text{ or } \frac{V_1}{V_2}=\frac{300}{327}=\mathbf{\frac{100}{109}}$$

14. $\frac{P_1V_1}{T_1}=\frac{P_2V_2}{T_2}$. Since temperature is constant, $P_1V_1 = P_2V_2$.

$\therefore \quad P_2=\frac{P_1V_1}{V_2}=\frac{760\times10}{(10+9)}=$ **400 mm of Hg**

15. $P=\frac{1}{3}mN\frac{\overline{v^2}}{V}$; Mass of gas, $M = mN \quad \therefore \quad P=\frac{1}{3}\frac{M\overline{v^2}}{V}$

At constant volume and temperature, $P \propto M$. Therefore, if the mass of the gas is doubled, the pressure will also be doubled *i.e.* **150 cm of Hg.**

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. Initially, *A* contains 3.00 m^3 of an ideal gas at a temperature of 250 K and pressure of 5.00×10^4 Pa while *B* contains 7.20 m^3 of the same gas at 400 K and 2.00×10^4 Pa [See Fig. 24.11]. Find the pressure after the connecting tap has been opened and the system has reached equilibrium assuming that *A* is kept at 250 K and *B* at 400 K.

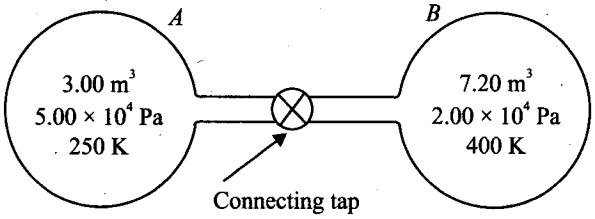

Fig. 24.11

Hint. When the tap is opened, some gas moves from *A* to *B* thereby reducing the pressure in *A* and increasing it in *B*. This process continues till equilibrium is reached and the pressure in *A* becomes equal to pressure in *B*. Let this final pressure be *P* in SI units. The total mass of the gas and, therefore, the total number of moles is the same after the tap is opened as it was before.

Now $PV = nRT \quad \therefore \quad n=\frac{PV}{RT}$

Number of moles initially in $A = \frac{5.00\times10^4\times3.00}{250R}=\frac{600}{R}$

Number of moles initially in $B = \frac{2.00\times10^4\times7.20}{400\,R}=\frac{360}{R}$

$\therefore$ Total number of moles initially $=\frac{600}{R}+\frac{360}{R}=\frac{960}{R}$

Number of moles finally in $A = \frac{P\times3.00}{250R}=\left(1.20\times10^{-2}\right)\frac{P}{R}$

Number of moles finally in $B = \frac{P\times7.20}{400\,R}=\left(1.80\times10^{-2}\right)\frac{P}{R}$

$\therefore$ Total number of moles finally $= (1.20+1.80)\times10^{-2}\times\frac{P}{R} = 3.00\times10^{-2}\times\frac{P}{R}$

Since the total number of moles does not change,

$\therefore\ 3.00\times10^{-2}\times\frac{P}{R}=\frac{960}{R}$ or $P=\frac{960}{3.00\times10^{-2}}=$ **3.20×10^4 Pa**

2. Use the gas law to determine the mass of air contained in 50.0 cm^3 flask at 9.33 × 10^4 Pa pressure and 20°C. Air consists of approximately 80% N_2 and 20% O_2 by mass. Given R= 8314 J/k mol·K.

Hint. Let us first find the molecular mass of air. We know that M (N_2)=28 kg/k mol and M (O_2) = 32 kg/k mol. Since air contains 80% N_2 and 20% O_2 by mass,

$\therefore$ M (air) = 0.80 × 28 + 0.20 × 32 = 29 kg/k mol

Now $PV = nR\,T$ or $PV = \frac{m}{M}R\,T$ $\therefore$ $m = \frac{PVM}{RT}$

Here P = 9.33 × 10^4 Pa ; V = 50.0 cm^3 = 50.0 × 10^{-6} m^3 ; M= 29 kg /k mol ;

R = 8314 J/k mol · K ; T = 273 + 20 = 293 K

$\therefore$ $m = \frac{(9.33\times10^4)\times(50.0\times10^{-6})\times29}{8314\times293}$ = **5.5 × 10^{-5} kg**

3. A diesel engine has a compression ratio of 18 : 1. This means that when the engine is running, the piston changes the cylinder volume from the initial volume V_1 to a final volume $V_2 = V_1 / 18$. Suppose the gaseous fuel-air mixture enters the cylinder at 300 K at a pressure of 740 torr when the cylinder volume is V_1 .What is the temperature of the gas when the piston has changed the cylinder volume to V_2 and the pressure has risen to 37000 torr?

Hint. Although P, V and T are all changing, they do so in such a way that PV/T remains constant (*i.e.*, $PV/T = nR$).

Now $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ $\therefore$ $T_2 = \frac{P_2}{P_1}\times\frac{V_2}{V_1}\times T_1 = \left(\frac{37000}{740}\right)\times\frac{1}{18}\times300$ = **833 K**

Note. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$

You must always use the Kelvin scale for temperature. This is because T_1 and T_2 are related by *additive* numbers which would not cancel in a ratio. All other quantities (P, V and n) can be used in any units in ratios, since the multiplying conversion factors will cancel.

4. A cylinder containing 19 kg of compressed air at a pressure 9.5 times that of the atmosphere is kept in a store at 7°C. When it is moved to a workshop where the temperature is 27°C, a safety valve on the cylinder operates, releasing some of the air. If the valve allows air to escape when its pressure exceeds 10 times that of the atmosphere, calculate the mass of air that escapes.

Hint. $PV = nRT$ or $n = \frac{PV}{RT}$

Suppose P_a is the atmospheric pressure and V is the volume of the cylinder.

Initial number of moles of air is $n_1 = \frac{(9.5\times P_a)\times V}{R\times280}$ ($\because T$ = 273 + 7 = 280 K)

Final number of moles of air is $n_2 = \frac{(10\times P_a)\times V}{R\times300}$ ($\because$ T = 273 + 27 = 300 K)

$\therefore$ $\frac{n_2}{n_1} = \frac{10\times280}{9.5\times300}$

Suppose m kg is the mass of air left in the cylinder. Since mass is proportional to the number of moles,

$\therefore$ $\frac{m}{19\text{ kg}} = \frac{n_2}{n_1}$ $\therefore$ $\frac{m}{19} = \frac{10\times280}{9.5\times300}$

or $$m = \left(\frac{10\times280}{9.5\times300}\right)\times19 = 18.67 \text{ kg}$$

$\therefore$ Mass of air escaped = 19 – 18.67 = **0.33 kg**

5. Four molecules of a gas have speeds 2, 4, 6 and 8 km/s respectively. Calculate their (*i*) average speed (*ii*) root mean square speed.

Hint. Here $v_1 = 2$ km/s ; $v_2 = 4$ km/s ; $v_3 = 6$ km/s ; $v_4 = 8$ km/s

(*i*) Average speed, $\overline{v} = \frac{v_1+v_2+v_3+v_4}{4} = \frac{2+4+6+8}{4} =$ **5 km/s**

(*ii*) R.M.S. speed, $v_{r.m.s.} = \sqrt{\frac{v_1^2+v_2^2+v_3^2+v_4^2}{4}} = \sqrt{\frac{(2)^2+(4)^2+(6)^2+(8)^2}{4}}$

= **5.4772 km/s**

6. At what temperature is the root mean square speed of oxygen molecules equal to the escape speed from earth's surface? Escape speed = 11.2 km/s; R= 8.3 J/K gm·mole.

Hint. $$v_{r.m.s.} = \sqrt{\frac{3RT}{M}} \quad \text{or} \quad T = \frac{M\times v^2_{r.m.s.}}{3R}$$

Here $M = 32$ (for oxygen) ; $v_{r.m.s.} = 11.2$ km/s $= 11.2 \times 10^5$ cm/s ; $R = 8.3$ J/K gm· mole $= 8.3 \times 10^7$ ergs /K gm · mole

$\therefore$ $$T = \frac{32\times(11.2\times10^5)^2}{3\times(8.3\times10^7)} = \textbf{1.61} \times 10^5 \text{ K}$$

7. An electric bulb of volume 250 cm^3 was sealed off during manufacture at a pressure of 10^{-3} mm of Hg at 27°C. Find the number of molecules in the bulb. [I.I.T.]

Hint. Let N be the number of molecules. The perfect gas equation in terms of the number of gas molecules is given by ;

$$P\,V = N\,k\,T$$

Here $P = 10^{-3}$ mm of Hg $= \frac{10^{-3}}{760}$ atmospheres $= \frac{10^{-3}}{760} \times 1.013 \times 10^5$ N /m^2

$V = 250$ cm^3 $= 250 \times 10^{-6}$ m^3 ; $k = 1.38 \times 10^{-23}$ JK^{-1} ; $T = 273 + 27 = 300$ K

$\therefore$ Number of molecules, $N = \frac{PV}{kT} = \frac{10^{-3}\times1.013\times10^5\times250\times10^{-6}}{760\times1.38\times10^{-23}\times300} =$ **8.05 × 10^{15}**

8. In a certain region of space, there are 5 molecules per cm^3 on an average. The temperature there is 3 K. What is the average pressure of this very dilute gas?

Hint. If N is the number of molecules present in this dilute gas, then perfect gas equation is given by ;

$$P\,V = NkT \quad \text{or} \quad P = \frac{N}{V} \times kT$$

Here $N/V = 5$ molecules per cm^3 $= 5 \times 10^6$ molecules per m^3 ;

$k = 1.38 \times 10^{-23}$ JK^{-1} ; $T = 3$ K

$\therefore$ $P = (5 \times 10^6) \times (1.38 \times 10^{-23}) \times 3 =$ **2.07 × 10^{-16} N/m^2**

9. A tank of volume 0.3 m^3 contains 2 moles of helium gas at 20°C. Assuming the helium behaves as an ideal gas :

(*i*) Find the total internal energy of the system.

(*ii*) What is the average kinetic energy per atom?

(*iii*) Determine the r.m.s. speed of the atoms.

Hint. (*i*) Total energy of the system is $E = \frac{3}{2} nRT$

Here n = 2 moles ; R = 8.31 J/mole ·K ; T = 273 + 20 = 293 K

$$\therefore \quad E = \frac{3}{2} \times 2 \times 8.31 \times 293 = \mathbf{7.30 \times 10^3\ J}$$

(*ii*) The average K.E. per atom is

$$\overline{\text{K.E.}} = \frac{3}{2} k T = \frac{3}{2} \times (1.38 \times 10^{-23}) \times 293 = \mathbf{6.07 \times 10^{-21}\ J}$$

(*iii*) Since the molecular mass of helium = 4 g/mol = 4×10^{-3} kg/mol, the r.m.s. speed is

$$v_{r.m.s.} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8.31 \times 293}{4 \times 10^{-3}}} = \mathbf{1.35 \times 10^3\ ms^{-1}}$$

Note that the same result is obtained using $v_{r.m.s.} = \sqrt{3kT/m}$

10. The mass of hydrogen molecules is 3.32×10^{-27} kg. If 10^{23} hydrogen molecules strike a fixed wall of area 2 cm^2 at an angle of 45° to the normal and rebound elastically at a speed of 10^3 ms^{-1}, calculate the pressure exerted on the wall.

Hint. Figure 24.12 shows the conditions of the problem. The molecules strike the wall along *AO* and rebound elastically along *OB* such that :

$$\angle AON = \angle BON = 45°$$

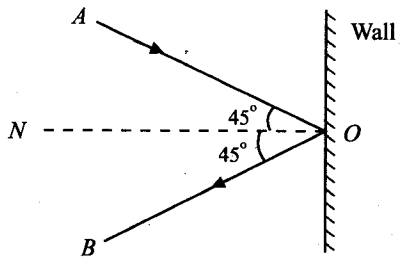

Fig. 24.12

Momentum of each incident molecule is mv.

Change in momentum normal to the wall per molecule is

$$\Delta p = 2mv \cos 45°$$
$$= 2 \times (3.32 \times 10^{-27}) \times 10^3 \times \cos 45°$$
$$= 4.67 \times 10^{-24} \text{ kg ms}^{-1}$$

Number of collisions = 10^{23}

Time between consecutive collisions, $\Delta t = \frac{1}{10^{23}} = 10^{-23}$ s

∴ Force exerted by molecules on wall is

$$F = \frac{\Delta p}{\Delta t} = \frac{4.67 \times 10^{-24}}{10^{-23}} = 0.467 \text{ N}$$

$$\therefore \text{ Pressure on the wall} = \frac{\text{Force}}{\text{Area}} = \frac{0.467}{2 \times 10^{-4}} = \mathbf{2.335 \times 10^3\ N/m^2}$$

11. If 2.1212 g of a monoatomic gas occupies 1.49 litres when the temperature is 0°C and the pressure 810.6 kPa, what is the gas?

Hint. $$PV = \frac{m}{M} RT$$

Here m = mass of gas ; M = molecular mass of gas

$$\therefore \quad M = mR \times \frac{T}{PV}$$

Here m = 2.1212 g = 2.1212×10^{-3} kg ; R= 8.31 J/mol ·K ; T = 273 K ;

P = 810.6 k Pa= 810.6×10^3 Pa ; V= 1.49 litres = 1.49×10^{-3} m^3

$$\therefore \quad M = (2.1212 \times 10^{-3}) \times 8.31 \times \frac{273}{(810.6 \times 10^3) \times 1.49 \times 10^{-3}}$$

$$= 3.99 \times 10^{-3} \text{ kg /mol} = 3.99 \text{ g/mol}$$

Therefore, gas is **helium.**

12. Show that the constant-temperature bulk modulus K of an ideal gas is the pressure P of the gas.

Hint. We know that a substance subjected to a pressure increase ΔP will undergo a small fractional volume decrease $\Delta V/V$ that is related to bulk modulus K by ;

$$K = \frac{\Delta P}{-\Delta V / V} \qquad ...(i)$$

In the case of an ideal gas at constant temperature before compression,

$$PV = \frac{m}{M} RT \qquad (ii)$$

Here M = molecular mass of gas

After compression at constant temperature, we have,

$$(P + \Delta P)(V + \Delta V) = \frac{m}{M} RT \qquad ...(iii)$$

From eqs. (*ii*) and (*iii*), we have,

$$P V = (P + \Delta P)(V + \Delta V)$$

or $$P V = P V + P \Delta V + V \Delta P + \Delta P \Delta V$$

or $$-\frac{*\Delta V}{V} = \frac{\Delta P}{P}\left(1+\frac{\Delta V}{V}\right)$$

We are concerned with only very small fractional changes. Therefore, $\Delta V/V$ is much smaller than 1. As a result, we can neglect $\Delta V/V$ as compared to 1.

$\therefore$ $$-\frac{\Delta V}{V} = \frac{\Delta P}{P}$$

Substituting this value for $\Delta V/V$ in eq. (*i*), we get, $K = \dfrac{\Delta P}{\Delta P / P} = P$

Thus, we find that the bulk modulus of an ideal gas is equal to the pressure of the gas in a compression carried out at constant temperature.

13. A vessel contains 14 g of hydrogen and 96 g of oxygen at S.T.P. Find the volume of the vessel.

Hint. The number of moles of hydrogen is $n_1 = \dfrac{14\ g}{2\ g} = 7$

The number of moles of oxygen is $n_2 = \dfrac{96\ g}{32\ g} = 3$

$\therefore$ Total number of moles in the vessel is $n = n_1 + n_2 = 7 + 3 = 10$

Now $$PV = n\,R\,T$$

$\therefore$ $$V = \frac{nRT}{P} = \frac{10 \times 8.3 \times 273}{1.0 \times 10^5} = \mathbf{0.23\ m^3}$$

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* You can arrive at this step after a little algebra.

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 13 : KINETIC THEORY OF GASES

Q.1. Estimate the fraction of molecular volume to the actual volume occupied by oxygen gas at STP. Take the diameter of an oxygen molecular to be 3Å.

Ans. Radius of oxygen molecule, $r = 3Å/2 = 1.5 \times 10^{-10}$ m

Volume of oxygen molecule, $v = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1.5\times10^{-10})^3 = 1.414 \times 10^{-29}$ m^3

The number of molecules in one mole of oxygen is equal to Avogadro's number ($N_A = 6.023 \times 10^{23}$). Therefore, molecular volume (V_m) of one mole of oxygen gas is

$$V_m = N_A\ v = (6.023 \times 10^{23}) \times 1.414 \times 10^{-29} = 8.52 \times 10^{-6}\ \text{m}^3$$

Now actual volume (V_a) occupied by one mole of oxygen at S.T.P. is $V_a = 22.4$ litres $= 22400$ cm^3 $= 22400 \times 10^{-6}$ m^3.

$$\therefore \quad \frac{\text{Molecular volume}}{\text{Actual volume}} = \frac{V_m}{V_a} = \frac{8.52\times10^{-6}}{22400\times10^{-6}} = 3.8 \times 10^{-4}$$

Q.2. Molar volume is the volume occupied by 1 mol of any (ideal) gas at standard temperature and pressure (STP : 1 atmospheric pressure, 0°C). Show that it is 22.4 litres.

Ans. $PV = nRT$. For one mole (*i.e.*, $n = 1$) of gas, $PV = RT$.

$$\therefore \quad V = \frac{RT}{P}$$

Here $R = 8.31$ J mol^{-1} K^{-1} ; $T = 273$ K ; $P = 1$ atm. $= 1.013 \times 10^5$ N/m^2

$$\therefore \quad V = \frac{8.31\times273}{1.013\times10^5} = 22.4 \times 10^{-3}\ \text{m}^3 = 22.4 \text{ litres}$$

Q.3. Figure 13.01 shows plot of *PV/T* versus *P* for 1.00×10^{-3} kg of oxygen gas at two different temperatures.

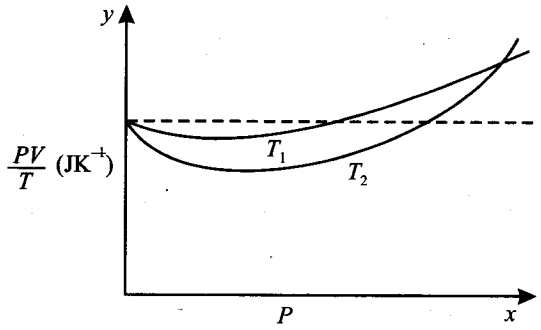


Fig. 13.01

(*a*) What does the dotted plot signify ?

(*b*) Which is true : $T_1 > T_2$ or $T_1 < T_2$?

(*c*) What is the value of *PV/T* where the curves meet on the *y*-axis ?

(*d*) If we obtained similar plots for 1.00×10^{-3} kg of hydrogen, would we get the same value of *PV/T* at the point where the curves meet on the *y*-axis ? If not, what mass of hydrogen yields the same value of *PV/T* (for low pressure, high temperature region of the plot) ? (Molecular mass of $H_2 = 2.02$ u, of $O_2 = 32.0$ u, $R = 8.31$ J mol^{-1} K^{-1}).

Ans. (*a*) The dotted plot shows that PV/T ($= nR$) = constant with the change in P. Therefore, this plot represents the behaviour of an ideal gas.

(*b*) As the temperature of a gas is increased, its behaviours approaches that of an ideal gas. Since the plot of T_1 is closer to the dotted plot (ideal-gas behaviour), $T_1 > T_2$.

(*c*) Where the two curves meet, the value of *PV/T* on *y*-axis is equal to *nR*.

Now $PV = nRT$ or $\frac{PV}{T} = nR$

Here n = number of moles of oxygen = $\frac{m}{M} = \frac{1.00\times10^{-3}}{32\times10^{-3}}$

$\therefore$ $\frac{PV}{T} = \frac{1.00\times10^{-3}}{32\times10^{-3}}\times8.31 = 0.26 \text{ J K}^{-1}$

(*d*) If we obtained similar plots for 1.00×10^{-3} kg of hydrogen, we will not get the same value of *PV/T* at the point where the two curves meet the *y*-axis.

For the same value of *PV/T* (= 0.26 J K^{-1}), we can find the mass of hydrogen required as under :

$\frac{PV}{T} = nR$ or $0.26 = \frac{m}{M}\times8.31$

$\therefore$ $0.26 = \frac{m}{2.02}\times8.31$ or $m = \frac{2.02\times0.26}{8.31}$ gm $= 6.32 \times 10^{-2}$ gram

Q.4. An oxygen cylinder of volume 30 litres has an initial gauge pressure of 15 atm and a temperature of 27°C. After some oxygen is withdrawn from the cylinder, the gauge pressure drops to 11 atm and its temperature drops to 17°C. Estimate the mass of oxygen taken out of the cylinder (R = 8.3 J mol^{-1} K^{-1}, molecular mass of O_2 = 32 u).

Ans. Initial conditions. Initial conditions in the oxygen cylinder are :

$V_1 = 30$ litres $= 30 \times 10^{-3}$ m^3 ; $P_1 = 15$ atm. $= 15 \times 1.01 \times 10^5$ Pa ; $T_1 = 27 + 273 = 300$ K

If the cylinder contains n_1 moles of oxygen gas, then, $P_1V_1 = n_1 RT_1$.

$\therefore$ $n_1 = \frac{P_1V_1}{RT_1} = \frac{(15\times1.01\times10^5)\times(30\times10^{-3})}{8.3\times300} = 18.253$

Now molecular weight of oxygen is M = 32 g.

$\therefore$ Initial mass of oxygen in the cylinder is

$m_1 = n_1 M = 18.253 \times 32 = 584.1$ g

Final conditions. Final conditions in the oxygen cylinder are :

$V_2 = 30 \times 10^{-3}$ m^3 ; $P_2 = 11 \times 1.01 \times 10^5$ Pa ; $T_2 = 17 + 273 = 290$ K

If n_2 is the number of moles of oxygen left in the cylinder, then,

$n_2 = \frac{P_2V_2}{RT_2} = \frac{(11\times1.01\times10^5)\times(30\times10^{-3})}{8.3\times290} = 13.847$

$\therefore$ Final mass of oxygen in the cylinder is

$m_2 = n_2M = 13.847\times 32 = 443.1$ g

$\therefore$ Mass of oxygen taken out = $m_1 - m_2 = 584.1 - 443.1 = 141.0$ g

Q.5. An air bubble of volume 1.0 cm^3 rises from the bottom of a lake 40 m deep at a temperature of 12°C. To what volume does it grow when it reaches the surface, which is at a temperature of 35°C ? Given 1 atm = 1.01×10^5 Pa.

Ans. Air bubble at depth of 40 m

$T_1 = 12 + 273 = 285$ K ; $V_1 = 1$ cm^3 $= 1 \times 10^{-6}$ m^3 ;

$P_1 = 1$ atm $+ h\rho g$

$= 1.01 \times 10^5 + 40 \times 10^3 \times 9.8 = 4.93 \times 10^5$ Pa

Air bubble at surface

$T_2 = 35 + 273 = 308$ K ; $V_2 = ?$; $P_2 = 1$ atm $= 1.01 \times 10^5$ Pa

Now $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$

$\therefore$ Volume of air bubble at the surface of the lake is

$V_2 = \frac{P_1V_1}{T_1}\times\frac{T_2}{P_2} = \frac{4.93\times10^5\times1\times10^{-6}}{285}\times\frac{308}{1.01\times10^5} = 5.275 \times 10^{-6}$ m^3

Q.6. Estimate the total number of air molecules (inclusive of oxygen, nitrogen, water vapour and other constituents) in a room of capacity 25.0 m^3 at a temperature of 27°C and 1 atm pressure.

Ans. Let N be the total number of air molecules in 25.0 m^3 of air. Then according to perfect gas equation,

$$PV = N k T$$

Here $P = 1.01 \times 10^5$ Pa ; $V = 25.0$ m^3 ; $k = 1.38 \times 10^{-23}$ J K^{-1} ; $T = 273 + 27 = 300$ K

$$\therefore \quad N = \frac{PV}{kT} = \frac{1.01\times10^5\times(25.0)}{(1.38\times10^{-23})\times300} = 6.1 \times 10^{26}$$

Q.7. Estimate the average thermal energy of a helium atom at (*i*) room temperature (27°C) (*ii*) the temperature on the surface of the Sun (6000 K) and (*iii*) the temperature of 10 million kelvin (the typical core temperature in the case of a star). Given $k = 1.38 \times 10^{-23}$ J molecule^{-1} K^{-1}.

Ans. Average thermal energy, $\overline{\text{K.E.}} = \frac{3}{2}kT$

(*i*) **At $T = 27 + 273 = 300$ K;** $\overline{\text{K.E.}} = \frac{3}{2}\times1.38\times10^{-23}\times300 = 6.21 \times 10^{-21}$ J

(*ii*) **At $T = 6000$ K ;** $\overline{\text{K.E.}} = \frac{3}{2}\times1.38\times10^{-23}\times6000 = 1.24 \times 10^{-19}$ J

(*iii*) **At $T = 10$ million K $= 10^7$ K;** $\overline{\text{K.E.}} = \frac{3}{2}\times1.38\times10^{-23}\times10^7 = 2.07 \times 10^{-16}$ J

Q.8. Three vessels of equal capacity have gases at the same temperature and pressure. The first vessel contains neon (monoatomic), the second contains chlorine (diatomic) and the third contains uranium hexafluoride (polyatomic). Do the vessels contain equal number of respective molecules ? Is the root mean square speed of molecules the same in the three cases ? If not, in which case is $v_{r.m.s.}$ the largest.

Ans. Yes. The three gases have the same volume and are at the same temperature and pressure. Therefore, according to Avogadro's law, the three vessels will contain equal number of respective molecules.

The root means square speed of molecules is given by ;

$$v_{r.m.s.} = \sqrt{\frac{3kT}{m}} \quad \text{or} \quad v_{r.m.s.} \propto \sqrt{\frac{1}{m}} \qquad \text{...for given } T$$

Thus for a given temperature, the r.m.s. speed depends upon the mass of the molecule of the gas. Therefore, the r.m.s. speed of the molecules will not be the same in the three cases. As the neon gas molecules are the lightest, the r.m.s. speed of molecules will be largest in case of neon gas.

Q.9. At what temperature is the root mean square speed of an atom in an argon gas cylinder equal to the r.m.s. speed of a helium gas atom at –20°C ? (atomic mass of Ar = 39.9 u, of He = 4.0 u).

Ans. Suppose $v_{r.m.s.}$ and $v'_{r.m.s.}$ are the speeds of argon and helium atoms at temperatures T and T' respectively. Then,

$$v_{r.m.s.} = \sqrt{\frac{3RT}{M}} \quad ; \quad v'_{r.m.s.} = \sqrt{\frac{3RT'}{M'}}$$

Here $M = 39.9$; $T' = 273 + (-20) = 253$ K ; $M' = 4.0$; $T = ?$

Since $v_{r.m.s.} = v'_{r.m.s.}$, we have,

$$\sqrt{\frac{3RT}{M}} = \sqrt{\frac{3R\times253}{4.0}}$$

or $$T = \frac{253\times39.9}{4.0} = 2523.7 \text{ K}$$

Q. 10. Estimate the mean free path and collision frequency of a nitrogen molecule in a cylinder containing nitrogen at 2.0 atm and temperature 17°C. Take the radius of a nitrogen molecule to be roughly 1.0 Å. Compare the collision time with the time the molecule moves freely between two successive collisions (Molecular mass of N_2 = 28.0 u).

Ans. Let n be the number of molecules per unit volume of the gas.

Now $PV = N\,k\,T$ ...N = No. of molecules in volume V of the gas

$\therefore$ $n = \dfrac{N}{V} = \dfrac{P}{kT}$

Here $P = 2 \times 1.01 \times 10^5$ N/m^2 ; $k = 1.38 \times 10^{-23}$ J/K ; $T = 273 + 17 = 290$ K

$\therefore$ $n = \dfrac{2 \times 1.01 \times 10^5}{(1.38 \times 10^{-23}) \times (290)} = 5.05 \times 10^{25}$ molecules/m^3

Mean free path, $\lambda = \dfrac{1}{4\pi\sqrt{2}\,r^2 n}$

Here $r = 1.0$ Å $= 1.0 \times 10^{-10}$ m ; $n = 5.05 \times 10^{25}$ molecules/m^3

$\therefore$ $\lambda = \dfrac{1}{4\pi\sqrt{2} \times (1.0 \times 10^{-10})^2 \times 5.05 \times 10^{25}} = 1.11 \times 10^{-7}$ m

Now $v_{r.m.s.} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 290}{28 \times 10^{-3}}} = 508.14$ m/s

$\therefore$ Collision frequency, $f = \dfrac{v_{r.m.s.}}{\lambda} = \dfrac{508.14}{1.11 \times 10^{-7}} = 4.58 \times 10^9$ **collisions/second**

Q.11. A metre long narrow bore held horizontally (and closed at one end) contains a 76 cm long mercury thread, which traps a 15 cm column of air. What happens if the tube is held vertically with the open end at the bottom ?

Ans. Let the cross sectional area of the tube be A cm^2. When the tube is held horizontally [See Fig. 13.02 (*i*)],

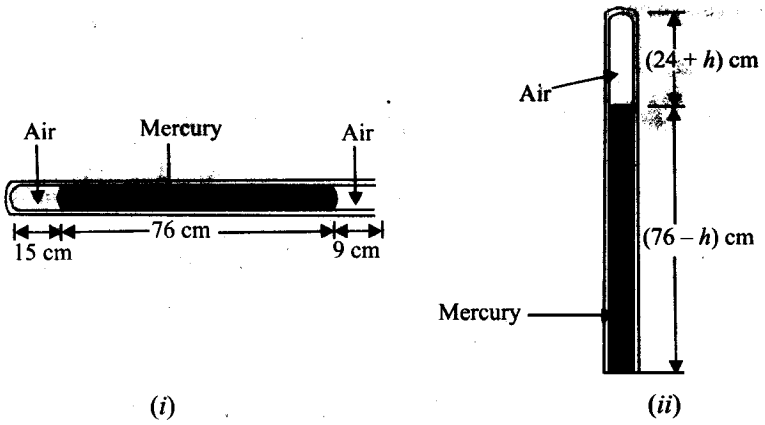


Fig. 13.02

Pressure of air enclosed, $P_1 = 1$ atm = 76 cm of Hg

Volume of air enclosed, $V_1 = A \times 15$ cm^3

When the tube is held vertically. When the tube is held vertically [See Fig. 13.02 (*ii*)], the mercury first reaches the open end and length of air column becomes 15 + 9 = 24 cm. Since the pressure exerted by 24 cm of air and 76 cm of Hg is greater than atmospheric pressure, some mercury will flow out of the tube till pressures become equal. Let h cm of mercury flow out of the tube. Then length of mercury column decreases to $(76 - h)$ cm and that of air column increases to $(24 + h)$ cm.

$\therefore$ Pressure of air column, $P_2 = 76 - (76 - h) = h$ cm of Hg

Volume of air column, $V_2 = A \times (24 + h)$ cm^3

If temperature remains constant, then from Boyle's law,

$$P_1V_1 = P_2V_2$$

or $$76 \times A \times 15 = h \times A \times (24 + h)$$

or $h^2 + 24\,h - 1140 = 0$

or $$h = \frac{-24 \pm \sqrt{576+4560}}{2} = \frac{-24 \pm \sqrt{5136}}{2}$$

$$= \frac{-24 \pm 71.6}{2} = 23.8 \text{ cm or } -47.8 \text{ cm}$$

Since h cannot be negative, h = **23.8 cm.** Therefore, when the tube is held vertically, the mercury thread will decrease in length by 23.8 cm.

Q.12. From a certain apparatus, the diffusion rate of hydrogen has an average value of 28.7 cm^3 s^{-1}. The diffusion of another gas under the same conditions is measured to have an average rate of 7.2 cm^3 s^{-1}. Identify the gas.

Ans. According to Graham's law of diffusion,

$$\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}$$

where r_1 = Rate of diffusion of hydrogen = 28.7 cm^3 s^{-1}

r_2 = Rate of diffusion of unknown gas = 7.2 cm^3 s^{-1}

M_1 = Molecular weight of hydrogen = 2u

M_2 = Molecular weight of the unknown gas = ?

$$\therefore \quad \frac{28.7}{7.2} = \sqrt{\frac{M_2}{2}} \text{ or } M_2 = \left(\frac{28.7}{7.2}\right)^2 \times 2 \simeq 32$$

This is the molecular weight of oxygen. Therefore, the unknown gas is oxygen.

Q.13. A gas in equilibrium has uniform density and pressure throughout its volume. This is strictly true only if there are no external influences. A gas column under gravity, for example, does not have uniform density (and pressure). As you might expect, its density decreases with height. The precise dependence is given by the so-called law of atmosphere :

$$n_2 = n_1 \exp\,[-mg(h_2 - h_1)/k_BT]$$

where n_2, n_1 refer to number density at heights h_2 and h_1 respectively. Use this relation to derive the equation for sedimentation equilibrium of a suspension in a liquid column :

$$n_2 = n_1 \exp\,[-mg\,N_A(\rho - \rho')\,(h_2 - h_1)/(\rho\,RT)]$$

where ρ is the density of the suspended particle and ρ′ that of surrounding medium. [N_A is Avogadro's number and R the universal gas constant.]

Ans. According to the law of atmospheres,

$$n_2 = n_1 \exp.\left[-\frac{mg}{k_BT}(h_2 - h_1)\right] \qquad ...(i)$$

where n_2 and n_1 represent the number density of particles at heights h_2 and h_1 respectively.

When we consider the sedimentation equilibrium of suspended particle in a liquid column, the weight mg of the suspended particle has to be replaced by its apparent weight. If V is the average volume of the suspended particle, then apparent weight of the particle is

Apparent weight = mg − Upthrust

$$= mg - V\rho' g = mg - (m/\rho)\,\rho' g$$

$$= mg\left(\frac{\rho - \rho'}{\rho}\right)$$

Also Boltzmann's constant $k_B = \dfrac{R}{N_A}$

Putting $mg\left(\dfrac{\rho - \rho'}{\rho}\right)$ in place of mg and the value of k_B in eq. (*i*), we have,

$$n_2 = n_1 \exp.\left[-\frac{mg\,N_A}{\rho\,RT}(\rho - \rho')\,(h_2 - h_1)\right]$$

Q.14. Given below are densities of some solids and liquids. Give rough estimates of the size of their atoms :

Substance	Atomic Mass (u)	Density (10^3 kg m^{-3})
Carbon (diamond)	**12.01**	**2.22**
Gold	**197.00**	**19.32**
Nitrogen (liquid)	**14.01**	**1.00**
Lithium	**6.94**	**0.53**
Fluorine (liquid)	**19.00**	**1.14**

Ans. If r is the radius of atom, then volume of each atom is $v = (4/3)\pi r^3$.

$\therefore$ Volume of atoms in one mole of the substance is

$$V = N_A v = N_A \times \frac{4}{3}\pi r^3 \quad \text{...}N_A \text{ is Avogadro's number}$$

If M is the atomic mass and ρ is the density of the substance, then,

$$V = \frac{M}{\rho}$$

$$\therefore \quad N_A \times \frac{4}{3}\pi r^3 = \frac{M}{\rho} \quad \text{or} \quad r = \left(\frac{3M}{4\pi\rho N_A}\right)^{1/3}$$

For carbon. $M = 12.01 \times 10^{-3}$ kg ; $\rho = 2.22 \times 10^3$ kg/m^3

$$\therefore \quad r = \left(\frac{3\times 12.01\times 10^{-3}}{4\pi\times 2.22\times 10^3\times 6.023\times 10^{23}}\right)^{1/3} = 1.29 \times 10^{-10} \text{ m} = 1.29 \text{ Å}$$

Proceeding in a manner as above, we have,

For gold, r = **1.59 Å** ; For nitrogen, r = **1.77 Å** ; For lithium, r = **1.73 Å** ; For fluorine, r = **1.88 Å**.

Unit XI

OSCILLATIONS AND WAVES

- **Oscillations**
- **Waves**
- **N.C.E.R.T. Textbook Exercises**

25

Oscillations

INTRODUCTION

So far we have studied translational and rotational motion and we have used them in examining several areas of physics. There is another important kind of motion that we now introduce: oscillatory or vibratory motion. When an object moves back and forth repeatedly over the same path, it is said to be oscillating or vibrating. An important type of oscillatory motion is **simple harmonic motion (S.H.M.)** for which displacement-time graph is a sine curve or cosine curve. The motion of a mass vibrating up and down on a spring, the motion of the prongs of a vibrating tuning fork and the motion of a swinging pendulum bob are the familiar examples of motion that is simple harmonic motion or very nearly so. The study of simple harmonic motion involves no new laws or principles of physics. Instead, we use Newton's second law, kinematics and energy principles to analyse this simple type of motion.

25.1. PERIODIC AND OSCILLATORY MOTION

Before discussing simple harmonic motion, it is desirable to discuss periodic motion and oscillatory motion.

1. Periodic motion. *The motion which repeats itself after a regular interval of time is called* **periodic motion**. *The regular time interval is called the time period of the periodic motion.*

Examples. A few examples of periodic motion are given below :

(*i*) The revolution of earth around the sun is a periodic motion. Its period of revolution is 1 year.

(*ii*) The revolution of moon around the earth is a periodic motion. Its period of revolution is 27.3 days.

(*iii*) The motion of the hands of a clock is a periodic motion.

2. Oscillatory motion. *If a body in periodic motion moves along the same path to and fro about a definite point (equilibrium position), then the motion of the body is* **vibratory motion** *or* **oscillatory motion**.

All objects that vibrate or oscillate have one thing in common : each object is subjected to a *restoring force* that increases with the increase in displacement from the equilibrium position. A restoring force is one that tries to pull or push a displaced object back to its equilibrium position. It may be noted that *all oscillatory motions are periodic motions but all periodic motions are not oscillatory.* For example, the motion of moon around the earth is periodic but not oscillatory.

Examples. A few examples of oscillatory motion are given below :

(*i*) The motion of the bob of swinging simple pendulum is oscillatory motion.

(*ii*) When a loaded spring is pulled and then released, the load attached to the spring executes oscillatory motion.

(*iii*) When a piece of wood floating over the surface of a liquid (*e.g.*, water) is pressed down and then released, it executes oscillatory motion.

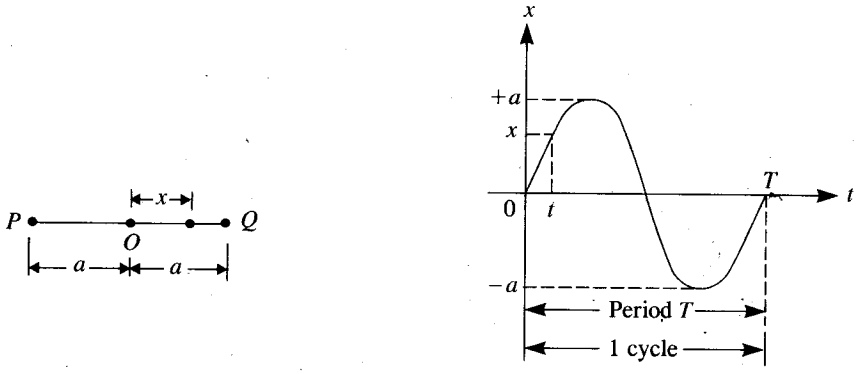
25.2. SIMPLE HARMONIC MOTION (S.H.M.)

A simple harmonic motion (S.H.M.) is an important type of oscillatory motion and may be defined as under :

A particle is said to execute **S.H.M.** *if it moves to and fro about the mean (equilibrium) position in a straight line and the graph between displacement of the particle from equilibrium position and time is a sine curve (or cosine curve).*

In simple harmonic motion, the oscillatory motion of the particle is in a straight line such that its displacement from the *equilibrium position (or mean position) varies sinusoidally (or cosinusoidally) with time. Since this is the simplest type of periodic motion (or harmonic motion), it is called simple harmonic motion.

Analysis of S.H.M. Consider a particle [See Fig. 25.1 (*i*)] executing simple harmonic motion about point O (equilibrium position) with an amplitude a (*i.e.*, maximum displacement from O).



(*i*) **Fig. 25.1** (*ii*)

The particle is said to have completed **1 vibration** (or oscillation) if starting from O, it has moved through positions $O - Q - P - O$ *i.e.*, when it has returned to its starting position and is moving in the *same* direction. The time taken to complete 1 vibration is called **time period** T of the motion. The number of complete oscillations in one second is called the **frequency** f of the motion. Clearly, $f = 1/T$. If the motion is taken to **start from the equilibrium position (*i.e.*, $x = 0$ at $t = 0$), then by definition, displacement x of the particle from the equilibrium position varies with time (t) according to the relation :

$$x = a \sin \omega t = a \sin \frac{2\pi t}{T}$$

where a and ω (or T) are constants of the motion. In order to give physical significance to these constants, it is convenient to plot $x - t$ (displacement – time) graph as in Fig. 25.1 (*ii*).

(*i*) Since $\sin\omega t$ varies between +1 and −1, the displacement x varies between $+ a$ and $- a$. The constant a is called the *amplitude* of the simple harmonic motion; it is simply the *maximum displacement* of the particle from the equilibrium position in either the positive or negative x – direction.

(*ii*) The constant ω is called the *angular frequency* of simple harmonic motion and as we shall see, its value depends on the rate of the oscillations. Let us interpret

* Equilibrium position is the position at which the particle would come to rest if it were to lose all of its energy.

** If the motion is taken to start at the extreme position (*i.e.*, $x = a$ at $t = 0$), then, $x = a \cos \omega t$. Note that when $t = 0$, $x = a$.

constant ω. The displacement x must return to its initial value after one period T of the motion. Thus in one cycle, the quantity ωt increases by 2π rad while the time t increases by T *i.e.*,

$$\omega(t+T) = \omega t + 2\pi$$

or $$\omega T = 2\pi$$

$$\therefore \quad \omega = \frac{2\pi}{T} = 2\pi f \qquad \left(\because f = \frac{1}{T}\right)$$

We note that time period T is inversely proportional to ω. This means that larger the angular frequency ω, the smaller the time period T and the more quickly the particle completes a cycle.

The reader may wonder what type of force can cause a body to execute simple harmonic motion. Since in S.H.M., the body periodically passes through the equilibrium position, one can expect that a *restoring force* must act on the body to bring it back to the equilibrium position time and again. We shall talk about this restoring force in later sections.

25.3. PERIODIC FUNCTIONS

Periodic functions are those which represent periodic motion. The sines and cosines functions of time are simple periodic functions. A function is said to be periodic if it repeats itself after time period T *i.e.* the same function is obtained when the variable t is changed to $t + T$. Consider the following periodic functions :

$$f(t) = \sin\frac{2\pi}{T}t$$

and $$g(t) = \cos\frac{2\pi}{T}t$$

Here T is the time period of the periodic motion and is equal to 2π radians. We shall see that if the variable t is changed to $t + T$, the same function results.

$$f(t+T) = \sin\left[\frac{2\pi}{T}(t+T)\right] = \sin\left[\frac{2\pi t}{T} + 2\pi\right] = \frac{*\sin 2\pi}{T}t$$

$$\therefore \quad f(t+T) = f(t)$$

Similarly, $$g(t+T) = g(t)$$

It can be easily verified that :

$$f(t+nT) = f(t)$$

and $$g(t+nT) = g(t) \qquad \text{where } n = 1, 2, 3, \ldots\ldots$$

Therefore, infinite sets of periodic function of period T may be represented as :

$$f_n(t) = \sin\frac{2\pi nt}{T}$$

and $$g_n(t) = \cos\frac{2\pi nt}{T}$$

Further, a linear combination of sine and cosine functions like :

$$f(t) = a\sin\omega t + b\cos\omega t \qquad \ldots(i)$$

is also a periodic function with a period T.

Let $a = A\cos\phi$ and $b = A\sin\phi$

$\therefore$ Eq. (*i*) can be written as : $f(t) = A\sin(\omega t + \phi)$

$$\text{where } A = \sqrt{a^2+b^2} \; ; \; \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

* $\because \sin(\theta + 2\pi) = \sin\theta$

25.4. ANALYSIS OF S.H.M. IN TERMS OF UNIFORM CIRCULAR MOTION

It is sometimes quite helpful to analyse simple harmonic motion (S.H.M.) in terms of uniform circular motion. Let us see how this is done. Suppose a particle is moving (anticlockwise) with a uniform angular speed ω along the circumference of a circle of radius a and centre O. When the particle is at point P, then the foot of the perpendicular drawn from the particle on the diameter AA' of the circle is at the point M [See Fig. 25.2 (*i*)]. Let us study the motion of M (*i.e.*, foot of the perpendicular drawn from the particle on the diameter AA') as the particle moves along the circle. When the particle was at point B, the foot of the perpendicular was at the point O. As the particle moves along the arc BA, the foot of the perpendicular (M) moves along the diameter OA. When the particle reaches point A, the foot of the perpendicular is also at A. When the particle moving along the arc AB' reaches point B', the foot of the perpendicular moving along diameter AO reaches point O. As the particle moves along the arc $B'A'B$, the foot of the perpendicular moves from O to A' and then from A' to O. Note that as the particle completes one circle, the foot of the perpendicular (M) completes one vibration. Note that M moves to and fro along the straight line AA'; O being the equilibrium position.

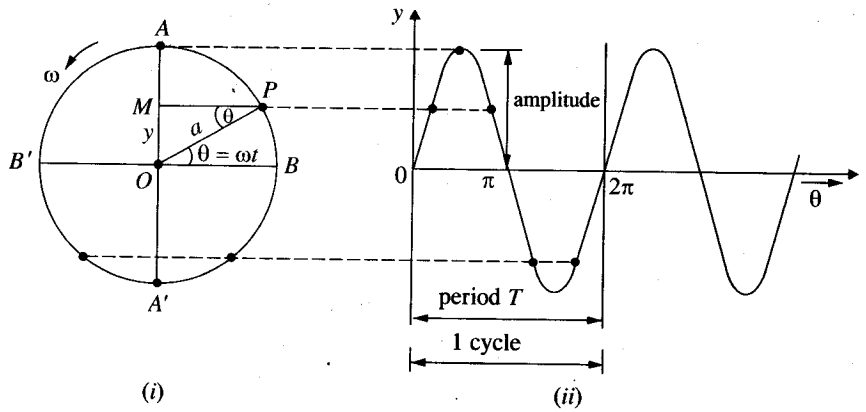


Fig. 25.2

The motion of M is the simple harmonic motion. To test it, let us find the equation of motion of M. Suppose the particle starts from point B and rotates through an angle θ rad in time t. Then $\theta = \omega t$. At this instant, the displacement of M is OM ($= y$).

Now $$\frac{OM}{OP} = \sin\theta = \sin\omega t$$

or $$OM = OP \sin\omega t$$

or $$y = a \sin\omega t \qquad (\because\ OP = a = \text{radius of circle})$$

This is the displacement equation of simple harmonic motion. The circle is called the *reference circle* of simple harmonic motion. Note that the particle is executing circular motion but the foot of the perpendicular drawn from the particle on the diameter AA' is executing SHM. We arrive at an important conclusion :

A particle travelling around a reference circle of radius a *at constant angular speed* $\omega(=2\pi f)$ *can be used to represent simple harmonic motion of amplitude* a *and frequency* f.

Because of this relationship between uniform circular motion and simple harmonic motion (S.H.M.), we often use the terminology of circular motion in discussing S.H.M. For example, ω is often called the angular frequency of S.H.M.

Now $$y = a \sin\omega t = a \sin 2\pi f t = a \sin\frac{2\pi}{T} t$$

where $$\omega = 2\pi f = \frac{2\pi}{T}$$

Note that S.I. units of f and ω are Hz (Hertz). Let us look at the graphical representation shown in the right-hand side in Fig. 25.2 (*ii*). As the particle moves along the circle, the displacement –time graph of M (foot of perpendicular from the particle on diameter AA') traces a sine curve. To emphasise the circular aspect of S.H.M., the horizontal axis is labelled $\theta(=\omega t)$. We call θ the **phase** of S.H.M. At start of the cycle, θ is 0°. Halfway through the cycle, θ is 180° (or π rad), when $\theta = 270°$ (or $3\pi/2$ rad), the cycle is three-fourth done. Therefore, when we say that "the phase is 90°" (or $\pi/2$ rad), it means that 1/4 cycle has been completed.

25.5. GENERAL DISPLACEMENT EQUATION OF S.H.M.

The displacement equation of S.H.M., $y = a \sin \omega t$ represents the condition when the motion is taken to start from the equilibrium position (*i.e.*, at $t = 0$; $y = 0$). This corresponds to point B on the reference circle of S.H.M. [See Fig. 25.3]. The general case requires that timing may commence when the particle is at any point of its oscillation. Suppose the time is measured from the instant when the particle moving on the reference circle was at a point different from B, say at P_0 (See Fig. 25.3) where $\angle P_0OB = \phi$. In time t, the particle has reached point P such that $\angle POP_0 = \omega t$. Therefore, displacement of M from the equilibrium position is OM $(= y)$.

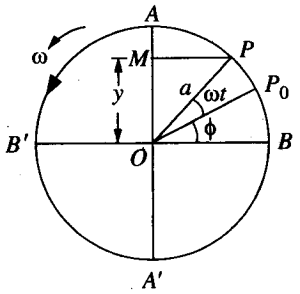


Fig. 25.3

$$\therefore \quad \frac{OM}{OP} = \sin MPO = \sin POB$$

$$\text{or} \quad OM = OP \sin(\omega t + \phi) \qquad (\because \angle MPO = \angle POB = \omega t + \phi)$$

$$\text{or} \quad y = a \sin(\omega t + \phi) \qquad (\because OP = a) \qquad ...(i)$$

Eq. (*i*) is the *general displacement equation of S.H.M. Note that if time is measured from the instant the particle is at the equilibrium position, then $\phi = 0$ so that $y = a \sin \omega t$. Note that ϕ is called the initial phase (or phase constant) or epoch of M.

Finally, it should be noted that cosines instead of sines could have been used throughout our discussion because :

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

This difference in representation is equivalent to adding a $\pi/2$ phase constant. This causes no difficulty because the choice of phase is arbitrary when only one oscillator is being described. However, once a choice has been made, that choice must be adhered to throughout the problem at hand.

Note. $y = a \sin(\omega t + \phi)$

(*i*) If $\phi = 0°$, then $y = a \sin \omega t$. This means that at $t = 0$, the particle is at the mean position.

(*ii*) If $\phi = 90°$, $y = a \sin(\omega t + 90°) = a \cos \omega t$. This means that at $t = 0$, the particle is at the extreme position. Therefore both $y = a \sin \omega t$ and $y = a \cos \omega t$ are valid representation of sinusoidal motion. Motion described by a sine or cosine function of time is known as *sinusoidal motion.* Sinusoidal motion and S.H.M. are the same.

Example 25.1. A simple harmonic oscillation is represented by ;

$$x = 0.34 \cos (3000\ t + 0.74)$$

where x and t are in mm and sec respectively. Determine (*i*) the amplitude (*ii*) the frequency and the angular frequency (*iii*) the time period.

Solution. $x = 0.34 \cos (3000\ t + 0.74)$ ...(*i*)

* Note that ϕ can be negative also if motion starts before point B.

The standard displacement equation of S.H.M. is

$$x = a \cos(\omega t + \phi) \quad ...(ii)$$

Comparing eqs. (*i*) and (*ii*), we have,

(*i*) Amplitude, $a = 0.34$ mm

(*ii*) Angular frequency, $\omega = 3000$ Hz

Now $\omega = 2\pi f$

$\therefore$ Frequency, $f = \dfrac{\omega}{2\pi} = \dfrac{3000}{2\pi} = \dfrac{1500}{\pi}$ Hz

(*iii*) Time period, $T = \dfrac{1}{f} = \dfrac{\pi}{1500}$ sec

Example 25.2. **An object executes S.H.M. with an amplitude of 0.17 m and a period of 0.84 s. Determine (*i*) the frequency and (*ii*) the angular frequency of the motion. Write down expression for the displacement equation.**

Solution. Amplitude of S.H.M., $a = 0.17$ m

Period of S.H.M., $T = 0.84$ s

(*i*) Frequency, $f = \dfrac{1}{T} = \dfrac{1}{0.84} = 1.2$ Hz

(*ii*) Angular frequency, $\omega = 2\pi f = 2\pi \times 1.2 = 7.5$ Hz

The displacement equation of S.H.M. is

$$x = a \sin(\omega t + \phi)$$

Here $a = 0.17$ m ; $\omega = 7.5$ Hz ; $\phi = 0$ ($\because$ Initial phase is not given)

$\therefore$ $x = 0.17 \sin 7.5\, t$

Example 25.3. **The equation of S.H.M. is given as :**

$$x = 6 \sin 10\pi t + 8 \cos 10\pi t$$

where x is in cm and t in second. Find (*i*) period (*ii*) amplitude and (*iii*) initial phase of motion.

Solution.

$$x = 6 \sin 10\pi t + 8 \cos 10\pi t \quad ...(i)$$

The standard displacement equation of S.H.M. is

$$x = a \sin(\omega t + \phi)$$

or $$x = a \cos\phi \sin\omega t + a \sin\phi \cos\omega t \quad ...(ii)$$

Comparing eqs. (*i*) and (*ii*), $\omega = 10\pi$ and

$$a \cos\phi = 6 \quad ...(iii)$$

$$a \sin\phi = 8 \quad ...(iv)$$

(*i*) Period, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{10\pi} = 0.2$ s

(*ii*) Squaring eqs. (*iii*) and (*iv*) and then adding, we have,

$$(a \cos\phi)^2 + (a \sin\phi)^2 = 6^2 + 8^2$$

or $$a^2 = 36 + 64 = 100$$

$\therefore$ Amplitude, $a = \sqrt{100} = 10$ cm

(*iii*) Dividing eq. (*iv*) by eq. (*iii*), we have,

$$\tan\phi = 8/6 = 1.333$$

$\therefore$ $$\phi = \tan^{-1} 1.333 = 53.12^\circ$$

PROBLEMS FOR PRACTICE

1. A body oscillates with simple harmonic motion along the x axis. Its displacement varies with time according to the equation ;

$$x = 4.0\cos\left(\pi t + \frac{\pi}{4}\right)$$

where x is in m and t is in sec. Determine the amplitude, frequency and period of the motion. **[4.0 m ; 0.50 Hz; 2.0 s]**

2. A body oscillates with simple harmonic motion according to the following equation :

$$x = 0.40 \cos (0.70\ t - 0.30)\ \text{m}$$

Find (*i*) the amplitude (*ii*) the frequency (*iii*) the period and (*iv*) the phase constant.

[(*i*) 0.40 m (*ii*) 0.111 Hz (*iii*) 8.97 s (*iv*) 17.2°]

3. The displacement of a particle is given by ;

$$x = 4 \cos (3\pi t + \pi)$$

where x is in metres and t is in seconds.

Determine (*i*) the frequency and period of the motion (*ii*) the amplitude of the motion (*iii*) the phase constant and (*iv*) the position of the particle at $t = 0$.

[(*i*) 1.5 Hz ; 0.67 s (*ii*) 4 m (*iii*) π rad (*iv*) – 4 m]

25.6. CHARACTERISTICS OF SIMPLE HARMONIC MOTION

Consider a particle moving (anticlockwise) with uniform angular speed ω along the circumference of a reference circle of radius a and centre O. Suppose the particle starts from point B and rotates through an angle θ rad in time t and assumes the position P. Then $\theta = \omega t$ [See Fig. 25.4 (*i*)]. When the particle is at the point P, then the foot of the perpendicular drawn from the particle on the diameter AA' of the reference circle is at the point M. As the particle moves along the circumference of the reference circle, M executes simple harmonic motion along the diameter AA'. Note that O is the equilibrium position for this S.H.M.

(*i*) Displacement. *The displacement of a particle executing simple harmonic motion at any instant is the distance of the particle from the equilibrium position at that instant.* Thus in Fig. 25.4 (*i*), M is executing S.H.M. along the diameter AA'. At the considered instant, the displacement of M is $OM = y$.

(*ii*) Displacement equation. When the particle is at point P, M has a displacement $OM = y$.

Now $$\frac{OM}{OP} = \sin \omega t$$

or $$OM = OP \sin \omega t$$

or $$y = a \sin \omega t \qquad (\because OP = a) \qquad ...(i)$$

Equation (*i*) is the displacement equation of simple harmonic motion. It is clear that the displacement-time graph for the motion of M [See Fig. 25.4 (*ii*)] is a sine curve — a test for S.H.M. In general, if the time is measured from the instant when the particle is at point P' between B and A such that $\angle P'OB = \phi$, then displacement equation becomes :

$$y = a \sin(\omega t + \phi)$$

(*iii*) Amplitude. *The maximum displacement of the particle executing S.H.M. on either side of the equilibrium position is called amplitude of the motion.* Thus referring to Fig. 25.4 (*i*), the displacement of M at any time t is given by ;

$$y = a \sin \omega t$$

The value of y will be maximum when $\sin \omega t$ is maximum. Now the maximum value of $\sin \omega t$ (or $\cos \omega t$) is 1.

$$\therefore \quad \text{Amplitude} = a \times 1 = a$$

Note that when the particle reaches point A, the displacement of M is maximum ($= OA = a$ = radius of reference circle). Thus amplitude is equal to the radius of the reference circle.

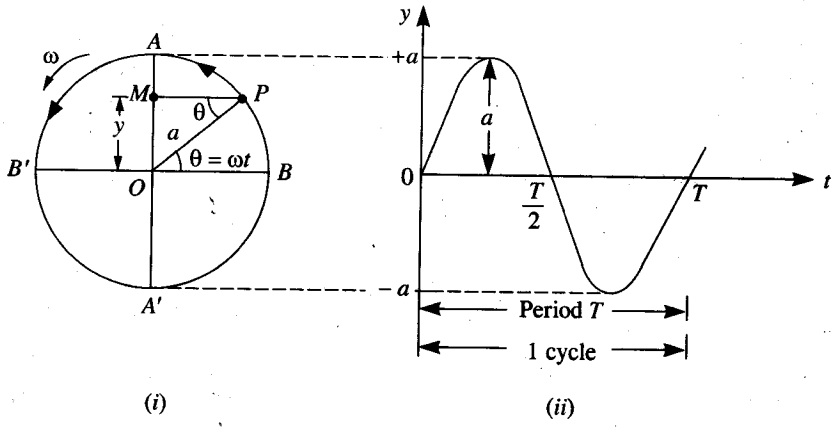


Fig. 25.4

(*iv*) **Phase.** *The phase angle of a particle executing S.H.M. is its position at* $t = 0$. The phase constant (or phase angle) tells us what the displacement was at $t = 0$. If at $t = 0$, the particle is at point B (starting point), $\phi = 0$. If at $t = 0$, the particle is at point A, $\phi = \pi/2$ rad.

$$\therefore \quad y = a \sin (\omega t + \phi)$$

The quantity $(\omega t + \phi)$ is called the *phase* of motion *i.e.*, displacement of particle in time t and is useful in comparing the motion of two systems of particles. Note that function y is *periodic* and repeats itself when ωt increases by 2π radians.

(*v*) **Velocity.** *The velocity of a particle executing S.H.M. at any instant is the time rate of change of its displacement at that instant.* Thus referring to Fig. 25.4 (*i*), M is executing S.H.M. along the diameter AA'. At the considered instant, its displacement is given by ;

$$y = a \sin \omega t$$

$$\therefore \quad \text{Velocity, } V = \frac{dy}{dt} = \frac{d}{dt}(a \sin \omega t) = a\omega \cos \omega t$$

$$= a\omega \sqrt{1 - \sin^2 \omega t} = a\omega \sqrt{1 - y^2/a^2}$$

$$\therefore \quad V = \omega \sqrt{a^2 - y^2} \qquad \ldots(ii)$$

Eq. (*ii*) gives the instantaneous velocity of the particle executing S.H.M. Thus the velocity of the particle executing S.H.M. changes with the displacement y of the particle.

At equilibrium position, $y = 0$ so that $V = \omega a$

At extreme positions, $y = a$ so that $V = 0$

Thus the velocity of a particle executing S.H.M. is maximum when $y = 0$ *i.e.*, when the particle passes through its equilibrium position. However, when the displacement is maximum ($y = a$), the velocity of the particle is zero. Note that maximum value of velocity is called *velocity amplitude* in S.H.M.

(*vi*) **Acceleration.** *The acceleration of a particle executing S.H.M. at any instant is the time rate of change of velocity at that instant.* Thus referring to Fig. 25.4 (*i*), M is executing S.H.M. along the diameter AA'. At the considered instant,

$$y = a \sin \omega t$$

$$\therefore \quad \text{Velocity, } V = \frac{dy}{dt} = \frac{d}{dt}(a \sin \omega t) = a\omega \cos \omega t$$

$$\therefore \quad \text{Acceleration, } A = \frac{dV}{dt} = \frac{d}{dt}(a\omega \cos \omega t) = -\omega^2 a \sin \omega t$$

$$\therefore \quad A = -\omega^2 y \qquad (\because a \sin \omega t = y) \qquad ...(iii)$$

Eq. (*iii*) gives the instantaneous acceleration of the particle executing S.H.M. Note that acceleration of the particle executing S.H.M. changes with the displacement y of the particle.

At equilibrium position, $y = 0$ so that $A = 0$

At extreme positions, $y = a$ so that $A = -\omega^2 a$

Thus the acceleration of a particle executing S.H.M. is zero when $y = 0$ *i.e.*, when the particle passes through its equilibrium position. However, when the displacement of the particle is maximum, its acceleration is also maximum. Note that maximum value of acceleration is called *acceleration amplitude* in S.H.M.

Another definition of S.H.M. We have seen that the acceleration of a particle executing S.H.M. is given by ;

$$A = -\omega^2 y$$

or $$A \propto -y \qquad (\because \omega \text{ is constant})$$

Thus we find that the acceleration of a particle executing S.H.M. is proportional to its displacement y and its direction (negative sign) is opposite to the direction of displacement. This fact provides another definition of S.H.M.

A particle is said to execute S.H.M. if its acceleration at any instant is directly proportional to its displacement from the equilibrium position and is always directed towards the equilibrium position.

(*vii*) **Time period.** *The time taken by a particle executing S.H.M. to complete *one vibration is called time period (T) of the motion. This means that the value of y at any time t is equal to the value of y at time t + T. We can show that time period of motion is given by* $T = 2\pi/\omega$ by using the fact that phase increases by 2π radians in a time T :

$$\omega t + 2\pi = \omega(t + T)$$

or $$T = \frac{2\pi}{\omega} \qquad ...(iv)$$

Neglecting the negative sign from eq. (*iii*), we have,

$$A = \omega^2 y$$

or $$\omega = \sqrt{A/y}$$

$$\therefore \quad \text{Time period, } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{y}{A}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

$$\text{Frequency, } f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

Note that time period (T) and frequency (f) of S.H.M. do not depend on the amplitude of the motion.

* Thus referring to Fig. 25.4 (*i*), M is said to have completed 1 vibration (or cycle of motion) if starting from O, it has moved through positions $O - A - A' - O$ *i.e.*, when it has returned to its starting position and is moving in the *same* direction.

25.7. GRAPHICAL REPRESENTATION OF S.H.M.

Figure 25.5 shows the displacement – time, velocity – time and acceleration – time graphs for a simple harmonic motion. Let us suppose that :

$$y = a \cos \omega t = a \cos \theta \qquad ...(i)$$

Note that we choose a phase constant (ϕ) equal to zero for the displacement.

(*i*) Velocity, $V = \frac{dy}{dt} = \frac{d}{dt}(a \cos \omega t) = -a\omega \sin \omega t$

$\therefore$ $$V = a\omega \cos\left(\omega t + \frac{\pi}{2}\right) \qquad ... (ii)$$

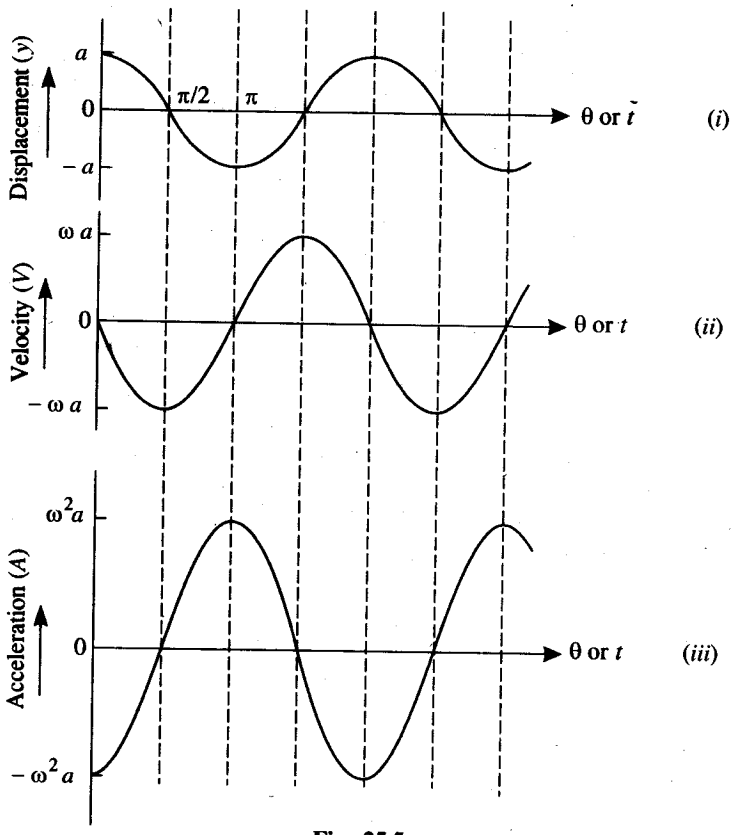


Fig. 25.5

Comparing eqs. (*i*) and (*ii*), we note that phase of the velocity differs from the phase of the displacement by $\pi/2$ rad or 90°. This means that when y is a maximum or minimum, the velocity is zero. Likewise, when $y = 0$, the velocity is maximum. Note that velocity of the particle at a particular time can be determined by taking the slope of displacement – time graph at that time.

(*ii*) Acceleration, $A = \frac{dV}{dt} = -a\omega^2 \cos \omega t = a\omega^2 \cos(\omega t + \pi)$

$\therefore$ $$A = a\omega^2 \cos(\omega t + \pi) \qquad ...(iii)$$

Comparing eqs. (*i*) and (*iii*), we note that acceleration (A) is 180° out of phase with the displacement. Note that acceleration-time graph looks like the displacement-time graph but with opposite sign. This is expected because in S.H.M., acceleration is proportional to the negative of the displacement. In other words, when y is positive, acceleration is directed towards $-y$.

Note that each differentiation changes the multiplication factor by the factor ω; y oscillates between a and $-a$, V oscillates between ωa and $-\omega a$ and A oscillates between $\omega^2 a$ and $-\omega^2 a$. Another effect of each differentiation is a change in phase by $\pi/2$ rad or 90°.

Important points. The following are the important properties of a particle executing S.H.M. :

(*i*) *The particle moves to and fro about the mean (equilibrium) position in a straight line.*

(*ii*) *The displacement, velocity and acceleration all vary *sinusoidally with time but are not in phase.*

(*iii*) *The acceleration of the particle is proportional to the displacement but in the opposite direction.*

(*iv*) *The frequency and period of motion are independent of the amplitude of the motion.*

Example 25.4. **The periodic-time of a body executing S.H.M. is 2 seconds. After how much time interval from time $t = 0$ will its displacement be half of its amplitude?**

Solution. Let t be the time interval for the displacement to be half the amplitude.

Now $$y = a\sin\omega t = a\sin\frac{2\pi t}{T}$$

Here $$y = a/2\ ;\ T = 2\text{ s}$$

$\therefore$ $$\frac{a}{2} = a\sin\frac{2\pi}{2}t$$

or $$\frac{1}{2} = \sin\pi t \quad \text{or} \quad \sin\frac{\pi}{6} = \sin\pi t$$

$\therefore$ $$\frac{\pi}{6} = \pi t \quad \text{or} \quad t = \frac{1}{6}\text{ second}$$

Example 25.5. **A particle executes S.H.M. of amplitude 25 cm and time period 3s. What is the minimum time required for the particle to move between two points 12.5 cm on either side of the mean position?**

Solution. Let t be the time taken by the particle to move from mean position to a point 12.5 cm from it.

Now $$y = a\sin\frac{2\pi}{T}t$$

Here $y = 12.5$ cm ; $a = 25$ cm ; $T = 3$ s

$\therefore$ $$12.5 = 25\sin\frac{2\pi}{3}t$$

or $$\sin\frac{2\pi}{3}t = \frac{12.5}{25} = \frac{1}{2} = \sin\pi/6$$

or $$\sin\frac{2\pi}{3}t = \sin\frac{\pi}{6} \quad \text{or} \quad t = \frac{1}{4}\text{ s}$$

$\therefore$ $$\text{Required time} = 2t = 2\times\frac{1}{4} = 0.5\text{ s}$$

Example 25.6. **A particle in S.H.M. is described by the displacement function :**

$$x = A\cos(\omega t + \phi),\ \omega = \frac{2\pi}{T}$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is π cm/s, what are its amplitude and phase angle? The angular frequency of the particle is $\pi\ \text{s}^{-1}$.

* The derivative of a sinusoidally varying function is itself a sinusoidally varying function with the same frequency.

Solution. $x = A \cos(\omega t + \phi)$

At $t = 0$, $x = 1$ cm ; $\omega = \pi s^{-1}$

$\therefore \quad 1 = A \cos(\pi \times 0 + \phi)$

or $\quad 1 = A \cos\phi \quad ...(i)$

Now $\quad V = \dfrac{dx}{dt} = -A\,\omega \sin(\omega t + \phi)$

At $t = 0$, $V = \pi$ cm s^{-1} ; $\omega = \pi s^{-1}$

$\therefore \quad \pi = -A\,\pi \sin(\pi \times 0 + \phi)$

or $\quad -1 = A \sin\phi \quad ...(ii)$

Squaring and adding eqs. (*i*) and (*ii*), we have,

$$A^2 \cos^2\phi + A^2 \sin^2\phi = 1 + 1 = 2$$

or $\quad A^2 = 2 \quad$ or $\quad A = \sqrt{2}$ cm

Dividing eq. (*ii*) by eq. (*i*), we have,

$$\tan\phi = -1 \quad \text{or} \quad \phi = 3\pi/4$$

Example 25.7. The time period of a particle executing S.H.M. is 2 seconds and it can go to and fro from equilibrium position at a maximum distance of 5 cm. If at the start of the motion, the particle is in the position of maximum displacement towards the right of the equilibrium position, then write the displacement equation of the particle.

Solution. The general equation for the displacement in S.H.M. is

$$y = a \sin(\omega t + \phi) \quad ...(i)$$

The time period of particle is $T = 2\pi/\omega = 2$ sec. Therefore, $\omega = \pi$ rad s^{-1}.

Now, $y = 5$ cm ; $a = 5$ cm ; $\omega = \pi$ rad s^{-1} ; $t = 0$

$\therefore \quad 5 = 5 \sin(\pi \times 0 + \phi)$ or $\sin\phi = 1$ or $\phi = \pi/2$ rad

Putting the value of a and ϕ in eq. (*i*), the displacement equation of the particle is

$$y = 5 \sin(\pi t + \pi/2)$$

or $\quad y = 5 \sin\pi\left(t + \dfrac{1}{2}\right)$

Example 25.8. In what time after its motion begins, will a particle oscillating according to the equation $y = 7 \sin 0.5\,\pi t$ move from the mean position to maximum displacement?

Solution. $y = 7 \sin 0.5\,\pi t \quad ...(i)$

The standard displacement equation of S.H.M. is

$$y = a \sin \omega t \quad ...(ii)$$

Comparing eqs. (*i*) and (*ii*), we have, $a = 7$; $\omega = 0.5\,\pi$.

Let t be the time taken by the particle in moving from mean position to maximum displacement position.

$\therefore \quad y = 7 \sin 0.5\,\pi t$

Here $y = a = 7$

$\therefore \quad 7 = 7 \sin 0.5\,\pi t$

or $\quad \sin 0.5\pi t = \dfrac{7}{7} = 1 = \sin\dfrac{\pi}{2}$

or $\quad 0.5\pi t = \dfrac{\pi}{2} \quad \therefore \quad t = 1$ s

Example 25.9. The vertical motion of a huge piston in a machine is approximately simple harmonic with a frequency of 0.5 s^{-1}. A block of 10 kg is placed on the piston. What is the maximum amplitude of the piston's S.H.M. for the block and piston to remain together?

Solution. The block will remain in contact with the piston if maximum acceleration (A_{max}) of S.H.M. does not exceed g *i.e.*, A_{max} is at the most equal to g.

Now $A_{max} = \omega^2 a = (2\pi f)^2 \times a$ where a = amplitude

$\therefore$ $g = (2\pi f)^2 \times a$

or $a = \dfrac{g}{(2\pi f)^2} = \dfrac{9.8}{(2\pi \times 0.5)^2} = \mathbf{0.99\ m}$

Example 25.10. A particle executing S.H.M. has a maximum displacement of 4 cm and its acceleration at a distance of 1 cm from its mean position is 3 cm s^{-2}. What will be its velocity when it is at a distance of 2 cm from its mean position?

Solution. The acceleration of a particle executing S.H.M. is given by ;

$$A = \omega^2 y$$

Here $A = 3$ cm s^{-2}; $y = 1$ cm.

$\therefore$ $3 = \omega^2 \times 1$ or $\omega = \sqrt{3}$ rad s^{-1}

The velocity of a particle executing S.H.M. is given by ;

$$V = \omega\sqrt{a^2 - y^2}$$

Here $\omega = \sqrt{3}$ rad s^{-1} ; $a = 4$ cm ; $y = 2$ cm.

$\therefore$ $V = \sqrt{3} \times \sqrt{4^2 - 2^2} = \mathbf{6\ cm\ s^{-1}}$

Example 25.11. A particle executing S.H.M. along a straight line has a velocity of 4 ms^{-1} when its displacement from mean position is 3 m and 3 ms^{-1} when the displacement is 4 m. Find the time taken to travel 2.5 m from the positive extremity of its oscillation.

Solution. $V^2 = \omega^2(a^2 - y^2)$

For the first case, $16 = \omega^2(a^2 - 9)$; For the second case, $9 = \omega^2(a^2 - 16)$

$\therefore$ $\dfrac{16}{9} = \dfrac{a^2 - 9}{a^2 - 16}$ or $7a^2 = 175$ $\therefore$ $a = \sqrt{\dfrac{175}{7}} = 5$ m

Also $16 = \omega^2(25 - 9)$ $\therefore$ $\omega = 1$ rad/s

When the particle is 2.5 m from the positive extreme position, its displacement from the mean position is $y = 5 - 2.5 = 2.5$ m. Since the time is measured when the particle is at extreme position,

$\therefore$ $y = a \cos \omega t$ or $2.5 = 5 \cos (1 \times t)$

$\therefore$ $\cos t = \dfrac{2.5}{5} = \dfrac{1}{2} = \dfrac{\cos \pi}{3}$ $\therefore$ $t = \dfrac{\pi}{3} = \mathbf{1.047\ s}$

Example 25.12. A particle is moving with S.H.M. in a straight line. When the distance of the particle from mean position has values x_1 and x_2, the corresponding values of velocities are u_1 and u_2, Show that time period of oscillation is given by ;

$$\boldsymbol{T = 2\pi\left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}}$$

Solution. $V^2 = \omega^2(a^2 - y^2)$

For the first case, $u_1^2 = \omega^2(a^2 - x_1^2)$ ... (*i*)

For the second case, $u_2^2 = \omega^2(a^2 - x_2^2)$ ... (*ii*)

Subtracting eq. (*ii*) from eq. (*i*), we have,

$$u_1^2 - u_2^2 = \omega^2(a^2 - x_1^2) - \omega^2(a^2 - x_2^2) = \omega^2(x_2^2 - x_1^2)$$

$$\therefore \quad \omega^2 = \frac{u_1^2 - u_2^2}{x_2^2 - x_1^2} \quad \text{or} \quad \omega = \left[\frac{u_1^2 - u_2^2}{x_2^2 - x_1^2}\right]^{1/2}$$

$$\therefore \quad \text{Time period, } T = \frac{2\pi}{\omega} = 2\pi\left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}$$

PROBLEMS FOR PRACTICE

1. The displacement equation of a particle executing simple harmonic motion is : $y = 0.01 \sin 100\pi\ (t + 0.005)$ metre where y is the displacement of the particle at any time t. Calculate the maximum velocity and the displacement at the time of the start of the motion. **[3.14 m s^{-1} ; 0.01 m]**
2. The acceleration of a particle executing S.H.M. is 12 cm s^{-2} at a distance of 3 cm from the mean position. Calculate its time period. **[3.14 s]**
3. A body is vibrating with S.H.M. with amplitude 15 cm and frequency 4 Hz. Calculate the maximum acceleration and the velocity when the displacement is 10 cm. **[94.82 m s^{-2} ; 2.81 m s^{-1}]**
4. The maximum velocity of a particle executing S.H.M. is 100 cm s^{-1} and the maximum acceleration is 157 cm s^{-2}. Determine the time period of the motion. **[4 seconds]**
5. If $y = a \cos\omega t + b \sin\omega t$, show that it represents S.H.M. Obtain its amplitude. $\left[\sqrt{a^2 + b^2}\right]$
6. A horizontal platform vibrates up and down with S.H.M. of amplitude 20 cm. At what frequency will an object be kept on the platform so as to lose contact with the platform? $g = 9.8$ ms^{-2}. **[1.1 Hz]**
7. The velocity of a particle executing S.H.M. is 16 cm s^{-1} at a distance of 8 cm from mean position and 8 cm s^{-1} at a distance of 12 cm from mean position. Determine the amplitude of the motion. **[0.1306 m]**
8. If a particle executes S.H.M. of period 4 s and amplitude 2 cm, find its maximum velocity and that at half its full displacement. **[3.14 cm s^{-1} ; 2.72 cm s^{-1}]**
9. A particle executing S.H.M. along a straight line has a velocity of 4 m s^{-1} when at a distance of 3 m from the mean position and 3 m s^{-1} when at a distance of 4 m from mean position. Find the time it takes to travel 2.5 m from the positive extreme position. **[1.048 s]**

25.8. TOTAL ENERGY IN S.H.M.

When a body oscillates in simple harmonic motion, it is acted upon by a restoring force which tends to bring it to the equilibrium position. Due to this force, there is a potential energy in the body. Moreover, as the body is in motion, it has kinetic energy. During oscillation of the body, its energy continuously interchanges between kinetic energy and potential energy but their sum remains constant (taking friction negligible).

Potential energy. Consider a particle of mass m executing S.H.M. with amplitude a and constant angular frequency ω. Suppose at any time t, the displacement of the particle from the

*equilibrium position is y. If A is the acceleration of the particle at that instant, then by definition of S.H.M., we have,

$$A = -\omega^2 y$$

Magnitude of the restoring force acting on the particle at that instant is

$$F = m A = -m\omega^2 y = -k y$$

where $m\omega^2 = k$ = force constant of S.H.M.

If the particle undergoes a further very small displacement dy, the small work done against the restoring force is

$$dW = -F\,dy = -(-k y)\,dy = k y\,dy$$

$\therefore$ Total work done for displacement y is

$$W = \int_0^y k y\,dy = \frac{1}{2}k y^2$$

This work done is equal to the potential energy E_P of the particle at displacement y *i.e,*

$$E_P = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2 y^2 \qquad (\because k = m\omega^2) \qquad ...(i)$$

Kinetic energy. Suppose V is the velocity of the particle when its displacement is y. Then kinetic energy E_K of the particle is

$$E_K = \frac{1}{2}mV^2$$

Now $V = \omega\sqrt{a^2 - y^2}$ where a = amplitude of S.H.M.

$\therefore$ $$E_K = \frac{1}{2}m\omega^2(a^2 - y^2) \qquad ...(ii)$$

Total energy. The total energy E of the particle for displacement y is given by ;

$$E = E_P + E_K = \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}m\omega^2(a^2 - y^2)$$

or $$E = \frac{1}{2}ma^2\omega^2 \qquad ...(iii)$$

We know that $\omega = 2\pi f$ where f is the frequency of S.H.M.

$\therefore$ $$E = \frac{1}{2}ma^2(2\pi f)^2$$

or $$E = 2m\pi^2 f^2 a^2 \qquad ...(iv)$$

Discussion. The following points may be noted :

(*i*) For a given particle in S.H.M., m, ω and a are constants [See eq. (*iii*)]. Thus the total energy ($E_P + E_K$) of a particle executing S.H.M. remains constant at all times.

(*ii*) The total energy of a particle executing S.H.M. is directly proportional to the square of the amplitude (a^2) and to the square of the frequency (f^2). It is also directly proportional to the mass (m) of the particle.

(*iii*) The energy of the particle continuously interchanges between kinetic energy and potential energy but their sum at all times remains constant. Thus in the position of maximum displacement (when $y = a$), the total energy is in the form of potential energy while in the equilibrium position (when $y = 0$), the total energy is in the form of kinetic energy.

* Unless stated otherwise, the particle in S.H.M. is assumed to start ($t = 0$) motion from the mean position.

This can be seen by putting $y = a$ and $y = 0$ in eq. (*i*) and eq. (*ii*) respectively. In other positions, both forms of energy exist.

Graphical representation. Fig. 25.6 shows the graph of kinetic energy and potential energy versus displacement. Each curve is a parabola centred at $y = 0$.

$$E_P = \frac{1}{2}m\omega^2 y^2 \; ; \; E_K = \frac{1}{2}m\omega^2(a^2 - y^2)$$

When the particle is at the mean position, $y = 0$ so that $E_P = 0$ and $E_K = \frac{1}{2}m\omega^2 a^2$. In this position, the total energy of the particle is in the form of kinetic energy since $E_P = 0$. This is quite clear from Fig. 25.6. Since the kinetic energy of the particle passing through mean position is maximum, it is clear that at this position, the speed of the particle will be maximum.

When the particle is at the extreme positions, $y = a$ so that $E_P = \frac{1}{2}m\omega^2 a^2$ and $E_K = 0$. In this position, the total energy of the particle is in the form of potential energy since $E_K = 0$. This is also clear from Fig. 25.6. Since the kinetic energy of the particle at the extreme positions is zero, the speed of the particle at these positions will be zero.

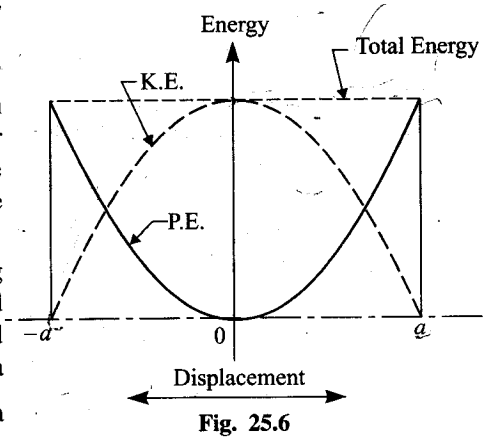


Fig. 25.6

It is clear from Fig. 25.6 that energy is being continuously transferred between potential energy and kinetic energy of the particle. Note that both E_K and E_P are always positive and their sum at all times is a constant equal to $\frac{1}{2}m\omega^2 a^2$. This is represented by a straight line parallel to the displacement axis.

Example 25.13. A mass of 0.5 kg connected to a light spring of force constant 20 Nm^{-1} oscillates on a horizontal frictionless surface.

(*i*) Calculate the total energy of the system and the maximum speed of the mass if the amplitude of the motion is 3 cm.

(*ii*) What is the velocity of the mass when the displacement is equal to 2 cm?

(*iii*) Find the kinetic and potential energies of the system when the displacement equals 2 cm.

Solution. (*i*) The total energy E of the system is equal to the initial energy stored in the spring for amplitude a (= 3 cm = 3 × 10^{-2} m).

$$\therefore \quad E = \frac{1}{2}ka^2 = \frac{1}{2}\times 20\times(3\times10^{-2})^2 = \mathbf{9.0 \times 10^{-3}\ J}$$

When the mass is at $x = 0$ (equilibrium position), $E_P = 0$ and E_K (= E) has the maximum value.

$$\therefore \quad \frac{1}{2}mV_{max}^2 = 9.0\times10^{-3}$$

or
$$V_{max} = \sqrt{\frac{2\times9.0\times10^{-3}}{0.5}} = \mathbf{0.19\ m\ s^{-1}}$$

(*ii*) Total energy, $E = E_K + E_P$

or
$$\frac{1}{2}ka^2 = \frac{1}{2}mV^2 + \frac{1}{2}kx^2$$

or
$$V = \pm\sqrt{\frac{k}{m}(a^2 - x^2)}$$

Here $k = 20$ N m^{-1} ; $m = 0.5$ kg ; $a = 3 \times 10^{-2}$ m ; $x = 2 \times 10^{-2}$ m

$$\therefore \quad V = \pm\sqrt{\frac{20}{0.5}(3^2 - 2^2)\times 10^{-4}} = \pm\ 0.14 \text{ m s}^{-1}$$

The positive and negative signs indicate that the mass could be moving to the right or left at this instant.

(*iii*) The K.E. of the mass when displacement is 2 cm is

$$E_K = \frac{1}{2}mV^2 = \frac{1}{2}\times 0.5\times(0.14)^2 = 5.0 \times 10^{-3} \text{ J}$$

The P.E. of the spring when the displacement is 2 cm is given by ;

$$E_P = \frac{1}{2}kx^2 = \frac{1}{2}\times 20\times(2\times 10^{-2})^2 = 4.0 \times 10^{-3} \text{ J}$$

Note that the sum $E_K + E_P$ is equal to total energy E.

Example 25.14. A small mass of 0.2 kg is attached to one end of a spring and produces an extension of 15 mm. The mass is now set into vertical oscillations of amplitude 10 mm. Find (*i*) the period of oscillations (*ii*) the maximum kinetic energy of the mass (*iii*) potential energy of the spring when the mass is 5 mm below the centre of oscillations.

Solution. (*i*) Force constant k of the spring is

$$k = \frac{F}{y} = \frac{mg}{y} = \frac{0.2\times 9.8}{0.015} \text{ N m}^{-1}$$

$$\text{Period of oscillations, } T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2\times 0.015}{0.2\times 9.8}} = 0.25 \text{ s}$$

(*ii*) $$\text{Maximum K.E.} = \frac{1}{2}mV_{max}^2$$

For S.H.M., $V_{max} = \omega a$ where a = amplitude of motion

$$\therefore \quad \text{Maximum K.E.} = \frac{1}{2}m\omega^2 a^2$$

$$= \frac{1}{2}\times 0.2\times\frac{9.8}{0.015}\times(0.01)^2 \qquad \left[\because \omega^2 = \frac{k}{m}\right]$$

$$= 6.5 \times 10^{-3} \text{ J}$$

(*iii*) The potential energy of the spring is given by $\frac{1}{2}kx^2$ where k is the force constant and x is the extension from its *original* length. The centre of oscillations is 15 mm below the unstretched length. Therefore, 5 mm below the centre of oscillations corresponds to an extension of 20 mm or 0.02 m.

$\therefore$ Potential energy of the spring is given by ;

$$E_P = \frac{1}{2}kx^2$$

$$\text{Here } k = \frac{0.2\times 9.8}{0.015} \text{ Nm}^{-1} ; \quad x = 20 \text{ mm} = 0.02 \text{ m}$$

$$\therefore \quad E_P = \frac{1}{2}\times\frac{0.2\times 9.8}{0.015}\times(0.02)^2 = 2.6 \times 10^{-2} \text{ J}$$

Example 25.15. A particle executes S.H.M. of amplitude *a*. At what distance from the mean position its K.E. is equal to its P.E.?

Solution. $\text{K.E.} = \frac{1}{2}m\omega^2(a^2 - y^2)$; $\text{P.E.} = \frac{1}{2}m\omega^2 y^2$

As K.E. = P.E. $\therefore \frac{1}{2}m\omega^2(a^2 - y^2) = \frac{1}{2}m\omega^2 y^2$

or $a^2 - y^2 = y^2$ or $y^2 = a^2/2$ or $y = a/\sqrt{2} = 0.707\ a$

Example 25.16. A body of mass 1 kg is executing S.H.M. given by ;

$$x = 6.0 \cos\left(100t + \frac{\pi}{4}\right) \text{cm}$$

Determine the maximum kinetic energy of the body.

Solution. $x = 6.0 \cos\left(100\ t + \frac{\pi}{4}\right) \text{cm}$

Comparing it with standard equation of S.H.M. $x = a \cos(\omega t + \phi)$,

$$a = 6.0 \text{ cm} = 0.060 \text{ m} ; \quad \omega = 100 \text{ s}^{-1}$$

$$\text{Maximum K.E.} = \frac{1}{2}mV_{max}^2 = \frac{1}{2}m(a\omega)^2$$

$$= \frac{1}{2} \times 1 \times (0.06 \times 100)^2 = 18 \text{ J}$$

PROBLEMS FOR PRACTICE

1. A particle of mass 0.25 kg vibrates with a period of 2.0 s. If its greatest displacement is 0.4 m, what is its maximum kinetic energy? **[0.2 J]**
2. A spring with a 1.6×10^4 N m^{-1} force constant and a mass of 0.10 kg at its end has a total vibrational energy of 3.2 J. (*i*) Calculate its amplitude of vibration (*ii*) calculate its maximum speed. **[(*i*) 0.02 m (*ii*) 8.0 m s^{-1}]**
3. A 3.0 kg ball is attached to a spring of negligible mass and with a spring constant k = 40 N m^{-1}. The ball is displaced 0.10 m from equilibrium and then released from rest. What is the maximum speed of the ball as it undergoes simple harmonic motion? **[0.37 m s^{-1}]**
4. Two joules are required to extend a spring by 0.25 m. What is the spring constant? **[64 Nm^{-1}]**
5. The force constant of a weightless spring is 16 N m^{-1}. A body of mass 1.0 kg is suspended from it. The body is pulled down through 5 cm and then left free. Determine (*i*) period of oscillations of the spring (*ii*) maximum K.E. of the mass. **[(*i*) 1.57 s (*ii*) 2×10^{-2} J]**

25.9. DYNAMICS OF SIMPLE HARMONIC MOTION

We have defined simple harmonic motion (S.H.M.) but we have not described its cause. Let us investigate what causes S.H.M. Consider a particle of mass m executing S.H.M. with amplitude a and constant angular frequency ω. Then acceleration A of the particle is given by ;

$$A = -\omega^2 y$$

Here y is the displacement of the particle from the mean position. The negative sign shows that acceleration is always directed towards the mean position. Therefore, the particle is subjected

to a force F which tries to bring the particle back to the mean position. For this reason, it is called *restoring force.*

$$\therefore \quad \text{Restoring force, } F = \text{Mass} \times \text{Acceleration} = m \times (-\omega^2 y)$$

$$\text{or} \quad F = -m\omega^2 y$$

$$\text{or} \quad F = -ky \qquad ...(i)$$

where k $(= m\omega^2)$ is a constant of proportionality called **force constant* for the system. **Eq. (*i*) is the general expression to produce S.H.M.** In other words, the motion of any system governed by a force of the form of eq. (*i*) is called simple harmonic motion.

If a body is subjected to a restoring force (F) that is proportional to its displacement from the mean position (y) and is directed opposite to the direction of displacement [minus sign in eq. (i)], **then the body will execute S.H.M.**

Once we find the value of force constant k of the system executing S.H.M., the time period (T) and frequency (f) of the system can be readily obtained.

Time period and frequency of S.H.M. The time period and frequency of S.H.M. can be expressed in the following two alternate forms :

(*i*) As shown above, the force constant (k) of the particle executing S.H.M. is

$$k = \omega^2 m \quad \text{or} \quad \omega = \sqrt{k/m}$$

$$\therefore \quad \text{Time period, } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\text{Mass}}{\text{Force constant}}}$$

$$\therefore \quad \text{Frequency, } f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{\text{Force constant}}{\text{Mass}}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}} \; ; \; f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Thus time period (T) and frequency (f) of S.H.M. depend on the *mass* (m) of the body and its force constant (k).

(*ii*) The time period and frequency of S.H.M. can also be expressed in the following alternate form :

$$\text{Time period, } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\text{Mass}}{\text{Force constant}}}$$

Now,
$$k = \frac{F}{y} = \frac{\text{Force}}{\text{Displacement}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{\text{Force/Displacement}}} = 2\pi\sqrt{\frac{m \times \text{Displacement}}{m \times \text{Acceleration}}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

$$\therefore \quad \text{Frequency, } f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

* The force constant (k) is the force required to give unit displacement to the particle (or body).

Example 25.17. The resultant force acting on a particle executing S.H.M. is 4 N when it is 5 cm away from the mean position. Find the force constant.

Solution. The equation of simple harmonic motion is

$$F = -ky$$

$$\therefore \quad \text{Force constant, } k = \left|\frac{F}{y}\right| = \frac{4\text{ N}}{5\times10^{-2}\text{ m}} = 80\text{ N/m}$$

25.10. TYPICAL EXAMPLES OF S.H.M.

There are many examples of bodies executing S.H.M. *The test for S.H.M. is that force (F) acting on the body must be in the form : F = – k y where y is the displacement of the body from the mean position and k is the force constant.* We shall consider the following cases of S.H.M. :

(*i*) Oscillations of a loaded spring

(*ii*) Oscillations of a simple pendulum

(*iii*) Oscillations of a liquid in a U-tube

(*iv*) Oscillations of a floating cylinder

(*v*) Body dropped in a tunnel along earth diameter.

We shall derive expressions for time period (*T*) and frequency (*f*) of the motion in each case.

25.11. OSCILLATIONS OF A LOADED SPRING

If load attached to a spring is pulled a little from its mean position and then released, the load will execute S.H.M. We shall consider the following two cases :

1. Vibrations (oscillations) of a horizontal spring
2. Vibrations of a vertical spring.

1. Vibrations of a horizontal spring. Fig. 25.7 (*i*) shows a block of mass *m* attached to one end of a horizontal spring while the other end of the spring is fixed to a rigid support. The block is at rest but is free to move along a frictionless horizontal surface. In Fig. 25.7 (*i*), the block is at rest at the mean position $x = 0$. If the block is displaced through a small distance x to the right [See Fig. 25.7 (*ii*)], the spring gets stretched. According to Hooke's law, the spring exerts a restoring force F to the left given by ;

$$F = -kx \qquad ...(i)$$

Here k is the force constant (generally called *spring constant) and x is the displacement of mass m from the mean position. Clearly, eq. (*i*) satisfies the condition to produce S.H.M. Therefore, if the block is released from the displaced position and left to itself, the block will execute S.H.M. The time period (*T*) and frequency (*f*) of the vibrations is given by ;

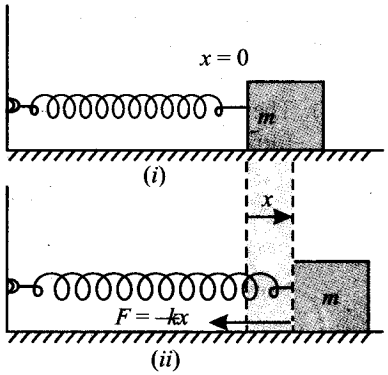


Fig. 25.7

$$T = 2\pi\sqrt{\frac{m}{k}} \; ; \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

We see that increasing the m or using a less stiff spring (smaller k) will increase the time period of oscillations.

2. Vibrations of a vertical spring. Suppose that a light spring with a spring constant k is suspended vertically from a rigid support as shown in Fig. 25.8 (*i*). Initially, the spring is neither

* Note that force constant is given different names for different systems executing S.H.M. In case of spring system, force constant is called spring constant.

compressed nor stretched. Suppose a mass m is suspended from the lower end. Due to weight mg, the spring stretches and comes to equilibrium as shown in Fig. 25.8 (*ii*). In this position, the spring is stretched by an amount l and exerts a restoring force $F_1 = -kl$* on the mass m. Since the mass m is in equilibrium, the sum of the forces on it is zero *i.e.*,

$$-kl + mg = 0 \qquad ...(i)$$

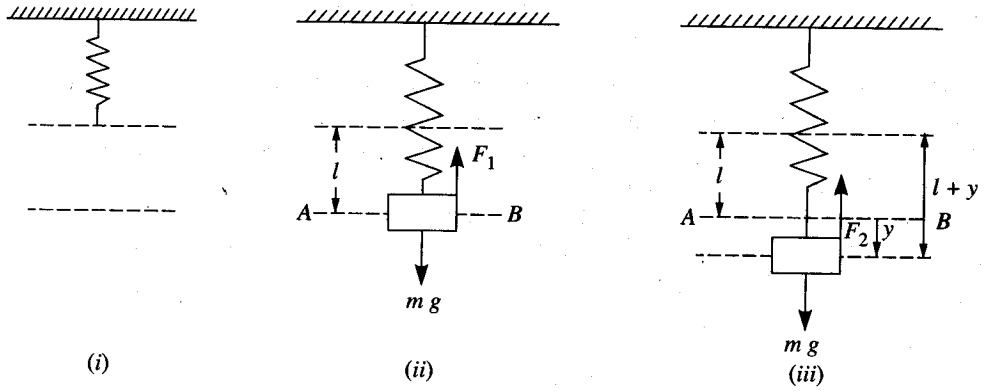


Fig. 25.8

If the mass is displaced downward an *additional* distance y, then the restoring force exerted by the spring is $F_2 = -k(l + y)$ as shown in Fig. 25.8 (*iii*). Now the net force F on the mass is

$$\text{Net force, } F = -k(l + y) + mg = -kl - ky + mg$$

But $-kl + mg = 0$ so that :

$$F = -ky$$

We note that restoring force is directly proportional to displacement and is directed towards the equilibrium position. Therefore, if the pull from the mass is released, it will execute S.H.M. about the horizontal dashed line AB. According to Newton's second law, $F = ma$ so that

$$ma = -ky$$

or $$a = -\frac{k}{m}y$$

Now $$\text{Time period, } T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{y}{a}}$$

$$= 2\pi\sqrt{\frac{y}{**ky/m}} = 2\pi\sqrt{\frac{m}{k}}$$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k}} \qquad ...(ii)$$

$\therefore$ $$\text{Frequency, } f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad ...(iii)$$

Note that the expressions for T and f are exactly the same as we obtained for a horizontally vibrating mass. The only difference is that here the mass oscillates about an equilibrium point displaced a distance l from the end of the unstretched spring. Therefore, we conclude that *gravity shifts the equilibrium point but does not otherwise effect the vibration.*

* Downward forces are being considered positive.

** Magnitude of acceleration.

Alternate formula. From eq. (*i*), $k = m\,g\,/l$. Putting this value of k in eq. (*ii*), we have,

$$T = 2\pi\sqrt{\frac{m}{mg/l}} = 2\pi\sqrt{\frac{l}{g}}$$

Note that in this formula, l is the increase in length of the spring due to mass m. The time period (and hence frequency) can be determined by this formula also.

25.12. SPRINGS IN SERIES AND PARALLEL

In many practical situations, springs are connected in series as well as in parallel. We shall determine the period of oscillation of the mass suspended from them in each case.

(*i*) Springs in series. Consider two springs S_1 and S_2 of force constants k_1 and k_2 connected in series as shown in Fig. 25.9. When weight mg is attached to this series combination, the restoring force F in each spring will be the same. Since force constants of the two springs are different, the extensions in their lengths will be different. Let the extension in the length of spring S_1 be y_1 and that of S_2 be y_2. Then,

$$F = -k_1 y_1 \quad \text{and} \quad F = -k_2 y_2$$

$$\therefore \quad y_1 = -\frac{F}{k_1} \quad \text{and} \quad y_2 = -\frac{F}{k_2}$$

Total extension y in the series combination is

$$y = y_1 + y_2 = \left(-\frac{F}{k_1}\right) + \left(-\frac{F}{k_2}\right) = -\left(\frac{1}{k_1} + \frac{1}{k_2}\right)F$$

$$= -\left(\frac{k_1 + k_2}{k_1 k_2}\right)F$$

$$\therefore \quad F = -\left(\frac{k_1 k_2}{k_1 + k_2}\right)y$$

or $$F = -ky$$

where $k = \dfrac{k_1 k_2}{k_1 + k_2}$ = Force constant of series combination

The period of oscillation of the mass m is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{(k_1 + k_2)m}{k_1 k_2}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{(k_1 + k_2)m}{k_1 k_2}} \qquad ...(i)$$

If the two springs are identical, $k_1 = k_2$. Therefore, the force constant of the series combination is $k = k_1/2$ and time period is

$$T = 2\pi\sqrt{\frac{2m}{k_1}}$$

The reader may note that when the springs are connected in series, the force constant of the combination is decreased. Therefore, the mass m will oscillate with a longer period as compared with the case for a single spring.

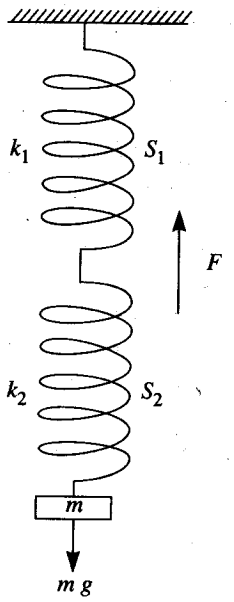


Fig. 25.9

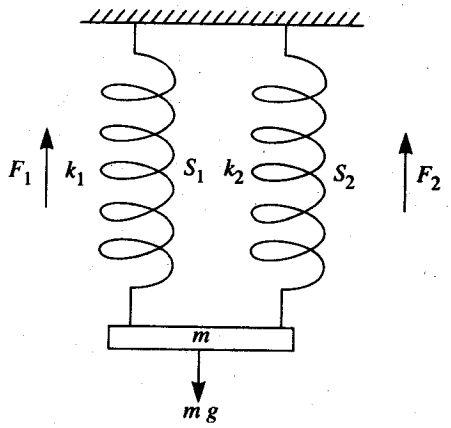


Fig. 25.10

(*ii*) **Springs in parallel.** Consider two springs S_1 and S_2 of force constants k_1 and k_2 connected in parallel as shown in Fig. 25.10. When weight $m\,g$ is attached at the middle of a short horizontal connecting bar, each spring will undergo the same amount of extension y. Since the force constants of the two springs are different, the restoring force exerted by each spring will be different. Let F_1 and F_2 be the restoring forces exerted by springs S_1 and S_2 respectively. Then,

$$F_1 = -k_1 y \quad \text{and} \quad F_2 = -k_2 y.$$

Both the restoring forces will be in the same direction (opposite to displacement). Therefore, the resultant restoring force F acting on mass m is

$$F = F_1 + F_2 = (-k_1 y) + (-k_2 y) = -(k_1 + k_2)\, y$$

or $$F = -ky$$

where $k = k_1 + k_2$ = Force constant of parallel combination

The period of oscillation of mass m is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \qquad \ldots(ii)$$

If the two springs are identical, $k_1 = k_2$. Therefore, the force constant of the parallel combination is $k = k_1 + k_1 = 2\,k_1$ and time period is

$$T = 2\pi\sqrt{\frac{m}{2k_1}}$$

Note that when the springs are connected in parallel, the force constant of the combination is increased. Therefore, the period of a parallel system is less than that for a single spring.

Another case. Consider two springs S_1 and S_2 of force constants k_1 and k_2 attached to a mass m and two fixed supports as shown in Fig. 25.11. When the mass is pulled downward, then length of the spring S_1 will be extended by y while that of spring S_2 will be compressed by y. Since the

force constants of the two springs are different, the restoring force exerted by each spring will be different. Let F_1 and F_2 be the restoring forces exerted by springs S_1 and S_2 respectively. Then,

$$F_1 = -k_1 y \quad \text{and} \quad F_2 = -k_2 y$$

Both the restoring forces will be directed upward (opposite to displacement). Therefore, the resultant restoring force F is

$$F = F_1 + F_2 = (-k_1 y) + (-k_2 y) = -(k_1 + k_2) y$$

or $$F = -k y$$

where $k = k_1 + k_2$ = Effective force constant of the system

The period of oscillation of mass m is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \qquad ...(iii)$$

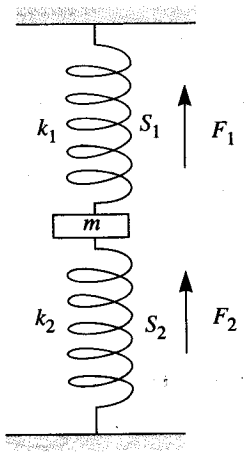


Fig. 25.11

Note that this spring-mass system is effectively a parallel system of springs.

Example 25.18. A body of mass 12 kg is suspended by a coil spring of natural length 50 cm and force constant 2.0×10^3 N m^{-1}. What is the stretched length of the spring? If the body is pulled down further stretching the spring to 59 cm and then released, what is the frequency of oscillations of the suspended mass? (Neglect the mass of the spring).

Solution. If a force F produces an increase of l in the length of a spring, then force-constant of the spring is given by ;

$$k = \frac{F}{l} = \frac{mg}{l}$$

$$\therefore \quad l = \frac{mg}{k} = \frac{12 \times 9.8}{2.0 \times 10^3} = 0.0588 \text{ m} = 5.88 \text{ cm}$$

Natural length of the spring = 50 cm

$\therefore$ Stretched length of the spring = 50 + 5.88 = **55.88 cm**

Frequency of oscillations is given by ;

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2.0 \times 10^3}{12}} = \mathbf{2.055\ s^{-1}}$$

Note that time period (and hence frequency) of oscillations of a loaded spring does not change by further stretching the spring.

Example 25.19. For the motion of the mass suspended by a coil spring in example 25.18,

(*i*) What is the net force on the suspended mass at its lowermost position?

(*ii*) What is the elastic restoring force on the mass due to spring at its uppermost position?

Solution. (*i*) The lowermost position of the spring corresponds to the position when it is stretched to a length of 59 cm.

$\therefore$ Extension produced in the spring in this position is

$$y = 59 - 50 = 9 \text{ cm} = 0.09 \text{ m}$$

Restoring force acting upward = $k\,y = 2.0 \times 10^3 \times 0.09 = 180$ N

Weight of suspended mass = $M\,g = 12 \times 9.8 = 117.6$ N

$\therefore$ Net force on the suspended mass at the lowermost position is

$$F = 180 - 117.6 = \mathbf{62.4\ N}\ (\textit{upwards})$$

(*ii*) Original length of spring = 50 cm

Stretched length of spring = 50 + 5.88 = 55.88 cm

When the suspended mass is pulled to stretch the spring to a length of 59 cm and released, the spring oscillates about its equilibrium length of 55.88 cm.

$\therefore$ Amplitude of oscillations = 59 – 55.88 = 3.12 cm

Length of spring corresponding to uppermost position = 55.88 – 3.12 = 52.76 cm

$\therefore$ Extension of the spring corresponding to uppermost position is

$$y' = 52.76 - 50 = 2.76 \text{ cm}$$

Restoring force at the uppermost position

$$= k\,y' = 2.0 \times 10^3 \times 2.76 = \mathbf{55.2\ N}$$

Example 25.20. A spring stretches 0.150 m when a 0.300 kg mass is hung from it. The spring is then stretched an additional 0.100 m from this equilibrium point and released. Determine (*i*) the spring constant *k* (*ii*) the amplitude of oscillations (*iii*) the maximum velocity (*iv*) the maximum acceleration of the mass (*v*) the period *T* and frequency *f* (*vi*) the displacement *y* as a function of time (*vii*) the velocity at *t* = 0.150 s.

Solution. (*i*) Spring constant, $k = \dfrac{F}{y} = \dfrac{mg}{y} = \dfrac{0.300 \times 9.8}{0.150} = \mathbf{19.6\ N\ m^{-1}}$

(*ii*) Since the spring is stretched 0.100 m from equilibrium, it is given no initial speed.

$\therefore$ Amplitude, $a = \mathbf{0.100\ m}$

(*iii*) The maximum velocity is given by ;

$$V_{max} = \omega a = \sqrt{\frac{k}{m}} \times a = \sqrt{\frac{19.6}{0.300}} \times 0.100 = \mathbf{0.808\ m\ s^{-1}}$$

(*iv*) Since $F = m\,a$, the maximum acceleration occurs when the force is maximum *i.e.*, when $y = a = 0.100$ m.

$\therefore$
$$A_{max} = \frac{F_{max}}{m} = \frac{ka}{m} = \frac{19.6 \times 0.100}{0.300} = \mathbf{6.53\ m\ s^{-2}}$$

(*v*)
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.300}{19.6}} = \mathbf{0.777\ s}$$

$$f = \frac{1}{T} = \frac{1}{0.777} = \mathbf{1.29\ Hz}$$

(*vi*) The motion begins at a point of maximum displacement downward. If we take y positive upward, then $y = -a$ at $t = 0$. So we need a sinusoidal curve that has its maximum negative value at $t = 0$; this is just a negative cosine.

$\therefore$
$$y = -a\cos 2\pi f t$$

which gives us $y = -a \cos 0 = -a$ at $t = 0$.

Therefore, the displacement y as function of time t for this motion is

$$y = -a\cos 2\pi f t$$

Here $a = 0.100$ m ; $f = 1.29$ Hz

$\therefore$ $y = -0.100 \cos 2\pi \times 1.29 \times t$

or $y = \mathbf{-0.100 \cos 8.10\ t}$

Note that t is in seconds and y is in metres.

(*vii*) The velocity of the mass at any time t is

$$V = \frac{dy}{dt} = \frac{d}{dt}(-a\cos 2\pi f t) = -a(-2\pi f)\sin 2\pi f t$$

$$= 0.810 \sin 8.10\ t$$

At $t = 0.150$ s ; $V = 0.810 \sin 8.10 \times 0.150 = \mathbf{0.761\ ms^{-1}}$

Example 25.21. A tray of mass 12 kg is supported by two identical springs as shown in Fig. 25.12. When the tray is pressed down slightly and released, it executes S.H.M. with a time period of 1.5 s. What is the force constant of each spring? When a block of mass M is placed on the tray, the period of S.H.M. changes to 3.0s. What is the mass of the block?

Solution. If k is the force constant of the parallel combination of springs, then period of the tray is given by ;

$$T = 2\pi\sqrt{\frac{m}{k}}$$

or $$T^2 = \frac{4\pi^2 m}{k}$$

or $$k = \frac{4\pi^2 m}{T^2}$$

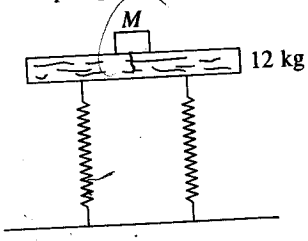


Fig. 25.12

Here m= 12 kg ; T = 1.5 s

$$\therefore \quad k = \frac{4\pi^2 \times 12}{(1.5)^2} = 2.1\times10^2 \text{ Nm}^{-1}$$

Since the force constant for the parallel combination is the sum of the force constants for individual springs,

$\therefore$ Force constant of each spring $= \frac{2.1\times10^2}{2} = 1.05 \times 10^2 \text{ N m}^{-1}$

Again $$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{or} \quad T^2 = \frac{4\pi^2 m}{k} \quad \text{or} \quad m = \frac{T^2 k}{4\pi^2}$$

Here T = 3.0 s ; $k = 2.1 \times 10^2$ N m^{-1}

$$\therefore \quad m = \frac{(3.0)^2 \times 2.1\times10^2}{4\pi^2} = 48 \text{ kg}$$

$\therefore$ Mass of block, $M = 48 - 12 = 36$ kg

Example 25.22. Two masses m_1 = 1.0 kg and m_2 = 0.5 kg are suspended by a weightless spring of force constant k = 12.5 N m^{-1}. When they are in equilibrium position, m_1 is gently removed. Calculate the angular frequency and amplitude of m_2. Take g = 10 m s^{-2}.

Solution. Suppose y is the extension in the length of the spring when both m_1 and m_2 are suspended. Then,

$$F = (m_1 + m_2)\, g = k\, y$$

Suppose the extension is reduced to y' when m_1 is removed. Then,

$$m_2 g = k\, y'$$

$$\therefore \quad y - y' = \frac{(m_1 + m_2)g - m_2 g}{k} = \frac{m_1 g}{k}$$

This will be the amplitude a of the oscillations of m_2.

$$\therefore \quad a = y - y' = \frac{m_1 g}{k} = \frac{1\times10}{12.5} = 0.8 \text{ m}$$

The angular frequency of oscillations of m_2 is

$$\omega = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{12.5}{0.5}} = 5 \text{ s}^{-1}$$

Example 25.23. A spring of force constant 1200 N m^{-1} is mounted on a horizontal table [See Fig. 25.13]. A mass of 3.0 kg attached to the free end of the spring is pulled sideways to a distance of 2.0 cm and released. What is the angular frequency of oscillations of

the mass ? What is (*i*) the speed of the mass when the spring is compressed by 1.0 cm (*ii*) potential energy of the mass when it momentarily comes to rest and (*iii*) total energy of the oscillating mass?

Solution. Here $m = 3$ kg ; $k = 1200$ Nm^{-1} ; $a = 0.02$ m

Frequency of oscillations of the mass is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{1200}{3}} = 3.18 \text{ s}^{-1}$$

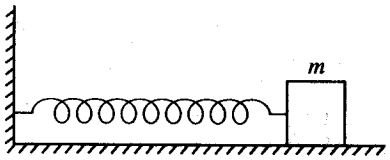


Fig. 25.13

$\therefore$ Angular frequency of oscillations of the mass is

$$\omega = 2\pi f = 2\pi \times 3.18 = \mathbf{20 \text{ rad s}^{-1}}$$

(*i*) Speed of the mass is given by ;

$$V = \omega\sqrt{a^2 - y^2} \qquad \text{(Here } y = 1.0 \text{ cm} = 0.01 \text{ m)}$$

$$= 20\sqrt{(0.02)^2 - (0.01)^2} = \mathbf{0.35 \text{ m/s}}$$

(*ii*) The oscillating mass momentarily comes to rest at the extreme position *i.e.*, when $y = a = 0.02$ m.

$\therefore$ P.E. of mass at extreme position = Total energy $= \frac{1}{2} m\omega^2 a^2$

$$= \frac{1}{2} \times 3.0 \times (20)^2 \times (0.02)^2 = \mathbf{0.24 \text{ J}}$$

(*iii*) At the extreme position, the potential energy of the mass is equal to the total energy. Therefore, total energy of oscillating mass is **0.24 J**.

PROBLEMS FOR PRACTICE

1. A mass of 0.1 kg is attached to the free end of a vertical spring whose upper end is fixed and the spring extends by 0.04 m. Now the mass is pulled down a small distance 0.02 m and then released. Find (*i*) its period (*ii*) the maximum force acting on it during the oscillations. Take $g = 10$ m s^{-2}. **[(*i*) 0.4 s (*ii*) 0.5 N]**
2. The pan attached to a spring balance has a mass of 1 kg. A mass of 2 kg when placed on the pan stretches the spring by 10 cm. What is the frequency with which the empty pan will oscillate? **[2.227 Hz]**
3. A piece of rubber is 45 cm long when a weight of 18.0 N hangs from it and is 68 cm long when a weight of 22.5 N hangs from it. What is the spring constant of this piece of rubber? **[19.56 N m^{-1}]**
4. A spring vibrates with a frequency of 2.4 Hz when 0.80 kg hangs from it. What will its frequency be if only 0.50 kg hangs from it? **[3 Hz]**
5. A spring compressed by 0.1 m develops a restoring force of 10 N. A body of mass 4 kg is placed on it. Find : (*i*) the force constant of the spring (*ii*) the depression of the spring under the weight of the body (take $g = 10$ N/kg) and (*iii*) the period of oscillations if the body is disturbed. **[(*i*) 100 N m^{-1} (*ii*) 0.4 m (*iii*) 1.26 s]**
6. A 2.0 kg body is suspended by a spring. When in addition to it, another body of mass 0.3 kg is suspended, the spring further stretches by 2.0 cm. If the second body is removed and the first is made to oscillate, then what will be the time period? **[0.73 s]**

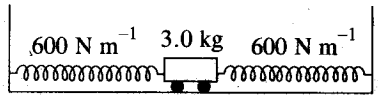


Fig. 25.14

7. A trolley of mass 3.0 kg is connected to two identical springs each of force constant 600 N m^{-1} as shown in Fig. 25.14. If the trolley is displaced from its equilibrium position by 5.0 cm and released, determine the period of oscillations and the maximum speed of the trolley. How much is the total energy dissipated as heat by the time the trolley comes to rest due to damping forces? **[0.314 s; 1.0 m s^{-1}; 1.5 J]**

25.13. SIMPLE PENDULUM

A simple pendulum consists of a small mass (the pendulum bob) suspended from the end of a light thread. The distance from the point of suspension to the centre of gravity of the bob is called the 'effective length' of the pendulum. The equilibrium or mean position [point *O* in Fig. 25.15 (*i*)] is where the pendulum hangs vertically. When the bob is displaced slightly to one side from its mean position and released, then it oscillates about the mean position in an arc of a circle. The linear displacement of the bob is the arc length [See Fig. 25.15 (*ii*)] from the mean position *O*. If we can show that the restoring force is proportional to the linear displacement from mean position and directed opposite to the displacement, we can conclude that the bob moves with S.H.M. Indeed, it is so.

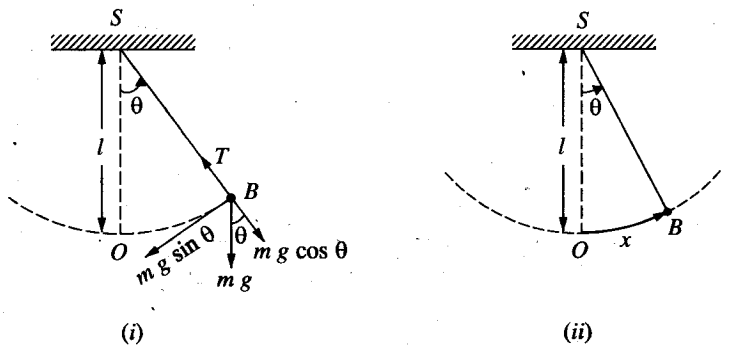


Fig. 25.15

Time period of simple pendulum. Suppose l is the effective length of a simple pendulum and m is the mass of its bob. The pendulum is suspended from point S and its mean position is O. Suppose at any instant, the vibrating bob is in the position B where its linear displacement is arc $OB = x$ and the thread makes an angle θ with the vertical. As shown in Fig. 25.15 (*i*), there are two forces acting on the mass m : the tension T and the weight mg. The tension T is always directed along the thread toward the point of suspension and mg is always vertically downward. The weight mg can be resolved into two rectangular components *viz.*

(*i*) the component $mg \cos\theta$ in line with the thread and directed opposite to T. The net radial force ($= T - mg \cos\theta$) along the thread causes the mass m to move in a circular arc whose radius is the length of the pendulum l.

(*ii*) the component $mg \sin\theta$ acts tangent to the arc along which the bob moves. This component restores the bob toward its equilibrium position and serves as a restoring force. Thus we can write :

$$\text{Restoring force, } F = -mg \sin\theta$$

The negative sign shows that the restoring force points opposite to the direction of the displacement.

If θ is small, $\sin\theta = \theta$ so that restoring force can be written as :

$$\text{Restoring force, } F = -mg\theta$$

The linear displacement x of the bob from its mean position is related to its angular displacement θ as :

$x = l\,\theta$ where θ must be in the units of radians.

$\therefore$ Restoring force, $F = -\dfrac{mgx}{l}$ $\left(\because \theta = \dfrac{x}{l}\right)$

$\therefore$ Acceleration of bob, $a = \dfrac{F}{m} = -\dfrac{mgx}{lm} = -\left(\dfrac{g}{l}\right)x$ ...(*i*)

Now g/l is constant for a given pendulum at a place.

$\therefore$ $*a \propto -x$

Thus the acceleration a of the bob is directly proportional to its displacement x and its direction is opposite to the displacement. *Therefore, the motion of the bob is simple harmonic.* The time period of a body executing S.H.M. is given by ;

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

From eq. (*i*) above, $\dfrac{\text{displacement } (x)}{\text{acceleration } (a)} = \dfrac{l}{g}$ (magnitude)

$\therefore$ $T = 2\pi\sqrt{\dfrac{l}{g}}$ ...(*ii*)

Eq. (*ii*) gives the expression for the time period of a simple pendulum. It is an interesting result. Note that *the period of simple pendulum does not depend on the mass of the pendulum bob.* It depends only on length l and on g.

Frequency of vibrations of simple pendulum is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \quad ...(iii)$$

Discussion. The following points may be noted :

(*i*) The frequency (and hence time period) of a simple pendulum depends *only* on the length l and on g. This fact provides a simple method to measure g. A pendulum of known length (l) is made to oscillate. The frequency f (no. of vibrations per second) is recorded. Then g can be easily calculated from eq. (*iii*).

(*ii*) On the moon, g is about one-sixth that on the earth. From eq. (*ii*) , we see that a pendulum of given length on the moon would have a period over twice as long as on the earth.

Second's pendulum. *A simple pendulum whose time period is two seconds is called second's pendulum.* It is called second's pendulum because it makes a swing from one side to the other in one second. Let us find the length of such a pendulum.

Now, $T = 2\pi\sqrt{\dfrac{l}{g}}$

or $l = \dfrac{T^2 g}{4\pi^2}$

Here $T = 2$ s ; $g = 9.8$ m s^{-2}

$\therefore$ $l = \dfrac{(2)^2 \times 9.8}{4\pi^2} = 0.993 \text{ m} = 99.3 \text{ cm}$

Note that the length of second's pendulum depends upon the value of g; the value of length changes from place to place due to change in g.

Example 25.24. A small bob of mass 50 g oscillates as a simple pendulum with an amplitude 5 cm and time period 2 s. Find the velocity of the bob and the tension in the supporting thread when the velocity of the bob is maximum.

* For S.H.M., $F = -kx$ or $ma = -kx$. Since m and k are constants, $a \propto -x$.

Solution. The velocity of the bob will be maximum when it passes through the mean position.

Here m = 50 g = 0.05 kg ; a = 5 cm = 0.05 m ; T = 2 s

$$\text{Angular frequency, } \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ s}^{-1}$$

$$\text{Maximum velocity of bob, } V_{max} = \omega a = \pi \times 0.05 = \mathbf{0.05\ \pi\ m\ s^{-1}}$$

Suppose F is the tension in the supporting thread. Then,

$$F - mg = \frac{mV_{max}^2}{l} \quad \text{where } l = \text{length of the pendulum}$$

or $$F = mg + \frac{mV_{max}^2}{l}$$

Now $$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad l = \frac{T^2 g}{4\pi^2} = \frac{(2)^2 \times 9.8}{4\pi^2}$$

$\therefore$ $$F = 0.05 \times 9.8 + \frac{0.05 \times (0.05\pi)^2 \times 4\pi^2}{(2)^2 \times 9.8}$$

$$= 0.49 + 0.001 = \mathbf{0.49\ N}$$

Example 25.25. A pendulum clock shows correct time. If the length increases by 0.1%, find the error in time per day.

Solution. Correct number of seconds per day, f = 24 × 60 × 60 = 86400

Let error introduced per day be x seconds.

$\therefore$ Incorrect number of seconds per day, f' = 86400 + x

If l is the original length, then its new length is l' = l + 0.1% of l = (1 + 0.001)l

$$\text{Now, frequency, } f = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \quad \text{or} \quad f \propto \frac{1}{\sqrt{l}}$$

$\therefore$ $$\frac{f'}{f} = \sqrt{\frac{l}{l'}} \quad \text{or} \quad \frac{86400 + x}{86400} = \sqrt{\frac{l}{(1+0.001)\,l}}$$

$\therefore$ $$1 + \frac{x}{86400} = (1+0.001)^{-1/2} = 1 - \frac{1}{2} \times 0.001 = 1 - 0.0005$$

$\therefore$ $$x = -\ 0.0005 \times 86400 = \mathbf{-\ 43.2\ s}$$

The negative sign indicates that the clock will run slow and lose 43.2 seconds per day.

Example 25.26. Two simple pendulums of lengths 1.44 m and 1.00 m start swinging at the same time. After how much time they will be (*i*) out of phase and (*ii*) in phase again? Take g = 10 ms^{-2}.

Solution. Let T_1 and T_2 be the time periods of larger (length) and smaller pendulums respectively and L_1 and L_2 be the corresponding lengths.

$\therefore$ $$T_1 = 2\pi\sqrt{\frac{L_1}{g}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{L_2}{g}}$$

$\therefore$ $$\frac{T_1}{T_2} = \sqrt{\frac{L_1}{L_2}} = \sqrt{\frac{1.44}{1.00}} = 1.2$$

(*i*) The pendulum of larger length (L_1) will have smaller frequency and greater time period. Suppose they get out of phase in time t. The two pendulums will be out of phase if the pendulum of smaller length makes 1/2 vibration extra compared to the pendulum of larger length. If

the pendulum of larger length makes f vibrations in time t, then the other pendulum will make $(f + 1/2)$ vibrations.

$$\therefore \quad t = fT_1 = \left(f + \frac{1}{2}\right)T_2$$

$$\text{or} \quad \frac{T_1}{T_2} = \frac{f + \frac{1}{2}}{f} \quad \text{or} \quad 1.2 = \frac{f + \frac{1}{2}}{f} \quad \therefore f = 2.5 \text{ vibrations}$$

$$\therefore \quad t = fT_1 = 2.5 \times 2\pi\sqrt{\frac{1.44}{10}} = 5.96 \text{ s}$$

(*ii*) The two pendulums will be again in phase when the pendulum of larger length completes f' vibrations and pendulum of smaller length completes $(f' + 1)$ vibrations in time t'.

$$\therefore \quad t' = f'T_1 = (f'+1)T_2$$

$$\text{or} \quad \frac{T_1}{T_2} = \frac{f'+1}{f'} \quad \text{or} \quad 1.2 = \frac{f'+1}{f'} \quad \therefore f' = 5 \text{ vibrations}$$

$$\therefore \quad t' = f'T_1 = 5 \times 2\pi\sqrt{\frac{1.44}{10}} = 11.92 \text{ s}$$

PROBLEMS FOR PRACTICE

1. Calculate the period of oscillations of a simple pendulum of length 1.8 m with a bob of mass 2.2 kg. **[0.2 s]**
2. How long must be a simple pendulum if it is to make one complete vibration in one second? **[0.248 m]**
3. The time taken by a simple pendulum to perform 1000 vibrations is 8 minutes 9 seconds in Mumbai and 8 minutes 20 seconds in Pune. Calculate the ratio of acceleration due to gravity in Mumbai and Pune. **[1.0455]**
4. Two simple pendulums are to have frequencies such that $f_1 = 3f_2$. What should be the ratio l_1/l_2 of their lengths? **[0.111]**
5. A simple pendulum of length 860 mm gives a time period of 4.6 s on the surface of the moon. Find acceleration due to gravity on the surface of the moon. **[1.6 m s^{-2}]**

25.14 OSCILLATIONS OF A LIQUID IN A U-TUBE

Consider a liquid filled in a vertical U-tube of uniform area of cross-section upto a height h as shown in Fig. 25.16. In other words, initially (equilibrium position), the liquid level is at the same height in both the limbs (at B and C). On pressing down level C upto D through a distance y, the level B rises up the same distance upto E. As a result, in the left limb of the U-tube, there is an additional liquid column $D'E$ of length $2y$ as shown in Fig. 25.16. When the pressed level of the liquid is left free, the levels of the liquid in the two limbs will oscillate about their respective initial (undisturbed) positions B, C. Note that the weight of the liquid column of height $2y$ serves as the restoring force and tends to restore the liquid column to its equilibrium position. If the mass of the liquid per unit length of the tube is m, then restoring force F acting on the liquid is

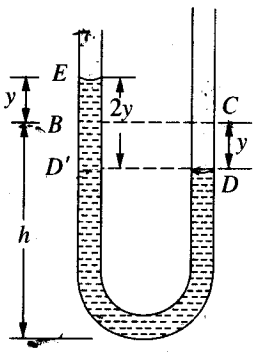


Fig. 25.16

$$\text{Restoring force, } F = -(m \times 2y)\, g$$

The negative sign indicates that the restoring force points opposite to the direction of the displacement of the liquid.

The mass of the whole liquid in the tube is $m \times 2h$ where $2h$ is the length of the whole liquid column.

$\therefore$ Acceleration a of the liquid column is

$$a = \frac{\text{Force}}{\text{Mass}} = \frac{-(m \times 2y)g}{m \times 2h} = -\left(\frac{g}{h}\right)y \qquad ...(i)$$

Since g/h is constant, $a \propto -y$.

Thus the acceleration a of the liquid column is directly proportional to its displacement y and its direction is opposite to the displacement. *Therefore, motion of the liquid is simple harmonic.* The time period for S.H.M. is

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

From eq. (i) above, $\dfrac{\text{displacement } (y)}{\text{acceleration } (a)} = \dfrac{h}{g}$ (magnitude only)

$$\therefore \qquad T = 2\pi\sqrt{\frac{h}{g}} \quad ...(ii) \; ; \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{h}} \qquad ...(iii)$$

It is an interesting result. *The period of oscillation does not depend on the density of the liquid and the cross-sectional area of the tube.* It depends upon the length of liquid column and on g. In practice, the oscillations are heavily damped due to friction, which we have ignored.

25.15. OSCILLATIONS OF A FLOATING CYLINDER

Consider a wooden cylinder of mass m and cross-sectional area A floating in a liquid of density ρ. At equilibrium, the cylinder is floating with a depth h submerged [See Fig. 25.17]. If the cylinder is pushed downward a small distance y and then released, it will move up and down with S.H.M. It is desired to find the time period and the frequency of oscillations.

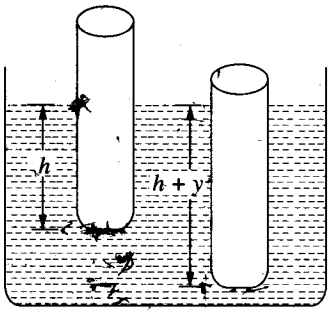


Fig. 25.17

According to the principle of floatation, the weight of the liquid displaced by the immersed part of the body is equal to the weight of the body. Therefore, at equilibrium,

Weight of cylinder = Weight of liquid displaced by the immersed part of cylinder

or $\qquad m\,g = (\rho\,A\,h)\,g$

$\therefore$ Mass of cylinder, $m = \rho\,A\,h$

When the cylinder is pushed down an additional distance y, the restoring force F (upward) equal to the weight of additional liquid displaced acts on the cylinder.

$\therefore$ Restoring force, F = – (weight of additional liquid displaced)

or $\qquad F = -(\rho\,A\,y)\,g$

The negative sign indicates that the restoring force acts opposite to the direction of the displacement.

Acceleration a of the cylinder is given by ;

$$a = \frac{F}{m} = \frac{-\rho A y g}{\rho A h} = -\left(\frac{g}{h}\right)y \qquad ...(i)$$

Since g/h is constant, $a \propto -y$.

Thus the acceleration a of the body (wooden cylinder) is directly proportional to the displacement y and its direction is opposite to the displacement. *Therefore, motion of the cylinder is simple harmonic.* The time period for S.H.M. is

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

From eq. (*i*) above, $\dfrac{\text{displacement } (y)}{\text{acceleration } (a)} = \dfrac{h}{g}$ (magnitude only)

$$\therefore \quad \text{Time period, } T = 2\pi\sqrt{\frac{h}{g}} \qquad ...(ii)$$

$$\therefore \quad \text{Frequency, } f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{h}} \qquad ...(iii)$$

These very interesting results show that time period and frequency have the same form as that of the simple pendulum. The submerged depth at equilibrium takes the place of the length of the pendulum.

25.16. BODY DROPPED IN A TUNNEL ALONG EARTH DIAMETER

Suppose earth to be a sphere of radius R with centre O. Let a tunnel be dug along the diameter of the earth as shown in Fig. 25.18. If a body of mass m is dropped at one end of the tunnel, the body will execute S.H.M. about the centre O of the earth. Suppose at any instant, the body in the tunnel is at a distance y from the centre O of the earth. Because the body is *inside* the earth, only the inner sphere of radius y will exert gravitational force F on the body. The force F serves as the restoring force that tends to bring the body to the equilibrium position O.

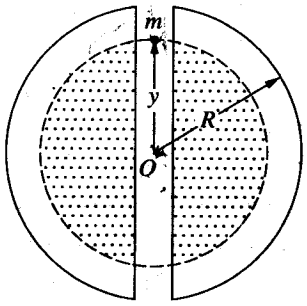


Fig. 25.18

$$\therefore \quad \text{Restoring force, } F = -G\times\frac{\left(\frac{4}{3}\pi y^3\rho\right)m}{y^2}$$

where ρ is the density of the earth. The negative sign is assigned because the force is of attraction.

$$\text{Acceleration of the body, } a = \frac{F}{m} = -\left(\frac{4}{3}\pi G\rho\right)y \qquad ...(i)$$

Now the quantity $(4/3)\,\pi G\rho$ is constant so that :

$$a \propto -y$$

Thus the acceleration of the body is directly proportional to the displacement y and its direction is opposite to the displacement. *Therefore, the motion of the body is simple harmonic.* The time period for S.H.M. is

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

From eq. (*i*) above, $\dfrac{\text{displacement } (y)}{\text{acceleration } (a)} = \dfrac{3}{4\pi G\rho}$ (magnitude only)

$$\therefore \quad \text{Time period, } T = 2\pi\sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{3\pi}{G\rho}} \quad \text{or} \quad T = \sqrt{\frac{3\pi}{G\rho}} \qquad ...(ii)$$

Example 25.27. A vertical *U*-tube of uniform cross-section contains water upto 80 cm. Find the frequency of oscillations of water when one side is depressed and then released.

Solution. The frequency of oscillations is given by ;

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{h}}$$

Here $g = 9.8\ \text{ms}^{-2}$; $h = 80\ \text{cm} = 0.8\ \text{m}$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{9.8}{0.8}} = 0.56\ \text{Hz}$$

Example 25.28. Imagine a tunnel is dug along a diameter of the earth. Show that a body dropped from one end of the tunnel executes simple harmonic motion. What is the time period of this motion? Assume the earth to be a sphere of uniform mass density = 5520 kg m^{-3} and $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Solution. For proof, refer to Art. 25.16.

The time period T of the body is given by ;

$$T = \sqrt{\frac{3\pi}{G\rho}}$$

Here $G = 6.67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$; $\rho = 5520\ \text{kg m}^{-3}$

$$\therefore \quad T = \sqrt{\frac{3\pi}{6.67\times10^{-11}\times5520}} = 5059\ \text{s}$$

Example 25.29. A cubical body (side 0.1 m and mass 0.002 kg) floats in water. It is pressed and then released so that it oscillates vertically. Find the time period. The density of water is 1000 kg m^{-3}.

Solution. Mass of the body, $m = 0.002$ kg

Cross-sectional area of body, $A = 0.1 \times 0.1 = 0.01\ \text{m}^2$

Density of water, $\rho = 1000\ \text{kg m}^{-3}$

Suppose at equilibrium, the body is floating with a depth h submerged. According to the principle of floatation,

Weight of body = Weight of water displaced by the immersed part of the body

or $$m\,g = (\rho A\,h)\,g$$

$$\therefore \quad h = \frac{m}{\rho A} = \frac{0.002}{1000\times0.01} = 2\times10^{-4}\ \text{m}$$

The time period of oscillations is given by ;

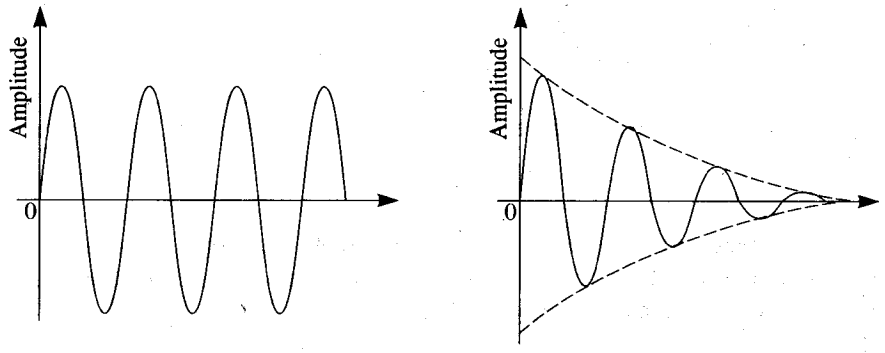
$$T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{2\times10^{-4}}{9.8}} = 0.028\ \text{second}$$

25.17. UNDAMPED AND DAMPED OSCILLATIONS

A system executing simple harmonic motion is called a *harmonic oscillator.* A harmonic oscillator produces sinusoidal oscillations. The sinusoidal oscillations can be of two types *viz., undamped oscillations* and *damped oscillations.*

(*i*) **Undamped oscillations.** *The oscillations whose amplitude remains constant with time are called* **undamped oscillations**. Fig. 25.19 (*i*) shows waveform of undamped oscillations. Such oscillations can occur if frictional forces are absent. For example, if the bob of a pendulum is

displaced in vacuum and then released, the bob will continue to execute S.H.M. of constant amplitude.



(*i*) (*ii*)

Fig. 25.19

(*ii*) **Damped oscillations.** *The oscillations whose amplitude goes on decreasing with time are called* ***damped oscillations** [See Fig. 25.19 (*ii*)]. In real oscillating systems, forces like friction are always present that dissipate the oscillating energy. Unless energy is somehow added, dissipation eventually brings the system to rest at equilibrium.

We can define damped motion only if we have a mathematical expression for the damping force. In many systems—especially those involving friction associated with slow motion through a viscous fluid—the damping force is approximately proportional to the velocity and acts opposite to motion.

$$\text{Damping force, } F_d = -bV = -b\frac{dx}{dt}$$

where b is a constant giving the strength of damping. We can write Newton's law, now including damping force along with the restoring force. For a spring-mass system, we have,

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

or
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

We will not solve this equation but its solution will be stated. Provided the damping force is not too large, the solution of this equation is

$$x = ae^{-bt/2m}\cos(\omega t + \phi) \qquad ...(i)$$

Eq. (*i*) describes sinusoidal motion whose amplitude (a) decreases exponentially with time. How fast the amplitude drops depends on the damping constants b and m.

The frequency of this damped motion is given by ;

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

If the frictional forces are absent, $b = 0$ so that :

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{...undamped oscillations}$$

25.18. FREE, FORCED AND RESONANT OSCILLATIONS

(*i*) **Free oscillations.** When a body capable of oscillating is displaced from its equilibrium position and then left free, it begins to oscillate with a definite frequency. This frequency depends

* Damp means to diminish.

upon the intrinsic properties (shape, elasticity etc.) of the body and is called *natural frequency* of the body. Such oscillations are called free oscillations.

When a body capable of oscillating is made to vibrate with its own natural frequency, it is said to execute **free oscillations**.

For example, when a simple pendulum is displaced from its mean position and then left free, it executes free oscillations. The natural frequency of the simple pendulum depends upon its length and is given by ;

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

Examples. (*a*) When a tuning fork is struck with a rubber pad, then its prongs produce free oscillations. The frequency of free oscillations (*i.e.*, natural frequency) depends on tuning fork's length, its thickness and the material of which it is made.

(*b*) If we pluck the string of a sitar and then left free, it executes free oscillations. The natural frequency depends upon the length, density and tension of the string.

(*c*) When the bob of a simple pendulum is displaced from its mean position and then left free, it executes free oscillations. The natural frequency depends upon the length of the pendulum.

In the absence of frictional forces, the amplitude of free oscillations would remain constant (undamped oscillations). However, in actual practice, the amplitude of free oscillations decreases gradually with time due to the frictional forces. When whole of the originally imparted energy to the body is used up in overcoming frictional forces, the body stops executing oscillations and comes to rest at the mean position.

(*ii*) **Forced oscillations.** *When a body is maintained in a state of oscillations by an external periodic force of frequency other than the natural frequency of the body, the oscillations are called* **forced oscillations**.

The frequency of forced oscillations is equal to the frequency of the periodic force. The external applied force on the body is called the *driver* and the body set into oscillations is called *driven oscillator.*

Examples. (*a*) When the stem of a vibrating tuning fork is held in hand, only a feeble sound is heard. However, if the stem is pressed against a table top, the sound becomes louder. It is because the tuning fork *forces* the table to vibrate with fork's frequency. Since the table has a large vibrating area than the tuning fork, these *forced oscillations* produce a more intense sound.

(*b*) A vibrating violin string stretched tightly between two clamps does not produce intense sound. When the string is stretched across the wooden bridge of a violin, the wooden bridge is *forced to vibrate at the same frequency. Since the wooden bridge has a much larger area than the string, these forced oscillations produce a more intense sound.

Mathematical analysis. Most of the oscillations that occur in systems (*e.g.*, machinery) are forced oscillations; oscillations that are produced and sustained by an external force. The simplest driving force is one that itself oscillates as a sine or a cosine. Suppose such an external force F_{ext} is applied to an oscillator that moves along x axis such as a block connected to a spring. We can represent the external force as :

$$F_{ext} = F_0 \cos \omega t$$

where F_0 is the maximum magnitude of the force and ω $(= 2\pi f)$ is the angular frequency of the force. Then the equation of motion (with damping) is

$$ma = -kx - bV + F_0 \cos \omega t$$

---

* These two examples are certainly not the case of resonant vibrations (*i.e.*, resonance). It is because loud sound will always be heard for different frequencies of the tuning fork or violin string. The wooden table or bridge is forced to vibrate at the frequency of the tuning fork or violin string.

This equation can be written as :

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0\cos\omega t$$

or $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos\omega t \quad ...(i)$$

The solution of eq. (*i*) is

$$x = A_0\cos(\omega t + \phi)$$

where $$A_0 = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}} \quad ...(ii)$$

and $\omega_0 = \sqrt{k/m}$ is the frequency of undamped ($b = 0$) oscillator *i.e.,* natural frequency.

Eq. (*ii*) reveals the following interesting features :

(*a*) The amplitude of the motion A_0 is directly proportional to the amplitude of the driving force F_0.

(*b*) The amplitude of motion (A_0) depends strongly on the difference between the applied frequency (ω) and natural frequency (ω_0). *The amplitude is a maximum when the frequency of the driving force equals the natural frequency i.e., when* $\omega = \omega_0$. It is because the denominator in eq. (*ii*) is a minimum when $\omega = \omega_0$.

This condition is called *resonance.* When the frequency of the driving force equals ω_0, the oscillator is said to be in resonance with the driving force.

***(iii)* Resonant oscillations.** *When a body is maintained in a state of oscillations by a periodic force having the same frequency as the natural frequency of the body, the oscillations are called* **resonant oscillations**. The phenomenon of producing resonant oscillations is called resonance. Note that resonance is a particular case of forced osciallations. We have seen that the amplitude (A_0) of motion of forced oscillations (with damping) is given by ;

$$A_0 = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}} \quad ...(iii)$$

At resonance, $\omega = \omega_0$ and $A_0 = \frac{F_0/m}{\sqrt{(b\omega/m)^2}} = \frac{F_0}{b\omega}$ ...(*iv*)

Fig. 25.20 shows the graph of forced oscillations as a function of ω. At $\omega = \omega_0$, the value of A_0 is maximum. Note that a smaller value of b, which corresponds to less damping, leads to a large value of A_0 at resonance. The natural vibration frequency of a system is called its **resonant frequency**. If $b = 0$ [See eq. (*iv*)], the resonant peak (of A_0) becomes infinite; in such a case, energy is being continuously transferred into the system and none is dissipated. For real systems, b is never precisely zero and the resonant peak is finite; the peak does not occur precisely at $\omega = \omega_0$ [because of the term

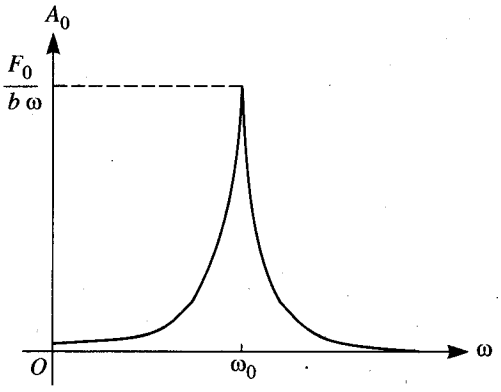


Fig. 25.20

$(b\omega/m)^2$ in the denominator in eq. (*iii*)], although it is quite close to ω_0 unless the damping is very large.

Examples. (*a*) A tuning fork is often mounted on a sound box. The box is constructed in such a way that the natural frequency of air column inside it has the same frequency as that of the fork. When the tuning fork is struck, it sets the air column into resonant vibrations. This reinforces the sound of the fork.

(*b*) Soldiers are asked to break step while crossing a bridge. If the soldiers march in step, there is a possibility that the frequency of the footsteps may become equal to natural frequency of the bridge. Due to resonance, the bridge may start vibrating violently, thereby damaging itself

(*c*) During earthquake, resonance can cause disaster. If the natural frequency of a building matches the frequency of earthquake, the building will begin to vibrate with a large amplitude. Consequently, the building may collapse.

25.19. DEMONSTRATION OF FREE, FORCED AND RESONANT VIBRATIONS

The distinction among free, forced and resonant vibrations can be beautifully illustrated by suspending four simple pendulums *A, B, C* and *D* from a rubber band *XY* as shown in Fig. 25.21. The lengths of pendulums *A* and *B* are equal while those of *C* and *D* are different. If the pendulum *'A'* is set into free vibrations [natural frequency, $f = (1/2\pi)\,(g/l)^{1/2}$], it is observed that pendulum *B* executes resonant vibrations with a large amplitude while pendulums *C* and *D* undergo forced vibrations with a small amplitude.

Explanation. When pendulum *A* executes free vibrations, it exerts a periodic force on the rubber band which transmits it to the other three pendulums *B, C* and *D.*

(*i*) The periodic force imparted to pendulum *B* through the rubber band has the *same time period as that of pendulum *A*. Consequently, pendulum *B* begins to vibrate with a large amplitude. In other words, pendulum *B* executes resonant vibrations.

(*ii*) The pendulums C and *D* have their time periods different from that of pendulum *A* (or pendulum *B*). Therefore, the imparted impulses do not reach them (*i.e.*, *C* and *D*) fitting

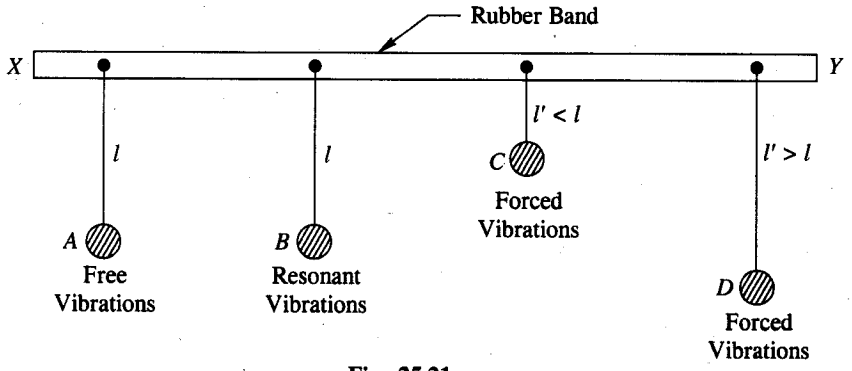


Fig. 25.21

with their natural time period. Consequently, pendulums C and *D* are forced to vibrate with the frequency of pendulum *A*. In other words, pendulums C and *D* execute forced vibrations.

It may be noted that pendulum *B* will vibrate with a large amplitude due to resonance while pendulums C and *D* undergo forced vibrations with a small amplitude. The forced vibrations, however, last for a short time.

* Since the lengths of pendulums *A* and *B* are the same, they have the same natural frequency or natural time period.

25.20. COUPLED OSCILLATIONS

A system of two bodies connected by a spring so that both can oscillate in a straight line along the length of the string is called a **coupled oscillator**.

If the spring is stretched and then released, the two bodies will oscillate along the length of the spring. Such oscillations are called **coupled oscillations**. It can be proved that oscillations produced by such a system are S.H.M. in nature.

Fig. 25.22 (*i*) shows two masses m_1 and m_2 connected at the ends of a massless spring of length l and the system is placed on a frictionless horizontal surface. When the spring is stretched and then released, the system oscillates in S.H.M. We can describe the motion of the system in terms of the separate motions of the two masses. Suppose at any time :

x_1 = distance of mass m_1 from the origin O

x_2 = distance of mass m_2 from the origin O

The relative separation $x_1 - x_2$ gives the length of the spring at any time. If the spring is stretched by an amount x, then,

$$x = (x_1 - x_2) - l \qquad ...(i)$$

The magnitude of force exerted by spring on each mass is $F = k\,x$ where k is spring constant. If the spring exerts a force $-F$ on m_1, it exerts a force $+F$ on m_2. Let us apply Newton's second law of motion separately to the two masses.

$$m_1 \frac{d^2x_1}{dt^2} = -kx \quad ...(ii) \qquad m_2 \frac{d^2x_2}{dt^2} = +kx \quad ...(iii)$$

Multiplying eq. (*ii*) by m_2 and eq. (*iii*) by m_1 and then subtracting eq. (*iii*) from eq. (*ii*), we have,

$$m_1 m_2 \frac{d^2x_1}{dt^2} - m_1 m_2 \frac{d^2x_2}{dt^2} = -m_2 k x - m_1 k x$$

or $$m_1 m_2 \frac{d^2}{dt^2}(x_1 - x_2) = -k(m_1 + m_2)x$$

or $$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{dt^2}(x_1 - x_2) = -kx \qquad ...(iv)$$

Taking second derivative of both sides of eq. (*i*), we have,

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(x_1 - x_2)$$

Therefore, eq. (*iv*) becomes :

$$\frac{m_1 m_2}{m_1 \quad m_2} \frac{d^2x}{dt^2} = -kx \qquad ...(v)$$

The quantity $\frac{m_1 m_2}{m_1 \quad m_2}$ has the dimensions of mass and is called *reduced mass* μ.

$$\therefore \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Note that reduced mass is always smaller than either mass. Therefore, eq. (*v*) becomes :

$$\mu \frac{d^2x}{dt^2} = -kx$$

or $$\frac{d^2x}{dt^2} = -\frac{k}{\mu}x$$

or Acceleration, $a = -\frac{k}{\mu}x$ ...(*vi*)

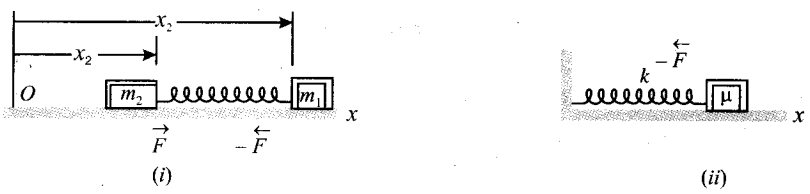


Fig. 25.22

Eq. (*vi*) is identical in form as for the motion of single mass connected to a spring. Therefore, the system shown in Fig. 25.22. (*i*) can be replaced by a single mass equal to the reduced mass $\mu(= m_1m_2/ m_1+ m_2)$ as shown in Fig. 25.22 (*ii*). If one of the masses is very much smaller than the other, then μ is roughly equal to the smaller mass. If the masses are equal, then μ is half as large as either mass.

Since k/μ is constant, acceleration, $a \propto -x$. Therefore, the acceleration of the motion of the system is directly proportional to the relative displacement of the masses from the mean position. Further, negative sign shows that acceleration is always directed towards the mean position. *Therefore, coupled oscillations are simple harmonic in nature.* The time period of S.H.M. is given by ;

$$\text{Time period, } T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{x}{\frac{k}{\mu}x}} = 2\pi\sqrt{\frac{\mu}{k}}$$

Note that time period of the system is identical in form as for a single oscillating mass except that now mass is equal to the reduced mass of the system.

CONCEPTUAL QUESTIONS

Q.1. What is the origin of the name simple harmonic motion?

Ans. Of all the periodic or harmonic motions (*e.g.*, circular motion, elliptical motion etc.), it is the simplest periodic motion. For this reason, it is called simple harmonic motion.

Q.2. What are the characteristics of simple harmonic motion?

Ans. The characteristics of simple harmonic motion are : (*i*) It is a periodic motion (*ii*) The particle moves to and fro about the mean position in a straight line (*iii*) The acceleration of the particle is directly proportional to the displacement from mean position and its direction is opposite to the displacement.

Q.3. What type of force causes a particle to execute S.H.M.?

Ans. When the force on a particle is directly proportional to its displacement from the mean position and acts opposite to the displacement, the particle will execute S.H.M.

Q.4. Is the motion of a simple pendulum strictly simple harmonic?

Ans. It is not strictly simple harmonic because we make the assumption that $\sin\theta = \theta$ which is nearly valid only if θ is very small.

Q.5. Why do we say that velocity and acceleration of a body executing S.H.M. are out of phase?

Ans. When a body is executing S.H.M., its acceleration is zero at the mean position and maximum at the extreme positions. On the other hand, the velocity is maximum at the mean position and zero at the extreme positions. Further, acceleration is always directed towards the mean position. For this reason, the velocity and acceleration of a body executing S.H.M. are out of phase.

Q.6. The amplitude of a simple harmonic oscillator is doubled. How does this effect (*i*) the period (*ii*) the total energy (*iii*) the maximum velocity of the oscillator?

Ans. (*i*) The time period of a simple harmonic oscillator is independent of the amplitude of the oscillations. Therefore, time period is not affected.

(*ii*) Total energy of oscillator is $E = (1/2)\ m\ \omega^2\ a^2$ where a is the amplitude. When a is doubled, E becomes four times.

(*iii*) $V_{max} = \omega\ a$. When a is doubled, V_{max} is also doubled.

Q.7. What is the relation between displacement and velocity of a body executing S.H.M. ?

Ans. The velocity of a body executing S.H.M. is maximum at the point where displacement is zero (mean position); it decreases with displacement and becomes zero at the extreme position.

Q.8. What is the relation between uniform circular motion and S.H.M.?

Ans. Uniform circular motion can be thought of as two simple harmonic motions operating at right angles.

Q.9. What is the most important characteristic of simple harmonic motion?

Ans. The most important characteristic of S.H.M. is that acceleration is directly proportional to the displacement and is directed opposite to the displacement. Mathematically, $A = -\omega^2 y$.

Q.10. Can simple pendulum experiment be done inside a satellite?

Ans. The time period of a simple pendulum is given by ;

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Inside a satellite, a body is in a state of weightlessness so that the effective value of g for it is zero. When $g = 0$, $T \to \infty$. Therefore, inside the satellite, the pendulum does not oscillate at all. Consequently, the simple pendulum experiment cannot be performed inside a satellite.

Q.11. Which of the following is not a sufficient condition for S.H.M. and why?
(*i*) acceleration $\propto$ displacement (*ii*) restoring force $\propto$ displacement.

Ans. Both the conditions are insufficient because the minus sign is missing. The missing minus sign implies that the acceleration increases as displacement increases and the object would then never return to its mean position.

Q.12. Show that time period of S.H.M. is given by $2\pi/\omega$ where ω is the angular frequency of S.H.M.

Ans. The displacement-time equation of S.H.M. is $y = a \sin(\omega t + \phi)$

Note that function y is periodic *i.e.* value of y at time t equals the value at time $t + T$ where T is the time period of the motion. Since the phase increases by 2π radians in time T,

$$\therefore \qquad \omega t + \phi + 2\pi = \omega(t + T) + \phi \quad \therefore \quad T = \frac{2\pi}{\omega}$$

Q.13. The acceleration of a spring-mass system is given by ;

$$a = -\frac{k}{m}x$$

What conclusions you draw from this equation?

Ans. If the mass is displaced a maximum distance a (amplitude) from the mean position and then released, its *initial* acceleration will be $-ka/m$ (*i.e.* it has a maximum negative value). When it passes through the mean position ($x = 0$), the acceleration is zero. At this instant, the velocity is maximum. It will then travel to the other side of mean position and finally

reaches $x = -a$ at which time its acceleration is ka/m (maximum positive) and its velocity is zero. In one full cycle of motion, the mass travels a distance of 4 a.

Q.14. Show that the total energy of a body executing S.H.M. is independent of time.

Ans. Let us discuss the total energy of mass-spring system. Assuming initial phase to be zero,

$$y = a \sin \omega t, \; V = \frac{dy}{dt} = \frac{d}{dt}(a \sin \omega t) = a\,\omega \cos \omega t$$

$$\therefore \quad \text{K.E.} = \frac{1}{2}mV^2 = \frac{1}{2}m\omega^2 a^2 \cos^2 \omega t = \frac{1}{2}ka^2 \cos^2 \omega t \qquad (\because m\omega^2 = k)$$

$$\text{P.E.} = \frac{1}{2}ky^2 = \frac{1}{2}ka^2 \sin^2 \omega t \qquad (\because y = a \sin \omega t)$$

Note that K.E. and P.E. are always positive quantities.

$$\text{Total energy} = \text{K.E.} + \text{P.E.} = \frac{1}{2}ka^2 (\cos^2 \omega t + \sin^2 \omega t) = \frac{1}{2}ka^2$$

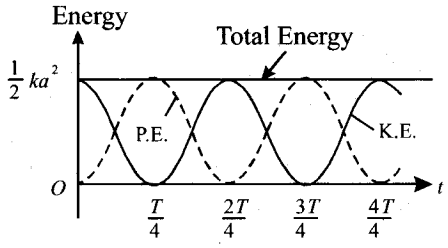


Fig. 25.23

Thus total mechanical energy is always constant and is equal to $k\,a^2/2$. Note that total energy is independent of time. Fig. 25.23 shows kinetic energy and potential energy versus time for the mass-spring system. The potential energy oscillates with time and has a maximum value of $ka^2/2$. Similarly, K.E. oscillates with time and has a maximum value of $k\,a^2/2$. At any instant, K.E. + P.E. = constant = $k\,a^2/2$. Note that K.E. or P.E. oscillates at double the frequency of S.H.M.

Q.15. You have a light spring, a metre scale and a known mass. How will you find the time period of oscillations of mass without the use of a clock?

Ans. Suspend the spring from a rigid support and attach the mass at its lowest end. Measure the extension (l) in the spring with a metre scale. If k is the force constant of the spring, then restoring force F is

$$F = -kl = -mg \quad \therefore \quad \frac{m}{k} = \frac{l}{g}$$

$$\text{Time period, } T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}$$

Q.16. Given some practical examples of S.H.M.

Ans. Some practical examples of S.H.M. are :

(*i*) Motion of piston in a gas-filled cylinder.

(*ii*) Atoms vibrating in a crystal lattice.

(*iii*) Motion of helical spring.

VERY SHORT ANSWER QUESTIONS

Q.1. What is a periodic motion?

Ans. The motion which repeats itself after a regular interval of time is called a periodic motion *e.g.*, motion of earth around the sun, motion of moon around the earth etc.

Q.2. What is an oscillatory motion?

Ans. It is a periodic motion (*i.e.*, having definite time period) in which the body moves to and fro along the same path about a fixed point (called mean position) *e.g.*, a pendulum swinging back and forth.

Q.3. What are the characteristics of oscillatory motion?

Ans. (*i*) It is a periodic motion (*ii*) It is to and fro motion along the same path about a fixed point called equilibrium position or mean position (*iii*) The body is confined within well defined limits called extreme positions.

Q.4. What is simple harmonic motion (S.H.M.)?

Ans. When the oscillatory motion of a body is along straight line and the graph between displacement of the body from the mean position and time is a sine curve (or cosine curve), the body is said to execute S.H.M.

Q.5. Define (*i*) displacement (*ii*) amplitude of S.H.M.

Ans. (*i*) The distance of the body at any time from the mean position is called displacement of the body at that time.

(*ii*) The maximum displacement of the body from its mean position is called amplitude of the body.

Q.6. What is the test for S.H.M.?

Ans. The displacement-time graph for a body executing S.H.M. is a sine curve or cosine curve.

Q.7. Why is S.H.M. called so?

Ans. It is the simplest type of oscillatory motion.

Q.8. Is the motion of earth around the sun periodic or oscillatory?

Ans. Periodic. Oscillatory motion is to and fro motion about a mean position.

Q.9. A particle is executing S.H.M. What are its velocity and acceleration at (*i*) mean position (*ii*) extreme positions?

Ans. (*i*) Velocity = maximum; Acceleration = zero (*ii*) Velocity = 0; Acceleration = maximum.

Q.10. How is time period of simple harmonic motion related to its acceleration?

Ans. Time period, $T = 2\pi\sqrt{\dfrac{\text{displacement}}{\text{acceleration}}}$.

Q.11. How are time period and frequency of S.H.M. affected by the amplitude of motion?

Ans. The time period (T) and frequency (f) of S.H.M. are independent of amplitude of motion.

Q.12. A particle is executing S.H.M. What are its K.E. and P.E. at (*i*) mean position (*ii*) extreme positions?

Ans. (*i*) K.E. = maximum ; P.E. = 0 (*ii*) P.E. = maximum ; K.E. = 0.

Q.13. A particle executing S.H.M. has amplitude 0.01 m and frequency 60 Hz. What is the maximum acceleration of the particle?

Ans. Amplitude, $a = 0.01$ m ; $\omega = 2\pi f = 2\pi \times 60$ rad s^{-1}

$\therefore$ Max. acceleration of particle $= \omega^2 a = (2\pi \times 60)^2 \times 0.01 = 144\ \pi^2$ m/s^2.

Q.14. The acceleration of a particle performing S.H.M. is 12 cm/s^2 at a distance of 3 cm from the mean position. What is its time period?

Ans. Acceleration, $A = \omega^2 y$ or $\omega^2 = A/y = 12/3 = 4$ $\therefore$ $\omega = 2$ rad/s

$\therefore$ Time period, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi = 3.14$ s.

Q.15. The length of a simple pendulum is doubled. How will its time period be affected?

Ans. Time period, $T \propto \sqrt{l}$. Therefore, the time period will become $\sqrt{2}$ times the original value.

Q.16. What is a seconds pendulum?

Ans. A simple pendulum that has a time period of 2 seconds is called a seconds pendulum.

Q.17. Why does a swinging simple pendulum eventually stop?

Ans. Due to friction between air and bob, the amplitude of the pendulum goes on decreasing and eventually it comes to rest.

Q.18. A swinging simple pendulum is placed in a lift which is falling freely. What is the frequency of the pendulum?

Ans. The time period of a simple pendulum falling with acceleration a is

$T = 2\pi\sqrt{\frac{l}{g-a}}$. For a freely falling lift, $a = g$ so that $T = \infty$. Hence $f = 0$.

Q.19. A swinging simple pendulum is placed in a lift which is accelerating downwards. How is its time period affected?

Ans. The time period of a simple pendulum falling downwards with acceleration a is

$T = 2\pi\sqrt{\frac{l}{g-a}}$. Therefore, the time period will increase because the effective acceleration due to gravity decreases from g to $(g - a)$.

Q.20. A swinging simple pendulum is placed in a lift which is accelerating upwards. How is its time period affected?

Ans. The time period of a simple pendulum moving upwards with acceleration a is

$T = 2\pi\sqrt{\frac{l}{g+a}}$. Therefore, the time period will decrease because the effective acceleration due to gravity increases from g to $(g + a)$.

Q.21. Is the tension in the string of a simple pendulum constant throughout the oscillations?

Ans. No. The tension in the string is $T = mg \cos \theta$ where θ is the angle which the string makes with the vertical. As θ varies, the tension T in the string also varies.

Q.22. When is tension (*i*) maximum (*ii*) minimum in the string of a simple pendulum?

Ans. (*i*) Tension in the string is $T = mg \cos \theta$. Therefore, tension in the string is maximum when $\cos \theta = 1$ or $\theta° = 0°$ *i.e.*, when the bob is at the mean position.

(*ii*) Tension in the string is minimum when $\cos \theta$ is minimum or θ is maximum *i.e.*, when the bob is at either extreme position.

Q.23. Will a pendulum clock gain or lose time when taken to the top of a mountain?

Ans. Time period, $T = 2\pi\sqrt{l/g}$. As g is less at the top of mountain, value of T will increase. Therefore, the pendulum will take more time to complete one vibration. As a result, pendulum clock will lose time.

Q.24. A wrist watch is taken to the top of a mountain. Will it give correct time?

Ans. Yes. It is because the time period of wrist watch (spring controlled) does not depend upon the value of g but depends upon the potential energy stored in the spring.

Q.25. Will a simple pendulum vibrate at the centre of earth?

Ans. No. It is because $g = 0$ at the centre of earth.

Q.26. If the metal bob of a simple pendulum is replaced by a wooden bob, what will be the change in the time period of the pendulum?

Ans. No change. It is because time period $T(= 2\pi\sqrt{l/g})$ of a simple pendulum is independent of the mass of the bob.

Q.27. The length of a seconds pendulum is *l*. If the length is halved, what will be its time period?

Ans. The time period T of seconds pendulum is 2s. Now $T \propto \sqrt{l}$.

$$\therefore \quad \frac{T_2}{T_1} = \sqrt{\frac{l_2}{l_1}} \text{ or } \frac{T_2}{2} = \sqrt{\frac{l_1/2}{l_1}} = \frac{2}{\sqrt{2}} \text{s}$$

Q.28. A particle is executing S.H.M. What is the phase difference between (*i*) velocity and displacement (*ii*) acceleration and velocity?

Ans. (*i*) The velocity leads the displacement by $\pi/2$ radians.

(*ii*) The acceleration leads the velocity by $\pi/2$ radians.

Q.29. A body is executing S.H.M. with amplitude *a*. In time equal to its period, what is the (*i*) distance moved (*ii*) displacement?

Ans. (*i*) The body reaches its initial position after a time equal to its period. Therefore, the distance covered in one time period = $4 \times a = 4a$.

(*ii*) Since body comes to its initial position after one time period, displacement is zero.

Q.30. What is spring constant?

Ans. In a spring, the magnitude of restoring force is $F = kx$ where k $(= F/x)$ is the spring constant. It is defined as the force required to produce unit extension or compression in the spring.

Q.31. What is the SI unit and dimensional formula of spring constant?

Ans. The SI unit of spring constant k $(= F/x)$ is Nm^{-1} and its dimensional formula is $[ML^0T^{-2}]$.

$$[k] = \frac{[F]}{[x]} = \frac{[MLT^{-2}]}{[L]} = [M\ L^0\ T^{-2}]$$

Note that spring constant is *dimensional constant.*

Q.32. A body of mass 5 kg hangs from a spring and oscillates with a time period of 2π seconds. What is the force constant of the spring?

Ans. Time period, $T = 2\pi\sqrt{\frac{m}{k}}$ or $2\pi = 2\pi\sqrt{\frac{5}{k}}$ $\therefore\ k = 5\ \text{Nm}^{-1}$.

Q.33. A mass *m* is suspended from a spring of negligible mass and the system oscillates with a frequency *f*. What will be the frequency of oscillations if a mass of 4 m is suspended from the same spring?

Ans. Frequency, $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. Therefore, $f \propto \sqrt{\frac{1}{m}}$.

Now $\frac{f'}{f} = \sqrt{\frac{m}{m'}}$ or $\frac{f'}{f} = \sqrt{\frac{m}{4m}} = \frac{1}{2}$ $\therefore\ f' = \frac{f}{2}$

Q.34. A spring of spring constant *K* is cut into three equal pieces. What will be the spring constant of each part?

Ans. Let k be the spring constant of each part. We can consider the original spring of spring constant K to be made of three springs in series, each of spring constant k.

$$\therefore \quad \frac{1}{K} = \frac{1}{k} + \frac{1}{k} + \frac{1}{k} = \frac{3}{k} \text{ or } k = 3K$$

This means that spring constant of each part will become three times that of the full length of the spring.

Q.35. Two springs of force constants 100 N/m and 400 N/m are connected in series. What will be the effective spring constant of the spring system?

Ans. The spring constant k of the system is given by ;

$$\frac{1}{k} = \frac{1}{100} + \frac{1}{400} \text{ or } k = \frac{100 \times 400}{100 + 400} = 80\ \text{N/m}$$

Q.36. Two unequal springs of the same material are loaded with same load. Which one will have a larger value of time period?

Ans. Time period, $T = 2\pi\sqrt{\frac{m}{k}}$. The longer spring will have smaller value of k and its time period will be larger.

Q.37. There are two springs, one delicate and other stout one. For which spring, the time period will be less?

Ans. Time period, $T = 2\pi\sqrt{\frac{m}{k}}$. For stout (stiffer) spring, the spring constant will be larger than the delicate spring. Therefore, time period for stout spring will be less.

Q.38. What will be the change in the time period of a loaded spring when taken to moon?

Ans. No change. It is because the time period of a loaded spring is independent of acceleration due to gravity.

Q.39. The spring-mass system oscillating vertically has a time period T. What will be the effect on the time period if the spring is made to oscillate horizontally?

Ans. The time period will remain the same ($\because T = 2\pi\sqrt{k/m}$).

Q.40. Is the oscillation of a mass suspended by a spring S.H.M.?

Ans. Yes, because it satisfies the conditions of S.H.M.

SHORT ANSWER QUESTIONS

Q.1. A restoring force is a must for a body to execute S.H.M. Explain.

Ans. In S.H.M., the body oscillates back and forth about the mean position. For this to happen, a restoring force must act on the body to permit it to perform back and forth journey about the mean position. When the body is at the extreme position, this restoring force brings it to the mean position. When the body reaches mean position, it has momentum and overshoots. The restoring force slows it down till the velocity of the body becomes zero at the other extreme position.

Q.2. What should be the characteristics of the restoring force in S.H.M.?

Ans. (*i*) The direction of the restoring force should be in a direction opposite to the direction of displacement of the body.

(*ii*) The magnitude of the restoring force should be directly proportional to the displacement of the body.

Q.3. Give mathematical equation of restoring force in S.H.M. How is it provided?

Ans. In S.H.M., the restoring force (F) can be expressed mathematically as :

$$F = -ky$$

Here y is the displacement of the body and k is a constant of proportionality. The negative sign shows that restoring force acts opposite to displacement. The restoring force may be provided due to gravity, elasticity etc.

Q.4. What are the characteristics of S.H.M.?

Ans. The important characteristics of S.H.M. are :

(*i*) The particle moves to and fro about the mean (equilibrium) position in a straight line.

(*ii*) The displacement, velocity and acceleration all vary sinusoidally with time but are not in phase.

(*iii*) The acceleration is always directed towards the mean position.

(*iv*) The restoring force is directly proportional to displacement but acts in a direction opposite to displacement.

Q.5. Show that in S.H.M., the acceleration of the particle is directly proportional to the displacement at the given instant.

Ans. In S.H.M., the displacement of the particle at any instant is

$$y = a \sin \omega t$$

$$\therefore \quad \text{Velocity, } V = \frac{dy}{dt} = \frac{d}{dt}(a \sin \omega t) = a\omega \cos \omega t$$

$$\therefore \quad \text{Acceleration, } A = \frac{dV}{dt} = \frac{d}{dt}(a\omega \cos \omega t) = -\omega^2 a \sin \omega t$$

or $$A = -\omega^2 y \quad (\because a \sin \omega t = y)$$

Thus in S.H.M., acceleration (A) is directly proportional to the displacement *i.e.*, $A \propto y$.

Q.6. Show that velocity and displacement of a body executing S.H.M. are out of phase by $\pi/2$ radians.

Ans. In S.H.M., displacement equation is $y = a \sin \omega t$ ...(*i*)

$$\text{Velocity, } V = \frac{dy}{dt} = \frac{d}{dt}(a \sin \omega t) = a\omega \cos \omega t$$

$$\therefore \quad V = a\omega \sin(\omega t + \pi/2) \qquad \text{...(ii)}$$

From eqs. (*i*) and (*ii*), it is clear that velocity (V) and displacement (y) are out of phase by $\pi/2$ radians.

Q.7. The displacement of a particle in S.H.M. may be given by ;

$$y = a \sin (\omega t + \phi)$$

Show that if the time t is increased by $2\pi/\omega$, the value of y remains the same.

Ans. $y = a \sin (\omega t + \phi)$. Therefore, displacement at any time $(t + 2\pi/\omega)$ is

$$\begin{aligned} y &= a \sin [\omega(t + 2\pi/\omega) + \phi] \\ &= a \sin [(\omega t + \phi) + 2\pi] \\ &= a \sin (\omega t + \phi) \qquad [\because \sin (2\pi + \theta) = \sin \theta] \end{aligned}$$

Therefore, the displacements at time t and $(t + 2\pi/\omega)$ are the same.

Q.8. A simple pendulum hangs from the ceiling of a car. If the car accelerates with uniform acceleration, will the frequency of the pendulum increase or decrease?

Ans. Frequency of simple pendulum, $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ $\therefore f \propto \sqrt{g}$.

As the car accelerates with uniform acceleration a, the resultant acceleration = $\sqrt{a^2 + g^2}$. Since $\sqrt{a^2 + g^2}$ is greater than $\sqrt{g}$, frequency of the simple pendulum will increase.

Q.9. A S.H.M. of amplitude a has a time period T. What will be the acceleration of the oscillator when (*i*) at mean position (*ii*) extreme position (*iii*) its displacement is half of the amplitude ?

Ans. Acceleration, $A = -\omega^2 y$; $\omega = 2\pi/T$.

(*i*) At mean position, $y = 0$ so that $A = 0$.

(*ii*) At extreme position, $y = a$ so that $A = -\omega^2 a = -\frac{4\pi^2}{T^2}a$.

(*iii*) At $y = a/2$, $A = -(2\pi/T)^2 \times \frac{a}{2} = -\frac{2\pi^2 a}{T^2}$.

Q.10. A S.H.M. of amplitude a has time period T. What will be the velocity of the oscillator at (*i*) equilibrium position (*ii*) half amplitude?

Ans. Velocity, $V = \omega\sqrt{a^2 - y^2}$; $\omega = 2\pi/T$.

(*i*) At equilibrium position, $y = 0$ so that $V = a\omega = a\,2\pi/T$.

(*ii*) At $y = a/2$, $V = \frac{2\pi}{T}\sqrt{a^2 - a^2/4} = \frac{\sqrt{3}\,a\pi}{T}$.

Q.11. A particle executing S.H.M. has maximum speed when passing through the mean position. Why?

Ans. P.E., $E_P = \frac{1}{2}m\omega^2 y^2$; K.E., $E_K = \frac{1}{2}m\omega^2(a^2 - y^2)$

At mean position, $y = 0$ so that $E_P = 0$ and $E_K = \frac{1}{2}m\omega^2 a^2$.

Since K.E. of the particle at mean position is maximum, its speed at this position will also be maximum.

Q.12. A particle executes S.H.M. with amplitude a. At what distance from the mean position its K.E. is equal to its P.E.?

Ans. Suppose K.E. of the particle is equal to its P.E. at a distance y from the mean position.

$\therefore$ $\quad$ K.E. $= \frac{1}{2}m\omega^2(a^2 - y^2)$; P.E. $= \frac{1}{2}m\omega^2 y^2$

Now $\quad \frac{1}{2}m\omega^2(a^2 - y^2) = \frac{1}{2}m\omega^2 y^2$ or $a^2 - y^2 = y^2$ $\therefore$ $y = a/\sqrt{2}$

Q.13. The bob of a simple pendulum is a hollow sphere full of water. If a fine hole is made at the bottom of the sphere, how will time period of the pendulum be affected?

Ans. Time period of simple pendulum, $T = 2\pi\sqrt{l/g}$. Here l is the distance between point of suspension

and centre of gravity (C.G.) of the sphere. Initially, when the sphere is full of water, C.G. is at the centre of sphere. As the water starts flowing out of sphere, the C.G. of sphere will first move down and then comes up. Finally, when all water flows out, the C.G. shifts to the centre of the sphere. Due to this, length l and hence time period T of the pendulum changes. Therefore, time period T first increases, then decreases and finally becomes equal to the initial value.

Q.14. A simple pendulum has metallic bob. What will be the effect on its time period if (*i*) metallic bob is replaced by ice bob (*ii*) metallic bob pendulum is taken to moon?

Ans. Time period of simple pendulum, $T = 2\pi\sqrt{\frac{l}{g}}$

(*i*) $T \propto \sqrt{l}$. Here l is the distance between the point of suspension and centre of gravity (C.G.) of the ice bob. If on melting of ice, l increases, T will increase and vice-versa.

(*ii*) $T \propto \sqrt{1/g}$. At moon, g is less (1/6th that on earth) so that T increases.

Q.15. The amplitude of an oscillating simple pendulum is doubled. How will it affect (*i*) time period (*ii*) total energy (*iii*) maximum velocity?

Ans. Suppose a is the amplitude of the pendulum.

(*i*) $T = 2\pi\sqrt{l/g}$. There will be no affect on the time period because it is independent of amplitude (a) of oscillation provided it is small.

(*ii*) Total energy, $E = \frac{1}{2}m\omega^2 a^2$. If amplitude ($a$) is doubled, E will become four times.

(*iii*) Max. velocity, $V_{max} = \omega a$. If amplitude (a) is doubled, V_{max} will become 2 times.

Q.16. In S.H.M., if displacement is half of the amplitude, then which part of total energy will be the kinetic energy?

Ans. In S.H.M., K.E., $E_K = \frac{1}{2}m\omega^2(a^2 - y^2)$; Total energy, $E = \frac{1}{2}m\omega^2 a^2$

For $y = \frac{a}{2}$; $E_K = \frac{1}{2}m\omega^2\left(a^2 - \frac{a^2}{4}\right) = \frac{3}{8}m\omega^2 a^2$ $\therefore \frac{E_K}{E} = \frac{3}{4}$

Q.17. Two identical springs of same spring constant of 10 N m^{-1} are connected in (*i*) series (*ii*) parallel. A mass of 5 kg is suspended in each case. What is the effective spring constant and elongation in each case?

Ans. (*i*) When springss are connected in series, the effective spring constant, $k = k_1k_2/k_1 + k_2 = 10 \times 10/10 + 10 = 5$ N m^{-1}.

$\therefore$ Elongation $= \frac{mg}{k} = \frac{5\times9.8}{5} = 9.8$ m

(*ii*) When springs are connected in parallel, effective spring constant $k = k_1 + k_2 = 10 + 10 = 20$ N m^{-1}.

$\therefore$ Elongation $= \frac{mg}{k} = \frac{5\times9.8}{20} = 2.45$ m

Q.18. The S.H.M. of a particle is given by ; $y = 3 \sin \omega t + 4 \cos \omega t$ What is the amplitude of motion?

Ans. $y = 3 \sin \omega t + 4 \cos \omega t$

Let $3 = a \cos \phi$ and $4 = a \sin \phi$

$\therefore$ $y = a\cos\phi \sin\omega t + a\sin\phi\cos\omega t = a\sin(\omega t + \phi)$

$\therefore$ Amplitude of motion, $a = \sqrt{3^2 + 4^2} = 5$.

Q.19. A large horizontal surface moves up and down in S.H.M. with an amplitude of 1 cm. If mass of 10 kg (which is placed on the surface) is to remain continuously in contact with it, what should be the maximum frequency of S.H.M.?

Ans. Amplitude, $a = 1$ cm $= 0.01$ m. The mass will remain in contact if the restoring force is equal to the weight of 10 kg mass *i.e.*,

$$mg = m\omega^2 a \quad \text{or} \quad \omega = \sqrt{\frac{g}{a}} = \sqrt{\frac{9.8}{0.01}} = 31.3 \text{ rad/s}$$

Now $\omega = 2\pi f$ $\therefore$ $f = \frac{\omega}{2\pi} = \frac{31.3}{2\pi} \simeq 5$ Hz

Q.20. The time period of a mass suspended from a spring (spring constant k) is T. If the spring is cut into three equal parts and the same mass is suspended from one piece, what will be the time period?

Ans. Time period, $T = 2\pi\sqrt{\frac{m}{k}}$. When the spring is cut into 3 equal parts, the effective spring constant of each piece becomes $3k$.

$\therefore$ Time period, $T' = 2\pi\sqrt{\frac{m}{3k}}$

$\therefore$ $\frac{T'}{T} = \frac{1}{\sqrt{3}}$ or $T' = \frac{T}{\sqrt{3}}$

Q.21. The kinetic energy of a particle executing S.H.M. is 16 J when it is in its mean position. If the amplitude of oscillations is 25 cm and mass of the particle is 5.12 kg, what is the period of oscillations?

Ans. In S.H.M., the K.E. of the particle is maximum at the mean position and is equal to total energy.

$\therefore$ $16 = \frac{1}{2} m\omega^2 a^2$ or $\omega^2 = \frac{16\times 2}{m\times a^2} = \frac{16\times 2}{5.12\times(0.25)^2} = 100$

$\therefore$ $\omega = 10$ rad/s. Now $T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}$ s

Q.22. A particle is executing S.H.M. of amplitude 4 cm and $T = 4$ sec. What is the time taken by it to move from positive extreme position to half the amplitude?

Ans. When the time is measured from extreme position, $y = a\cos\omega t$.

Since $y = \frac{a}{2}$; $\frac{a}{2} = a\cos\omega t$ or $\frac{1}{2} = \cos\frac{2\pi}{T}t$

$\therefore$ $\frac{1}{2} = \cos\frac{2\pi}{4}t$ or $\frac{2\pi}{4}t = \cos^{-1}\left(\frac{1}{2}\right) = \pi/3$ $\therefore$ $t = \frac{\pi}{3}\times\frac{4}{2\pi} = \frac{2}{3}$ sec

Q.23. A girl is swinging a swing in the sitting position. What will be the effect on the time period of the swing if she stands up?

Ans. The girl swinging a swing is like an oscillating pendulum. Therefore, time period of the swing is

$$T = 2\pi\sqrt{\frac{l}{g}} \text{ or } T \propto \sqrt{l}$$

Here l is the distance between the point of suspension of the swing and centre of gravity (C.G.) of the girl-swing system. As the girl stands up, the C.G. of the oscillating system is raised. Consequently, the value of l and hence time period decreases.

Q. 24. A man is standing on a platform moving up and down as a S.H.M. Will there be any change in his weight as recorded by a weighing machine on the platform?

Ans. Yes. In S.H.M., acceleration is always directed towards the mean position. Therefore, when the platform moves from the mean position towards the upper extreme position and vice-versa, the acceleration of S.H.M. is vertically downward and hence weight of man recorded by the machine will decrease. When the platform moves from mean position to lower extreme position and vice-versa, the acceleration of S.H.M. acts vertically upward. Hence weight of man increases.

Q.25. The maximum acceleration of a simple harmonic oscillator is A_0 and maximum velocity is V_0. What is the displacement amplitude?

Ans. Let a be the displacement amplitude and ω the angular velocity of the oscillator.

$\therefore$ $A_0 = \omega^2 a$; $V_0 = \omega a$

Now $\frac{V_0^2}{A_0} = \frac{(\omega a)^2}{\omega^2 a} = a$ $\therefore$ $a = \frac{V_0^2}{A_0}$

Q.26. Water in a U-tube executes S.H.M. Will the time period for mercury filled upto the same height in the U-tube be lesser or greater than that in case of water?

Ans. The time period of oscillations of a liquid in a U-tube is

$$\text{Time period, } T = 2\pi\sqrt{\frac{h}{g}}$$

Here h is the height of the liquid column in the U-tube before the start of oscillations. In the given situation, the time period (T) of oscillations will remain the same because it is independent of density of the liquid.

Q.27. A simple pendulum is executing S.H.M. with a time period T. If the length of the pendulum is increased by 21%, what will be the %age increase in time period.

Ans. Time period, $T = 2\pi\sqrt{\frac{l}{g}}$ or $T \propto \sqrt{l}$

Now $$\frac{T_2}{T_1} = \sqrt{\frac{l_2}{l_1}} = \sqrt{\frac{1.21 l_1}{l_1}} = 1.1 \text{ or } T_2 = 1.1\ T_1$$

$\therefore$ % increase in time period $= \frac{T_2 - T_1}{T_1} \times 100 = \frac{1.1 T_1 - T_1}{T_1} \times 100 = 10\%$.

Q.28. When will the motion of a simple pendulum be S.H.M.?

Ans. The motion of a simple pendulum will be S.H.M. if the *amplitude of the oscillating pendulum is small.* In a simple pendulum,

Restoring force, $F = -mg \sin\theta$

If θ is small, $\sin\theta = \theta$ so that $F = -mg\,\theta$.

If x is displacement of the bob, then $x = l\,\theta$ where l is the length of the pendulum.

$\therefore$ Restoring force, $F = -\frac{mgx}{l}$ $\left(\because \theta = \frac{x}{l}\right)$

or $F \propto -x$

Hence restoring force is directly proportional to the displacement x of the bob and its direction is opposite to the displacement. Therefore, motion of pendulum is S.H.M.

LONG ANSWER QUESTIONS

1. Explain (*i*) periodic motion (*ii*) oscillatory motion and (*iii*) S.H.M. with examples. **[Refer to Arts. 25.1 and 25.2]**
2. Explain displacement, velocity, acceleration and time period in S.H.M. **[Refer to Art. 25.6]**
3. Derive expressions for velocity and acceleration of a particle executing S.H.M. **[Refer to Art. 25.6]**
4. What are the characteristics of S.H.M.? **[Refer to Art. 25.6]**
5. Derive expressions for time period and frequency of oscillations for a mass m attached to a vertical spring of force constant k. **[Refer to Art. 25.11]**
6. Find the total energy of a particle executing S.H.M.? **[Refer to Art. 25.8]**
7. Derive an expression for the time period of a simple pendulum. **[Refer to Art. 25.13]**
8. Derive expressions for time period and frequency (*i*) of oscillations of a liquid in a U-tube (*ii*) of oscillations of a floating cylinder. **[Refer to Arts. 25.14 and 25.15]**
9. Explain the terms (*i*) undamped and damped oscillations (*ii*) free, forced and resonant oscillations. **[Refer to Arts. 25.17 and 25.18]**
10. Explain coupled oscillations. **[Refer to Art. 25.20]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Objects that vibrate have an *equilibrium (or mean) position* and are subjected to a *restoring force* that tries to return the object to its equlibrium position.
2. The mean or equilibrium position of a vibrating body is the position at which the body would come to rest if it were to lose all of its energy.
3. In simple harmonic motion (S.H.M.), the body moves to and fro about the mean position in a straight line and the force acting on the body at any instant is directly proportional to the displacement of the body from the mean position at that instant and this force is always directed towards the mean position.
4. For a body executing S.H.M., its displacement from the mean position at any time t is given by ;
$$y = a \sin (\omega t + \phi) \qquad ...(i)$$
where a is the amplitude of the motion (*i.e.* maximum displacement from the mean position), ω is the angular frequency and ϕ is the initial phase of the body. Note that eq. (i) is for the case the body starts its motion from the mean position. If the body starts its motion from the extreme position, then,
$$y = a \cos (\omega t + \phi)$$
5. For a body executing S.H.M., the time for one complete cycle or vibration is called time period T and is given by ;
$$T = \frac{2\pi}{\omega}$$
6. The velocity and acceleration at any time t of a body executing S.H.M.are given by;
$$\text{Velocity, } V = a\,\omega \cos (\omega t + \phi) = \omega\sqrt{a^2 - y^2} \qquad \text{(magnitude)}$$
$$\text{Acceleration, } A = a\,\omega^2 \sin (\omega\, t + \phi) = \omega^2 y \qquad \text{(magnitude)}$$
The maximum velocity of the body is $\omega\, a$ (*i.e.* when $y = 0$) and occurs at the mean position. The velocity of the body is zero at the extreme positions where $y = \pm a$. The maximum acceleration of the body is $\omega^2 a$ (when $y = a$) and occurs at the extreme positions. The acceleration is zero (when $y = 0$) at the mean position.
7. The time period (T) and frequency (f) of S.H.M. do not depend on the amplitude of the motion.
8. The acceleration of a body executing S.H.M. is given by ;
$$\text{Acceleration} = -\omega^2 y$$
The negative sign shows that acceleration is directed towards the mean position.
9. Restoring force F acting on the body executing S.H.M. is
$$F = \text{mass} \times \text{acceleration} = m \times (\omega^2 y) = m\,\omega^2 y$$
It is maximum at the extreme positions where $y = \pm a$ and zero at the mean position where $y = 0$.
10. The kinetic energy of a body in S.H.M.at any instant is
$$E_K = \frac{1}{2}mV^2 = \frac{1}{2}m\omega^2(a^2 - y^2) = \frac{1}{2}m\omega^2 a^2 \cos^2 \omega t$$
The kinetic energy of the body is zero at the extreme positions where $y = \pm a$ and is maximum at the mean position where $y = 0$.

11. The potential energy of a body in S.H.M. at any instant is :

$$E_P = \frac{1}{2}m\omega^2 y^2 = \frac{1}{2}m\omega^2 a^2 \sin^2 \omega t$$

The potential energy of the body is maximum at the extreme positions where $y = \pm a$ and is zero at the mean position where $y = 0$.

12. The total energy of a body executing S.H.M. is

$$E = E_K + E_P = \frac{1}{2}m\omega^2(a^2 - y^2) + \frac{1}{2}m\omega^2 y^2 = \frac{1}{2}m\omega^2 a^2$$

At the mean position (where $y = 0$), the total energy of the body is in the form of K.E. and at the extreme positions (where $y = \pm a$), the total energy of the body is in the form of P.E.

13. In S.H.M., $y = a \cos \omega t$; $V = \frac{dy}{dt} = a\,\omega \cos(\omega t + \pi/2)$

Acceleration, $A = \frac{dV}{dt} = a\,\omega^2 \cos(\omega t + \pi)$

Therefore, V leads y by $\pi/2$; A leads V by $\pi/2$; A leads y by π.

14. In S.H.M., Time period, $T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$

15. The time period T of a spring - mass system is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

where k is force constant and m is the mass attached to the spring.

16. If two springs of force constants k_1 and k_2 are connected in series, then force constant k of the series combination is

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

17. If two springs of force constants k_1 and k_2 are connected in parallel, then force constant k of the parallel combination is

$$k = k_1 + k_2$$

18. A simple pendulum of length l exhibits S.H.M. for small angular displacements from the vertical with time period T given by ;

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Note that time period is independent of the suspended mass.

19. If the time period of a pendulum clock increases, it runs slow and vice-versa.

20. If acceleration due to gravity decreases by $b\%$, the time period of simple pendulum increases by $b/2\%$.

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. A particle executing S.H.M. has amplitude 0.01 m and frequency 60 Hz. The maximum acceleration of the particle is **[AFMC Pune 2001]**

(a) 144 π^2m/s^2 (b) 120 π^2m/s^2

(c) 80 π^2m/s^2 (d) 60 π^2m/s^2

2. A simple pendulum is executing simple harmonic motion with a time period T. If the length of the pendulum is increased by 21%, the increase in the time period is **[AIIMS 2001]**

(a) 21% (b) 10%

(c) 30% (d) 50%

3. The frequency of oscillations of the springs shown in Fig. 25.24 will be **[AIIMS 2001]**

Fig. 25.24

(a) $\frac{1}{2\pi}\sqrt{\frac{k_1}{m}}$ (b) $\frac{1}{2\pi}\sqrt{\frac{(k_1+k_2)m}{k_1k_2}}$

(c) $2\pi\sqrt{\frac{k_2}{m}}$ (d) $\frac{1}{2\pi}\sqrt{\frac{k_1k_2}{(k_1+k_2)m}}$

4. A hollow sphere filled with water forms the bob of a simple pendulum. A small hole at the bottom of the bob allows the water to slowly flow out as it is set into small oscillations and its period of oscillations is measured. The time period will **[AFMC - Pune 2000]**

(a) increase (b) decrease

(c) remain constant (d) first (a), then (b)

5. A simple pendulum hangs from the ceiling of a car. If the car accelerates with uniform acceleration, the frequency of the simple pendulum will **[Pb PMT 2000]**

(a) increase (b) decrease

(c) become infinite (d) remain constant

6. A body is vibrating in S.H.M. with an amplitude of 0.06 m and frequency of 15 Hz. The maximum velocity and acceleration of the body are **[AFMC Pune 1999]**

(a) 5.65 m/s, 5.32×10^2 m/s^2

(b) 6.82 m/s, 7.62×10^2 m/s^2

(c) 8.91 m/s, 8.21×10^2 m/s^2

(d) 9.82 m/s, 9.03×10^2 m/s^2

7. The total energy of a body performing S.H.M. depends on **[CBSE PMT 2001]**

(a) k, a, m (b) k, a, x

(c) k, a (d) k, x

8. The bob of a simple pendulum having length l is displaced from the mean position to an angular position θ w.r.t. the vertical. It is then released. The velocity of the bob at equilibrium position is **[CBSE PMT 2000]**

(a) $\sqrt{2gl(1+\cos\theta)}$ (b) $\sqrt{2gl(1-\cos\theta)}$

(c) $\sqrt{2gl\cos\theta}$ (d) $\sqrt{2gl}$

9. Two springs of constants k_1 and k_2 have equal maximum velocities when executing simple harmonic motion. The ratio of their amplitudes (masses are equal) will be **[Pb PMT 1999]**

(a) $\frac{k_1}{k_2}$ (b) $\frac{k_2}{k_1}$

(c) $\left(\frac{k_1}{k_2}\right)^{1/2}$ (d) $\left(\frac{k_2}{k_1}\right)^{1/2}$

10. A particle executes simple harmonic motion of amplitude A. At what distance from the mean position its kinetic energy is equal to its potential energy? **[AFMC Pune 1999]**

(a) 0.51 A (b) 0.61 A

(c) 0.71 A (d) 0.81 A

11. Two springs A and B ($k_A = 3k_B$) are stretched by the same suspended weight. Then ratio of work done in stretching is **[CBSE PMT 1999]**

(a) 1 : 2 (b) 1 : 3

(c) 1: 4 (d) 2 : 1

12. A mass m suspended separately from two different springs of spring constants k_1 and k_2 gives time periods t_1 and t_2 respectively. If the same mass is connected to both the springs as shown in Fig. 25.25, then time period t is given by ; **[CBSE PMT 2002]**

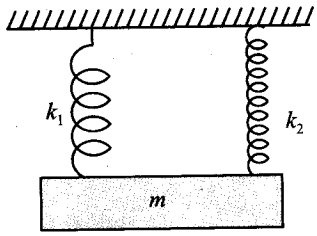


Fig. 25.25

(a) $t = t_1 + t_2$ (b) $t = \frac{t_1t_2}{t_1+t_2}$

(c) $t^2 = t_1^2 + t_2^2$ (d) $t^{-2} = t_1^{-2} + t_2^{-2}$

13. A body of mass 5 kg hangs from a spring and oscillates with a time period of 2π seconds. If the body is removed, the length of the spring will decrease by **[CBSE PMT 1994]**

(a) 2π metres (b) g metres

(c) g/k metres (d) k/g metres

14. Two simple pendulums having lengths 0.5 m and 2.0 m are displaced linearly a little at the same time. They will be in the same phase when shorter pendulum completes oscillations equal to **[CBSE PMT 1998]**
(*a*) 2 (*b*) 5
(*c*) 3 (*d*) 1

15. A particle is executing S.H.M. with an amplitude of 4 cm and time period 12 seconds. The time taken by the particle in going 2 cm from the mean position is T_1 seconds. The time taken from this displaced position to reach the extreme position on the same side is T_2 seconds.Then T_1/T_2 will be **[CBSE PMT 1993]**
(*a*) 2 (*b*) 1
(*c*) 1/3 (*d*) 1/2

16. A simple pendulum with a bob of mass m oscillates from A to C and back to A [See Fig. 25.26] such that PB is H. If the acceleration due to gravity is g then velocity of the bob as it passes through B is **[CBSE PMT 1995]**

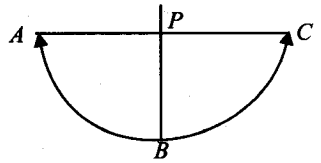


Fig. 25.26

(*a*) zero (*b*) $2gH$
(*c*) $\sqrt{2gH}$ (*d*) mgH

17. A mass m is suspended from a spring of negligible mass and the system oscillates with a frequency n. What will be the frequency of oscillations if a mass of 4 m is suspended from the same spring? **[CBSE PMT 1998]**
(*a*) $\frac{n}{4}$ (*b*) $2n$
(*c*) $\frac{n}{2}$ (*d*) $4n$

18. Frequency of a simple pendulum in a freely falling lift is **[CBSE PMT 1999]**
(*a*) zero (*b*) infinite
(*c*) finite (*d*) cannot say

19. In a simple harmonic motion, if the displacement is half of the amplitude, then which part of total energy will be kinetic energy? **[CBSE PMT 1995]**
(*a*) 1/3 (*b*) 3/4
(*c*) 2/3 (*d*) 1/4

20. A particle is performing simple harmonic motion along x-axis with amplitude 4 cm and time period 1.2 sec. The minimum time taken by the particle to move from x = + 2 cm to x = + 4 cm and back again is **[AIIMS 1995]**
(*a*) 0.4 s (*b*) 0.3 s
(*c*) 0.2 s (*d*) 0.6 s

21. A particle moves such that its acceleration a is given by $a = -bx$ where x is the displacement from equilibrium position and b is a constant. The period of oscillations is **[CPMT 1991]**
(*a*) $\frac{4\pi}{\sqrt{b}}$ (*b*) $2\pi\sqrt{\frac{1}{b}}$
(*c*) $2\pi\sqrt{b}$ (*d*) $\frac{2\pi}{b}$

22. A body is executing S.H.M. When its displacement from the mean position is 4 cm and 5 cm, the corresponding velocities are 10 cm/sec and 8 cm/sec. Then time period of the body is **[CPMT 1991]**
(*a*) $\pi/2$ sec (*b*) 2π sec
(*c*) $3\ \pi/2$ sec (*d*) π sec

23. A particle has simple harmonic motion. Its equation of motion is:
$x = 5\sin(4t - \pi/6)$ cm where x is the displacement. If the displacement of the particle is 3 cm, then its velocity in cm/s is **[MP PMT 1994]**
(*a*) 20 (*b*) $\frac{5}{6}\pi$
(*c*) 16 (*d*) $\frac{2}{3}\pi$

24. The displacement x (in metres) of a particle performing S.H.M. is related to time t (in seconds) as : $x = 0.05\cos\left(4\pi t + \frac{\pi}{4}\right)$.
The frequency of the motion will be **[MP PMT 1998]**
(*a*) 0.5 Hz (*b*) 1.0 Hz
(*c*) 1.5 Hz (*d*) 2.0 Hz

25. If metal bob of a simple pendulum is replaced by a wooden bob, then its time period will **[AIIMS 1998]**
(*a*) increase (*b*) decrease
(*c*) remain the same (*d*) first (*a*), then (*b*)

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*a*)	**2.** (*b*)	**3.** (*d*)	**4.** (*d*)	**5.** (*a*)
6. (*a*)	**7.** (*c*)	**8.** (*b*)	**9.** (*d*)	**10.** (*c*)
11. (*b*)	**12.** (*d*)	**13.** (*b*)	**14.** (*a*)	**15.** (*d*)
16. (*c*)	**17.** (*c*)	**18.** (*a*)	**19.** (*b*)	**20.** (*a*)
21. (*b*)	**22.** (*d*)	**23.** (*c*)	**24.** (*d*)	**25.** (*c*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Amplitude, $a = 0.01$ m ; $\omega = 2\pi f = 2\pi \times 60$

$\therefore$ Max. acceleration of particle $= \omega^2 a = (2\pi \times 60)^2 \times 0.01 = 144\ \pi^2$ m/s^2

2. Time period of simple pendulum, $T = 2\pi\sqrt{\dfrac{l}{g}}$ or $T \propto \sqrt{l}$

$\therefore \quad \dfrac{T_2}{T_1} = \sqrt{\dfrac{l_2}{l_1}} = \sqrt{\dfrac{1.21\,l}{l}} = 1.1$ or $T_2 = 1.1\,T_1 = 1.1\,T$

$\therefore$ % Increase in time period $T = \dfrac{1.1T - T}{T} \times 100 = 10\%$

3. Since the springs are connected in series, the force constant of the series combination is

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

$\therefore$ Frequency of oscillations, $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1 k_2}{(k_1 + k_2)m}}$

4. Time period of simple pendulum, $T = 2\pi\sqrt{l/g}$. Here l = distance between point of suspension and centre of gravity of sphere. Initially, when the sphere is filled with water, C.G. is at the centre of the sphere. As the water starts flowing out, the C.G. of sphere starts lowering down and l starts increasing. Finally, when all the water flows out, C.G. shifts to centre of the sphere. Therefore, time period of this pendulum first increases and then decreases.

5. Frequency of simple pendulum, $f = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{l}}$ $\therefore$ $f \propto \sqrt{g}$. As the car accelerates with uniform acceleration a, the resultant acceleration $= \sqrt{a^2 + g^2}$. Since $\sqrt{a^2 + g^2}$ is greater than $\sqrt{g}$, frequency of the simple pendulum increases.

6. Amplitude, $a = 0.06$ m ; Frequency, $f = 15$ Hz

Max. velocity of the body, $V_{max} = \omega\, a = (2\pi f)a = (2\pi \times 15) \times 0.06 = 5.65$ m/s

Max. acceleration of the body is

$$A_{max} = \omega^2\, a = (2\pi f)^2 a = (2\pi \times 15)^2 \times 0.06 = 5.32 \times 10^2 \text{ m/s}^2$$

7. The total energy E of a body performing S.H.M. is given by ;

$E = \dfrac{1}{2}ka^2$ .Therefore, total energy depends on force constant (k) and amplitude (a)

8. Fig. 25.27 shows the conditions of the problem.
According to law of conservation of energy,
Decrease in P.E. = Increase in K.E.

or $m\,gl\,(1 - \cos\theta) = \frac{1}{2}\,mv^2$

$\therefore \quad v = \sqrt{2gl(1-\cos\theta)}$

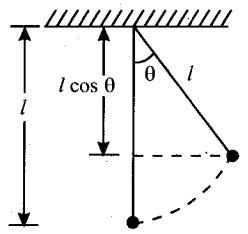


Fig. 25.27

9. Angular frequency of spring, $\omega = \sqrt{\frac{k}{m}}$ or $\omega \propto \sqrt{k}$

For equal maximum velocities, $a_1\omega_1 = a_2\omega_2$.

$\therefore \quad \frac{a_1}{a_2} = \frac{\omega_2}{\omega_1} = \sqrt{\frac{k_2}{k_1}} = \left(\frac{k_2}{k_1}\right)^{1/2}$

10. Suppose K.E. of the particle is equal to its P.E. at a displacement y from the mean position.

$$\text{K.E.} = \frac{1}{2}k(A^2 - y^2)\,; \quad \text{P.E} = \frac{1}{2}ky^2$$

$\therefore \quad \frac{1}{2}k(A^2 - y^2) = \frac{1}{2}ky^2$ or $A^2 - y^2 = y^2$ $\therefore$ $y = \frac{A}{\sqrt{2}} = \mathbf{0.71\ A}$

11. Work done, $W = \frac{1}{2}k\,x^2$ where k is force constant.

Restoring force, $F = -\,k\,x = -\,mg$ or $x = mg/k$

$\therefore \quad W = \frac{1}{2}k\left(\frac{mg}{k}\right)^2$ or $W \propto 1/k$ for same suspended weight.

$\therefore \quad \frac{W_A}{W_B} = \frac{k_B}{k_A} = \frac{1}{3}$ $\therefore$ $W_A : W_B = \mathbf{1 : 3}$

12. Time period, $T = 2\pi\sqrt{\frac{m}{k}}$ or $k = \frac{4\pi^2 m}{T^2}$

For the first spring, $k_1 = \frac{4\pi^2 m}{t_1^2}$; For the second spring, $k_2 = \frac{4\pi^2 m}{t_2^2}$

When the springs are connected in parallel, $k = \frac{4\pi^2 m}{t^2}$

Now $\quad k = k_1 + k_2 \quad$ or $\quad \frac{4\pi^2 m}{t^2} = \frac{4\pi^2 m}{t_1^2} + \frac{4\pi^2 m}{t_2^2}$

$\therefore \quad \frac{1}{t^2} = \frac{1}{t_1^2} + \frac{1}{t_2^2} \quad$ or $\quad \mathbf{t^{-2} = t_1^{-2} + t_2^{-2}}$

13. Time period, $T = 2\pi\sqrt{\frac{m}{k}}$ or $2\pi = 2\pi\sqrt{\frac{5}{k}}$ $\therefore$ $k = 5$ N/m

Now $\quad mg = -\,k\,x \quad$ or $\quad 5g = -5x \quad$ or $\quad x = \mathbf{-g\ metres}$

14. For $l_1 = 0.5$ m, $T_1 = 2\pi\sqrt{\frac{0.5}{g}}$; For $l_2 = 2.0$ m, $T_2 = 2\pi\sqrt{\frac{2.0}{g}}$

$\therefore \quad \frac{T_2}{T_1} = \sqrt{\frac{2.0}{0.5}} = 2$ or $T_2 = 2\,T_1$

Therefore, both the pendulums will be in phase when the shorter length pendulum completes 2 oscillations.

15. Amplitude of S.H.M, a = 4 cm ; Time period, T= 12 sec

Value of T_1 : $x = a \sin \omega t$ or $2 = 4\sin\frac{2\pi}{T}T_1$

$\therefore$ $\sin\frac{\pi}{6}T_1 = \frac{1}{2}$ or $\frac{\pi}{6}T_1 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\therefore$ $T_1 = \frac{\pi}{6}\times\frac{6}{\pi} = 1$ sec

Value of T_2 : Time taken from mean position to extreme position = $T/4$ = 12/4 = 3 sec.Therefore, time taken by the particle from this displaced position to reach the extreme position is T_2 = 3 –1 = 2 sec. Therefore T_1/T_2 = **1/2.**

16. By law of conservation of energy, increase in K.E. = decrease in P.E.

$\therefore$ $\frac{1}{2}mv^2 = mgH$ or $v = \sqrt{2gH}$

17. For the first case, $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$; For the second case, $n' = \frac{1}{2\pi}\sqrt{\frac{k}{4m}}$

$\therefore$ $\frac{n'}{n} = \sqrt{\frac{m}{4m}} = \frac{1}{2}$ or $n' = \frac{n}{2}$

18. The time period of a simple pendulum falling with acceleration a is $T = 2\pi\sqrt{\frac{l}{g-a}}$. For a freely falling lift, $a = g$ and $T = \infty$ so that f = **0.**

19. In S.H.M, K.E. of particle, $E_K = \frac{1}{2}m\,\omega^2\,(a^2 - y^2)$; Total energy, $E = \frac{1}{2}m\omega^2 a^2$

For $y = a/2$; $E_K = \frac{1}{2}m\omega^2\left(a^2 - \frac{a^2}{4}\right) = \frac{3}{8}m\omega^2 a^2$ $\therefore$ $\frac{E_K}{E} = \frac{3}{4}$

20. Amplitude of S.H.M., a = 4 cm ; Time period, T = 1.2 sec

If t sec is the time taken by the particle to move from mean position to +2 cm, then,

$$x = a \sin \omega t \text{ or } 2 = 4\sin\frac{2\pi}{T}t \text{ or } \frac{1}{2} = \sin\frac{2\pi}{1.2}t$$

$\therefore$ $\frac{2\pi}{1.2}t = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\therefore$ $t = \frac{\pi}{6}\times\frac{1.2}{2\pi} = 0.1$ sec

Time taken by the particle to move from mean position to +4 cm = $T/4$ = 1.2/4 = 0.3 sec. Therefore, time taken by the particle to move from x = + 2 cm to x =+4 cm = 0.3 – t = 0.3 – 0.1 = 0.2 sec. Time taken by the particle to move from x = + 2 cm to x = + 4 cm and back again = 2 × 0.2 = **0.4 sec.**

21. Given that $a = -bx$. Since $a \propto x$ and is directed opposite to x, the particle moves in S.H.M.

Now $a = b\,x$ (in magnitude) or $\frac{x}{a} = \frac{1}{b}$ or $\frac{\text{Displacement}}{\text{Acceleration}} = \frac{1}{b}$

$\therefore$ $T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{1}{b}}$

22. In S.H.M., velocity of the body at a displacement y from mean position is $V = \omega\sqrt{a^2 - y^2}$.

For the first case, $100 = \omega^2 (a^2 - 4^2)$; For the second case, $64 = \omega^2 (a^2 - 5^2)$

$\therefore \quad 100 - 64 = \omega^2 (a^2 - 4^2) - \omega^2 (a^2 - 5^2) = \omega^2 (5^2 - 4^2) = 9\omega^2$

$$\therefore \quad \omega^2 = \frac{100-64}{9} = \frac{36}{9} \quad \text{or} \quad \omega = \sqrt{\frac{36}{9}} = 2$$

Time period, $T = 2\pi/\omega = 2\pi/2 = \pi$ sec

23. Here $\omega = 4$ rad/s ; $a = 5$ cm ; $x = 3$ cm

Therefore, velocity, $V = \omega\sqrt{a^2 - x^2} = 4\sqrt{5^2 - 3^2} = 16$ cm/sec

24. Comparing it with standard equation of S.H.M. *viz* $x = a \cos(\omega t + \phi)$, $\omega = 4\pi$ or $2\pi f = 4\pi$. Therefore, $f = 2$ Hz.

25. Time period, $T = 2\pi\sqrt{\frac{l}{g}}$. The time period of simple pendulum is independent of mass.

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A point particle of mass 0.1 kg is executing S.H.M. of amplitude 0.1 m. When the particle passes through the mean position, its K.E. is 8×10^{-3} J. The equation of motion of this particle if its initial phase of oscillation is 45° is **[Roorkee 1991]**

   (*a*) $y = 0.1 \sin\left(\frac{t}{4} + \frac{\pi}{4}\right)$

   (*b*) $y = 0.1 \sin\left(\frac{t}{2} + \frac{\pi}{4}\right)$

   (*c*) $y = 0.1 \sin\left(4t - \frac{\pi}{4}\right)$

   (*d*) $y = 0.1 \sin(4t + \pi/4)$

2. A particle of mass 200 g executes S.H.M. The restoring force is provided by a spring of force constant 80 N/m. The time period of oscillations is **[MP PET 1994]**

   (*a*) 0.15 sec (*b*) 0.02 sec

   (*c*) 0.31 sec (*d*) 0.05 sec

3. A spring has a certain mass suspended from it and its period of vertical oscillations is T. The spring is now cut into two halves and the same mass is suspended from one of the halves. The period of vertical oscillations is now **[MP PET 1995]**

   (*a*) $T/\sqrt{2}$ (*b*) $2T$

   (*c*) $\sqrt{2}\,T$ (*d*) $T/2$

4. A particle is executing S.H.M. of amplitude 4 cm and $T = 4$ sec. The time taken by it to move from positive extreme position to half the amplitude is **[BHU 1995]**

   (*a*) $\sqrt{3/2}$ sec (*b*) 1 sec

   (*c*) 1/3 sec (*d*) 2/3 sec

5. A particle starts S.H.M. from the mean position. Its amplitude is A and time period is T. At the time its speed is half of the maximum speed, its displacement y from the mean position is **[Haryana CET 1996]**

   (*a*) $\frac{A}{2}$ (*b*) $\frac{A}{\sqrt{2}}$

   (*c*) $\frac{A\sqrt{3}}{2}$ (*d*) $\frac{2A}{\sqrt{3}}$

6. The kinetic energy of a particle executing S.H.M. is 16 J when it is in its mean position. If the amplitude of oscillations is 25 cm and the mass of the particle is 5.12 kg, the time period of oscillations is **[Haryana CET 1996]**

   (*a*) 2π sec (*b*) $20\,\pi$ sec

   (*c*) 5π sec (*d*) $\frac{\pi}{5}$ sec

7. The acceleration of a particle performing S.H.M. is 12 cm/sec^2 at a distance of 3 cm from the mean position. Its time period is **[MP PET 1996]**

   (*a*) 2.0 sec (*b*) 3.14 sec

   (*c*) 0.5 sec (*d*) 1.0 sec

8. What will be the force constant of the spring system shown in Fig. 25.28? **[Rajasthan PET 1996]**

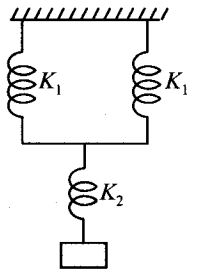


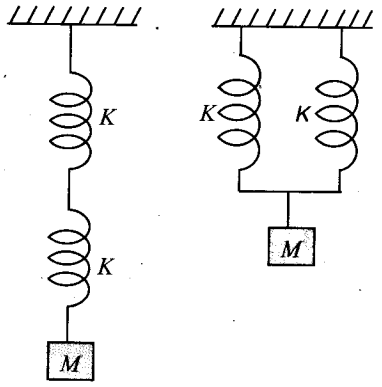
Fig. 25.28

(a) $\frac{K_1}{2}+K_2$ (b) $\left[\frac{1}{2K_1}+\frac{1}{K_2}\right]^{-1}$

(c) $\frac{1}{2K_1}+\frac{1}{K_2}$ (d) $\left[\frac{2}{K_1}+\frac{1}{K_1}\right]^{-1}$

9. A particle of mass 10 g is executing S.H.M. with an amplitude of 0.5 m and periodic time of $\pi/5$ second. The maximum value of force acting on the particle is **[MP CEE 1999]**
(a) 25 N (b) 5 N
(c) 2.5 N (d) 0.5 N

10. Two identical springs of spring constant K are connected in series and parallel as shown in Fig. 25.29. A mass M is suspended from them. The ratio of frequencies of their vertical oscillations will be **[BHU 1997]**



(i) (ii)

Fig. 25.29

(a) 2 : 1 (b) 1 : 2
(c) 1 : 4 (d) 4 : 1

11. A particle executes simple harmonic motion with a frequency f. The frequency with which kinetic energy oscillates is **[MP PET 2000]**
(a) f (b) $4f$
(c) $2f$ (d) $f/2$

12. The acceleration of a particle in S.H.M. is **[CET Karnataka 1997]**
(a) always constant
(b) maximum at extreme positions
(c) maximum at mean position
(d) always zero

13. A large horizontal surface moves up and down in S.H.M. with an amplitude of 1 cm. If mass of 10 kg (which is placed on the surface) is to remain continuously in contact with it, the maximum frequency of S.H.M. will be **[CET Karnataka 1996]**
(a) 5 Hz (b) 0.5 Hz
(c) 1.5 Hz (d) 10 Hz

14. A system exhibiting S.H.M. must possess **[CET Haryana 1994]**
(a) elasticity as well as inertia
(b) elasticity only
(c) inertia only
(d) elasticity, inertia and external force

15. The S.H.M. of a particle is given by the equation $y = 3 \sin \omega t + 4 \cos \omega t$. The amplitude of motion is **[PET Rajasthan 1997]**
(a) 7 (b) 12
(c) 1 (d) 5

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (d)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (c)
6. (d)	**7.** (b)	**8.** (b)	**9.** (d)	**10.** (b)
11. (c)	**12.** (b)	**13.** (a)	**14.** (a)	**15.** (d)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. K.E. at mean position $=\frac{1}{2}m\omega^2a^2$ $\therefore$ $\omega=\left[\frac{2\times8\times10^{-3}}{0.1\times(0.1)^2}\right]^{\frac{1}{2}}=4$

$\therefore y=a\sin(\omega t+\phi)=0.1\sin\left(4t+\frac{\pi}{4}\right)$

2. Time period, $T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{0.2}{80}}$ = 0.31 s

3. $T = 2\pi\sqrt{\frac{m}{k}}$ or $T \propto \sqrt{\frac{1}{k}}$. When the spring is half in length, the spring constant is $2k$. Therefore, $\frac{T_1}{T} = \sqrt{\frac{k}{2k}}$ or $T_1 = \frac{T}{\sqrt{2}}$

4. When the time is measured from extreme position, $y = a \cos \omega t$. Since $y = a/2$,

$$\therefore \quad \frac{a}{2} = a\cos\omega t \quad \text{or} \quad \frac{1}{2} = \cos\frac{2\pi}{T}t \quad \text{or} \quad \frac{1}{2} = \cos\frac{2\pi}{4}t$$

$$\therefore \quad \frac{2\pi}{4}t = \cos^{-1}\left(\frac{1}{2}\right) = \pi/3 \qquad \therefore\ t = \frac{\pi}{3}\times\frac{4}{2\pi} = \frac{2}{3}\ \text{sec}$$

5. Speed in S.H.M, $V = \omega\sqrt{A^2 - y^2}$. The speed is maximum when the particle passes through the mean position (*i.e.* $y = 0$). Therefore, $V_{max} = \omega A$. As per question,

$$\frac{\omega A}{2} = \omega\sqrt{A^2 - y^2} \quad \text{or} \quad \frac{\omega^2 A^2}{4} = \omega^2\left(A^2 - y^2\right) \quad \therefore\ y^2 = A^2 - \frac{A^2}{4} \quad \text{or} \quad y = \frac{\sqrt{3}A}{2}$$

6. In S.H.M., the K.E. of the particle is maximum at the mean position and is equal to total energy.

$$\therefore \quad 16 = \frac{1}{2}m\omega^2 a^2 \quad \text{or} \quad \omega^2 = \frac{16\times 2}{m\times a^2} = \frac{16\times 2}{5.12\times(0.25)^2} = 100$$

$$\therefore \quad \omega = 10\ \text{rad/s. Now } T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}\text{sec}$$

7. Magnitude of acceleration, $A = \omega^2 y$ or $\omega^2 = \frac{A}{y} = \frac{12}{3} = 4 \quad \therefore\ \omega = 2$ rad/sec

Time period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ sec = **3.14 sec**

8. The two parallel springs each of spring constant K_1 give an equivalent spring constant of $2K_1$ and this in series with spring of spring constant K_2 gives an equivalent spring constant equal to

$$\frac{1}{K} = \frac{1}{K_2} + \frac{1}{2K_1} \quad \text{or} \quad K = \left[\frac{1}{K_2} + \frac{1}{2K_1}\right]^{-1}$$

9. $\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi/5} = 10$ rad/sec. Max. acceleration,

$A_{max} = \omega^2 a = (10)^2 \times 0.5 = 50\ \text{ms}^{-2}$

Maximum force on the particle, $F_{max} = m \times A_{max} = (10 \times 10^{-3}) \times 50 =$ **0.5 N**

10. For series springs, the equivalent spring constant $= \frac{K\times K}{K+K} = \frac{K}{2}$

For parallel springs, the equivalent spring constant $= K + K = 2K$

Now $f = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$ or $f \propto \sqrt{K} \quad \therefore\ \frac{f_s}{f_p} = \sqrt{\frac{K/2}{2K}} = \sqrt{\frac{1}{4}}$ or $f_s : f_p = 1:2$

11. K.E. or P.E. oscillates at double the frequency of S.H.M.

12. Magnitude of acceleration, $A = \omega^2 y$. Acceleration is **maximum** when $y = \pm a$ *i.e.*, at the **extreme positions**.

13. Amplitude, $a = 1$ cm $= 0.01$ m. The mass will remain in contact if the restoring force is equal to the weight of 10 kg mass *i.e.*

$$mg = m\omega^2 a \quad \text{or} \quad \omega = \sqrt{\frac{g}{a}} = \sqrt{\frac{9.8}{0.01}} = 31.3\ \text{rad/s} \quad \therefore\ f = \frac{\omega}{2\pi} = \frac{31.3}{2\pi} \simeq 5\ \text{Hz}$$

14. The system must possess **elasticity as well as inertia.**

15. $y = 3 \sin \omega t + 4 \cos \omega t$. Let $3 = a \cos \phi$ and $4 = a \sin \phi$.
Then, $y = a \cos \phi \sin \omega t + a \sin \phi \cos \omega t = a \sin (\omega t + \phi)$
$\therefore$ Amplitude of motion, $a = \sqrt{3^2 + 4^2} = 5$

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. **The displacement of an oscillating particle is given by ;**

$$x = a \sin \omega t + b \cos \omega t$$

where a, b and ω are constants.

(*i*) Show that motion of the particle is simple harmonic.

(*ii*) Show that amplitude A is given by;

$$A = \sqrt{(a^2 + b^2)}$$

Hint. (*i*) $x = a \sin \omega t + b \cos \omega t$

$\therefore$ $V = \dfrac{dx}{dt} = \dfrac{d}{dt}(a \sin \omega t + b \cos \omega t) = \omega (a \cos \omega t - b \sin \omega t)$

Now, acceleration $= \dfrac{dV}{dt} = \dfrac{d}{dt}[\omega(a \cos \omega t - b \sin \omega t)]$

$= -\omega^2 (a \sin \omega t + b \cos \omega t)$

$\therefore$ acceleration $= -\omega^2 x$ $[\because x = a \sin \omega t + b \cos \omega t]$

Therefore, motion of the particle is simple harmonic.

(*ii*) We know that $x = A \sin (\omega t + \phi)$

Now $A \sin (\omega t + \phi) = a \sin \omega t + b \cos \omega t$

or $(A \cos \phi - a) \sin \omega t + (A \sin \phi - b) \cos \omega t = 0$

Thus $A \cos \phi - a = 0$ or $A \cos \phi = a$

And $A \sin \phi - b = 0$ or $A \sin \phi = b$

$\therefore$ $A = \sqrt{(a^2 + b^2)}$

2. **A particle moves in a straight line, with simple harmonic motion, making seven complete oscillations in 11 seconds. The velocity of the particle is 1.2 ms^{-1} when its distance from the centre of oscillation is 12.5 cm. Find the amplitude of motion, the maximum velocity and maximum acceleration.**

Hint. Frequency, $f = 7/11$ Hz ; $\omega = 2\pi f = 2\pi \times 7/11 = 4$ rad/s

If V is the velocity of the particle when at a distance y from the centre of oscillation, then,

$$V^2 = \omega (a^2 - y^2)$$

or $(1.2)^2 = (4)^2 [a^2 - (0.125)^2]$

$\therefore$ Amplitude, $a = 0.325$ m = **32.5 cm**

Maximum velocity of the particle occurs when $y = 0$.

$\therefore$ $V_{max} = \omega a = 4 \times 0.325 =$ **1.3 ms^{-1}**

The acceleration of the particle is maximum when $y = a$.

$\therefore$ $A_{max} = \omega^2 y = \omega^2 a = (4)^2 \times 0.325 =$ **5.2 ms^{-2}**

3. **A particle which is attached to a spring oscillates horizontally with simple harmonic motion with a frequency of (1/π) Hz and total energy of 10 joules. If**

the maximum speed of the particle is 0.4ms^{-1}, what is the force constant of the spring? What will be the maximum potential energy of the spring during the motion? [M.N.R.]

Hint. Let m be the mass of the particle in SI units.

The maximum speed of the particle is V_{max}= 0.4 ms^{-1}.

Total energy = Maximum kinetic energy = $\frac{1}{2}mV_{max}^2$

or $$10 = \frac{1}{2} m \times (0.4)^2 \quad \text{or} \quad m = \frac{10\times 2}{(0.4)^2} = 125 \text{ kg}$$

Angular frequency, $\omega = 2\pi f = 2\pi\times\frac{1}{\pi} = 2$ rad/s

Force constant of the spring is given by;

$$k = m\omega^2 = 125 \times (2)^2 = 500 \text{ Nm}^{-1}$$

Maximum P.E. = Total energy = Maximum K.E. = 10 joules

4. **A body of mass 0.5 kg suspended by an ideal spring oscillates up and down. The amplitude of oscillations is 0.5 m and periodic time is 1.57 seconds. Determine (*i*) maximum speed of the body (*ii*) maximum kinetic energy (*iii*) total energy (*iv*) force constant of the spring.**

Hint. (*i*) The maximum speed of a body executing S.H.M.is given by ;

$$V_{\max} = \omega a = \frac{2\pi}{T}\times a = \frac{2\pi}{1.57}\times 0.5 = 2 \text{ ms}^{-1}$$

(*ii*) Maximum K.E. of the body is given by ;

$$(E_K)_{max} = \frac{1}{2}mV_{max}^2 = \frac{1}{2}0.5\times(2)^2 = 1 \text{ J}$$

(*iii*) The total energy of the body is equal to its maximum kinetic energy.

$\therefore$ Total energy = 1 J

(*iv*) The periodic time of a body of mass m suspended by a spring of force constant k is given by ;

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{or} \quad k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2\times 0.5}{(1.57)^2} = 8 \text{ Nm}^{-1}$$

Alternatively, $$k = m\omega^2 = m\times\left(\frac{2\pi}{T}\right)^2 = 0.5\times\left(\frac{2\pi}{1.57}\right)^2 = 8 \text{ Nm}^{-1}$$

5. **A point particle of mass 0.1 kg is executing S.H.M. of amplitude 0.1 m. When the particle passes through the mean position, its kinetic energy is 8 × 10^{-3} J. Obtain the equation of motion of this particle if the initial phase of oscillation is 45°.**

Hint. The equation of S.H.M. is $y = a \sin(\omega t + \phi)$

Here $a = 0.1$ m ; $\theta = 45°$

$\therefore$ $$y = 0.1 \sin(\omega t + 45°) \quad \text{...(i)}$$

K.E. at mean position = Maximum K.E.

or $$8 \times 10^{-3} = \frac{1}{2}m\omega^2 a^2$$

or $$\omega^2 = \frac{2\times 8\times 10^{-3}}{m\times a^2} = \frac{2\times 8\times 10^{-3}}{0.1\times(0.1)^2} = 16 \quad \therefore \quad \omega = \sqrt{16} = 4 \text{ rads}^{-1}$$

Putting the value of ω in eq. (*i*), the equation of S.H.M. is

$$y = 0.1 \sin(4t + 45°)$$

6. A person of mass 50 kg stands on a platform. The platform oscillates with a frequency of 2 Hz. The amplitude of oscillations is 0.05 m. Calculate (*i*) the maximum and (*ii*) the minimum weight of the person recorded by a machine of the platform.

Hint. $a = 0.05$ m ; $f = 2$ Hz $\therefore \omega = 2\pi f = 2\pi \times 2 = 4\pi$ rad/s

Maximum acceleration $= a\omega^2 = 0.05 \times (4\pi)^2 = 7.9$ ms^{-2}

Variation of weight $= m\,(a\omega^2) = 50 \times 7.9 = 395$ N

(*i*) Maximum reading $= 50 \times 9.8 + 395 = 885$ N

(*ii*) Minimum reading $= 50 \times 9.8 - 395 = 95$ N

7. A block of mass 5 kg executes simple harmonic motion under the restoring force of a spring. The amplitude and the time period of motion are 0.1 m and 3.14 s respectively. Find the maximum force exerted by the spring on the block.

Hint. The spring will exert maximum force on the block when the block is at the extreme position.

Max. force on block $= ka$

Now, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{3.14} = 2$ rads^{-1} ; $k = m\omega^2 = 5 \times (2)^2 = 20$ N/m

$\therefore$ Max. force on block $= 20 \times 0.1 = 2$ N

8. Springs of spring constants k, $2k$, $4k$, $8k$,... .are connected in series. A mass M kg is attached to the lower end of the last spring and the system is allowed to vibrate. What is the time period of oscillations?

Hint. Since the springs are connected in series, the effective spring constant k_{eff} is given by;

$$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \ldots = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right]$$

$$= \frac{1}{k}\left[\frac{1}{1 - \frac{1}{2}}\right] = \frac{2}{k} \quad \therefore \quad k_{eff} = \frac{k}{2}$$

$$\text{Time period, } T = 2\pi\sqrt{\frac{M}{k_{eff}}} = 2\pi\sqrt{\frac{M}{k/2}} = 2\pi\sqrt{\frac{2M}{k}}$$

9. A body of mass 1.0 kg is suspended from a weightless spring having a spring constant 600 N/m. Another body of mass 0.5 kg moves vertically upward with a velocity of 3.0 ms^{-1} hits the suspended body and gets embedded in it. Find the frequency of oscillations and the amplitude of the system. **[Roorkee]**

Hint. The frequency of oscillations is given by ;

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_1 + m_2}} = \frac{1}{2\pi}\sqrt{\frac{600}{(1.0 + 0.5)}} = 3.18 \text{ s}^{-1}$$

The initial momentum of the system $= m_2 v_2 = 0.5 \times 3.0 = 1.5$ kg ms^{-1} (vertically upward). If after hitting v is the velocity of the combined $(m_1 + m_2)$, then momentum conservation requires that :

$$m_2 v_2 = (m_1 + m_2)\, v$$

$$\therefore \quad v = \frac{m_2 v_2}{m_1 + m_2} = \frac{1.5}{1.0 + 0.5} = 1.0 \text{ ms}^{-1}(\uparrow)$$

The K.E. of the combined body is $(m_1 + m_2)\,(v^2/2)$. As the combined body moves upward, its K.E. is converted into P.E. of compression $ka^2/2$ of the spring where a is the amplitude (maximum compression) of the system.

$$\therefore \quad \frac{1}{2}(m_1+m_2)\,v^2 = \frac{1}{2}ka^2$$

or $$a = v\sqrt{\frac{m_1+m_2}{k}} = 1.0\times\sqrt{\frac{1.0+0.5}{600}} = 0.05 \text{ m} = 5 \text{ cm}$$

10. Two simple harmonic motions are represented by the equations :

$$x_1 = 5\sin\left(2\pi t+\frac{\pi}{4}\right);\ x_2 = 5\sqrt{2}(\sin 2\pi t+\cos 2\pi t)$$

What is the ratio of their amplitudes? **[Roorkee]**

Hint. $$x_1 = 5\sin\left(2\pi t+\frac{\pi}{4}\right) \quad ...(i)$$

$$x_2 = 5\sqrt{2}(\sin 2\pi t+\cos 2\pi t) \quad \text{or} \quad x_2 = 10\left(\sin 2\pi t\cos\frac{\pi}{4}+\cos 2\pi t\sin\frac{\pi}{4}\right)$$

or $$x_2 = 10\sin\left(2\pi t+\frac{\pi}{4}\right) \quad ...(ii)$$

From eqs. (*i*) and (*ii*), ratio of amplitudes = 1 : 2.

11. A simple pendulum consists of a small sphere of mass *m* suspended by a thread of length *l*. The sphere carries a positive charge *q*. The pendulum is placed in a uniform electric field of strength *E*, directed vertically upwards. With what period will the pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force? Assume that oscillations are small. **[I.I.T.]**

Hint. Upward force on bob = qE

Weight of bob acting downward = mg

Net downward force on bob = $mg - qE$

$$\therefore \quad \text{Acceleration, } a = \frac{mg-qE}{m} = \left(g-\frac{qE}{m}\right)$$

The time period of the oscillations is

$$T = 2\pi\sqrt{\frac{l}{a}} = 2\pi\sqrt{\frac{l}{(g-qE/m)}}$$

12. Consider two discs as shown in Fig. 25.30 with a massless spring of force constant 100 N/m. Calculate the frequency of oscillations of the spring (*i*) when the system is resting on a table and (*ii*) when the table is removed and the system is falling freely.

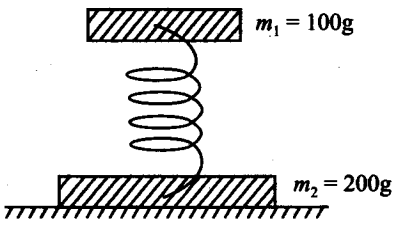


Fig. 25.30

Hint. (*i*) When the system is resting on the table, only the upper disc will vibrate. Therefore, the frequency of the oscillations of the system is

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}} = \frac{1}{2\pi}\sqrt{\frac{100}{0.1}} = 5 \text{ per second}$$

(*ii*) When the table is removed and the system is falling freely, both the masses vibrate. Therefore, the frequency of oscillations in this case is $f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$

where μ = reduced mass $= \frac{m_1m_2}{m_1+m_2} = \frac{100\times 200}{100+200} = \frac{200}{3}\text{ g} = \frac{2}{30}\text{ kg}$

$$\therefore \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{100}{2/30}} = 6.1 \text{ per second}$$

26

Waves

INTRODUCTION

One of the most important phenomena in nature is the transfer of energy from one point to another by **wave motion**. The word *wave* brings to mind a picture of water waves moving across the pond of water. When you throw a stone into a pond of water, circular waves are formed which move outward and finally reach the shore of the pond. It appears as if water is moving outward but that is not so. If you float a cork at a suitable place in the pond, the cork will simply move up and down without any forward or backward movement. This shows that it is not the water that moves outward but it is the water wave (*i.e.* disturbance) that travels outward. Therefore, a wave consists of oscillations that move without carrying matter with them. *An important property of waves is that they transfer energy.* A wave in water passing under a boat is momentarily able to lift the boat – increasing its gravitational potential energy. The boat also gains a little upward kinetic energy as the wave passes. The wave could not momentarily increase the boat's energy if it did not carry energy.

26.1. TYPES OF WAVES

A wave transfers energy from one point to another. There are two types of waves *viz.*

(*i*) Mechanical waves (*ii*) Electromagnetic waves

(*i*) **Mechanical waves.** *The waves which require material medium for their transmission are called* **mechanical waves** *or* **elastic waves**. For example, sound waves, ripples on water surface, vibrations in a string etc. are mechanical waves. Clearly, they cannot travel through vacuum. The speed of mechanical waves is very small as compared to electromagnetic waves. For example, the speed of sound waves in air is about 340 m/s.

(*ii*) **Electromagnetic waves.** *The waves which do not require material medium for their transmission are called* **electromagnetic waves**. They can travel through vacuum also. The light waves, X-rays etc. are electromagnetic waves. The speed of electromagnetic waves is very large. For example, the speed of electromagnetic waves in air/vacuum is 3×10^8 m/s. The transfer of energy by electromagnetic waves is due to the motion of electromagnetic fields.

In this chapter, we shall discuss only mechanical waves i.e. waves which travel only in a material medium.

26.2. MECHANICAL WAVES

A **mechanical wave** *is a form of disturbance that travels through a medium due to the repeated periodic motion of the particles of the medium about their mean positions ; the motion being transferred from particle to particle without the transfer of matter.*

In order to produce a mechanical wave, we need a **source** which produces disturbance and an **elastic medium**. When a particle of a medium is disturbed by some source, it begins to vibrate about its mean position. The disturbance of this particle is transferred to the next particle due to elasticity of the medium. The second particle also starts vibrating about its mean position and transfers its disturbance to the third particle and so on. In this way, disturbance (*i.e.* wave) is transferred from particle to particle while the particles continue to vibrate about their mean positions. Note that in wave motion, the disturbance travels through the matter but the particles of the medium do not move themselves along with the disturbance (*i.e.* wave).

Examples. We shall give two common examples to make clear how mechanical waves are formed and energy is transferred without the transference of the particles of the medium.

(*i*) **Waves on water surface.** Consider that a stone is dropped in still water in a pond. The waves spread out on the water surface around the point where the stone hits the water surface. Fig. 26.1 shows the mechanism of wave transmission on water surface. When a stone hits the water surface at *O*, the water molecule at *O* is displaced from its mean position and moves downwards. As it moves downwards, the restoring force starts acting on it due to elasticity of the medium. Consequently, the molecule comes to rest at point *A*. In this position, the energy of the molecule is in the form of potential energy. At point *A*, the restoring force on the molecule is maximum and hence it starts returning to the original mean position *O* ; during this journey, potential energy starts converting into kinetic energy. As the molecule reaches the mean position *O*, it overshoots the mark due to inertia and moves upto point *B* where again its velocity becomes zero. The molecule again starts downward journey. Thus the molecule at *O* begins to move up and down. Due to forces of cohesion among the water molecules, this upward and downward motion is passed on to the neighbouring molecules. Thus the disturbance (*i.e.* wave) travels outward from point *O*. *Note that water molecules move only up and down about their mean positions ; they do not move outward along with the disturbance.* The mechanical wave we have just described is called a **travelling wave** or **progressive wave** because the disturbance (*i.e.* wave) travels from one point to another through the medium.

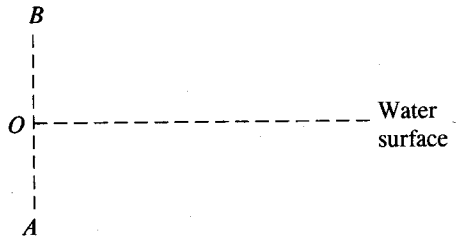


Fig. 26.1

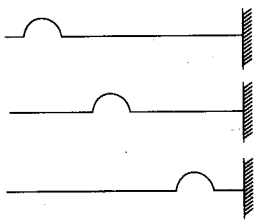


Fig. 26.2

(*ii*) **Waves on a rope.** When one end of a rope is fastened to a rigid support and the free end is moved up and down rapidly, each jerk travels to the other end of the rope as shown in Fig. 26.2. Each particle of the rope moves up and down while at the same time the disturbance (*i.e.* wave) moves along the length of the rope. Note that rope particles move only up and down about their mean positions ; they do not move along with the disturbance (*i.e.* wave). Clearly, it is a *travelling wave* or *progressive wave.*

26.3. TYPES OF MECHANICAL WAVES

In a mechanical wave, the particles of the medium vibrate about their mean positions while the disturbance (*i.e.* wave) moves on. Now, there can be two situations. First, the particles of the medium vibrate about their mean positions at right angles (**transverse**) to the direction of propagation of the wave. Secondly, the particles of the medium move back and forth (*i.e.* vibrate) about their mean positions (*i.e.* vibrations of the particles are **longitudinal**) along the direction of propagation of the wave. Accordingly, there are two types of mechanical waves *viz.*

1. Transverse waves 2. Longitudinal waves

The transverse waves can only be set up in solids or at the surface of liquids. However, longitudinal waves can be set up in all types of media *i.e.* solids, liquids as well as gases.

1. Transverse waves. *In a* **transverse wave**, *the particles of the medium vibrate about their mean positions at right angles to the direction of propagation of the wave.* The examples of transverse waves are :

(*i*) Ripples produced on the surface of water when stone is dropped into it.

(*ii*) Waves produced in a rope fixed at one end when the free end is moved up and down rapidly.

(*iii*) Waves in the stretched strings of musical instruments when the string is plucked.

A transverse wave (*e.g.* water wave) travels through a medium in the form of **crests** and **troughs** as shown in Fig. 26.3. When the particles of the medium move up from their mean positions, the medium is raised and a crest is formed. On the other hand, when the particles move down from their mean positions, the medium is depressed and a trough is formed. In Fig. 26.3, point *A* represents the centre of crest while point *B* represents the centre of trough. Thus when a transverse wave travels through a medium, the crests and troughs are formed alternately at each point along the wave.

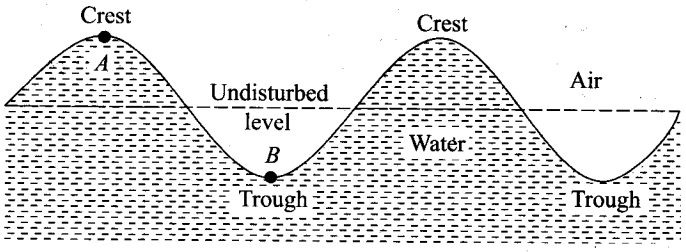


Fig. 26.3

2. Longitudinal waves. *In a* **longitudinal wave**, *the particles of the medium move back and forth about their mean positions along the direction of propagation of the wave.* The examples of longitudinal waves are :

(*i*) Sound waves travel through air in the form of longitudinal waves.

(*ii*) When a spring is alternately expanded and stretched, longitudinal waves are set up.

(*iii*) Vibrations of air column in organ pipes are longitudinal.

A longitudinal wave (*e.g.* sound waves in air) travels through a medium in the form of **compressions** and **rarefactions** as shown in Fig. 26.4. When the particles of the medium move longitudinally towards their mean positions, the medium is compressed and a *compression* is formed. Here the medium is air and at the site of compression, the volume of air decreases and its density increases. When the particles of the medium (air) move longitudinally away from their mean positions, the medium is rarefied and a *rarefaction* is formed. In the region of rarefaction, the volume of air increases and its density decreases. The compressions and rarefactions cause only slight variations in air pressure. Even for very loud sounds, the pressure variations are only about 0.01% of atmospheric pressure. The distance between the centres of two consecutive compressions (or rarefactions) is the wavelength λ of the wave.

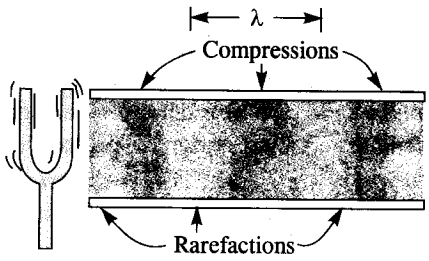


Fig. 26.4

26.4. COMPARISON OF TRANSVERSE AND LONGITUDINAL WAVES

Below is given the comparison between transverse and longitudinal waves in a tabular form.

S. No.	Transverse waves	Longitudinal waves
1.	The particles of the medium vibrate perpendicular to the direction in which the wave advances.	The particles of the medium vibrate in the same direction in which the wave advances.
2.	It is formed of crests and troughs.	It is formed of a series of compressions and rarefactions.
3.	It can propagate only in solids and at the surface of liquids.	It can propagate in all types of media (solid, liquid and gas).
4.	There is no pressure variation.	The pressure and density are maximum at compressions and minimum at rarefactions.

26.5. WAVE TERMINOLOGY

We generally consider a mechanical wave (transverse or longitudinal) to be of *sinusoidal shape. Fig. 26.5 shows the transverse wave (sine wave) produced on a string. In order to describe the wave, we usually use the following terms :

(*i*) **Amplitude.** *It is the maximum displacement of the medium from its equilibrium state when a mechanical wave passes through the medium.* The amplitude of wave in Fig. 26.5 is *a*. Note that it is only half the total vertical distance of the string.

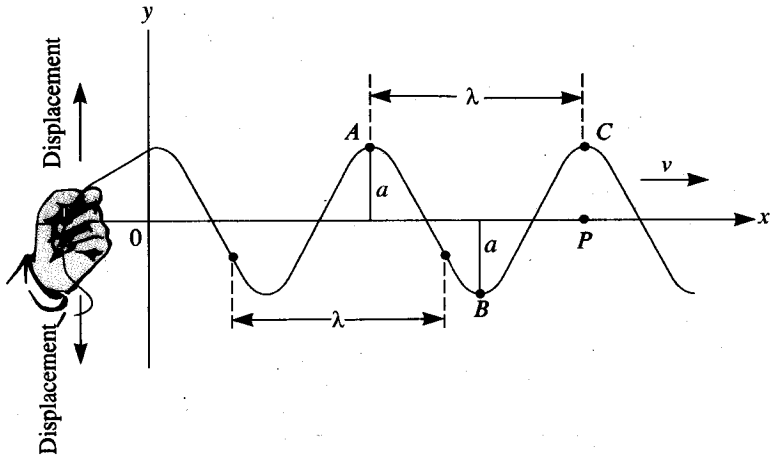


Fig. 26.5

(*ii*) **Wavelength.** *It is the distance travelled by the wave during the time any particle of the medium completes one vibration or cycle about its mean position.* It is denoted by λ (Greek letter lambda). We can also define wavelength as the distance between any two nearest particles of the medium vibrating in the same phase (See Fig. 26.5).

(*iii*) **Time period.** *The time period of a wave is the time taken by the wave source to complete 1 vibration or cycle.* It is denoted by *T*.

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* Although the wave need not have the shape of a sine function, we shall nevertheless restrict our attention mostly to sine waves. This may seem an arbitrary limiting choice. In fact it is not. It is because once we know how sine waves behave and propagate, we can predict the behaviour of waves of any shape.

At a fixed point (such as P in Fig. 26.5), the string repeatedly moves up and down as the wave moves to the right. During the time it takes the source to send out one wavelength, one wavelength must pass through P to make way for a new one. Therefore, P undergoes one complete cycle of motion in the same time the source takes to complete one cycle.

It follows that the period of the vibrating source is the same as the period of vibration of a point in the path of the wave. We can, therefore, define wavelength of a wave in another useful way : *The wavelength of a wave is the distance travelled by the wave in time period T of the wave.*

(*iv*) Frequency. *The frequency of a wave is the number of complete cycles that pass a given point in one second.* It is denoted by f. Thus referring to Fig. 26.5, frequency is equal to the number of wave crests passing through point P each second. As for a harmonic oscillator, the frequency of the wave is related to its time period as :

$$f = \frac{1}{T}$$

(*v*) Wave velocity. *The distance covered by a wave in one second is called wave velocity.* It is denoted by v and is measured in m s^{-1} in SI units.

$$\text{Wave velocity, } v = \frac{\text{distance covered by wave}}{\text{time taken}}$$

26.6. RELATION BETWEEN WAVE VELOCITY, FREQUENCY AND WAVELENGTH

Consider a mechanical wave passing through a medium possessing the properties of elasticity and inertia [See Fig. 26.6].

Let v = Velocity of the wave
f = Frequency of the wave
λ = Wavelength of the wave
T = Time period of the wave

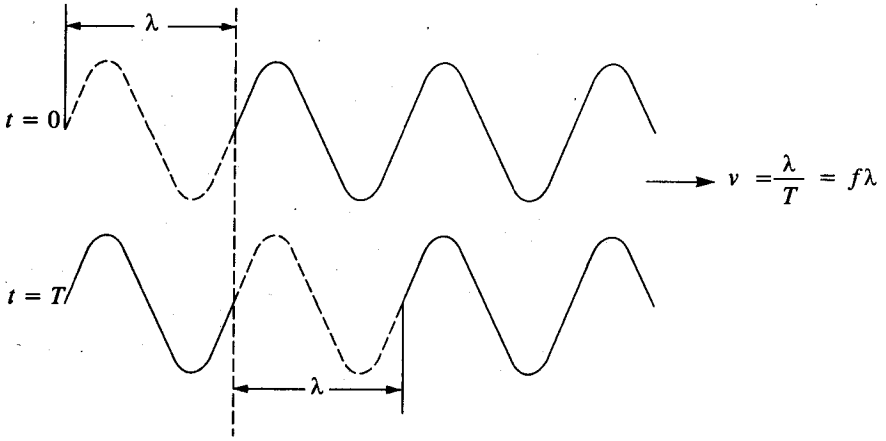

Fig. 26.6

By definition, wavelength (λ) is the distance travelled by the wave in time T (*i.e.* one time period).

$$\therefore \quad \text{Wave velocity, } v = \frac{\text{distance travelled by wave}}{\text{time taken}}$$

$$= \frac{\lambda}{T} = f\lambda \qquad \left(\because f = \frac{1}{T}\right)$$

$$\therefore \quad v = f\lambda$$

or **Wave velocity = Frequency × Wavelength**

This simple relation between wave velocity, frequency and wavelength is valid for all kinds of waves regardless of their physical nature and the mechanism by which they are generated or propagated. A mechanical wave travels at different speeds in different media. Note that in SI units, v is measured in ms^{-1}; f in hertz (Hz) and wavelength in metres (m).

Example 26.1. **A wave pulse on a string moves a distance of 10 m in 0.05 s. (*i*) What is the velocity of the pulse? (*ii*) What is the frequency of a periodic wave on the same string if its wavelength is 0.8 m?**

Solution. (*i*) Velocity of pulse, $v = \frac{\Delta x}{\Delta t} = \frac{10}{0.05} = 200 \text{ ms}^{-1}$

(*ii*) The periodic wave has the same velocity *i.e.*, 200 ms^{-1}. Using the relation $v = f\lambda$, the frequency of the periodic wave is given by ;

$$f = \frac{v}{\lambda} = \frac{200}{0.8} = 250 \text{ Hz}$$

Example 26.2. **How far does sound travel in air when a tuning fork of frequency 250 Hz completes 50 vibrations? The speed of sound in air is 340 ms^{-1} .**

Solution. $v = f\lambda$

$\therefore$ Wavelength, $\lambda = \frac{v}{f} = \frac{340}{250} = 1.36 \text{m}$

One wavelength is the distance travelled by the wave in one complete vibration of the tuning fork.

$\therefore$ Distance travelled by the sound wave in 50 vibrations

$$= 50 \times \lambda = 50 \times 1.36 = 68 \text{ m}$$

Example 26.3. **A source of sound produces waves of wavelength 0.80 m in air. The same source of sound produces waves of wavelength 4.0 m in water. If the velocity of sound in air = 332 ms^{-1}, find the velocity of sound in water.**

Solution. Since the source of sound is the same, the frequency of sound in the two cases is the same.

For water, $v_w = f\lambda_w$; For air, $v_a = f\lambda_a$

$\therefore$ $\frac{v_w}{v_a} = \frac{f\lambda_w}{f\lambda_a} = \frac{\lambda_w}{\lambda_a}$ or $v_w = v_a \times \frac{\lambda_w}{\lambda_a} = 332 \times \frac{4.0}{0.80} = 1660 \text{ m s}^{-1}$

Example 26.4. **The audible frequency range of a human ear is 20 Hz to 20 kHz. Convert this into corresponding wavelength range. Take the speed of sound in air at ordinary temperature to be 340 ms^{-1}.**

Solution. $\lambda = \frac{v}{f}$

Wavelength corresponding to 20 Hz frequency is

$$\lambda_1 = \frac{v}{f_1} = \frac{340}{20} = 17 \text{m}$$

Wavelength corresponding to 20 kHz (= 20×10^3 Hz) frequency is

$$\lambda_2 = \frac{v}{f_2} = \frac{340}{20 \times 10^3} = 17 \times 10^{-3} \text{m} = 17 \text{ mm}$$

$\therefore$ Corresponding wavelength range is 17 m to 17 mm.

Example 26.5. **A body vibrating with a certain frequency sends waves 2 m long through a medium *A* and 1.5 m long through medium *B*. The velocity of wave in medium *A* is 10 ms^{-1}. What is the wave velocity in medium *B*?**

Solution. The frequency of source is the same in the two media.

In medium *A* : $v_A = f\lambda_A$

$\therefore$ Frequency, $f = \frac{v_A}{\lambda_A} = \frac{10}{2} = 5$ Hz

In medium *B* : $f = 5$ Hz ; $\lambda_B = 1.5$ m ; $v_B = ?$

$\therefore$ $v_B = f\lambda_B = 5 \times 1.5 = \mathbf{7.5\ ms^{-1}}$

PROBLEMS FOR PRACTICE

1. A hospital uses an ultrasonic scanner to locate tumours in a tissue. What is the wavelength of sound in a tissue in which the speed of sound is 1.7 kms^{-1}? The operating frequency of the scanner is 4.2 MHz. **[0.405 mm]**
2. How far does sound travel in air when a tuning fork of frequency 480 Hz completes 50 vibrations? Velocity of sound in air is 348 m s^{-1}. **[36.25 m]**
3. A stone is dropped into a well 19.6 m deep and the impact of sound is heard after 2.056 seconds. Find the velocity of sound in air. **[350 m s^{-1}]**
4. If the frequency of radio waves is 600 kHz, find the wavelength of the waves. The speed of radio waves = 3×10^8 ms^{-1}. **[500 m]**
5. A bat emits ultrasonic sound of frequency 100 kHz in air. If this sound meets a water surface, what is the wavelength of (*i*) the reflected sound (*ii*) the transmitted sound? Speed of sound in air = 340 ms^{-1} and in water = 1486 ms^{-1}.

[(*i*) 3.4×10^{-3} m (*ii*) 1.486×10^{-3} m]

26.7. CHARACTERISTICS OF WAVE MOTION

By wave motion, we mean the transfer of energy from a source to a distant receiver without the transfer of matter between the two points. A transverse or longitudinal wave motion has the following characteristics :

(*i*) Wave motion is a disturbance travelling through a medium.

(*ii*) When a wave passes through a medium, energy is transferred and the medium particles execute vibrations only about their mean positions *i.e.* there is no net displacement of the particles of the medium.

(*iii*) The energy transfer in the medium takes place with a constant speed ($v = f\lambda$) which depends on the nature of the medium.

(*iv*) The wave velocity ($v = f\lambda$) is different from the velocity with which the particles of the medium vibrate. The wave velocity is constant throughout the medium while the velocity of the particles during their vibration is different at different *positions.

(*v*) When a wave travels in a medium, the particle which meets it first begins to vibrate a little earlier than the particle immediately after it. In other words, there is a continuous phase difference amongst the successive particles of the medium.

(*vi*) A wave motion is possible in a medium which possesses the properties of elasticity and inertia.

(*vii*) If the disturbance at the source is of repetitive nature, the wave is maintained. If it is not, the amplitude of vibration of each particle becomes progressively smaller and eventually the wave ceases to exist.

(*viii*) If the disturbance at the source is simple harmonic, a plot of the displacements of the particle at a single instant in time as a distance from the source is sinusoidal.

* For example, the velocity of a particle is maximum at the mean position and is zero at the extreme position.

26.8. SPEED OF TRANSVERSE WAVES

(*i*) The speed of transverse waves in a **solid** is given by ;

$$v = \sqrt{\frac{\eta}{\rho}}$$

where η = Modulus of rigidity of the material

ρ = Density of the material

We can verify that this expression is dimensionally correct.

Dimensions of L.H.S. = $[M^0LT^{-1}]$

Dimensions of R.H.S. = $\left[\frac{ML^{-1}T^{-2}}{ML^{-3}}\right]^{1/2} = [M^0LT^{-1}]$

We see that the expression is dimensionally correct. No other combination of η and ρ is dimensionally correct.

(*ii*) The speed of transverse waves in a **stretched string** is given by ;

$$v = \sqrt{\frac{T}{m}}$$

where *T = Tension in the string

m = Mass per unit length of the string (*i.e.* linear density of the string)

We shall again verify that this expression is dimensionally correct.

Dimensions of L.H.S. = $[M^0LT^{-1}]$

Dimensions of R.H.S. = $\sqrt{\frac{MLT^{-2}}{ML^{-1}}} = [M^0LT^{-1}]$

Therefore, the expression is dimensionally correct. No other combination of T and m is dimensionally correct.

26.9. SPEED OF LONGITUDINAL WAVES

(*i*) The speed of longitudinal waves in a **solid** is given by ;

$$v = \sqrt{\frac{K + \frac{4}{3}\eta}{\rho}}$$

where K is the bulk modulus, η is the modulus of rigidity and ρ is the density of the material of the solid.

When the solid is in the form of a long rod, the speed of longitudinal waves through the rod is given by ;

$$v = \sqrt{\frac{Y}{\rho}}$$

where Y is the Young's modulus of the material of the rod.

(*ii*) The speed of longitudinal waves in a **liquid** is given by ;

$$v = \sqrt{\frac{K}{\rho}}$$

* T = tension = force = $[MLT^{-2}]$

where K is the bulk modulus and ρ is the density of the liquid.

(*iii*) The speed of longitudinal waves in a **gas** is given by ;

$$v = \sqrt{\frac{K}{\rho}}$$

where K is the bulk modulus of the gas and ρ is its density.

Example 26.6. A steel wire 0.72 m long has a mass of 5.0×10^{-3} kg. If the wire is under a tension of 60 N, what is the speed of transverse waves in the wire?

Solution. Length of wire, $l = 0.72$ m ; Mass of wire, $M = 5.0 \times 10^{-3}$ kg

$$\text{Mass per unit length, } m = \frac{M}{l} = \frac{5.0\times10^{-3}}{0.72} = 6.94 \times 10^{-3} \text{ kg m}^{-1}$$

Tension in the wire, $T = 60$ N

The speed of transverse waves in the wire is given by ;

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{60}{6.94\times10^{-3}}} = 93 \text{ m s}^{-1}$$

Example 26.7. One steel pipe of length 660 m is struck a blow which produces loud sound. A listener at the other end hears two sounds at an interval of 1.89 sec ; one from the wave that has travelled along the metal portion of the pipe and the other from the wave that has travelled through air. If the density of steel is 8×10^3 kg/m^3, what is the bulk modulus of elasticity of steel?

Solution. Let t_1 and t_2 be the time taken by sound to cover a distance of 660 m (= l) in steel pipe and air respectively.

$$\text{Speed of sound in steel, } v_s = \sqrt{\frac{K}{\rho}}$$

where K is bulk modulus of elasticity of steel and ρ is its density.

$$\therefore \quad t_1 = \frac{l}{v_s} = l\sqrt{\frac{\rho}{K}}$$

Also $$t_2 = \frac{l}{v_a} = \frac{660}{330} = 2 \text{ s}$$

It is given : $t_2 - t_1 = 1.89$

or $$2 - l\sqrt{\frac{\rho}{K}} = 1.89 \quad \text{or} \quad l^2\frac{\rho}{K} = (2-1.89)^2 = (0.11)^2$$

$$\therefore \quad K = \frac{l^2\rho}{(0.11)^2} = \frac{(660)^2\times8\times10^3}{(0.11)^2} = 2.88 \times 10^{11} \text{ N/m}^2$$

Example 26.8. A long wire *PQR* is made by joining two wires *PQ* and *QR* of equal radii. *PQ* has length 4.8 m and a mass 0.06 kg while *QR* has length 2.56 m and mass 0.2 kg. The wire PQR is under a tension of 80 N. A sinusoidal wave pulse is sent along the wire *PQ* from the end *P*. Calculate the time taken by the wave pulse to reach the other end *R* of the wire.

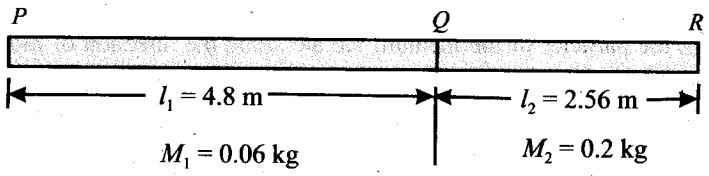

Fig. 26.7

Solution. Fig. 26.7 shows the conditions of the problem. Let v_1 and v_2 be the velocities of the sinusoidal wave through wires PQ and QR respectively. Then total time t taken by the wave to travel from P to R is

$$t = \frac{PQ}{v_1} + \frac{QR}{v_2}$$

For wire PQ, $m_1 = \frac{M_1}{l_1} = \frac{0.06}{4.8} = 1.25 \times 10^{-2}\ \text{kg m}^{-1}$; $T = 80$ N

$$\therefore \quad v_1 = \sqrt{\frac{T}{m_1}} = \sqrt{\frac{80}{1.25 \times 10^{-2}}} = 80\ \text{ms}^{-1}$$

For wire QR, $m_2 = \frac{M_2}{l_2} = \frac{0.2}{2.56} = 7.8125 \times 10^{-2}\ \text{kg m}^{-1}$; $T = 80$ N

$$\therefore \quad v_2 = \sqrt{\frac{T}{m_2}} = \sqrt{\frac{80}{7.8125 \times 10^{-2}}} = 32\ \text{ms}^{-1}$$

$$\therefore \quad t = \frac{4.8}{80} + \frac{2.56}{32} = 0.06 + 0.08 = \mathbf{0.14\ s}$$

PROBLEMS FOR PRACTICE

1. A steel wire 70 cm long has a mass of 7.0 g. If the wire is under a tension of 100 N, what is the speed of transverse wave in the wire? **[100 m s^{-1}]**
2. A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If a transverse jerk is struck at one end of the string, how long does the disturbance take to reach the other end? **[0.5 s]**
3. One end of a rubber tube is fastened to a fixed support. The other end passes over a pulley at a distance 8 m from the fixed end and carries a load of 2 kg. The mass of the tube, between the fixed end and the pulley, is 600 g.
   (*i*) What is the speed of a transverse wave in the tube?
   (*ii*) If a point on the string is given a transverse simple harmonic motion with a frequency of 20 Hz, what is the wavelength of the wave? **[(*i*) 16.2 m s^{-1} (*ii*) 0.81 m]**
4. You can often hear an approaching train by putting your ear to the track. How long does it take for the wave to travel down the steel track if the train is 1.0 km away? Y for steel $= 2 \times 10^{11}$ N m^{-2} and density of steel $= 7.8 \times 10^3$ kg m^{-3}. **[0.20 s]**
5. Determine the speed of sound waves in water. Given $K_{water} = (1/49) \times 10^{11}$ Pa. **[1478 m s^{-1}]**

26.10. SOUND WAVES

Sound waves are longitudinal waves that can travel through any *material medium (*i.e.* solids, liquids or gases) with a speed that depends on the properties of the medium. As sound travels through a medium, the particles of the medium vibrate along the direction of motion of the wave. This is in contrast to a transverse wave where the particle motion is perpendicular to the direction of wave motion. The displacements that occur as a result of sound waves involve the longitudinal displacements of individual molecules from their mean or equilibrium positions. This results in a series of high- and low-pressure regions called *compressions* and *rarefactions* respectively.

* Sound cannot travel through a vacuum because in a vacuum there is no material to transmit this mechanical wave (*i.e.* sound).

Sound waves *are longitudinal waves that travel through all media (solid, liquid or gas) in the form of compressions and rarefactions.*

Why sound waves travel in all media? The fact that longitudinal waves can travel through any material medium (solid, liquid or gas) can be easily explained. Since longitudinal waves travel through a medium in the form of compressions and rarefactions, there is a change in the volume of the medium. Therefore, longitudinal waves can only travel through a medium if the medium offers a resistance to change in volume. In other words, the medium should possess *volume elasticity.* We know that the three states of matter (solids, liquids and gases) possess volume elasticity *i.e.* they offer resistance to change in their volume. For this reason, longitudinal waves can be transmitted through all three states of matter. It may be noted that sound travels faster and with less energy loss in solids and liquids than in gas (*e.g.* air). This is why putting your ear to a steel railroad track allows you to hear the approach of a train long before you can hear in the air.

26.11. SOUND WAVES IN AIR

We are more often concerned with sound waves in air since that is the basis of our sense of hearing. Fig. 26.8 shows the production of a sound wave in air. The vibrating drum head alternately compresses and rarefies the air and produces a longitudinal wave that travels outward in the air.

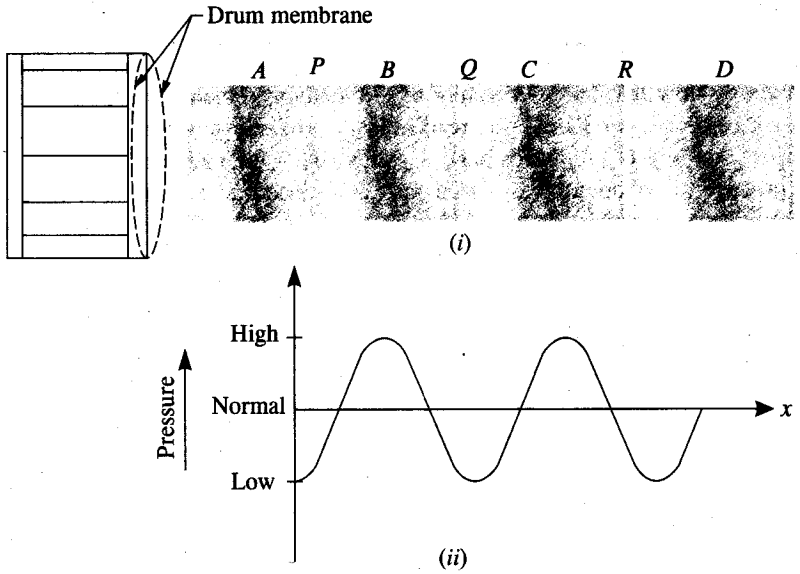

Fig. 26.8

When the drum membrane moves to the right, it compresses the air in front of it and a *compression* travels out through the air. An instant later, the drum membrane is moving to the left, leaving a region of decreased air pressure called a *rarefaction.* This disturbance also travels out from the drum. Hence a series of pressure disturbances, the compressions and rarefactions, travel out from the drum. The regions marked as *A, B, C* and *D* are compressions while the regions marked *P, Q* and *R* are rarefactions. A plot of air pressure along this sound wave at a given instant is also shown. The pressure at the horizontal line in this graph is the average atmospheric pressure. The compressions and rarefactions cause only slight variations in air pressure. Even for very loud sounds, the pressure variations are only about 0.01% of atmospheric pressure.

Audible and inaudible waves. When we wave our hands, we produce compressions and rarefactions in air. However, no audible sounds are produced because the frequency of such

vibrations is too low (< 20 Hz) to affect our auditory nerves. Likewise if the frequency of sound is high (> 20 kHz), no sound is heard by the human ear. It is because the vibrations are so rapid that auditory nerves do not respond to them.

(*i*) **Audible waves.** *Audible waves are sound waves that human ear can hear.* The range of human hearing is 20 Hz to 20 kHz. In other words, we cannot hear waves of frequency below 20 Hz or above 20 kHz. The audible waves can be generated in a variety of ways such as by musical instruments, human vocal cords and loudspeakers.

(*ii*) **Inaudible waves.** *Those waves which human ear cannot hear are called inaudible waves.* There are two types of inaudible waves *viz.*, **infrasonics** and **ultrasonics.** Infrasonics are longitudinal waves with frequencies below 20 Hz. Earthquake waves are an example. Ultrasonic waves are longitudinal waves with frequencies above 20 kHz. For example, they can be generated by inducing vibrations in a quartz crystal with an applied alternating electric field.

26.12. NEWTON'S FORMULA FOR VELOCITY OF SOUND

The velocity of sound in a gas (*e.g.*, air) is given by ;

$$v = \sqrt{\frac{K}{\rho}}$$

where K is the bulk modulus of the gas and ρ is its density.

Newton assumed that when sound waves travel through a gas, the compressions and rarefactions are formed so slow that the temperature of the medium remains the same. At the regions of compression where heat is produced, it is readily lost to the surroundings. Likewise, at regions of rarefaction where cooling is produced, heat is gained from the surroundings. Thus according to Newton, the propagation of sound waves in a gas takes place under *isothermal conditions* and the appropriate modulus of elasticity is the isothermal bulk modulus (K_{iso}).

$$\therefore \quad v = \sqrt{\frac{K_{iso}}{\rho}} \qquad ...(i)$$

Recall that bulk modulus is defined as the ratio of change in pressure (dP) to the resulting fractional change in volume ($-dV/V$).

$$\therefore \quad K_{iso} = -\frac{dP}{dV/V}$$

Note that K_{iso} is always positive since an increase in pressure (positive dP) results in a decrease in volume. Hence the ratio dP/dV is always negative.

At constant temperature, we have,

$$PV = \text{Constant}$$

Differentiating both sides, we have,

$$P\,dV + V\,dP = 0$$

or

$$P = -\frac{dP}{dV/V} = K_{iso}$$

Therefore, according to Newton, the speed of sound in a gas is given by ;

$$v = \sqrt{\frac{P}{\rho}} \qquad ...(ii)$$

Velocity of sound in air by Newton's formula. Let us find the speed of sound in air by applying Newton's formula. We know that for air at N.T.P.,

$P = 0.76 \times 13600 \times 9.8$ N m^{-2} ; $\rho = 1.293$ kg m^{-3}

$\therefore$ Speed of sound in air at N.T.P. is

$$v = \sqrt{\frac{0.76\times13600\times9.8}{1.293}} = 280 \text{ ms}^{-1}$$

This value is about 16% less than the experimental value (332 m s^{-1}) for the speed of sound in air at N.T.P. This large difference in the two values (experimental and theoretical) cannot be attributed to any experimental error. In 1817, Pierre Laplace, a French scientist, explained the reason for this discrepancy.

26.13. LAPLACE'S CORRECTION

Laplace pointed out that Newton was wrong in assuming that pressure variations (*i.e.* compressions and rarefactions) in a gas (*e.g.* air) during the passage of sound waves take place isothermally. According to Laplace, the compressions and rarefactions are formed so rapidly that *neither the heat is transferred to the surroundings during compression nor heat is taken from the surroundings during rarefaction. In other words, the propagation of sound waves in a gas takes place under *adiabatic conditions.* Therefore, the appropriate modulus of elasticity is the *adiabatic bulk modulus* (K_{adi}). Thus according to Laplace, the speed of sound in a gas is given by ;

$$v = \sqrt{\frac{K_{adi}}{\rho}} \qquad ...(i)$$

The pressure and volume of a gas during an adiabatic process are related by the expression :

$$PV^{\gamma} = \text{Constant} \qquad ...(ii)$$

where γ is a constant equal to the ratio of the specific heat at constant pressure to the specific heat at constant volume (*i.e.* $\gamma = C_p / C_v$). We can use eq. (*ii*) to obtain an expression for the adiabatic bulk modulus (K_{adi}). Differentiating both sides of eq. (*ii*), we have,

$$\gamma(PV^{\gamma-1})dV + V^{\gamma}dP = 0$$

or
$$dP = -\frac{\gamma P\, dV}{V}$$

Now
$$K_{adi} = -\frac{dP}{dV/V} = \gamma P$$

Therefore, the speed of sound in a gas is given by ;

$$v = \sqrt{\frac{\gamma P}{\rho}} \qquad ...(iii)$$

The value of γ depends upon the nature of gas. For air, the value of γ is 1.4.

$\therefore$ Speed of sound in air at N.T.P. is

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{(1.4)\times0.76\times13600\times9.8}{1.293}} = 331.2 \text{ ms}^{-1}$$

This result closely agrees with the experimental value.

26.14. FACTORS AFFECTING VELOCITY OF SOUND IN A GAS

The velocity v of sound in a gas is given by ;

$$v = \sqrt{\frac{\gamma P}{\rho}} \qquad ..(i)$$

* Most gases are poor heat conductors. Therefore, when a sound wave propagates through a gas, very little heat is transferred between compressions and rarefactions.

(*i*) **Effect of density.** If the value of γ for any two gases be the same, the speed of sound be v_1 and v_2 at the same temperature and pressure and the densities of the gases be ρ_1 and ρ_2 at this temperature and pressure, then,

$$v_1 = \sqrt{\frac{\gamma P}{\rho_1}} \quad \text{and} \quad v_2 = \sqrt{\frac{\gamma P}{\rho_2}}$$

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{\rho_1}{\rho_2}}$$

If the molecular weights of the gases be M_1 and M_2 respectively, then,

$$v_1 = \sqrt{\frac{\gamma RT}{M_1}} \quad \text{and} \quad v_2 = \sqrt{\frac{\gamma RT}{M_2}}$$

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{M_1}{M_2}}$$

Thus the speed of sound in a gas is inversely proportional to the square root of the density or the molecular weight of the gas.

For example, density of oxygen is 16 times that of hydrogen. Therefore, for the same temperature and pressure, the velocity of sound in hydrogen is 4 times than in oxygen.

$$\frac{v_H}{v_O} = \sqrt{\frac{\rho_O}{\rho_H}} = \sqrt{\frac{16}{1}} = 4 \quad \therefore \quad v_H = 4v_O$$

(*ii*) **Effect of temperature.** For one mole of a gas, $P\,V = R\,T$. If M is the molecular weight of the gas, then,

$$\rho = \frac{M}{V} \quad \text{or} \quad V = \frac{M}{\rho}$$

Substituting the value of V in the gas equation, we have,

$$P\frac{M}{\rho} = RT \quad \text{or} \quad \frac{P}{\rho} = \frac{RT}{M}$$

Now $$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}} \qquad \left(\because \frac{P}{\rho} = \frac{RT}{M}\right)$$

$$\therefore \quad v \propto \sqrt{T}$$

Hence the velocity of sound in a gas is directly proportional to the square root of its absolute temperature.

Thus if v_1 and v_2 are the velocities of sound in a gas at absolute temperatures T_1 and T_2 respectively, then,

$$\frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$$

Note. The velocity of sound in a gas is given by ;

$$v = \sqrt{\frac{\gamma\, RT}{M}}$$

It is interesting to compare this result with the r.m.s. speed of the molecules of a gas *viz.*,

$$v_{r.m.s.} = \sqrt{\frac{3RT}{M}}$$

The two results differ only by the factors γ and 3. It is known that γ lies between 1 and 1.67; hence the two speeds are nearly the same !

(*iii*) **Effect of pressure.** The velocity of sound in a gas is independent of gas pressure provided the temperature remains constant. For one mole of a gas,

$$P\,V = RT \text{ or } P = \frac{RT}{V}$$

$$\therefore \quad v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{\rho V}} = \sqrt{\frac{\gamma RT}{M}}$$

where $M = \rho\, V$ = Molecular mass of the gas

or $$v = \sqrt{\frac{\gamma RT}{M}}$$

For a given gas, γ, R and M are constants. Therefore, if temperature is constant, v is constant.

Alternatively : $P\,V$ = constant ... at constant temperature

or $$P\,\frac{M}{\rho} = \text{constant}$$

or $$\frac{P}{\rho} = \frac{\text{constant}}{M} = \text{constant}$$

Thus if the temperature of a gas remains constant, a change in pressure of the gas changes its density in the same ratio so that P/ρ remains unchanged. Consequently, *the velocity of sound is independent of the pressure of the gas provided temperature remains constant.*

(*iv*) **Effect of humidity.** Since the density of water vapour is less than that of dry air, the presence of moisture tends to decrease the density of the atmosphere. Hence the velocity of sound in moist air is greater than in dry air. Suppose

ρ_m = Density of moist air ; ρ_d = Density of dry air

v_m = Velocity of sound in moist air ; v_d = Velocity of sound in dry air

*Assuming that value γ for the moist air is the same as for dry air,

$$v_m = \sqrt{\frac{\gamma P}{\rho_m}} \quad \text{and} \quad v_d = \sqrt{\frac{\gamma P}{\rho_d}}$$

$$\therefore \quad \frac{v_m}{v_d} = \sqrt{\frac{\rho_d}{\rho_m}}$$

Since $\rho_m < \rho_d$, $v_m > v_d$

It is for this reason that the sirens and the whistle of trains are heard upto longer distances in the rainy season as compared in summer.

(*v*) **Effect of frequency.** There is no effect of frequency on the speed of sound in a medium. Sound waves of different frequencies travel with the same speed in air although their wavelengths in air are different. If the speed of sound were dependent on the frequency, then we could not have enjoyed orchestra.

(*vi*) **Effect of wind.** If air is blowing, the speed of sound changes. If the wind blows in the direction in which the sound travels, the velocity of sound increases. If the wind blows in the opposite direction, the velocity of sound decreases. Therefore, net velocity of sound becomes $(v \pm v_w)$ where v is the velocity of sound and v_w is the velocity of the wind. The positive sign is taken when both are in the same direction and negative sign is taken when they are in opposite direction.

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* In actual practice, the value of γ for moist air is slightly less than that for dry air but this difference is generally negligible.

Note. If the velocity of wind v_w is at an angle θ to the direction of sound, the component $v_w \cos\theta$ will be in the direction of sound. Therefore, net velocity of sound = $v + v_w \cos\theta$.

Example 26.9. The planet Jupiter has an atmosphere composed mainly of methane at a temperature of –130°C. Find the velocity of sound on the planet, assuming $\gamma_{gas} = 1.30$. Given $R = 8.36$ J mol^{-1} K^{-1}.

Solution. The velocity of sound in a gas is given by ;

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Here $\gamma = 1.30$; $R = 8.36$ J mol^{-} K^{-1} = 8.36×10^3 J k mol^{-1} K^{-1}

$T = -130 + 273 = 143$ K ; M = Molecular weight of methane (CH_4) = 12 + 4 = 16

$$\therefore \quad v = \sqrt{\frac{1.30\times8.36\times10^3\times143}{16}} = 311.6 \text{ ms}^{-1}$$

Example 26.10. At what temperature will the velocity of sound in air be double than the velocity in air at 0°C?

Solution. Let t°C be the required temperature.

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{273+t}{273}}$$

Here $v_2/v_1 = 2$ so that, we have, $\frac{273+t}{273} = 4 \quad \therefore \quad t = 273 \times 4 - 273 = 819$°C

Example 26.11. Show that the velocity of sound in a gas is given by ;

$$v_t = v_0 + \frac{v_0}{546}t$$

where v_0 is the velocity at 0°C and t is the temperature of the gas in °C. Assume $t << 273$.

Solution. The velocity of sound in a gas at temperature t is

$$v_t = \sqrt{\frac{\gamma RT}{M}} \qquad ...(i)$$

Now $$T = 273 + t = 273\left(1+\frac{t}{273}\right)$$

In our case $t/273 << 1$ so that :

$$\sqrt{T} = \sqrt{273\left(1+\frac{t}{273}\right)} \simeq \left(1+\frac{t}{2\times273}\right)\sqrt{273}$$

Putting the value of $\sqrt{T}$ in eq. (i), we have,

$$v_t = \sqrt{\frac{\gamma}{M}R(273)} \times\left(1+\frac{t}{546}\right)$$

But the quantity under the square root symbol is the velocity at 0°C (*i.e.* v_0).

$$\therefore \quad v_t = v_0\times\left(1+\frac{t}{546}\right) = v_0 + \frac{v_0}{546}t$$

Note. $$v_t = v_0 + \frac{v_0}{546}t \quad \text{or} \quad \frac{v_t - v_0}{t} = \frac{v_0}{546}$$

The quantity $(v_t - v_0)/t$ is called *temperature coefficient* of velocity of sound in gas (α_{gas}).

$$\therefore \quad \alpha_{gas} = \frac{v_0}{546} \text{ ms}^{-1} \text{ °C}^{-1}$$

If the gas is air, then, $\alpha_{air} = \frac{v_0}{546} = \frac{332}{546} = 0.61$ ms^{-1} °C^{-1}

Thus the velocity of sound in air increases by 0.61 ms^{-1} for every 1°C rise in temperature.

Example 26.12. A sound wave travels in helium. The equilibrium pressure of the gas is 1.2×10^5 Pa and its temperature is 310 K. Determine the speed of sound. Molecular weight of helium is 0.0040 kg/mol and $\gamma = 1.63$.

Solution. The speed of sound in a gas is given by ; $v = \sqrt{\frac{\gamma RT}{M}}$

Here $\gamma = 1.63$; $R = 8.31$ J mol^{-1}K^{-1}; $T = 310$ K ; $M = 0.0040$ kg/mol

$$\therefore \quad v = \sqrt{\frac{(1.63)\times(8.31)\times 310}{0.0040}} = \mathbf{1024\ ms^{-1}}$$

Example 26.13. At normal temperature and pressure, the speed of sound in air is 332 ms^{-1}. What will be the speed of sound in hydrogen (*i*) at normal temperature and pressure (*ii*) at 819°C and 4 atmospheric pressure? Air is 16 times heavier than hydrogen.

Solution. (*i*) If at normal temperature and pressure, the speed of sound in air be v_a and in hydrogen be v_h, then,

$$\frac{v_h}{v_a} = \sqrt{\frac{\rho_a}{\rho_h}} = \sqrt{\frac{16}{1}} = 4 \quad \therefore \ v_h = 4\ v_a = 4 \times 332 = \mathbf{1328\ ms^{-1}}$$

(*ii*) On changing the pressure of the gas, the speed of the sound remains the same. However, due to change in temperature of the gas, the speed of sound changes. If the speed of sound in hydrogen at normal temperature (0°C) is v_0, then at 819°C, it will be :

$$\frac{v_{819}}{v_0} = \sqrt{\frac{273+819}{273+0}} = \sqrt{\frac{1092}{273}} = \sqrt{4} = 2$$

$$\therefore \quad v_{819} = 2\ v_0 = 2 \times 1328 = \mathbf{2656\ ms^{-1}}$$

Example 26.14. Densities of oxygen and nitrogen are in the ratio 16 : 14. At what temperature the speed of sound in oxygen will be the same as at 15°C in nitrogen?

Solution. If the molecular weight of a gas be M and the absolute temperature be T, then speed of sound in the gas is given by ;

$$v = \sqrt{\frac{\gamma RT}{M}}$$ where R is the universal gas constant.

Suppose the speed of sound in oxygen at t°C is the same as in nitrogen at 15°C.

Speed of sound in oxygen at t°C $= \sqrt{\frac{\gamma R(273+t)}{M_O}}$

Speed of sound in nitrogen at 15°C $= \sqrt{\frac{\gamma R(273+15)}{M_N}}$

The two speeds must be equal. Note that γ is the same for the two gases.

$$\therefore \quad \sqrt{\frac{\gamma R(273+t)}{M_O}} = \sqrt{\frac{\gamma R(273+15)}{M_N}}$$

or $$\frac{M_O}{M_N} = \frac{273+t}{288}$$

or $$\frac{16}{14} = \frac{273+t}{288} \quad \therefore \ t = \mathbf{56.1°C} \qquad \left(\because \frac{M_O}{M_N} = \frac{16}{14}\right)$$

Example 26.15. A tuning fork of frequency 220 Hz produces sound waves of wavelength 1.5 m in air at N.T.P. Calculate the increase in wavelength when the temperature of air is 27°C.

Solution. Frequency of tuning fork, f = 220 Hz

Wavelength of sound waves at N.T.P., λ_0 = 1.5 m

$\therefore$ Velocity of sound waves at N.T.P., $v_0 = f\lambda_0 = 220 \times 1.5 = 330\ ms^{-1}$

Let λ_{27} and v_{27} be the wavelength and velocity of sound at 27°C respectively. Then,

$$\frac{v_{27}}{v_0} = \sqrt{\frac{273+27}{273}} = \sqrt{\frac{300}{273}}$$

or $$v_{27} = v_0\sqrt{\frac{300}{273}} = 330\times\sqrt{\frac{300}{273}}$$

Now $$v_{27} = f\lambda_{27} \quad \text{or} \quad 330\times\sqrt{\frac{300}{273}} = 220\times\lambda_{27}$$

$\therefore$ $$\lambda_{27} = \frac{330}{220}\times\sqrt{\frac{300}{273}} = 1.57\ m$$

$\therefore$ Increase in wavelength = $\lambda_{27} - \lambda_0 = 1.57 - 1.5$ = **0.07 m**

Example 26.16. The speed of sound in dry air at N.T.P. is 332 ms^{-1}. Assume air as composed of 4 parts of nitrogen and one part of oxygen, calculate the velocity of sound in oxygen under similar conditions when the densities of oxygen and nitrogen at N.T.P. are in the ratio 16 : 14.

Solution. Density of mixture, $\rho_{mix} = \dfrac{\text{Total mass}}{\text{Total volume}} = \dfrac{4V\times\rho_N + V\times\rho_O}{4V+V}$

$$= \frac{4\rho_N + \rho_O}{5} = \frac{\rho_O\left(4\frac{\rho_N}{\rho_O}+1\right)}{5} = \frac{\rho_O\left(4\times\frac{14}{16}+1\right)}{5}$$

$$= \frac{9}{10}\rho_O = 0.9\rho_O$$

$\therefore$ $$\frac{v_O}{v_{mix}} = \sqrt{\frac{\rho_{mix}}{\rho_O}} = \sqrt{\frac{0.9\rho_O}{\rho_O}} = \sqrt{0.9} = 0.9487$$

or $$v_O = 0.9487 \times v_{mix} = 0.9487 \times 332 = \mathbf{314.77\ ms^{-1}}$$

PROBLEMS FOR PRACTICE

1. At what temperature will the speed of sound in air become twice the speed of sound at 27°C? **[927°C]**
2. Calculate the increase in the velocity produced by 1°C rise of temperature of air if the velocity of sound in air at 0°C is 332 ms^{-1}. **[0.61 ms^{-1}]**
3. Calculate the temperature at which the sound travels in hydrogen with the same velocity as in helium at N.T.P. The density of helium is twice that of hydrogen. **[–136.5°C]**
4. If the speed of sound in helium at 0°C be 960 m s^{-1}, then what will be the speed of sound in hydrogen at the same temperature? The values of γ for He and H_2 are respectively 1.67 and 1.40 and the ratio of their molecular weights is 2 : 1. **[1243 ms^{-1}]**
5. Calculate the speed of sound in air at atmospheric pressure taking $P = 1.01 \times 10^5$ N m^{-2}, γ = 1.40 and ρ = 1.2 kg m^{-3}. **[343 ms^{-1}]**
6. The velocity of sound in air at 0°C is 332 m s^{-1}. Calculate the velocity when the temperature rises to 91°C and its pressure is doubled. **[383.4 ms^{-1}]**

26.15. WAVE FUNCTION

So far we have given only a verbal and graphical description of a travelling (progressive) wave. Let us now give a mathematical description of a travelling wave. We know that when a quantity y depends on another quantity x; y is said to be a function of x and is written as $y = f(x)$. For example, when a point mass moves along x-axis, its position x depends only on time t. To describe the motion of the particle, we need a function of one variable such as $x(t)$. However, for the case of motion of a wave, we need functions dependent on two variables *e.g.* x and t.

The functions involving x and t which describe mathematically the motion of a wave pulse are called **wave functions.**

General form of a wave function. Consider a wave pulse travelling to the right on a long stretched string with a constant speed v as shown in Fig. 26.9. The pulse moves along the x-axis (the axis of the string) and transverse displacement of the string is measured with the coordinate y.

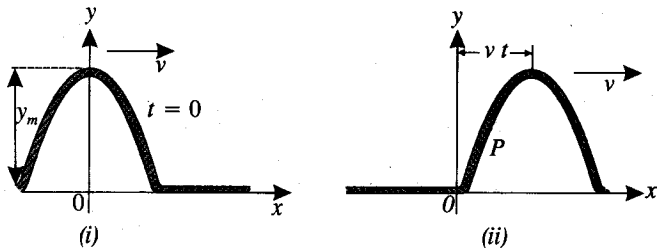


Fig. 26.9

Fig. 26.9 (*i*) represents the shape and position of the wave pulse at time $t = 0$. At this time, the shape of the pulse can be represented as $y = f(x)$. That is y is some definite function of x. The maximum displacement y_m is the *amplitude* of the wave. Since the speed of the wave pulse is v, it travels to the right a distance $v\,t$ in a time t [See Fig. 26.9 (*ii*)]. If the shape of the wave does not change with time, we can represent the displacement y for all later times measured in a stationary frame with origin at 0 as :

$$y = f(x - v\,t) \quad \text{..... wave travelling to the right}$$

Similarly, if the wave pulse travels to the *left,* its displacement y is given by :

$$y = f(x + v\,t) \quad \text{... wave travelling to the left}$$

The general form of the wave function is :

$$y = f(x \pm v\,t)$$

The function $y = f(x \pm v\,t)$ is called the *wave function.* Note that displacement y of the string depends on two variables x and t. For this reason, it is often written as $y(x, t)$ which is read "y as a function of x and t". Hence the wave function is a function of two variables.

It is important to understand the meaning of y. Consider a particular point P on the string identified by a particular value of its coordinates [See Fig. 26.9 (*ii*)]. As the wave passes the point P, the y coordinate of this point will increase, reach a maximum value and then decrease to zero. Therefore, *the wave function y represents the y coordinate of any point P at any time t.*

26.16. PERIODIC WAVE FUNCTION

A periodic wave function represents a periodic wave *i.e.* a wave which repeats itself after a fixed time interval. Consider that in a stretched string, identical wave pulses are produced by giving jerks at the free end O at regular intervals of time T (Time period as shown in Fig. 26.10).

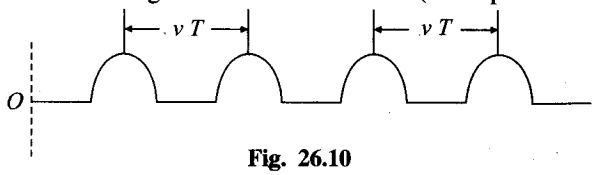


Fig. 26.10

If v is the velocity of the wave pulse along the string, then distance between any two successive wave pulses will be $= vT = \lambda$ where λ is the *wavelength* of the periodic wave and T is the time period. The general form of a wave function is :

$$y\ (x,\ t) = f\ (x \pm v\ t) \qquad ...(i)$$

In a periodic wave, the displacement repeats itself after a fixed periodic length λ *i.e.*

$$y\ [(x + n\lambda),\ t] = y\ (x,\ t) \qquad ...(ii)$$

where n is an integer.

Also, in a periodic wave, the displacement repeats itself after a fixed time period T *i.e.*

$$y\ [x,\ (t + mT)] = y\ (x,\ t) \qquad ...(iii)$$

where m is an integer.

If the wave function of eq. (*i*), satisfies the periodicity conditions of eq. (*ii*) and (*iii*), it would represent periodic wave function.

26.17. HARMONIC WAVE FUNCTION

Harmonic wave functions represent harmonic wave. An important case is *simple harmonic wave,* so called because the wave medium executes simple harmonic motion about its equilibrium position. A simple harmonic wave is sinusoidal (sine or cosine) in shape. Therefore, in general form of wave function we replace f by sine/cosine and argument $(v\,t - x)$ by $\frac{2\pi}{\lambda}(vt - x)$. Note that we choose argument of sine/cosine function so that it becomes 2π when $x = \lambda$. Therefore, the general harmonic wave function can be represented as :

$$y(x,\ t) = a \sin\left[\frac{2\pi}{\lambda}(vt - x) + \phi_0\right] \qquad ...(i)$$

or

$$y(x,\ t) = a \cos\left[\frac{2\pi}{\lambda}(vt - x) + \phi_0\right] \qquad ...(ii)$$

Here y = Displacement of the particle from mean position at time t

a = Maximum displacement of the particle from the mean position

v = Velocity of the wave

ϕ_0 = Initial phase of the wave

26.18. HARMONIC WAVE FUNCTION IS PERIODIC IN x AND t

(*i*) To prove that harmonic wave function represented by eq. (*i*) above is periodic in x, we calculate y when x is replaced by $(x + \lambda)$.

$$\therefore \quad y(x+\lambda,\ t) = a \sin\left[\frac{2\pi}{\lambda}\{vt - (x+\lambda)\} + \phi_0\right]$$

$$= a \sin\left[\frac{2\pi}{\lambda}(vt - x) - 2\pi + \phi_0\right]$$

$$= a \sin\left[\left\{\frac{2\pi}{\lambda}(vt - x) + \phi_0\right\} - 2\pi\right]$$

Since $\sin(\theta \pm 2\pi) = \sin\theta$

$$\therefore \quad y(x+\lambda,\ t) = a \sin\left[\frac{2\pi}{\lambda}(vt - x) + \phi_0\right] = y(x,t)$$

Therefore, the harmonic wave function represented by eq. (*i*) is periodic in x.

(*ii*) To prove that harmonic wave function represented by eq. (*i*) above is periodic in *t*, we calculate *y* when *t* is replaced by (*t* + *T*).

$$\therefore \quad y(x, t+T) = a \sin\left[\frac{2\pi}{\lambda}\{v(t+T)-x\}+\phi_0\right]$$

$$= a \sin\left[\left\{\frac{2\pi}{\lambda}(vt-x)+\phi_0\right\}+\frac{2\pi}{\lambda}vt\right]$$

$$= a \sin\left[\left\{\frac{2\pi}{\lambda}(vt-x)+\phi_0\right\}+2\pi\right] \qquad (\because vT=\lambda)$$

$$= a \sin\left[\frac{2\pi}{\lambda}(vt-x)+\phi_0\right] = y(x, t)$$

Therefore, the harmonic wave function represented by eq. (*i*) is periodic in *t*.

26.19. EQUATION OF SIMPLE HARMONIC TRAVELLING WAVE

A progressive or travelling wave is one which travels onwards through a given medium in a given direction without attenuation *i.e.* with constant amplitude. A travelling or progressive wave may be transverse or longitudinal. We shall assume the wave shape to be sinusoidal—one with the shape of a sine or cosine. The sinusoidal wave is the most important type of travelling wave for two principal reasons : (*i*) a sinusoidal wave is easily produced ; simply oscillate one part of the elastic medium in simple harmonic motion (*ii*) any wave shape can be regarded as the superposition of strictly sinusoidal waves.

Suppose a plane simple harmonic wave starts from the origin *O* and travels with velocity *v* along the positive direction of *X*-axis (*i.e.* from left to right). Fig. 26.11 shows the displacement curve of the wave at any time *t*. Let the time be counted from the instant when the particle at the origin *O* just passes through its mean position in the positive direction (*i.e.* upwards). The displacement *y* of the particle at *O* at any time *t* is given by ;

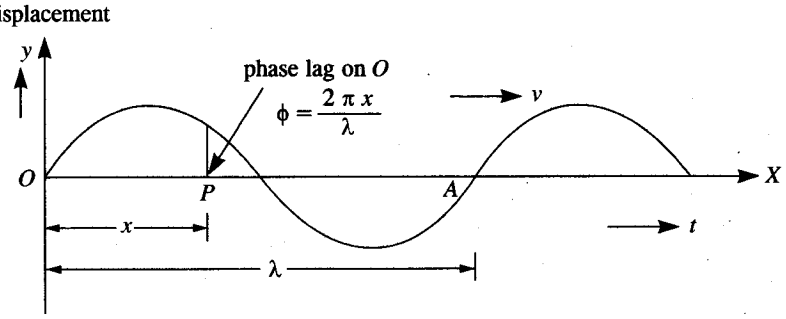


Fig. 26.11

$$y = a \sin \omega t$$

where a = Amplitude of S.H.M.

ω = Angular frequency of S.H.M.

Since the particles to the right of *O* start vibrating after definite time intervals from the instant particle at *O* starts vibrating, the phase lag goes on increasing in this direction. Let us find the displacement of the particle at point *P* situated at a distance *x* from the origin at any time *t*. If ϕ is the phase lag of the particle at *P w.r.t.* the particle at the origin *O*, then displacement of this particle at *P* at the same time *t* can be written as :

$$y = a \sin (\omega t - \phi) \qquad ...(i)$$

We can easily find the phase lag ϕ. A distance λ (wavelength of the wave) from O corresponds to a phase difference of 2π radians. Therefore, phase lag ϕ at P is given by ;

$$\phi = \frac{2\pi}{\lambda} \times x = \frac{2\pi x}{\lambda}$$

$\therefore$ Eq. (*i*) becomes : $y = a \sin\left(\omega t - \frac{2\pi x}{\lambda}\right)$ ...(*ii*)

Now $\omega = \frac{2\pi}{T}$ where T is the time period of vibrations.

$\therefore$ $y = a \sin\left(\frac{2\pi}{T} t - \frac{2\pi x}{\lambda}\right)$

or $y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$ ..(*iii*)

or $y = a \sin \frac{2\pi}{\lambda}\left(\frac{\lambda}{T} t - x\right)$

But $\frac{\lambda}{T} = v$ = velocity of the wave

$\therefore$ $y = a \sin \frac{2\pi}{\lambda}(vt - x)$ ...(*iv*)

Equation (*iv*) or (*iii*) represents a plane progressive simple harmonic wave travelling along the positive direction of X-axis. The negative sign in the bracket indicates that since the wave moves from left to right, the vibration at points such as P to the right of O will lag on that at O. A wave travelling in the *opposite direction* (*i.e.* along the negative direction of X-axis), from right to left, arrives at P before O. Thus the vibration at P leads that at O. Consequently, the wave travelling in the opposite direction is given by ;

$$y = a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) \quad ...(v)$$

Also $y = a \sin \frac{2\pi}{\lambda}(v t + x)$ ...(*vi*)

Discussion. The following points may be noted about the equation representing a travelling wave :

(*i*) The displacement y, sometimes called *wave function* depends on the two variables x and t. For this reason, it is often written as y (x, t) which is read "y as a function of x and t".

(*ii*) $y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$ ...wave travelling in + X direction

This form of wave function clearly shows the *periodic* nature of y. That is, at any given time t, y has the *same* value at the positions x, $x + \lambda$, $x + 2\lambda$ etc. Furthermore, at any given position x, y has the same value at times t, $t + T$, $t + 2T$ etc.

(*iii*) We can express the harmonic wave function in a convenient form by defining the quantity called *wave number* k (or propagation constant).

Wave number, $k = \frac{2\pi}{\lambda}$ = Coefficient of x in wave equation

Note that k gives the rate of change of phase with distance.

Now $y = a \sin\left(\frac{2\pi t}{T} - \frac{2\pi}{\lambda} x\right)$

or $y = a \sin(\omega t - kx)$ $\left(\because \frac{2\pi t}{T} = 2\pi f t = \omega t\right)$

This is a very useful form.

(*iv*) If ϕ_0 is the initial phase of the wave, then,

$$y = a \sin\left[\frac{2\pi}{\lambda}(vt - x) + \phi_0\right]$$ ...wave travelling in + X direction

(*v*) We can also use cosine function for the equation of a travelling wave. But we choose the sine function so that $y = 0$ when $x = 0$ and $t = 0$.

(*vi*) The acceleration of the wave is zero because speed of wave v $(= f\lambda)$ is constant.

26.20. PARTICLE VELOCITY AND ACCELERATION

(*i*) **Particle velocity.** The particle velocity (V) is different from the wave velocity (v). It is the velocity with which the particles of the medium vibrate about their mean positions. The equation of a plane progressive wave travelling in the + X direction is

$$y = a \sin(\omega t - kx) \quad ...(i)$$

This is the displacement at position x in time t. Differentiating eq. (*i*) *w.r.t.* time (t) keeping x constant, we get,

$$V = \frac{dy}{dt} = \frac{d}{dt}[a \sin(\omega t - kx)]$$

or $$V = \omega a \cos(\omega t - k x) .$$

or $$V = V_{max} \cos(\omega t - k x) \quad ...(ii)$$

where V_{max} = Maximum particle velocity = $\omega a = \frac{2\pi}{T} \times a$

$\therefore$ $$V_{max} = \frac{2\pi a}{T}$$

Note that particle velocity varies simple harmonically with time t. However, wave travels with a constant velocity in a medium. The following points may be noted :

(*a*) The particle velocity (V) is a varying quantity whereas the wave velocity (v) is a constant for a medium.

(*b*) The velocity amplitude is $2\pi/T$ times the displacement amplitude (a).

(*c*) It is clear that graph of V against x is 90° out of phase with the graph of y against x.

(*ii*) **Particle acceleration**

$$A = \frac{dV}{dt} = \frac{d}{dt}[\omega a \cos(\omega t - kx)]$$

$$= \omega a[-\sin(\omega t - kx)]\,\omega$$

$\therefore$ $$A = -\omega^2 a \sin(\omega t - kx)$$

or $$A = -A_{max} \sin(\omega t - k x) \quad ...(iii)$$

where A_{max} = acceleration amplitude = $\omega^2 a = \frac{4\pi^2}{T^2} a$

The negative sign in eq. (*iii*) shows that the acceleration is directed toward the mean position.

26.21. PHASE AND PHASE DIFFERENCE

The equation of a simple harmonic progressive wave travelling in the + X direction and having an initial phase ϕ_0 is given by ;

$$y = a \sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \phi_0\right]$$

The argument of sine function is called the phase ϕ. Thus the phase of the wave at position x and time t is given by ;

$$\phi = 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \phi_0$$

It is clear that the phase of a wave is a function of both x and t.

(*i*) **Variation of phase with time.** At a given position (*i.e.* for the fixed value of x), the phase changes periodically with time. For a given position, let ϕ_1 and ϕ_2 be the phases of a particle at times t_1 and t_2 respectively. Then,

$$\phi_1 = 2\pi\left(\frac{t_1}{T} - \frac{x}{\lambda}\right) + \phi_0$$

and

$$\phi_2 = 2\pi\left(\frac{t_2}{T} - \frac{x}{\lambda}\right) + \phi_0$$

$$\therefore \quad \phi_2 - \phi_1 = 2\pi\left(\frac{t_2}{T} - \frac{t_1}{T}\right) = \frac{2\pi}{T}(t_2 - t_1)$$

or

$$\Delta\phi = \frac{2\pi}{T}\Delta t$$

When $\Delta t = T$ (time period), $\Delta\phi = 2\pi$.

Note that a phase change of 2π means no change in phase. We can, therefore, define time period in another way :

It is the time in which the phase changes at a given point in the medium by 2π.

(*ii*) **Variation of phase with distance.** At a given time (*i.e.* for fixed value of t), phase changes periodically with distance x. At a given time, let ϕ_1 and ϕ_2 be the phases of the two particles at distances x_1 and x_2 from the origin respectively. Then,

$$\phi_1 = 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right) + \phi_0$$

and

$$\phi_2 = 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right) + \phi_0$$

$$\therefore \quad \phi_2 - \phi_1 = 2\pi\left(-\frac{x_2}{\lambda} + \frac{x_1}{\lambda}\right) = -\frac{2\pi}{\lambda}(x_2 - x_1)$$

or

$$\Delta\phi = -\frac{2\pi}{\lambda}\Delta x$$

The negative sign indicates the phase lag. As Δx increases, the phase lag also increases. In other words, when a particle is at larger distance from the origin, its phase lag is greater.

When $\Delta x = \lambda$ (wavelength) ; $\Delta\phi = -2\pi$

Thus wavelength may be defined as the distance between two consecutive points having a phase difference of 2π.

Example 26.17. A simple harmonic wave-train is travelling in a gas in the positive direction of *X*-axis. Its amplitude is 2 cm, velocity 45 ms^{-1} and frequency is 75 per second. Find out the displacement of the particle of the medium at a distance 135 cm from the origin in the direction of the wave at the instant t = 3 seconds.

Solution. The equation of a plane progressive wave travelling in the $+X$ direction is

$$y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad \ldots (i)$$

Here $a = 2$ cm ; $f = 75\ s^{-1}$; $v = 45\ m\ s^{-1}$; $x = 135$ cm

$$\therefore \quad T = \frac{1}{f} = \frac{1}{75} s \ ; \ \lambda = \frac{v}{f} = \frac{45}{75} = 0.6\ m = 60\ cm$$

Substituting the values of a, T, λ and x in eq. (i), we have,

$$y = 2\sin 2\pi\left(3\times 75 - \frac{135}{60}\right)$$
$$= 2\sin 2\pi(225 - 2.25)$$
$$= 2\sin(450\pi - 4.5\pi)$$
$$= 2\sin(-4.5\pi) = -2\sin\left(4\pi + \frac{\pi}{2}\right)$$
$$= -2\sin\frac{\pi}{2} = -2\ cm$$

Example 26.18. **A displacement wave is represented by ;**

$$y = 0.25 \times 10^{-3} \sin (500\ t - 0.025\ x)$$

where y, t and x are in cm, s and cm respectively. Determine (i) the amplitude (ii) the period (iii) the angular frequency (iv) the wavelength (v) amplitudes of particle velocity and acceleration.

Solution. $y = 0.25 \times 10^{-3} \sin (500\ t - 0.025\ x)$

Compare it with the general equation of wave motion :

$$y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

(i) Amplitude, $a = 0.25 \times 10^{-3}$ cm

(ii) $$\frac{2\pi}{T} = 500 \text{ or } T = \frac{\pi}{250} s$$

(iii) Angular frequency, $\omega = \frac{2\pi}{T} = 2\pi \times \frac{250}{\pi} = 500$ Hz

(iv) $$\frac{2\pi}{\lambda} = 0.025$$

$$\therefore \quad \text{Wavelength, } \lambda = \frac{2\pi}{0.025} = 80\pi\ cm = \frac{4\pi}{5} m$$

(v) Amplitude of particle velocity $= \omega\ a = 500 \times 0.25 \times 10^{-3} = 0.125\ cm\ s^{-1}$

Amplitude of particle acceleration $= \omega^2\ a = (500)^2 \times 0.25 \times 10^{-3} = 62.5\ cm\ s^{-2}$

Example 26.19. **The equation of a transverse wave travelling along a coil spring is**

$$y = 4.0 \sin \pi\ (0.010\ x - 2.0\ t)$$

where x and y are in cm and t in s. Find (i) the amplitude (ii) the wavelength (iii) the initial phase at the origin (iv) the speed and (v) the frequency of the wave.

Solution. $y = 4.0 \sin \pi\ (0.010\ x - 2.0\ t)$ ...(i)

The general equation of a travelling harmonic wave is

$$y = a\sin\left[\frac{2\pi}{\lambda}(x - vt) + \phi_0\right] \quad \ldots(ii)$$

Comparing eqs. (i) and (ii), we have,

(*i*) Amplitude, $a = 4.0 \text{ cm} = 4.0 \times 10^{-2} \text{ m}$

(*ii*) $\frac{2\pi}{\lambda} = 0.010\ \pi$ or Wavelength, $\lambda = \frac{2}{0.010} = 200 \text{ cm} = 2.0 \text{ m}$

(*iii*) Initial phase, $\phi_0 = 0°$

(*iv*) $\frac{2\pi v}{\lambda} = 2.0\ \pi \quad \therefore$ Speed, $v = \lambda = 2.0 \text{ m s}^{-1}$

(*v*) Frequency, $f = \frac{v}{\lambda} = \frac{2.0}{2.0} = 1.0 \text{ s}^{-1}$

PROBLEMS FOR PRACTICE

1. The equation for a progressive wave in a wire is given by ;

$$y = 0.15 \sin 2\pi\left(15t - \frac{x}{10}\right)$$

where the distances are in metre and time in second. Determine (*i*) amplitude (*ii*) frequency (*iii*) wavelength. **[(*i*) 0.15 m (*ii*) 15 Hz (*iii*) 10 m]**

2. A source of sound of frequency 50 Hz is producing longitudinal waves in air. The amplitude of vibrations of air particles is 5 mm and the speed of wave is 330 m s^{-1}. Find the distance-displacement equation for the wave.

$$\left[y = 0.5 \sin 2\pi\left(500t - \frac{x}{66}\right) \text{ where } y \text{ and } x \text{ are in cm}\right]$$

3. The amplitude, speed and frequency of a plane progressive wave are respectively 0.05 m, 330 m s^{-1} and 110 Hz. Find the equation of the wave. **[$y = 0.05 \sin 2\pi (110\ t - x/3)$]**

4. The equation of a simple harmonic progressive wave is given by ;

$$y = 8 \sin (0.628\ x - 12.56\ t)$$

where x and y are in cm and t in second. Calculate the amplitude, frequency and speed of the wave and the phase difference between two particles at a distance of 2.0 cm apart at any instant. **[8 cm ; 2.0 s^{-1} ; 20 cm s^{-1}; 72°]**

5. What is the amplitude, the wavelength and the velocity of the wave represented by the formula :

$$y\ (x,\ t) = 5 \sin (6\pi\ t - 4x)$$

where x, the distance and t, the time are in SI units? **[5 m ; π/2 m ; 3π/2 m s^{-1}]**

6. The equation of motion of a particle is given by ;

$$y = 1.2 \sin (3.5\ t + 0.5\ x)$$

where distances and time are respectively in metre and second. Determine the amplitude, frequency and maximum velocity of the particle. **[1.2 m ; 0.56 s^{-1} ; 4.2 m s^{-1}]**

26.22. REFLECTION OF WAVES

In order to understand the formation of stationary waves in strings and air columns, it is important to understand the reflection of transverse and longitudinal waves at the rigid and a free boundary.

1. Reflection of transverse waves. We shall consider the following two cases :

(*a*) Reflection at rigid support (denser medium). Whenever a travelling wave reaches a boundary, part or all of the wave will be reflected. For example, consider a pulse travelling on a string fixed at one end as shown in Fig. 26.12. When the pulse reaches the fixed wall, it will be reflected. Since the support attaching the string to the wall is assumed to be rigid, it does not transmit any part of the disturbance to the wall. Note that reflected pulse is inverted. This can be

explained as follows. When the pulse meets the fixed support, the string produces an upward force on the support. By Newton's third law, the support must then exert an equal and opposite (downward) reaction force on the string. This downward force causes the pulse to invert upon reflection *i.e.* a crest is reflected as a trough and *vice-versa.*

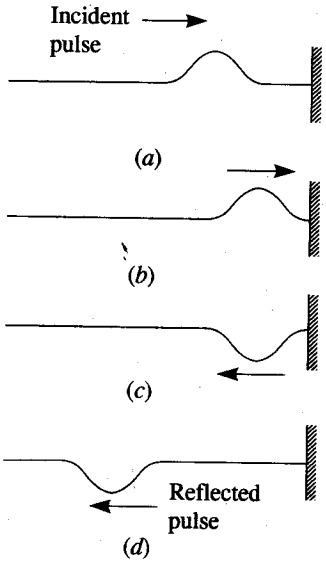


Fig. 26.12. Rigid support

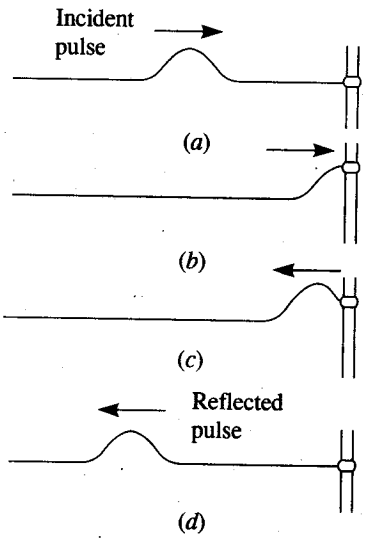


Fig. 26.13. Free boundary

Thus when a transverse wave is reflected at the rigid support (denser medium), a phase change of 180° takes place in the displacement i.e. crest is reflected as trough and vice-versa.

(*b*) **Reflection at free boundary (rarer medium).** Now consider another case where the pulse arrives at the end of a string that is free to move vertically as shown in Fig. 26.13. The tension at the free end is maintained by tying the string to a ring of negligible mass that is free to slide vertically on a smooth post. Again the pulse is reflected but this time its displacement is not inverted. As the pulse reaches the post, it exerts a force on the free end, causing the ring to accelerate upward. In the process, the ring "overshoots" the height of the incoming pulse and is then returned to its original position by the downward component of tension. Hence there is no change in phase on reflection from the free end *i.e.* crest is reflected back as crest.

Thus when a transverse wave is reflected at the free boundary (rarer medium), no phase change takes place in displacement i.e. crest is reflected as crest and vice-versa.

2. Reflection of longitudinal waves. Again we shall consider the following two cases :

(*a*) **Reflection at rigid support (denser medium).** A longitudinal wave travels in a medium in the form of compressions and rarefactions. Suppose a compression travelling from left to right reaches the rigid support. The compression exerts a force F on the rigid support. In turn, the support gives an equal and opposite reaction $R = -F$. The result is that the direction of compression is reversed *i.e.* a phase change of 180° takes place in the direction of compression.

Thus when a longitudinal wave is reflected from a rigid support (i.e., denser medium), a phase difference of 180° takes place in its direction i.e., compression is reflected back as compression and vice-versa.

(*b*) **Reflection at free boundary (rarer medium).** Suppose a compression travelling from left to right reaches the free boundary. The compression exerts a force F on the free end towards the

right. Being free, it has smaller resistance for expansion. As a result, the medium behind gets rarefied and a rarefaction travels from right to left *i.e.* a compression is reflected as a rarefaction.

Thus when a longitudinal wave is reflected at free boundary (rarer medium), a compression is reflected as a rarefaction and vice-versa.

Note. Sound waves are longitudinal waves and their reflection at rigid support (denser medium) and free boundary (rarer medium) is governed by above rules. We shall use this fact while discussing open and closed organ pipes.

26.23. SUPERPOSITION PRINCIPLE

Two or more progressive waves can travel simultaneously in a medium without affecting the motion of one another. Therefore, the resultant displacement of each particle of the medium at any instant is equal to the vector sum of the displacements produced by the two waves separately. This principle is called principle of superposition and may be stated as under :

When two or more waves pass through the same medium at the same time, the net displacement at any point is equal to the vector sum of their individual displacements at that point.

In general, if $\vec{y_1}$, $\vec{y_2}$, $\vec{y_3}$, ... $\vec{y_n}$ are the displacements at a point in a medium where n waves superpose on each other, then the resultant displacement $\vec{y}$ at that point is given by ;

$$\vec{y} = \vec{y_1} + \vec{y_2} + \vec{y_3} + ... + \vec{y_n}$$

Indeed, principle of superposition holds good for all types of waves, provided the waves are not of very large amplitude. Superposition is one of the major differences between a particle model and a wave model ; particles collide but waves superpose.

Illustration. Consider two waves A and B travelling in a medium. Suppose they meet at a point in the medium in such a way that crest of A falls on crest of B and trough falls on trough as shown in Fig. 26.14. Then according to superposition principle, the amplitude of the resultant wave R_1 is the sum of the amplitudes of the two waves A and B. This is an example of **constructive interference of waves**—the resultant displacement is greater than the displacement caused by either wave if moving alone in the medium.

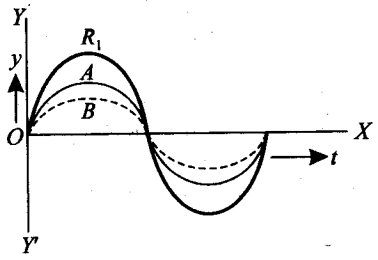


Fig. 26.14

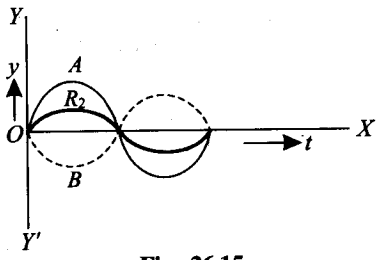


Fig. 26.15

Now consider a situation when the crest of wave A falls on the trough of wave B and trough falls on crest as shown in Fig. 26.15. In this case, the amplitude of the resultant wave R_2 is equal to the difference in the amplitudes of two waves A and B. This is an example of **destructive interference of waves**—the resultant displacement is smaller than the displacement caused by either wave if moving alone through the medium.

26.24. BEHAVIOUR OF WAVES IN A MEDIUM

The unique property of a wave is that it maintains its individuality (*i.e.* nature and shape) while travelling through a medium. Thus when two or more waves travel in a medium, each wave proceeds independently as if other waves are absent. When the waves meet at any point in the medium, the resultant displacement of the particle at that point is governed by superposition

principle *i.e.* resultant displacement of the particle at that point is equal to the vector sum of displacements caused by the individual waves. After the meeting, the waves emerge retaining original shape and travel with the same speed and in the same direction as if nothing had happened. It is only during meeting that each wave makes its own contribution to the displacement of the particle.

Illustration. Let us illustrate the above remarkable behaviour of waves in a medium. Consider two identical wave pulses P and Q of opposite signs moving towards each other along a string as shown in Fig. 26.16 (*i*). At the instant the two wave pulses superpose on each other, the resultant displacement becomes zero [See Fig. 26.16 (*ii*)] and the string becomes horizontal. It is because two equal-amplitude displacements are of opposite signs and cancel each other. After passing through the common region, the two pulses continue in the same direction with the same amplitude as though they had never met [See Fig. 26.16 (*iii*)].

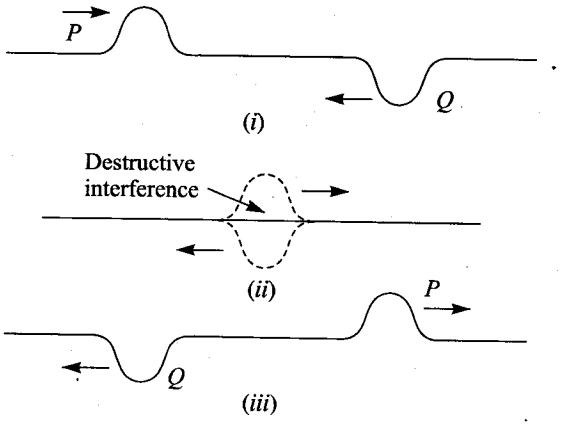


Fig. 26.16

Applications of superposition of waves. The following are the two important applications of superposition of waves :

(*i*) When two waves of the same frequency and having the same amplitude moving with the same speed in opposite directions superpose on each other, they give rise to **stationary** or **standing waves**.

(*ii*) When two waves of slightly different frequency moving with the same speed in the same direction superpose on each other, they give rise to **beats**.

We shall discuss these two applications of superposition of waves in the following sections.

26.25. STATIONARY WAVES OR STANDING WAVES

When two progressive waves of the same wavelength and amplitude travel with the same speed along the same straight line in the opposite directions, these waves interfere to produce **stationary waves** *or* **standing waves**.

In a stationary or standing wave, the amplitude varies from place to place along the wave. Fig. 26.17 shows the diagram of a stationary wave. Note that there are points where the amplitude of particles of the medium is zero. These points are called **nodes** (N). Midway between these nodes, there are points, where the amplitude of particles of the medium is maximum. These points are called **antinodes** (A). Inbetween the nodes and the antinodes, the amplitude of particles varies from zero to maximum.

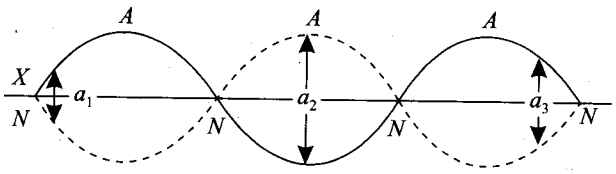


Fig. 26.17

These waves are called stationary waves because there is no transfer of energy along the wave. The reason is simple. The particles at the nodes do not move at all. Therefore, energy cannot be transmitted across them.

26.26. TYPES OF STATIONARY WAVES

Like the progressive waves, stationary waves may be (*i*) transverse stationary waves (*ii*) longitudinal stationary waves.

(*i*) **Transverse stationary waves.** The transverse stationary waves are formed due to the superposition of two identical transverse waves travelling in the opposite directions. For example, the stationary waves produced on the vibrating string of a sonometer are transverse stationary waves.

(*ii*) **Longitudinal stationary waves.** The longitudinal stationary waves are formed due to the superposition of two identical longitudinal waves travelling in the opposite directions. For example, the stationary waves produced in the organ pipes and in the air column of a resonance tube apparatus are longitudinal stationary waves.

26.27. FORMATION OF STATIONARY WAVES (GRAPHICAL METHOD)

Let us investigate how stationary waves are formed on a string fixed at both ends. Suppose a transverse wave is travelling along the string from left to right. On reaching the other end, it is reflected and travels on the string from right to left. The incident and reflected waves have the same frequency, amplitude and are travelling in opposite directions. Therefore, their superposition results in a stationary wave. The situation is shown graphically in Fig. 26.18. It shows two combining waves, one travelling to the left [26.18(*a*)] and the other travelling to the right [Fig. 26.18(*b*)]. Fig. 26.18(*c*) shows their sum obtained by applying superposition principle graphically.

(*i*) **At $t = 0$.** At $t = 0$, the two waves are in phase with each other and in accordance with superposition principle, the resultant wave will be as shown in Fig. 26.18(*c*). The points marked *N* (*node*) represent the zero displacement of the particles while points marked *A* (*antinode*) represent maximum displacement of the particles. Note that each wave has time period *T* and wavelength λ.

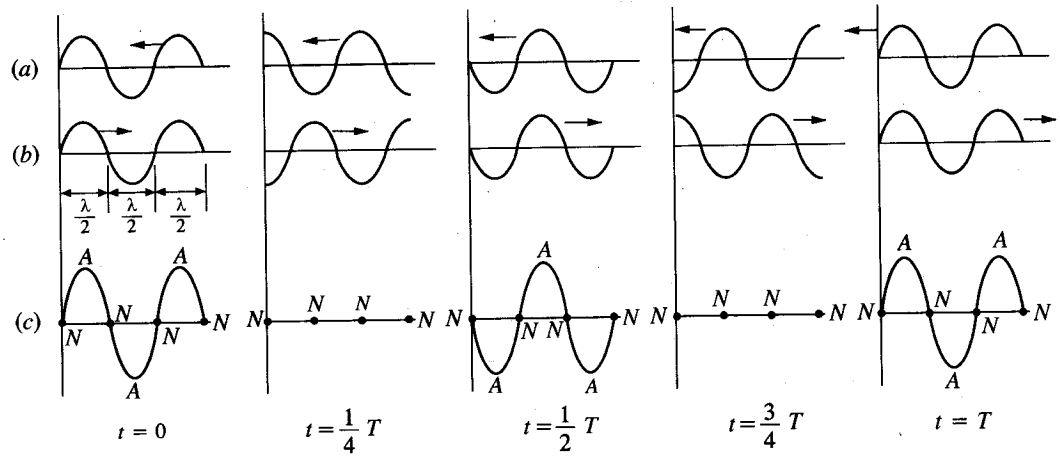


Fig. 26.18

(*ii*) **At $t = T/4$.** At $t = T/4$, one wave has travelled a distance λ/4 to the right and the other wave has travelled an equal distance λ/4 to the left on the string. At this instant, the waves are completely out of phase and (momentarily) destroy each other at every point. The resultant displacement at each point is zero and the string becomes horizontal. Thus at $t = T/4$, all the particles of the medium cross their mean positions simultaneously.

(*iii*) **At $t = T/2$.** At $t = T/2$, one wave has travelled a distance λ/2 to the right and the other wave has travelled an equal distance λ/2 to the left on the string. The waves are in phase with each other but the pattern is now inverted relative to the $t = 0$ pattern.

(*iv*) At $t = 3T/4$ and $t = T$. The reader can easily see that pattern for the resultant wave at $t = 3T/4$ and $t = T$ will be as shown.

Wave patterns shown in Fig. 26.18(*c*) are called **standing waves** because the patterns do not move *i.e.* locations of nodes and antinodes do not change. The name stationary wave is appropriate because it does not travel left or right and no energy is transferred left or right although energy is associated with the waves.

26.28. CHARACTERISTICS OF STATIONARY WAVES

The following are the important characteristics of stationary waves :

(*i*) The disturbance is confined to a particular region between the starting point and the reflecting point of the wave.

(*ii*) In the stationary waves, the disturbance does not move forward or backward. Therefore, disturbance is not transferred from one particle to other particle beyond this particular region.

(*iii*) The amplitude of vibration of different particles is different. It is maximum at the anti-nodes and decreases to zero at the nodes.

(*iv*) All the particles (except those at nodes) execute S.H.M. about their mean positions with the same time period.

(*v*) The wavelength (or frequency or time period) of the stationary wave is the same as that of the component waves forming the stationary waves.

(*vi*) The distance between adjacent nodes or adjacent antinodes in a stationary wave is $\lambda/2$.

Distance between adjacent nodes = Distance between adjacent antinodes = $\lambda/2$

It is clear that distance between a node and adjacent antinode is $\lambda/4$.

(*vii*) All the particles between two adjacent nodes are in phase *i.e.* they move in the same direction at the same time. Each particle of the next pair of adjacent nodes is out of phase with these by π radians or 180°. This fact is illustrated in Fig. 26.19.

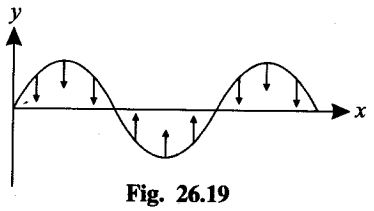


Fig. 26.19

(*viii*) The total energy associated with a stationary wave is twice the energy of each of incident and reflected wave.

(*ix*) In a stationary wave, the particles at nodes always remain at rest.

(*x*) Twice during each vibration, all the particles of the medium pass simultaneously through their mean positions.

(*xi*) The direction of motion of the particles is reversed after half a vibration.

26.29. EQUATION OF A STATIONARY WAVE

The displacement y_1 of a progressive sinusoidal wave at time t and at a distance x from the origin is given by ;

$$y_1 = a \sin(\omega t - kx)$$

where a = Amplitude of wave

ω = Angular frequency = $2\pi f$

k = Wave number = $2\pi / \lambda$

λ = Wavelength of wave

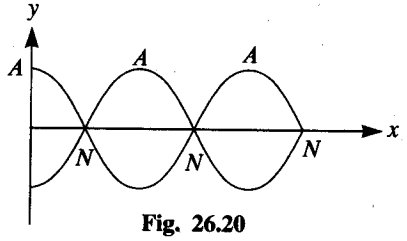


Fig. 26.20

The displacement y_2 of an identical wave travelling in the opposite direction is given by ;

$$y_2 = a\sin(\omega t + kx)$$

If these two waves are superposed, the resultant displacement y is given by ;

$$y = y_1 + y_2$$

or $$y = a\sin(\omega t - kx) + a\sin(\omega t + kx)$$

We use the trigonometric identity :

$$\sin A + \sin B = 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

in the equation above so that it becomes :

$$y = (2a\cos kx)\sin\omega t \qquad ...(i)$$

or $$y = A\sin\omega t \qquad ...(ii)$$

where A is the **amplitude** of the stationary wave.

Now $$A = 2a\cos kx = 2a\cos\frac{2\pi x}{\lambda} \qquad \left(\because k = \frac{2\pi}{\lambda}\right)$$

Eq. (*i*) or (*ii*) is the equation of a stationary wave. The following points are apparent from these equations :

(*i*) Each particle of the medium (except at nodes) vibrates in simple harmonic motion with amplitude $2a\cos kx$. The amplitudes are not equal for all the particles ; the amplitude depends on x.

(*ii*) The positions of zero amplitude are the nodes. The amplitude [$= 2a\cos kx = 2a\cos(2\pi x/\lambda)$] is always zero for those positions for which $\cos(2\pi x/\lambda)$ is zero. This implies that $2\pi x/\lambda$ is an odd multiple of $\pi/2$ radians. Therefore, nodes come at the positions :

$$\frac{2\pi x}{\lambda} = n\frac{\pi}{2} \qquad \text{where } n = 1, 3, 5...$$

Nodes : $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}....$

Note that adjacent nodes are separated by a distance of $\lambda/2$.

(*iii*) Antinodes are those positions for which the amplitude $2a\cos(2\pi x/\lambda)$ has its maximum value *i.e.* $\pm 2a$. This occurs for $\cos(2\pi x/\lambda) = \pm 1$ which in turn implies $2\pi x/\lambda$ is an integral multiple of π. Antinodes come at the positions :

$$\frac{2\pi x}{\lambda} = m\pi \qquad \text{where } m = 0, 1, 2, ...$$

Antinodes : $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}....$

Note that adjacent antinodes are separated by a distance $\lambda/2$. Further, the distance between a node and an adjacent antinode is $\lambda/4$.

(*iv*) In the equation of stationary wave, reference origin is taken at an antinode.

Note. A standing or stationary wave can also be represented as $y = (2a\sin kx)\cos\omega t = [2a\sin(2\pi x/\lambda)]\cos\omega t$. In this case, the reference origin is at a node.

Example 26.20. **Two waves travelling in opposite directions produce a standing wave. The individual wave functions are given by ;**

$$y_1 = 4\sin(3x - 2t)\text{ cm} \quad ; \quad y_2 = 4\sin(3x + 2t)\text{ cm}$$

where x and y are in cm.

(*i*) Find the maximum displacement of the motion at x = 2.3 cm.

(*ii*) Find the positions of nodes and antinodes.

Solution. When the two waves are summed up, the result is a standing wave whose wave function is given by ;

$$y = (2\ a \sin kx) \cos \omega t$$

(*i*) Here a = 4 cm ; k = 3 cm^{-1}

$$\therefore \qquad y = (2 \times 4 \sin 3\ x) \cos \omega\ t = (8 \sin 3x) \cos\omega t \text{ cm}$$

The *maximum* displacement of motion at x = 2.3 cm is given by ;

$$y_{max} = 8 \sin 3 \times 2.3 = 8 \sin (6.9 \text{ rad}) = 4.63 \text{ cm}$$

(*ii*) Nodes exist at those positions for which the amplitude 2 a sin kx is zero. This means that kx is an integral multiple of π. Therefore, nodes come at positions :

$$kx = n\pi \quad \text{where } n = 0, 1, 2, 3, \ldots$$

or $$3x = n\pi \quad \therefore \quad x = n\left(\frac{\pi}{3}\right) \text{cm}$$

Antinodes exist at those positions for which the amplitude $2a$ sin kx is maximum *i.e.* $\pm 2a$. This means that kx is an odd multiple of $\pi/2$. Therefore, antinodes come at positions:

$$kx = n\frac{\pi}{2} \quad \text{where } n = 1, 3, 5\ldots$$

or $$3x = n\frac{\pi}{2} \quad \text{or } x = n\left(\frac{\pi}{6}\right) \text{cm}$$

Example 26.21. **A standing wave on a string has nodes at x = 0 cm ; x = 4 cm ; x = 8 cm and x = 12 cm. What is the wavelength of the travelling waves that combine to make this standing wave?**

Solution. We know that the separation between two adjacent nodes of a standing wave is $\lambda/2$ where λ is the wavelengh of the travelling waves.

$\therefore$ Wavelength of travelling waves is

$$\lambda = 2 \text{ (separation between two adjacent nodes)} = 2\ (4) = 8 \text{ cm}$$

Example 26.22. **The equation of a standing wave is given by ;**

$$y\ (x, t) = 12 \cos\left(\frac{\pi x}{5}\right) \sin 20\ \pi\ t$$

where x and y are in cm and t in seconds. Find (*i*) frequency and wavelength (*ii*) velocity and amplitude of the progressive waves.

Solution. When two identical progressive waves travel in opposite directions with the same speed, then a stationary wave is formed. If the amplitude of each progressive wave be a, the time period T and wavelength λ, then the equation of the stationary wave produced is given by ;

$$y = 2\ a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{T} \qquad \ldots(i)$$

Given equation is $y = 12 \cos\left(\frac{\pi x}{5}\right) \sin 20\pi\ t$ $\qquad \ldots(ii)$

Comparing eqs. (*i*) and (*ii*), we have,

(*i*) Wavelength, $\lambda = 10$ cm ; Time period, $T = \frac{1}{10}$ sec.

$\therefore$ Frequency, $f = \frac{1}{T} = 10$ Hz

(*ii*) Velocity of progressive waves is

$$v = f\lambda = 10 \times 10 = 100 \text{ cm s}^{-1}$$

$$\text{Amplitude, } a = \frac{12}{2} = 6 \text{ cm}$$

PROBLEMS FOR PRACTICE

1. The constituent waves of a stationary wave have amplitude, frequency and velocity 8 cm, 30 Hz and 180 cm s^{-1} respectively. Write down the equation of the stationary wave.

[$y = 16 \cos \frac{\pi x}{3} \sin 60 \pi t$]

2. Two harmonic waves are described by ;

$$y_1 = 3 \sin \pi (x + 0.6\, t) \text{ cm} ; y_2 = 3 \sin \pi (x - 0.6\, t) \text{ cm}$$

Determine the maximum displacement of the motion at (*i*) $x = 0.25$ cm (*ii*) $x = 0.5$ cm and (*iii*) $x = 1.5$ cm. **[(*i*) 4.24 cm (*ii*) 6.0 cm (*iii*) 6.0 cm]**

3. The wave function for a standing wave in a string is given by ;

$$y = 0.3 \sin (0.25\, x) \cos (120\, \pi\, t)$$

where x is in m and t is in s. Determine the wavelength and frequency of the interfering travelling waves. **[25.1 m ; 60 Hz]**

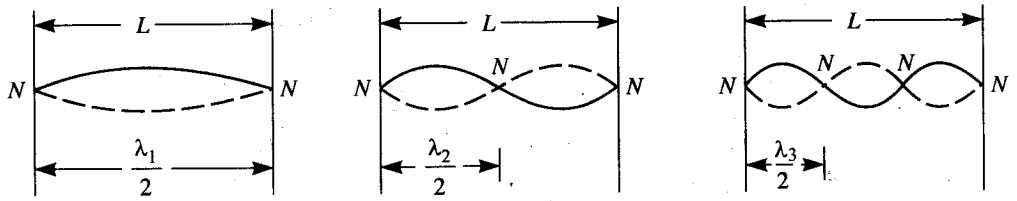
4. A standing wave is given by the expression :

$$y = 0.2 \cos (6.28\, t) \sin 18.8\, x$$

where x and y are in metres and t is in seconds. What are the wavelength and frequency of the two waves that give rise to this standing wave? **[0.334 m ; 1.0 Hz]**

26.30. STANDING WAVES IN A STRING FIXED AT BOTH ENDS

Consider a string of length L stretched between two fixed supports as shown in Fig. 26.21. When a wave is set up in the string by plucking it at some point, the wave will travel in both directions and will be reflected at each end. Standing waves are set up in the string by a continuous



1st harmonic (fundamental) (*i*) | 2nd harmonic (1st overtone) (*ii*) | 3rd harmonic (2nd overtone) (*iii*)

Fig. 26.21

superposition of waves incident and reflected from the ends. Because the ends of the string are fixed, they must correspond to zero displacement. In other words, the boundary conditions demand that the ends of the string must be nodes. We know that separation between adjacent nodes of a standing wave is $\lambda/2$ where λ is the wavelength of the component travelling waves. To meet our boundary conditions, we must, therefore, be able to fit exactly an integral number of half-wavelengths into the distance L, the distance between the fixed ends. In other words, the string length

L must be an integral multiple of half-wavelengths. Therefore, the allowed wavelengths for a standing wave on a string fixed at both ends are given by ;

$$*n\,\frac{\lambda}{2} = L \qquad \text{where } n = 1, 2, 3, \ldots \qquad \ldots(i)$$

Only those travelling waves combine to give standing waves that satisfy eq. (i). Any wave that does not correspond to eq. (i) cannot exist on the string. If such a wave does exist, it will die out ; its eneregy being dissipated at the boundaries.

(*i*) **First mode of vibration.** The first mode of vibration occurs for $n = 1$ and the corresponding wavelength λ_1 is given by ;

$$1 \times \left(\frac{\lambda_1}{2}\right) = L \quad \text{or} \quad \lambda_1 = 2L$$

Therefore, a wave whose wavelength is twice the string length L can form a stationary wave and represents the longest wavelength that can exist on the string. The corresponding frequency f_1 is given by ;

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L} \qquad (\because \lambda_1 = 2L)$$

where v = The velocity of either one of the progressive waves that have produced the stationary wave (Note that the velocity is the **same for all wavelengths).

The frequency f_1 is called the **fundamental frequency** or **first harmonic**. It is the minimum frequency that can set up the standing wave. Note that in this case, string vibrates in one segment or loop [See Fig. 26.21 (*i*)].

Now $$v = \sqrt{\frac{T}{m}}$$

where T = Tension in the string (N)

m = Mass per unit length of the string (kg m^{-1})

$$\therefore \qquad f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}} \qquad \ldots(ii)$$

(*ii*) **Second mode of vibration.** The second mode of vibration occurs for $n = 2$ and the corresponding wavelength λ_2 is given by ;

$$2 \times \left(\frac{\lambda_2}{2}\right) = L \quad \text{or} \quad \lambda_2 = L$$

Therefore, a wave whose wavelength is equal to the string length L can form a standing wave and the corresponding frequency f_2 is given by ;

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1 \qquad \left(\because f_1 = \frac{v}{2L}\right)$$

---

* We can also prove this relation from the equation of a standing wave : $y = 2a \sin kx \cos \omega t$. The amplitude $2a \sin kx$ must be zero at the string ends. This requirement is obviously met at $x = 0$. And at the end $x = L$, it means

$$2a \sin (kL) = 0$$

or $kL = n\pi$ where $n = 1, 2, 3, 4, \ldots$

But $k = 2\pi/\lambda$ so that this condition is

$$\frac{2\pi}{\lambda} L = n\pi \quad \text{or} \quad n\left(\frac{\lambda}{2}\right) = L$$

** Since the velocity v of the wave is dependent only upon the tension and the mass per unit length of the string, waves of all the permitted frequencies have the same velocity.

This is called **first overtone** or **second harmonic**. Its value is twice the fundamental frequency. Note that in this case in addition to the nodes at the ends, this standing wave has a node at the centre of the string. Therefore, the string vibrates in two segments [See Fig. 26.21 (*ii*)].

$$\text{Since} \quad v = \sqrt{\frac{T}{m}} \quad ; \quad f_2 = 2f_1 = \frac{2}{2L}\sqrt{\frac{T}{m}} \qquad ...(iii)$$

(*iii*) **Third mode of vibration.** The third mode of vibration occurs for $n = 3$ and the corresponding wavelength is given by ;

$$3 \times \left(\frac{\lambda_3}{2}\right) = L \quad \text{or} \quad \lambda_3 = \frac{2L}{3}$$

The corresponding frequency f_3 is given by ;

$$f_3 = \frac{v}{\lambda_3} = \frac{v}{2L/3} = 3\frac{v}{2L} = 3f_1 \qquad \left(\because f_1 = \frac{v}{2L}\right)$$

This is called **second overtone** or **third harmonic**.

Note that in this case the string vibrates in three segments [See Fig. 26.21 (*iii*)].

$$\text{Since} \quad v = \sqrt{\frac{T}{m}} \; ; \; f_3 = \frac{3}{2L}\sqrt{\frac{T}{m}} \qquad ...(iv)$$

(*iv*) **Frequency of nth harmonic.** If λ_n is the wavelength of the *nth* harmonic, then,

$$n\left(\frac{\lambda_n}{2}\right) = L \quad \text{or} \quad \lambda_n = \frac{2L}{n}$$

Therefore, the frequency f_n of the *nth* harmonic is

$$f_n = \frac{v}{\lambda_n} = \frac{v}{2L/n} = \frac{nv}{2L}$$

$$\text{Now} \quad v = \sqrt{\frac{T}{m}}$$

$$\therefore \quad *f_n = \frac{n}{2L}\sqrt{\frac{T}{m}} \qquad \text{where } n = 1, 2, 3, ... \qquad ...(v)$$

$$\text{Now} \quad f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

$$\therefore \quad f_n = n f_1 \qquad \text{where } n = 1, 2, 3, ...$$

This means that frequencies of the various overtones are whole number multiples of the fundamental frequency.

Position of nodes. Referring to Fig. 26.21, there are two nodes in first mode of vibration, three nodes in the second mode of vibration and so on. It follows that in the *n*th mode of vibration, there will be $(n + 1)$ nodes. These nodes are located at distances :

$$x = 0, \frac{L}{n}, \frac{2L}{n}, \frac{3L}{n} ..... L$$

Thus for first mode of vibration ($n = 1$), the nodes are located at $x = 0$, $x = L/1 = L$. For second mode of vibration ($n = 2$), nodes are located at $x = 0$, $x = L/2$ and $x = 2L/2 = L$.

Position of antinodes. Referring to Fig. 26.21, there is one antinode in first mode of vibration, two antinodes in second mode of vibration and so on. It follows that in the *n*th mode of

* In this case, the string will vibrate in *n* segments.

vibration, there will be n antinodes. Since antinodes are located inbetween nodes, their position is given by ;

$$x = \frac{L}{2n}, \frac{3L}{2n}, \frac{5L}{2n} \ldots\ldots \frac{(2n-1)L}{2n}$$

Thus for the first mode of vibration (n = 1), the antinode is located at $x = L/2 \times 1 = L/2$. For the second mode of vibration (n = 2), antinodes are located at $x = L/2 \times 2 = L/4$ and $x = 3L/2$.

Note. When a guitar string is plucked or piano string is struck, transverse waves travel along the string and are reflected on reaching its ends. The energy of any wave whose wavelength is such that it does not give rise to one of the allowed stationary waves is very quickly dissipated. The waves which remain have frequencies that are given by eq. (v) above and the string vibrates with all these frequencies simultaneously. The largest amplitude of vibration, and therefore the predominant frequency, is that of the fundamental. The relative amplitudes of the various overtones depend on the particular instrument being played and it is this that gives an instrument its characteristic sound.

Example 26.23. **A wire of length 140 cm and mass 0.52×10^{-3} kg is stretched by means of a load of 16 kg. Calculate the frequency of the fundamental note.**

Solution. The frequency of fundamental note is

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Here L = 140 cm = 1.4 m ; $T = 16 \times 9.8$ N ; $m = \frac{0.52\times10^{-3}}{1.4}$ kg m^{-1}

$$\therefore \quad f = \frac{1}{2\times1.4}\sqrt{\frac{16\times9.8\times1.4}{0.52\times10^{-3}}} = 232 \text{ Hz}$$

Example 26.24. **The length of a sonometer wire between two fixed ends is 110 cm. Where should the two bridges be placed so as to divide the wire into three segments whose fundamental frequencies are in the ratio 1 : 2 : 3?**

Solution. Let l_1, l_2 and l_3 be the lengths of the three parts of the wire and f_1, f_2 and f_3 be their respective frequencies.

Now $f_1 : f_2 : f_3 = 1 : 2 : 3$

Since T and m are fixed eqantities, $f \propto 1/l$

$$\therefore \quad f_1 l_1 = f_2 l_2 = f_3 l_3$$

$$\therefore \quad l_2 = \frac{f_1}{f_2} l_1 = \frac{l_1}{2} \quad \text{and} \quad l_3 = \frac{f_1}{f_3} l_1 = \frac{l_1}{3}$$

Now $l_1 + l_2 + l_3 = 110$ or $l_1 + \frac{l_1}{2} + \frac{l_1}{3} = 110$ or $\frac{11\, l_1}{6} = 110$

$$\therefore \quad l_1 = 60 \text{ cm} ; l_2 = 30 \text{ cm} ; l_3 = 20 \text{ cm}$$

Hence the bridges should be placed at 60 cm and 90 cm from the zero end of the wire.

Example 26.25. **A string of length 1 m and mass 5×10^{-4} kg fixed at both ends is under a tension of 20 N. It is plucked at a point situated 25 cm from one end. What would be the frequency of vibrations of the string?**

Solution. Fig. 26.22 shows the conditions of the problem. The plucking point of the string will be an *antinode* and each end will be a *node*. As nodes and antinodes are situated at equal distances, the string will vibrate in two segments *i.e.* n = 2.

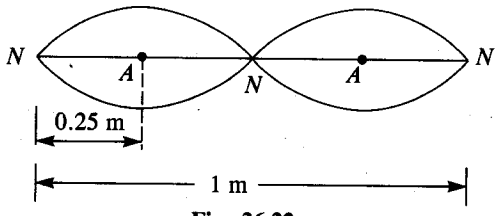


Fig. 26.22

Frequency of vibrations of the string is

$$f = \frac{n}{2L}\sqrt{\frac{T}{m}}$$

Here $n = 2$; $L = 1$ m ; $m = \frac{5\times10^{-4}}{1} = 5 \times 10^{-4}$ kg m^{-1} ; $T = 20$ N

$$\therefore \quad f = \frac{2}{2\times1}\sqrt{\frac{20}{5\times10^{-4}}} = 200 \text{ Hz}$$

Example 26.26. **The fundamental frequency of a sonometer wire increases by 5 Hz if its tension is increased by 21%. How will the frequency be affected if its length is increased by 10%?**

Solution. The fundamental frequency of the wire is given by ;

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}} \qquad ...(i)$$

On increasing the tension by 21%, the new tension will be = 1.21 T.

$$\therefore \quad f + 5 = \frac{1}{2L}\sqrt{\frac{1.21T}{m}} \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we get,

$$\frac{f+5}{f} = \sqrt{1.21} = 1.1$$

On solving, $f = 50$ Hz

When the length is increased by 10%, the new length = 1.1 L. Therefore, new frequency f' is

$$f' = \frac{1}{2(1.1\,L)}\sqrt{\frac{T}{m}} = \frac{1}{(1.1)\times2L}\sqrt{\frac{T}{m}}$$

$$= \frac{f}{1.1} = \frac{50}{1.1} = 45.45 \text{ Hz}$$

Example 26.27. **Two perfectly identical wires are in unison. When the tension in one wire is increased by 1%, then on sounding them together, 3 beats are heard in 2 seconds. Calculate the initial frequency of each wire.**

Solution. Let the initial tension in each wire be T and the frequency be f. Then, we have,

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}} \qquad ...(i)$$

When tension in one wire is increased by 1%, the new tension = $T + (T/100) = (101T/100)$. Clearly, the frequency of the wire will increase slightly. Since it produces 3 beats in 2 seconds (*i.e.* 1.5 beats in 1 second), its frequency will now be $(f + 1.5)$.

$$\therefore \quad f + 1.5 = \frac{1}{2L}\sqrt{\frac{101T}{100\,m}} = 1.005 \times \frac{1}{2L}\sqrt{\frac{T}{m}} \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we get,

$$\frac{f+1.5}{f} = 1.005$$

or $f = 300$ Hz

PROBLEMS FOR PRACTICE

1. What is the fundamental frequency of a guitar string whose tension is adjusted to give a wave speed of 143 m s^{-1} if the length of the string free to vibrate is 0.65 m ? **[110 Hz]**
2. A stretched wire emits a fundamental note of 256 Hz. Keeping the stretching force constant and reducing the length of wire by 10 cm, the frequency becomes 310 Hz. Calculate the original length of the wire. **[50 cm]**
3. A stretched wire 75 cm long emits a fundamental frequency of 300 Hz. By how much the length be changed so that it emits a fundamental note of 450 Hz? **[18.75 cm]**
4. A nylon string is stretched between supports 1.20 m apart. Given that speed of sound in the string is 800 m s^{-1}, find the frequency of the fundamental vibration and the first two overtones. **[333 Hz ; 666 Hz ; 1000 Hz]**
5. The vibrations from an 800 Hz tuning fork set up standing waves in a string clamped at both ends. The wave speed in the string is known to be 400 ms^{-1} for the tension used. The standing wave is observed to have four antinodes and an amplitude of 2.0 mm. How long is the string? **[1.00 m]**
6. A tuning fork of unknown frequency vibrates in unison with a wire of certain length stretched under a tension of 49 N. It produces 6 beats per second with the same wire when tension is changed to 44.1 N. Find the frequency of the tuning fork. **[117.65 Hz]**
7. Two identical steel wires are stretched between posts 1.2 m apart. When the same tension is applied to both, their fundamental frequency of vibration is 400 Hz. If tension on one is now increased by 2%, what beat frequency will be heard when both wires are set into vibration? **[4.0 Hz]**
8. A 90 cm long wire of a sitar has a fundamental frequency of 256 Hz. At what distance from the upper end should the wire be compressed so that a note of frequency 384 Hz is produced? **[30 cm]**
9. A sonometer wire has a length of 114 cm between its two fixed ends. Where should two bridges be placed so as to divide the wire into three segments whose fundamental frequencies are in the ratio of 1:3:4? **[At a distance 72 cm, 96 cm from one end]**
10. A 50 cm long wire is in unison with a tuning fork of frequency 256 Hz when stretched by a load of density 9 g cm^{-3} hanging vertically. The load is then immersed in water. By how much the length of the wire should be reduced to bring it again in unison with the same tuning fork? **[2.867 cm]**

26.31. STANDING WAVES IN AIR COLUMNS

Standing longitudinal waves can be set up in the tube of air, such as an organ pipe, as the result of interference between longitudinal waves travelling in opposite directions. Suppose that the air at one end of a pipe is caused to vibrate. The vibration produces a longitudinal wave which travels along the pipe and is reflected at its far end. Since the incident and reflected waves have the same speed, frequency and amplitude, a stationary wave results. As in case of a string fixed at both ends, only certain permitted frequencies form standing waves in a pipe. The modes of vibration of a pipe closed at one end are different from those of an open pipe. We now discuss the two cases separately.

26.32. STANDING WAVES IN A PIPE CLOSED AT ONE END

A closed or stopped organ pipe consists essentially of a metal pipe closed at one end Q and a blast of air is blown into the other end P as shown in Fig. 26.23 (*i*). A longitudinal wave thus travels towards Q and is reflected at the closed end Q. As a result, a stationary wave is formed.

The closed end of the pipe must be a node N since the layer of air in contact with it must be permanently at rest. At the open end, the air is free to vibrate so that open end is antinode A.

Modes of vibration. The equation of the standing wave is given by (reference origin at node) ;

$$y = 2\,a \sin kx \cos \omega t \qquad ...(i)$$

At the *closed end* of the pipe $x = 0$ so that $\sin kx = 0$. Therefore, from eq. (i), $y = 0$ *i.e.* a node is formed.

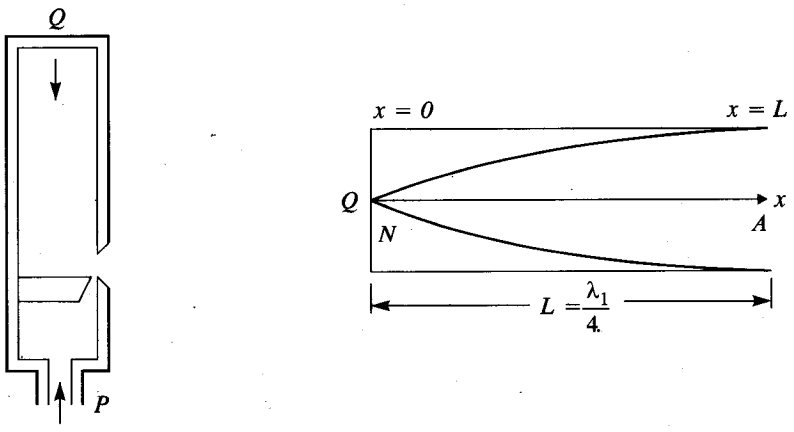
At the open end of the pipe of length L, an antinode is formed *i.e.* y is maximum. Now y will be maximum when $\sin kx$ is maximum (*i.e.* ± 1).

$$\therefore \quad \sin kL = \pm 1 = \sin (2n - 1)\frac{\pi}{2} \qquad \text{where } n = 1, 2, 3, ...$$

$$\text{or} \quad kL = (2n - 1)\frac{\pi}{2}$$

$$\text{or} \quad \frac{2\pi}{\lambda} L = (2n - 1)\frac{\pi}{2} \qquad \left(\because k = \frac{2\pi}{\lambda}\right)$$

$$\therefore \quad \lambda = \frac{4L}{(2n-1)} \qquad \text{where } n = 1, 2, 3, ... \qquad ...(ii)$$



(*i*) (*ii*)

Fig. 26.23

(*i*) **First mode of vibration.** The first mode of vibration occurs for $n = 1$ and the corresponding wavelength λ_1 is given by [using eq. (ii)],

$$\lambda_1 = \frac{4L}{2\times1-1} = 4L$$

Therefore, a wave whose wavelength is four times the pipe length L can form a stationary wave and represents the longest wavelength that can exist in a pipe closed at one end. The corresponding frequency f_1 is given by ;

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{4L} \qquad (\because \lambda_1 = 4L)$$

where v = velocity of sound in air.

The frequency f_1 is called the **fundamental frequency** (or **first harmonic**). It is the minimum frequency that can set up the standing wave. Note that in this case, one end is a node and the other is an antinode ; no other nodes or antinodes between the ends [See Fig. 26.23 (ii)].

(*ii*) **Second mode of vibration.** The second mode of vibration occurs for $n = 2$ and the corresponding wavelength λ_2 is given by [using eq. (ii)],

$$\lambda_2 = \frac{4L}{2\times2-1} = \frac{4L}{3}$$

Therefore, a wave whose wavelength is 4/3 times the pipe length L can form a stationary wave and the corresponding frequency f_2 is

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{4L/3} = 3\left(\frac{v}{4L}\right)$$

or $$f_2 = 3f_1 \qquad (\because f_1 = v/4L)$$

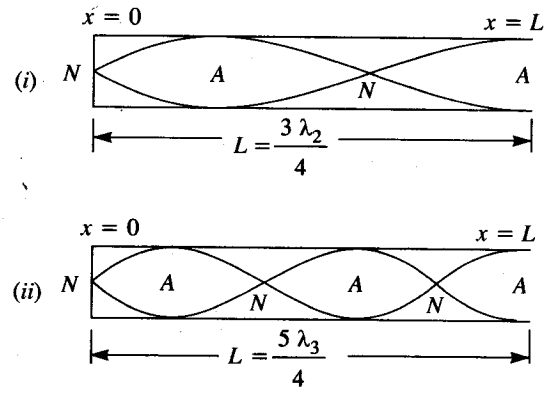


Fig. 26.24

This is called **first overtone**. Its frequency is three times the fundamental frequency. Therefore, it is called **third harmonic**. Note that in this case, both an antinode and a node exist between the ends [See Fig. 26.24 (*i*)].

(*iii*) **Third mode of vibration.** The third mode of vibration occurs for $n = 3$ and the corresponding wavelength λ_3 is given by [using eq. (*ii*)],

$$\lambda_3 = \frac{4L}{2\times 3-1} = \frac{4L}{5}$$

The corresponding frequency f_3 is given by ;

$$f_3 = \frac{v}{\lambda_3} = \frac{v}{4L/5} = 5\left(\frac{v}{4L}\right)$$

$$\therefore \qquad f_3 = 5f_1 \qquad (\because f_1 = v/4L)$$

This is called **second overtone** or **fifth harmonic**.

(*iv*) **Frequency of *n*th harmonic.** In a pipe closed at one end, the frequency of *n*th harmonic is given by ;

$$f_n = n\left(\frac{v}{4L}\right) \qquad \text{where } n = 1, 3, 5, 7, \ldots.$$

Therefore, a pipe closed at one end can produce only odd-numbered harmonics (i.e. f_1, $3f_1$, $5f_1$ ..... where f_1 is the fundamental frequency).

Musical instruments that behave somewhat like closed pipes are clarinet, trumpet, trombone and human voice. The closed end of these instruments is at the reed, mouthpiece and the vocal cords respectively.

26.33. STANDING WAVES IN AN OPEN PIPE

An 'open' pipe is one which is open at both ends. When air is blown into it at P, a wave p travels to the open end Q where it is reflected in the direction q on encountering the free air [See Fig. 26.25 (*i*)]. Therefore, a stationary wave is set up in the air in the pipe. Since the two ends of the pipe are open, they must both be antinodes.

Modes of vibration. If the reference origin is taken at an antinode, the equation of the stationary wave is given by ;

$$y = 2a \cos kx \sin \omega t$$

The amplitude $2a \cos kx$ must be maximum at the ends because antinodes are formed there. This requirement is obviously met at $x = 0$ and at the end $x = L$. It means that :

$$2a \cos kx = \pm 2a$$

or $$\cos \frac{2\pi}{\lambda} L = \pm 1 \qquad (\because x = L \text{ and } k = \frac{2\pi}{\lambda})$$

or $$\frac{2\pi L}{\lambda} = n\pi \qquad \text{where } n = 1, 2, 3, \ldots\ldots\ldots$$

or $$\lambda = \frac{2L}{n} \qquad \ldots(i)$$

(*i*) **First mode of vibration.** The first mode of vibration occurs for $n = 1$ and the corresponding wavelength λ_1 is given by [using eq. (*i*)] ;

$$\lambda_1 = \frac{2L}{n} = \frac{2L}{1} = 2L$$

Therefore, a wave whose wavelength is two times the pipe length L can form a stationary wave and represents the longest wavelength that can exist in an open pipe. The corresponding frequency f_1 is given by ;

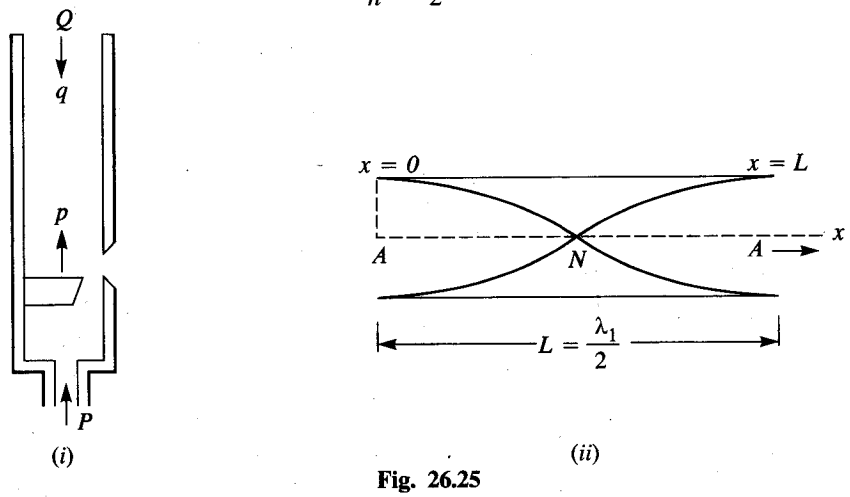
$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L} \qquad (\because \lambda_1 = 2L)$$

Here v = Velocity of sound in air

The frequency f_1 is called the **fundamental frequency** (or **first harmonic**). It is the minimum frequency that can set up the standing wave. Note that in this case, an antinode A is at each end and a node N at the centre of the pipe [See Fig. 26.25 (*ii*)].

(*ii*) **Second mode of vibration.** The second mode of vibration occurs for $n = 2$ and the corresponding wavelength λ_2 is given by [using eq. (*i*)] ;

$$\lambda_2 = \frac{2L}{n} = \frac{2L}{2} = L$$



(*i*) (*ii*)

Fig. 26.25

Therefore, a wave whose wavelength is equal to the length L of the pipe can form a stationary wave and the corresponding frequency f_2 is

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2\left(\frac{v}{2L}\right) = 2f_1$$

This is called **second harmonic** or **first overtone**. Note that frequency in this mode is twice the fundamental frequency. Note that in this case, the standing wave has two nodes [See Fig. 26.26(*i*)].

(*iii*) **Third mode of vibration.** The third mode of vibration occurs for $n = 3$ and the corresponding wavelength λ_3 is given by [using eq. (*i*)] ;

$$\lambda_3 = \frac{2L}{n} = \frac{2L}{3}$$

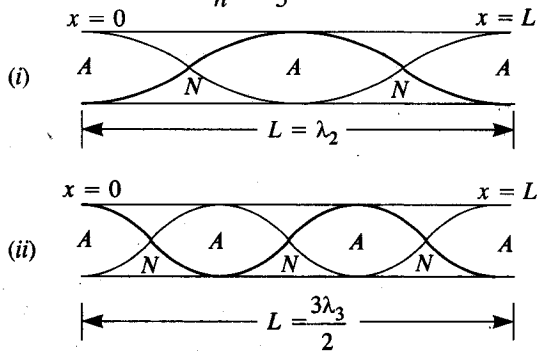


Fig. 26.26

The corresponding frequency f_3 is given by ;

$$f_3 = \frac{v}{\lambda_3} = \frac{v}{2L/3} = 3\left(\frac{v}{2L}\right) = 3f_1$$

This is called **third harmonic** or **second overtone**. Note that in this case, the standing wave has three nodes [See Fig. 26.26 (*ii*)].

(*iv*) **Frequency of *n*th harmonic.** If λ_n is the wavelength of the *nth* harmonic, then,

$$\lambda_n = \frac{2L}{n}$$

$$\therefore \quad f_n = \frac{v}{\lambda_n} = \frac{v}{2L/n} = n\left(\frac{v}{2L}\right) = nf_1$$

$$\therefore \quad f_n = nf_1 \qquad \text{where } n = 1, 2, 3 \text{ .....}$$

This means that frequencies of the overtones in an open pipe are whole numbered multiples of the fundamental frequency. This is just what is found for a string fixed at both ends.

Musical instruments that function as open pipes include the flute, recorder, oboe and basson.

26.34. CLOSED AND OPEN PIPES COMPARED

One end open, one closed : $\lambda_n = \frac{4L}{n}$; $f_n = \frac{nv}{4L}$ (n = odd integer)

Both ends open : $\lambda_n = \frac{2L}{n}$; $f_n = \frac{nv}{2L}$ (n = all integers)

It follows from the above that :

(*i*) The fundamental frequency of an open pipe is twice that of a closed pipe of the same length.

(*ii*) A closed pipe of length $L/2$ produces the same fundamental frequency as an open pipe of length L. Since both odd and even harmonics are present in the note from an open pipe, it gives a richer note.

(*iii*) The velocity v of sound in air is proportional to $\sqrt{T}$ (where T is the temperature of air in kelvin). Therefore, the frequencies produced by both closed and open pipes increase with temperature.

Example 26.28. What will be the fundamental frequencies and first three overtones for a 26 cm long organ pipe at 20°C if it is (*i*) open and (*ii*) closed? Given that speed of sound in air at 0°C is 331 ms^{-1}.

Solution. The speed of sound in air is given by ;

$$v_t = (v_0 + 0.60\ t)\ ms^{-1}$$

where v_0 = speed at 0°C ; t = rise of temperature in °C.

$$\therefore \quad v_{20} = 331 + 0.60 \times 20 = 343\ m\ s^{-1}$$

(*i*) For an open pipe, the fundamental frequency is

$$f_1 = \frac{v}{2L} = \frac{343}{2\times 0.26} = 600\ Hz$$

The first three overtones are **1320 Hz, 1980 Hz** and **2640 Hz.**

(*ii*) For a closed pipe, the fundamental frequency is

$$f_1 = \frac{v}{4L} = \frac{343}{4\times 0.26} = 300\ Hz$$

The first three overtones are **990 Hz, 1650 Hz** and **2310 Hz.**

Example 26.29. The third overtone of a closed pipe is found to be in unison with the first overtone of an open pipe. Find the ratio of the length of the pipes.

Solution. Let L_1 and L_2 be the lengths of the closed pipe and open pipe respectively and f_1 and f_2 their corresponding fundamental frequencies.

$$\therefore \quad f_1 = \frac{v}{4L_1} \quad \text{and} \quad f_2 = \frac{v}{2L_2}$$

Third overtone of closed pipe = $7f_1$; First overtone of open pipe = $2f_2$

It is given that $7f_1 = 2f_2$ or $7\times\frac{v}{4L_1} = 2\times\frac{v}{2L_2}$ $\therefore$ $\frac{L_1}{L_2} = 7/4$

Example 26.30. A closed pipe is suddenly opened and it is observed that the second overtone of the closed pipe differs by 200 vib/s from the first overtone of the open pipe. What is the fundamental frequency of the closed pipe?

Solution. Let f_1 and f_2 be the fundamental frequencies of the closed pipe and open pipe respectively. Since the pipe is same having the same length,

$$\therefore \quad f_2 = 2f_1 \qquad ...(i)$$

Now the second overtone of the closed pipe is $5f_1$ while the first overtone of the open pipe is $2f_2$.

It is given that $5f_1 - 2f_2 = 200$ ...(*ii*)

Solving eqs. (*i*) and (*ii*), we get, $f_1 = 200$ **Hz**

PROBLEMS FOR PRACTICE

1. What will be the frequency of the fundamental note emitted by a closed pipe 32.4 cm long if the velocity of sound in air is 332 m s^{-1}? **[256 Hz]**
2. An organ pipe 60 cm long is open at both ends. Assume the air in the pipe is at 20°C. Find the frequency of the fundamental resonance and that of first overtone. Given that speed of sound in air at 20°C = 343 m s^{-1}. **[286 Hz ; 572 Hz]**
3. Calculate the minimum length for a pipe that has a fundamental frequency of 240 Hz if the pipe is (*i*) closed at one end (*ii*) open at both ends. **[(*i*) 0.358 m (*ii*) 0.717 m]**
4. An organ pipe open at both ends is vibrating in its third harmonic with a frequency of 748 Hz. The length of the pipe is 0.7 m. Determine the speed of sound in air in the pipe. **[349 ms^{-1}]**
5. Assume that the auditory canal of the outer ear is a 3.0 cm long pipe closed at one end by the ear drum. Calculate the fundamental frequency of the pipe when (*i*) in air and (*ii*) in water. Speed of sound in air = 340 m s^{-1}, speed of sound in water = 1500 ms^{-1}. **[(*i*) 2800 Hz (*ii*) 17000 Hz]**

26.35. BEATS

We now consider the superposition of two waves of *slightly different frequencies* travelling in the *same direction.* If the waves are in phase at some time, the interference will be constructive and the intensity is large. But at some time later, because the frequencies are different, the waves will be out of phase and the interference will be destructive, resulting in low intensity. Therefore, if two sound sources of slightly different frequencies are sounded together, a periodic rise and fall in intensity will be heard. This is known as the phenomenon of *beats.*

The periodic variations in intensity at a given point due to the superposition of two sound waves of slightly different frequencies travelling in the same direction are called **beats.**

The frequency of the beats is the number of intense sounds heard per second. The time interval between two successive beats is called beat period. As we shall see, the beat frequency is equal to the difference in the frequencies of the two sources. For example, when two tuning forks of frequencies f_1 (= 266 Hz) and f_2 (= 262 Hz) are sounded together, then,

Beat frequency, $f_b = f_1 - f_2 = 266 - 262 = 4$ beats/s

The maximum beat frequency that the human ear can hear is about 10 beats per second. It is due to the persistence of hearing.

26.36. FORMATION OF BEATS

To see how beats are formed, consider two equal amplitude sound waves of frequencies $f_2 = 48$ Hz and $f_1 = 56$ Hz as shown in Fig. 26.27 (*i*) and Fig. 26.27 (*ii*) respectively. In 1 second, the 48 Hz source makes 48 vibrations whereas the 56 Hz source makes 56 vibrations.

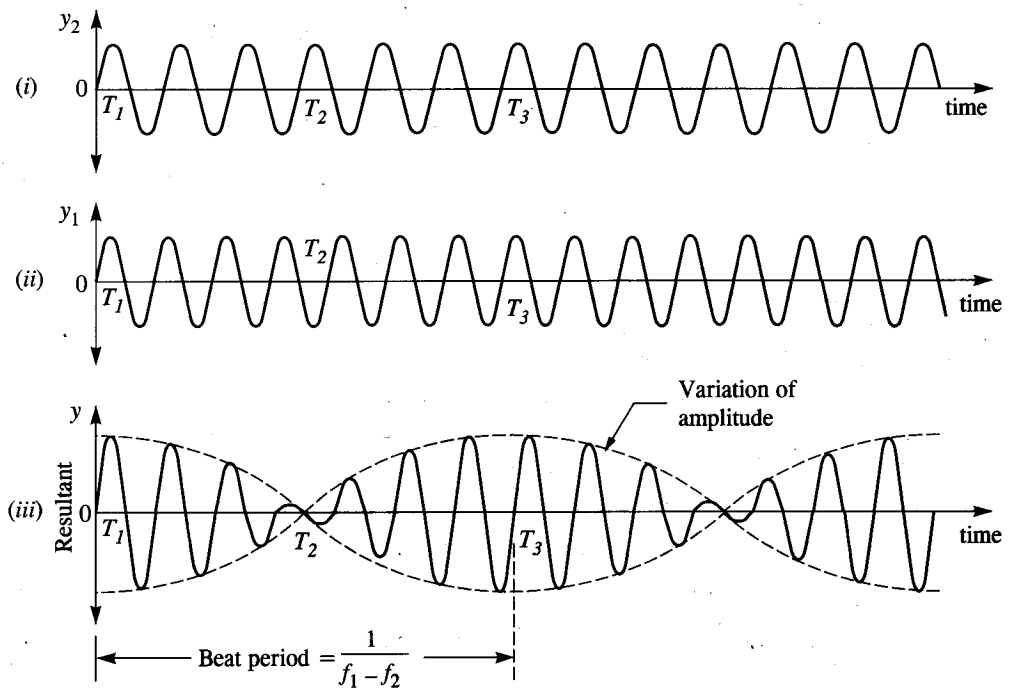


Fig. 26.27

The variation of the displacement y_2 due to 48 Hz source is shown in Fig. 26.27 (*i*) and the variation of displacement y_1 of 56 Hz source is shown in Fig. 26.27 (*ii*). According to the principle of superposition, the variation of the resultant displacement y is the algebraic sum of the two curves which varies in amplitude in the way shown in Fig. 26.27 (*iii*). To understand the variation

of y, suppose that displacements y_1, y_2 are in phase at some time T_1 [See Fig. 26.27]. Since the frequency of the curve in Fig. 26.27 (*i*) is 48 cycles per sec, the variation of y_2 undergoes 3 complete cycles in $\frac{1}{16}$th second ; in the same time, the variation y_1 undergoes $3\frac{1}{2}$ cycles since its frequency is 56 cycles per second. Thus y_1 and y_2 are 180° out of phase with each other at this instant and their resultant y is then a minimum (zero in this case because amplitudes of two waves are equal) at some instant T_2. Thus, T_1T_2 represents $\frac{1}{16}$th of a second in Fig. 26.27 (*iii*). In $\frac{1}{8}$th of a second from T_1, y_2 has undergone 6 complete cycles and y_1 has undergone 7 complete cycles. Hence the two waves are in phase again at T_3 where T_1T_3 represents $\frac{1}{8}$th of a second and their resultant at this instant is again a maximum [See Fig. 26.27 (*iii*)]. In this way it can be seen that a loud sound will be heard after every 1/8th of a second and thus the beat frequency in 8 cycles per second. This is the difference between the frequencies 56 Hz and 48 Hz of the two sounds. We also see that beats are phenomenon of interference.

Beat frequency formula. Suppose two sounding tuning forks A and B have frequencies f_1 and f_2 ($f_1 > f_2$) cycles per second respectively. Note that frequencies f_1 and f_2 should be close to each other. At some instant, the displacement of a particular layer of air near the ear due to each fork will be a maximum to the right. The resultant displacement is then a maximum and a loud sound or beat is heard. After this, vibrations of air due to each fork go out of phase and t seconds later, the displacement due to each fork is again a maximum to the right so that a loud sound or beat is again heard. The fork A has then made exactly one cycle more than the fork B. But the number of cycles made by forks A and B in t seconds is f_1t and f_2 t respectively.

$$\therefore \qquad f_1 t - f_2 t = 1$$

or
$$f_1 - f_2 = \frac{1}{t}$$

Now 1 beat has been made in t seconds so that $1/t$ is the number of beats per second *i.e.* beat frequency.

$$\therefore \qquad \text{Beat frequency, } f_b = f_1 - f_2$$

26.37. MATHEMATICAL TREATMENT OF BEATS

Suppose that y_1 and y_2 are the individual displacements of two sinusoidal wave motions whose frequencies are f_1 and f_2 respectively. If the amplitude of each wave is a, then at a single point,the displacements vary with time according to :

$$y_1 = a \sin (2\pi f_1 t) \quad \text{and} \quad y_2{}^* = a \sin (2\pi f_2 t) \qquad ...(i)$$

If the two waves are superposed, the resultant displacement y is given by ;

$$y = y_1 + y_2$$

or
$$y = a \sin (2\pi f_1 t) + a \sin (2\pi f_2 t)$$

It is convenient to write this in a form that uses the trigonometric identity :

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\therefore \qquad y = 2a \sin\left(\frac{2\pi f_1 t + 2\pi f_2 t}{2}\right)\cos\left(\frac{2\pi f_1 t - 2\pi f_2 t}{2}\right)$$

or
$$y = \left[2a\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad ...(ii)$$

---

* This assumes that the waves are in phase at $t = 0$ and simplifies the mathematics without invalidating the result.

Graphs demonstrating the individual waveforms as well as the resultant wave are shown in Fig. 26.28.

The following points may be noted from eq. (*ii*) :

(*i*) The resultant wave has an effective frequency equal to the average frequency of the two sources *i.e.* $(f_1 + f_2)/2$.

(*ii*) The resultant wave has an amplitude given by ;

$$\text{Amplitude} = 2\,a\cos 2\pi\frac{(f_1 - f_2)t}{2}$$

Note that the *amplitude varies in time* with a frequency given by $(f_1 - f_2)/2$. When f_1 is close to f_2, this amplitude variation is *slow as illustrated by the envelope (broken line) of the resultant waveform in Fig. 26.28. Note that a beat or loud sound (*i.e.* a maximum in amplitude) will be detected whenever :

$$\cos 2\pi\frac{(f_1 - f_2)t}{2} = +1 \text{ or } -1$$

That is two beats occur per cycle so that beat frequency is twice of $(f_1 - f_2)/2$ which is just $f_1 - f_2$.

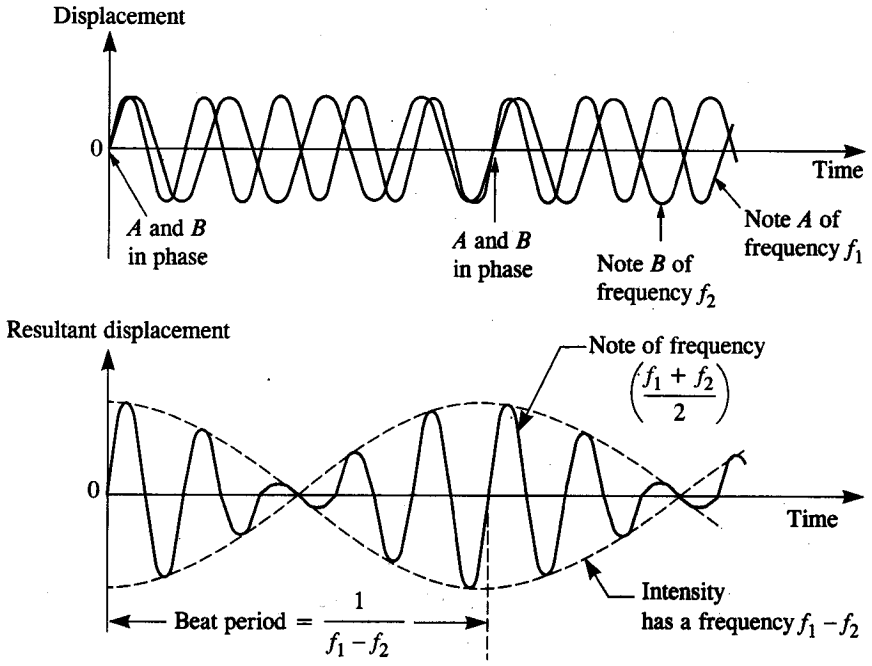


Fig. 26.28

$\therefore$ Beat frequency, $f_b = f_1 - f_2$

For example, if two tuning forks vibrate individually at frequencies of 442 Hz and 438 Hz, the resultant sound wave of the combination would have a frequency = $(f_1 + f_2)/2 = (442 + 438)/2 = 440$ Hz and a beat frequency $f_b = f_1 - f_2 = 442 - 438 = 4$ Hz. That is the listener would hear the 440 Hz sound wave go through an intensity maximum four times every second.

(*iii*) Beat Period, $t_b = \frac{1}{f_b} = \frac{1}{f_1 - f_2}$

* The cosine term is a *slowly* varying term and it is this which makes it possible for it to be regarded as the amplitude in eq. (*ii*).

26.38. USES OF THE PHENOMENON OF BEATS

There are a number of practical applications of the phenomenon of beats. However, we shall only discuss a few of them by way of illustration.

(*i*) **Measurement of unknown frequency.** The phenomenon of beats can be used to measure the unknown frequency f_1 of a note. For this purpose, a note of known frequency f_2 is used to provide beats with the unknown note. If f_2 is not very different from f_1, it is possible to determine the beat frequency f_b by counting the number of beats that occur in a given time. Since f_b is the difference between f_2 and f_1, it follows that $f_1 = f_2 - f_b$ or $f_1 = f_2 + f_b$. Thus suppose f_2 = 1000 Hz and the number of beats per second made with a tuning fork of unknown frequency f_1 is 4. Then f_1 = 1004 Hz or 996 Hz.

To decide which value of f_1 is correct, the end of the tuning fork prong is loaded with a small piece of wax which diminishes the frequency a little. The two notes are sounded again. If the beat frequency is **increased,* a little thought indicates that the unknown frequency in 996 Hz. If the beats are decreased, the unknown frequency is 1004 Hz. The tuning fork must not be overloaded as the frequency may decrease if it was 1004 Hz to a frequency such as 995 Hz in which case the significance of the beats can be wrongly interpreted.

(*ii*) **Tuning of stringed instruments.** Beats are also used to tune a stringed instrument (*e.g.* a piano) to a given note. As the frequency of instrument note approaches that of the given note, beats are produced. When the frequencies are identical, the beats disappear and the instrument may be regarded as tuned. The members of an orchestra tune up by listening for beats between their instruments and a standard tone produced by a piano or some other instrument.

(*iii*) **Beats are utilised by dolphins in surveying the motions of things around them.** When a dolphin sends out sound signals, beats may be produced when the echoes it receives interfere with the sound it sends. When there is no relative motion between the dolphin and the object returning the sound, the sending and the receiving frequencies are the same and no beats occur. But when there is a relative motion, the echo has a different frequency due to the Doppler effect and beats are produced when the echo and emitted sounds combine. Dolphins rely more on sound than light and it is nice that increases and decreases of beat frequencies are components of their acoustic images, just as similar differences in light frequencies are components of our visual images.

Example 26.31. When a tuning fork of unknown frequency is sounded with another tuning fork whose frequency is 384 Hz, 6 beats per second are produced. When wax is attached to the first fork, then on sounding it with the second, 4 beats per second are produced. Determine the unknown frequency.

Solution. The first tuning fork produces 6 beats per second with a tuning fork of 384 Hz frequency. Therefore, the possible frequencies of the first tuning fork are :

$$f = 384 \pm 6 = 390 \text{ Hz or } 378 \text{ Hz}$$

When the first tuning fork is loaded with wax, its frequency decreases. Now the number of beats per second is 4 *i.e.,* now the difference in the frequencies of the forks is less than before loading. This is possible only if the frequency of the first tuning fork is greater, before loading, than the first. Hence the frequency of the first tuning fork is **390 Hz.**

Example 26.32. Calculate the speed of sound in a gas in which two waves of wavelength 1.00 m and 1.01 m produce 30 beats in 10 seconds.

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* Had f_1 been 1004 Hz, then on loading it with wax, frequency would have decreased, say by 2 Hz, so that it would be 1002 Hz. Therefore, difference between f_1 and f_2 would decrease *i.e.* beat frequency would decrease. But actually beat frequency has increased. The fact that f_1 is 996 Hz is that on loading it with wax, its frequency has decreased, say by 2 Hz, so that it becomes 994 Hz. Clearly, the difference between f_1 and f_2 has increased *i.e.* beat frequency is increased.

Solution. Let v be the speed of sound in the gas and f_1 and f_2 be the frequencies of the two waves. Then,

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{1.00} \quad \text{and} \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{1.01}$$

Number of beats per second is

$$f_1 - f_2 = \frac{30}{10} = 3$$

Now $$f_1 - f_2 = \frac{v}{1.00} - \frac{v}{1.01}$$

or $$3 = v\left(\frac{1}{1.00} - \frac{1}{1.01}\right) = \frac{0.01v}{1.00 \times 1.01}$$

$\therefore$ $$v = \frac{3 \times 1.00 \times 1.01}{0.01} = \mathbf{303\ m\ s^{-1}}$$

Example 26.33. Two similar sonometer wires of the same material produce 2 beats per second. The length of one is 50 cm and that of the other is 50.1 cm. Calculate the frequencies of the two wires.

Solution. The frequency of the wire is given by ;

$$f = \frac{1}{2l}\sqrt{\frac{T}{m}} = \frac{k}{l} \qquad \text{where } k = \frac{1}{2}\sqrt{\frac{T}{m}}$$

$\therefore$ $$f_1 = \frac{k}{l_1} \quad \text{and} \quad f_2 = \frac{k}{l_2}$$

$\therefore$ $$f_1 - f_2 = \frac{k}{l_1} - \frac{k}{l_2} = k\left(\frac{1}{l_1} - \frac{1}{l_2}\right)$$

or $$2 = k\left(\frac{1}{50} - \frac{1}{50.1}\right) \quad \therefore \quad k = 50100$$

$\therefore$ $$f_1 = \frac{k}{l_1} = \frac{50100}{50} = \mathbf{1002\ Hz}\ ;\ f_2 = \frac{k}{l_2} = \frac{50100}{50.1} = \mathbf{1000\ Hz}$$

Example 26.34. Sixteen tuning forks are arranged in the order of increasing frequency. Any two successive forks give 8 beats per second when sounded together. If the last fork gives the octave of the first, then determine the frequency of the first fork.

Solution. Suppose the frequency of the first fork is f .Then the frequency of the second will be $(f + 8)$, of the third will be $(f + 16)$ ... and of the 16th will be $f + (16 - 1)\ 8 = (f + 120)$. It is given that :

$$f + 120 = 2f \quad \therefore \quad f = \mathbf{120\ Hz}$$

Example 26.35. The two parts of a sonometer wire divided by a movable knife differ by 2 mm and produce one beat per second when sounded together. Find their frequencies if the whole length of the wire is one metre.

Solution. Let l_1 and l_2 be the lengths of the two parts of a sonometer wire and f_1 and f_2 the corresponding frequencies. Then,

$$l_1 + l_2 = 100 \text{ cm and } l_1 - l_2 = 0.2 \text{ cm}$$

$\therefore$ $$l_1 = 50.1 \text{ cm and } l_2 = 49.9 \text{ cm}$$

The fundamental frequency of a stretched string $\propto \frac{1}{l}$.

$$\therefore \quad \frac{f_2}{f_1'} = \frac{l_1}{l_2} = \frac{50.1}{49.9} \quad \text{or} \quad f_2 = \frac{50.1}{49.9} f_1$$

Now $$f_2 - f_1 = 1 \quad \text{or} \quad \frac{50.1}{49.9} f_1 - f_1 = 1$$

$$\therefore \quad \frac{50.1 f_1 - 49.9 f_1}{49.9} = 1 \quad \text{or} \quad \frac{0.2 f_1}{49.9} = 1 \quad \therefore \quad f_1 = \frac{49.9 \times 1}{0.2} = \mathbf{249.5\ Hz}$$

And $$f_2 - f_1 = 1 \quad \text{or} \quad f_2 - 249.5 = 1 \quad \therefore f_2 = \mathbf{250.5\ Hz}$$

Example 26.36. At 16°C, two open end organ pipes when sounded together produce 51 beats in 3 seconds. How many beats per second will be produced if the temperature rises to 88°C? Neglect the increase in length of the pipes.

Solution. No. of beats produced/sec at 16°C = 51/3 = 17

Let l_1, l_2 = lengths of the two organ pipes

f_1, f_2 = fundamental frequencies emitted by pipes at 16°C

f_1', f_2' = fundamental frequencies emitted by pipes at 88°C

v_{16}, v_{88} = velocities of sound at 16°C and 88°C

$$\therefore \quad f_1 = \frac{v_{16}}{2l_1} \text{ and } f_2 = \frac{v_{16}}{2l_2}$$

Now $$f_1 - f_2 = 17 \text{... number of beats at 16°C}$$

or $$\frac{v_{16}}{2l_1} - \frac{v_{16}}{2l_2} = 17 \quad \text{or} \quad v_{16}\left(\frac{1}{2l_1} - \frac{1}{2l_2}\right) = 17 \qquad ...(i)$$

Let the number of beats produced per second at 88°C be n.

$$\therefore \quad f_1' - f_2' = n \quad \text{or} \quad \frac{v_{88}}{2l_1} - \frac{v_{88}}{2l_2} = n$$

or $$v_{88}\left(\frac{1}{2l_1} - \frac{1}{2l_2}\right) = n \qquad ...(ii)$$

Dividing eq. (*ii*) by eq. (*i*), we get,

$$\frac{v_{88}}{v_{16}} = \frac{n}{17} \quad \text{or} \quad \sqrt{\frac{88+273}{16+273}} = \frac{n}{17} \qquad \left(\because v \propto \sqrt{T}\right)$$

$$\therefore \quad n = 17 \times \sqrt{\frac{361}{289}} = \mathbf{19\ s^{-1}}$$

PROBLEMS FOR PRACTICE

1. Two tuning forks produce 12 beats in 2 seconds. The frequency of one fork is 288. When a little wax is attached to the prongs of the other fork, the beats disappear. Determine the frequency of the other fork. **[294]**
2. Two notes of wavelengths 2.04 m and 2.08 m produce 200 beats per minute in a gas. Find the velocity of sound in the gas. **[353.6 m s^{-1}]**
3. The frequency of a tuning fork is 256 Hz. When it is sounded with another tuning fork, three beats per second are heard. When the second tuning fork is loaded with a small quantity of wax and sounded again with the first tuning fork, again 3 beats per second are heard. What is the frequency of the second tuning fork? **[259 Hz]**

4. 26 tuning forks are arranged in the order of increasing frequency and any two successive forks give 4 beats per second when sounded together. If the frequency of the last fork is three times the frequency of the first fork, find the frequency of the first fork. **[50 Hz]**
5. Two tuning forks produce 4 beats per second when sounded together. One of them is in unison with a length of 120 cm of a sonometer wire and the other with 125 cm of the same wire under the same tension. Calculate the frequencies of the forks. **[100 Hz ; 96 Hz]**
6. The wavelengths of two waves are 49 cm and 50 cm. If the room temperature be 30°C, then how many beats per second will be produced due to these two waves? Velocity of sound at 0°C = 332 m s^{-1}. **[14 beats/second]**
7. The frequency of a tuning fork *A* is 256. When it is sounded with a tuning fork *B* of unknown frequency, 4 beats per second are produced. When the prong of *A* is loaded with a small quantity of wax, only 1 beat per second is heard. If the prong of *A* is further loaded with wax, the number of beats per second is still 1. Determine the frequency of *B*. What is the final frequency of *A*? **[252, 251]**

26.39. DOPPLER EFFECT

So far we have been studying situations in which the source of waves and the observer are both stationary. In this case, the frequency of the waves as noted by the observer is the same as the actual frequency of the waves. However, if there is a relative motion between the source of waves and the observer, the frequency of the waves as noted by the observer is different from the actual frequency of the waves. This apparent shift in frequency is called Doppler effect, after Christian Doppler (1803 – 1853), the Austrian physicist who first explained this phenomenon. The Doppler effect occurs for all types of waves whenever there is a relative motion between the source of waves and the observer.

The apparent change in the observed frequency of a wave due to the relative motion between the source of waves and the observer is called **Doppler effect.**

The Doppler effect occurs when a source of waves moves relative to the observer. You have certainly experienced this effect with sound. You are familiar with the rise and subsequent drop in pitch of an automobile horn as it approaches and then passes. In other words, the frequency of sound is *raised when the source of sound approaches you and lowered when the source is moving away from you. The reason is simple. As the car approaches a stationary listener, the sound waves are crowded together, causing a decrease in wavelength and increase in frequency of the sound heard. After the car has passed and moving away from the listener, the sound waves spread out. As a result, the wavelength is increased and the observed frequency is decreased. To understand the cause of the Doppler effect, study Fig. 26.29 which shows a source of sound moving to the right toward a stationary observer *B*. The source radiates spherical waves, shown here as circles. Each wave crest moves out as expanding sphere but since the source is moving, it emits each successive wave at a different location. As a result, the waves moving in the same direction as the source are crowded together while those moving in the opposite direction are spread farther apart. The wave speed is constant whether the source moves or not. Thus where the wavelength is shortened (for observer *B*), the frequency is increased and where the wavelength is lengthened (for observer *A*), the frequency is reduced.

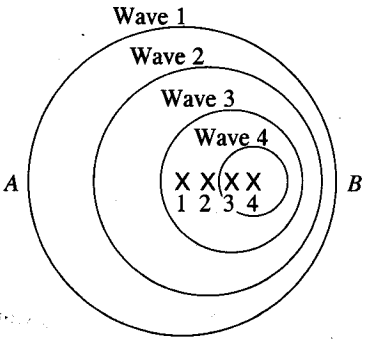


Fig. 26.29

* Note that the actual frequency of the source does not change. However, due to the relative motion between the source and the observer, the latter hears a change in frequency.

26.40. CALCULATION OF APPARENT FREQUENCY

Let V = Velocity of sound in air

u_S = Velocity of the source of sound S

u_O = Velocity of the observer O

f = Actual or true frequency of the source

(i) Source moving towards stationary observer. If the source S were stationary, then f waves sent out in one second towards the observer O would occupy a distance V and the wavelength would be V/f as shown in Fig. 26.30 (*i*). If S moves with a velocity u_S towards O, then f waves sent out occupy a distance $(V - u_S)$ because S has moved a distance u_S towards O in one second as shown in Fig. 26.30 (*ii*). Thus wavelength λ' of the wave reaching O is now

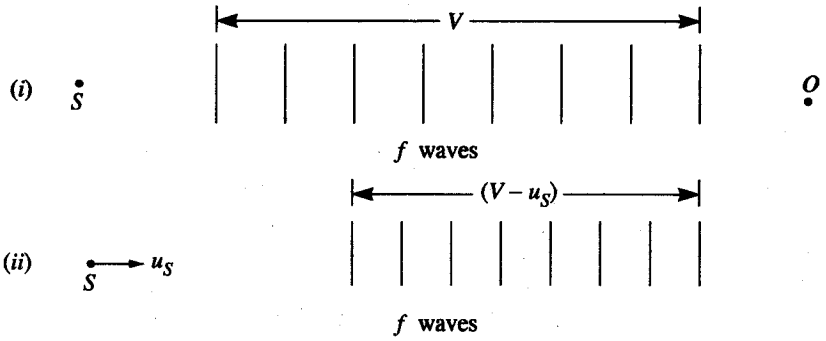
$$\lambda' = \frac{V - u_S}{f}$$

But velocity of sound waves = V

$$\therefore \quad \text{Apparent frequency, } f' = \frac{\text{Velocity of sound relative to } O}{\text{Wavelength of waves reaching } O}$$

$$= \frac{V}{\lambda'} = \frac{V}{(V - u_S)/f}$$

$$\therefore \quad f' = \frac{V}{V - u_S} \times f \qquad \ldots(i)$$



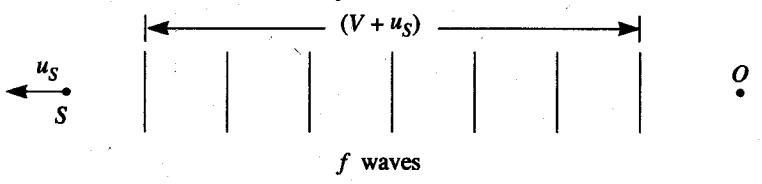
Source moving towards stationary observer

Fig. 26.30

Since $(V - u_S)$ is less than V, f' is greater than f. *Therefore, the apparent frequency increases when the source of sound moves towards the stationary observer.*

(ii) Source moving away from stationary observer. In this case, the f waves sent out towards O in one second occupy a distance $(V + u_S)$ as shown in Fig. 26.31. Therefore, the wavelength λ' of the waves reaching O is

$$\lambda' = \frac{V + u_S}{f}$$



Source moving away from stationary observer

Fig. 26.31

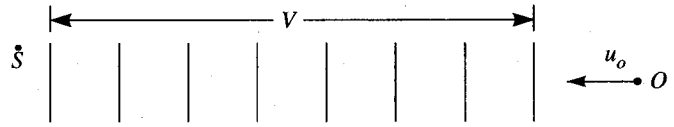
$$\therefore \quad \text{Apparent frequency, } f' = \frac{V}{\lambda'} = \frac{V}{(V+u_S)/f}$$

$$\therefore \quad f' = \frac{V}{V+u_S} \times f \qquad ...(ii)$$

Since $(V + u_S)$ is greater than V, f' is less than f. *Therefore, the apparent frequency decreases when the source of sound moves away from the stationary observer.*

(*iii*) **Source stationary and observer moving towards it.** Since the source is stationary, the f waves sent out by S towards the moving observer O occupy a distance V as shown in Fig. 26.32. Therefore, the wavelength of the waves reaching O is V/f. Note that unlike the cases already discussed, the wavelength is unaltered.

The velocity of the sound waves relative to O is not V because O is moving relative to the source. In this case, the velocity of sound waves relative to O is $(V + u_O)$.



f waves

Observer moving towards stationary source

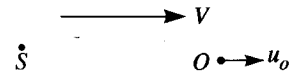
Fig. 26.32

$$\therefore \quad \text{Apparent frequency, } f' = \frac{\text{Velocity of sound relative to } O}{\text{Wavelength of waves reaching } O} = \frac{V+u_O}{V/f}$$

$$\therefore \quad f' = \frac{V+u_O}{V} \times f \qquad ...(iii)$$

Since $(V + u_O)$ is greater than V, f' is greater than f. *Thus when the observer is moving towards a stationary source of sound, the apparent frequency is increased.* Note that motion of the observer has no effect on the wavelength. An observer moving towards a stationary source hears an increase in frequency because he intercepts the crests more frequently than he would if he were stationary.

(*iv*) **Source stationary and observer moving away from it.** This case is shown in Fig. 26.33. As in the case just considered, the wavelength of the waves reaching O is unaltered and is equal to V/f.



Observer moving away from stationary source

Fig. 26.33

The velocity of sound waves relative to O is $(V - u_O)$.

$$\therefore \quad \text{Apparent frequency, } f' = \frac{\text{Velocity of sound relative to } O}{\text{Wavelength of waves reaching } O} = \frac{V-u_O}{V/f}$$

$$\therefore \quad f' = \frac{V-u_O}{V} \times f \qquad ...(iv)$$

As $(V - u_O)$ is less than V, f' is less than f. *Thus when the observer is moving away from a stationary source of sound, the apparent frequency is decreased.*

(*v*) **Source and observer both moving.** If the source and the observer are both moving, the apparent frequency f' can be found from the formula :

$$f' = \frac{V'}{\lambda'}$$

where V' = Velocity of sound waves relative to the observer

λ' = Wavelength of the waves reaching the observer

This formula can be used to find the apparent frequency in any of the cases considered before.

(*a*) **When source and observer are moving in the same direction.** Suppose that the observer has a velocity u_O, the source a velocity u_S and that both are moving in the same direction. Then,

$$V' = V - u_O \quad \text{and} \quad \lambda' = (V - u_S)/f$$

Note that λ' has already been deduced in eq. (*i*) above.

$$\therefore \quad f' = \frac{V'}{\lambda'} = \frac{V - u_O}{(V - u_S)/f}$$

or
$$f' = \frac{V - u_O}{V - u_S} \times f \qquad ...(v)$$

In this case, f' may be greater or less than f depending upon u_O is less than or greater than u_S. In case $u_O = u_S$ (*i.e.* both observer and source are moving with the same velocity) then,

$$f' = \frac{V - u_O}{V - u_O} \times f = f$$

(*b*) **When source and observer are moving towards each other.** In this case,

$$V' = V + u_O \quad \text{and} \quad \lambda' = (V - u_S)/f$$

$$\therefore \quad f' = \frac{V'}{\lambda'} = \frac{V + u_O}{(V - u_S)/f}$$

or
$$f' = \frac{V + u_O}{V - u_S} \times f \qquad ...(vi)$$

(*c*) **When source and observer are moving away from each other.** In that case,

$$V' = V - u_O \quad \text{and} \quad \lambda' = (V + u_S)/f$$

$$\therefore \quad f' = \frac{V'}{\lambda'} = \frac{V - u_O}{(V + u_S)/f}$$

or
$$f' = \frac{V - u_O}{V + u_S} \times f \qquad ...(vii)$$

Notes. (*i*) It should be noted that the motion of the observer affects only V', the velocity of the sound waves relative to the observer. However, the motion of the source affects only λ', the wavelength of the waves reaching the observer.

(*ii*) Suppose the wind is also blowing with a velocity v_w. If the wind is blowing in the same direction as sound waves, then change V to $V + v_w$ in all above formulas. If the wind is blowing in a direction opposite to the direction of sound waves, then change V to $V - v_w$ in all the above formulas.

Example 26.37. **A cyclist and a railway train are approaching each other. The cyclist is moving at 10 m s^{-1} and the train at 20 m s^{-1}. The engine driver sounds a warning siren at a frequency of 480 Hz. Calculate the frequency of the note heard by the cyclist (*i*) before and (*ii*) after the train has passed by. Speed of sound in air = 340 ms^{-1}.**

Solution. Here f = 480 Hz, V = 340 m s^{-1} ; u_O = 10 ms^{-1} ; u_S = 20 ms^{-1} .

(*i*) **Before train has passed**

Velocity of sound relative to observer, $V' = V + u_O = 340 + 10 = 350$ m s^{-1}

Wavelength of waves reaching observer, $\lambda' = \frac{V-u_S}{f} = \frac{340-20}{480} = \frac{320}{480}$ m

$\therefore$ Apparent frequency, $f' = \frac{V'}{\lambda'} = \frac{350\times480}{320} = 525$ Hz

(*ii*) **After the train has passed**

Velocity of sound relative to observer, $V' = V - u_O = 340 - 10 = 330$ m s^{-1}

Wavelength of waves reaching observer, $\lambda' = \frac{V+u_S}{f} = \frac{340+20}{480} = \frac{360}{480}$ m

$\therefore$ Apparent frequency, $f' = \frac{V'}{\lambda'} = \frac{330\times480}{360} = 440$ Hz

Example 26.38. A policeman on duty detects a drop of 10% in the pitch of the horn of a motor car as it crosses him. If the velocity of sound is 330 m s^{-1}, calculate the speed of the car.

Solution. Here V = 330 m s^{-1} ; f = 100 Hz ; f' = 90 Hz ; u_S = ?

Velocity of sound relative to the policeman is

$$V' = V$$

Wavelength of the waves reaching the policeman is

$$\lambda' = \frac{V+u_S}{f}$$

$\therefore$ $$f' = \frac{V'}{\lambda'} = \frac{V}{V+u_S}\times f$$

or $$90 = \frac{330}{330+u_S}\times100 \quad \text{or} \quad u_S = 36.67 \text{ m } s^{-1}$$

Example 26.39. A source of sound of frequency 256 Hz is moving rapidly towards a wall with a velocity of 5 ms^{-1}. How many beats per second will be heard if sound travels at a speed of 330 m s^{-1}?

Solution. Here V = 330 m s^{-1}; f = 256 Hz ; u_S = 5 m s^{-1} ; u_O = 5 m s^{-1}

$$\text{Apparent frequency, } f' = \frac{V+u_O}{V-u_S}\times f = \frac{330+5}{330-5}\times256 = 263.88 \text{ Hz}$$

$\therefore$ Number of beats per second = $f' - f$ = 263.88 − 256 = 7.88 Hz

Example 26.40. A train approaches a stationary observer, the velocity of train being 1/20 of the velocity of sound. A sharp blast is blown with the whistle of the engine at equal intervals of a second. Find the time interval between successive blasts as heard by the observer.

Solution. Let V be the velocity of sound.

$\therefore$ Velocity of train, $u_S = V/20$

Since the blast is blown at intervals of one second, actual frequency of blast, f = 1. Since the source of sound is approaching the stationary observer, the apparent frequency f' of blast as heard by the observer is

$$f' = \frac{V}{V-u_S}\times f = \frac{V}{V-\frac{V}{20}}\times1 = \frac{20}{19}$$

$\therefore$ Time interval between two successive blasts

$$= \frac{1}{f'} = \frac{19}{20} \text{ second}$$

Example 26.41. Find the velocity of source of sound when the frequency appears to be (*i*) double (*ii*) half the original frequency to a stationary listener.

Solution. Let V be the velocity of sound and u_S the velocity of source of sound.

(*i*) Since the apparent frequency is doubled, it is possible only if the source is approaching the stationary listener.

$$\text{Apparent frequency, } f' = \frac{V}{V - u_S} f$$

or $$2f = \frac{V}{V - u_S} f \quad \text{or} \quad u_S = V/2$$

Therefore, the source approaches the stationary listener with half the velocity of sound.

(*ii*) Since the apparent frequency is halved, it is possible only if the source of sound is moving away from the stationary listener.

$$\text{Apparent frequency, } f' = \frac{V}{V + u_S} f \quad \text{or} \quad \frac{f}{2} = \frac{V}{V + u_S} f \quad \therefore \; u_S = V$$

Therefore, the source moves away from the stationary listener with velocity equal to the velocity of sound.

Example 26.42. A whistle emitting a sound of frequency 440 Hz is tied to a string of 1.5 m length and rotated with an angular velocity of 20 rad s^{-1} in the horizontal plane. Calculate the range of frequencies heard by an observer stationed at a large distance from the whistle. Take velocity of sound = 330 ms^{-1}.

Solution. Fig. 26.34 shows the conditions of the problem. The observer is at point O situated at a large distance from the whistle revolving in a horizontal circular path of radius r (= 1.5 m) and centre C. When the whistle is at position A, it is moving away from the stationary observer, while when it is at position B, it is moving towards the stationary observer.

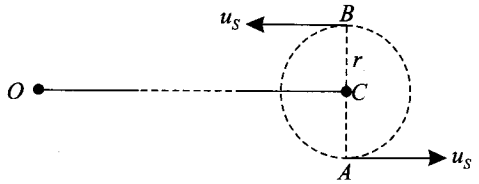


Fig. 26.34

Apparent frequency heard by the stationary observer when whistle is at position A is

$$f' = \frac{V}{V + u_S} f \qquad \text{...source moving away from stationary observer}$$

Here, V = 330 m s^{-1}; $u_S = r\omega$ = 1.5 × 20 = 30 ms^{-1}; f = 440 Hz

$$\therefore \qquad f' = \frac{330}{330 + 30} \times 440 = 403.3 \text{ Hz}$$

Apparent frequency heard by the stationary observer when the whistle is at position B is

$$f'' = \frac{V}{V - u_S} f = \frac{330}{330 - 30} \times 440 = 484 \text{ Hz}$$

Therefore, the stationary observer will hear the sound of whistle in the frequency range **403.3 Hz to 484 Hz.**

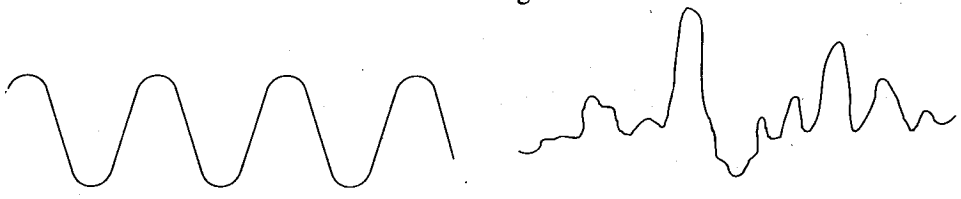
PROBLEMS FOR PRACTICE

1. A police car horn emits a 250 Hz frequency when sitting still. What frequency does a stationary observer hear if the police car sounds its horn while approaching at a speed of 27 m s^{-1}? What frequency is heard if the horn is sounded as the car is leaving at 27 m s^{-1}? Speed of sound in air = 340 m s^{-1}. **[272 Hz ; 232 Hz]**
2. A car travelling at 20 m s^{-1} sounds its horn which has a frequency of 600 Hz. What frequency is heard by a stationary distant observer as the car approaches? What frequency is heard after the car has passed? Velocity of sound = 340 ms^{-1}. **[1091 Hz ; 909 Hz]**
3. A train stands at a platform blowing a whistle of frequency 400 Hz in still air.
   (*i*) What is the frequency of the whistle heard by a man running (*a*) towards the engine at 10 ms^{-1} (*b*) away from the engine at 10 ms^{-1}?
   (*ii*) What is speed of sound in each case?
   (*iii*) What is the wavelength of sound received by the running man in each case?
   [(*i*) 411.8 Hz; 388.2 Hz (*ii*) 350 ms^{-1} ; 330 ms^{-1} (*iii*) 0.875 m ; unchanged]
4. An observer travelling with a constant velocity of 20 ms^{-1} passes close to a stationary source of sound and notices that there is a change of frequency of 50 Hz as he passes the source. What is the frequency of the source? Speed of sound in air = 340 ms^{-1}. **[425 Hz]**
5. A 440 Hz source has been sounding in air for a long time. (*i*) What frequency will you hear if you move away from it at 0.90 times the speed of sound? (*ii*) What frequency will you hear if you move away from it at the speed of sound? **[(*i*) 44 Hz (*ii*) You will not hear it]**
6. An observer is moving towards a wall at 2 m s^{-1}. He hears a sound from a source at some distance behind him directly as well as after reflection from the wall. Calculate the beat frequency between the two sounds if true frequency of the source is 680 Hz. Velocity of sound = 340 m s^{-1}. **[8 Hz]**
7. A whistling engine, whose frequency is 1240 Hz is going toward a hill with a speed of 72 km/h. The driver of the engine hears an echo of the whistle. Find the frequency of the echo as heard by him. **[1400 Hz]**

26.41. MUSICAL SOUND AND NOISE

All sounds which produce sensation of hearing may be roughly divided into two classes *viz.*, (*i*) musical sound and (*ii*) noise. The distinction between a musical sound and a noise is subjective *i.e.*, it depends upon the senses of a person. A sound which is musical to someone may be a noise to others.

(*i*) **Musical sound.** *It is a pleasant, continuous and uniform sound produced by regular and periodic vibrations e.g.* sound produced by a tuning fork, flute, piano etc. Fig. 26.35 (*i*) shows the graph of a typical musical sound. Note that musical sound is regular, periodic and continuous. There are no sudden changes in loudness.



(*i*) (*ii*)

Fig. 26.35

(*ii*) **Noise.** *It is an unpleasant, discontinuous and non-uniform sound produced by irregular succession of disturbances.* All sounds other than musical notes are noises *e.g.*, sound produced by a falling brick, sound produced by the clapping of two wooden blocks etc. Fig. 26.35 (*ii*) shows the graph of a typical noise. Note that a noise is irregular and discontinuous. There are sudden changes in loudness.

26.42. CHARACTERISTICS OF A MUSICAL SOUND

There are three fundamental characteristics of a musical sound *viz.*

1. Pitch 2. Loudness 3. Quality

One musical sound can be distinguished from another by one or more of the above characteristics. It may be noted that these characteristics (*i.e.* pitch, loudness and quality) of a musical sound have sensory magnitude *i.e.* only our sense of hearing can measure them. However, their objective measurements are made by other physical quantities.

Subjective measurement	Objective measurement
Pitch	Frequency
Loudness	Intensity
Quality	Waveform

1. Pitch. *It is the characteristic of a musical sound by which we can distinguish a shrill sound from a grave (hoarse) sound.*

Pitch is a sensation which determines the shrillness of a sound. Pitch is subjective and cannot be measured quantitatively. The pitch of a sound depends upon frequency (or wavelength). If the frequency of a sound is high, its pitch is also high and the sound is said to be shrill. If the frequency of sound is low, its pitch is low and the sound is said to be grave or flat. Thus, higher the frequency of a musical sound, higher is its pitch and vice-versa.

Examples (*i*) The voice of children and ladies is shrill because of higher pitch.

(*ii*) The voice of an old man is hoarse because of low pitch.

(*iii*) The sound produced by a mosquito is of high pitch and is, therefore, shrill.

2. Loudness. *It is the characteristic of a musical sound by which a loud sound can be distinguished from a faint sound even though the two have the same pitch.*

Loudness is a sensation which determines the degree of sound power produced in the ear. It is subjective and cannot be measured quantitatively. Loudness depends upon the intensity of sound near the ear. If the intensity of sound near the ear is high, loudness will be more. The intensity and hence loudness of sound depends upon the following factors :

(*i*) *Amplitude of vibration of the source.* The greater the amplitude of vibration of the source, the greater is the intensity (and hence loudness) of sound and vice-versa.

(*ii*) *Motion of the medium.* If wind is blowing in the direction of propagation of sound, loudness is increased. On the other hand, if wind is blowing in a direction opposite to the propagation of sound, loudness is decreased.

(*iii*) *Presence of other bodies.* The loudness of sound is increased due to the presence of other bodies near the source of sound. For example, sound appears to be much louder inside a hall than in open air. It is because the walls, roof, floor etc. reflect sound. Consequently, loudness is increased.

(*iv*) *Surface area of vibrating body.* The greater the surface area of the vibrating body, the greater is the loudness of sound.

(*v*) *Frequency of source.* The loudness of a vibrating body is directly proportional to the square of the frequency of the vibrating body.

3. Quality or Timbre. *It is the characteristic of a musical sound which enables us to distinguish between two sounds of the same pitch and loudness produced by two different sources.* It depends upon the waveform of the sound.

We can always distinguish the sounds produced by different instruments in an orchestra, even though they may be producing the same tone with equal loudness. This property of a musical sound is called its quality. The sound from an instrument or any other source does not contain note of a single frequency but contains vibrations of different frequencies and different amplitudes. The vibration of the lowest frequency has the greatest amplitude and is called *fundamental vibration* while vibrations of frequency integral multiple of it (*i.e.* fundamental frequency) are called **harmonics** or **subsidiary vibrations** since their amplitudes are very small. The waveform of a sound depends upon the presence of these harmonics. Thus, the quality of a sound depends upon the number of harmonics and their prominence. Different instruments emit different harmonics and hence the quality of sound produced is different.

Examples. (*i*) A note played on a piano has a large number of harmonics while the same note when played on a flute contains only a few harmonics. For this reason, musical sound from a piano is more rich (*i.e.*, of better quality) than that of a flute.

(*ii*) When stringed instruments (*e.g.*, violin, sitar etc.) are played, they are plucked near one end instead of in the middle. It is because plucking near the end produces more harmonics and gives a richer sound.

26.43. INTENSITY AND INTENSITY LEVEL OF SOUND

All waves carry energy along with them. Sound waves are no exception. Suppose a sound wave is travelling in the propagation direction as shown in Fig. 26.36. We define the intensity in terms of energy it carries. To be precise, we consider a unit area perpendicular to the propagation direction as shown. We define the intensity I of the wave to be the energy the wave carries per second through this unit area. Since power is energy produced per unit time, **Sound intensity** ***I*** *is the power passing through a unit area perpendicular to the direction of wave propagation.*

$$I = \frac{\text{Power}}{\text{Area}} = \frac{P}{A}$$

The SI unit of sound intensity is watt per square metre (W/m^2).

One of the many interesting properties of the ear is that its response to various levels of intensity is roughly proportional to the *logarithm* of I. That is our perception of the relative *loudness* of two sounds of intensities I_2 and I_1 is not I_2/I_1 but rather $\log_{10}(I_2/I_1)$. Thus a convenient scale for expressing loudness (called **intensity level** or **sound level**) is the **decibel** scale defined by ;

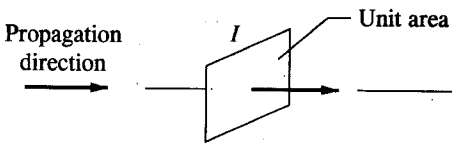


Fig. 26.36

$$\text{Intensity level in decibel (dB)} = 10 \log_{10} \frac{I}{I_0}$$

where I = Intensity (W/m^2) of sound under consideration

I_0 = Intensity of least audible sound, 10^{-12} W/m^2

$\therefore$ Intensity level of least audible sound is

$$10 \log_{10} \frac{I}{I_0} = 10 \log_{10} \frac{10^{-12}}{10^{-12}} = 0 \text{dB}$$

If a sound has intensity 1 W/m^2, its intensity level

$$= 10 \log_{10} \frac{I}{I_0} = 10 \log_{10} \frac{1}{10^{-12}} = 10 \log_{10} 10^{12} = 120 \text{ dB}$$

The scale compresses the 14 orders of magnitude into a scale of 0 to 120 dB. The adjoining table shows the values corresponding to the various intensities. Note that 1B (bel) = 10 dB and is named after Alexander Graham Bell, the inventor of telephone.

Intensity (W/m^2)	Intensity level (dB)
10^{-12}	0
10^{-11}	10
10^{-10}	20
10^{-9}	30
:	:
:	:
10^{-1}	110
1	120
10	130

Example 26.43. Find the sound level in decibels of a sound wave that has an intensity of 10^{-5} W/m^2.

Solution. Sound level (in dB) = $10 \log_{10} \frac{I}{I_0}$

Here $I = 10^{-5}$ W/m^2 ; $I_0 = 10^{-12}$ W/m^2

$\therefore$ Sound level (in dB) = $10\log_{10} \frac{10^{-5}}{10^{-12}} = 10\log_{10} 10^7 = 70$ dB

Example 26.44. Find the intensity level of a sound that has an intensity level of 50 dB.

Solution. Let I be the intensity of sound in W/m^2.

Sound level (in dB) = $10\log_{10}\left(\frac{I}{I_0}\right)$ or $50 = 10\log_{10}\left(\frac{I}{I_0}\right)$

$\therefore$ $\log_{10} \frac{I}{I_0} = 5$ or $\frac{I}{I_0} = 10^5$ $\therefore$ $I = I_0 \times 10^5 = 10^{-12} \times 10^5 = 10^{-7}$ W/m^2

26.44. ACOUSTICS OF BUILDINGS

The branch of physics which deals with the production of best sound effects in public buildings (e.g. cinema halls, lecture halls, auditoriums etc.) is called **acoustics of buildings.**

Practically little or nothing was known of acoustics of buildings prior to the year 1900. Big halls and auditoriums were built with the hope that they would prove suitable for the purpose for which they were intended. However, these halls and auditoriums proved failure from acoustic point of view *i.e.* they suffered from such defects as lengthening out of the sound of the speaker, low intelligibility of speech, echos etc. It was in the year 1900 that Wallace C. Sabine started a systematic study of acoustics of buildings in a bid to provide best sound effects in public buildings. The science of acoustics of buildings has now achieved a unique place in the design of modern buildings.

26.45. SOUND WAVES IN A HALL

When a speaker speaks or music is played in a hall, sound waves are produced which spread out until they strike the walls, ceiling or floor of the hall. Here, the sound waves are (*i*) partly reflected and (*ii*) partly absorbed as shown in Fig. 26.37. The amount of reflection and absorption depends upon the character of the surface. A hard and smooth surface reflects most of the sound and absorbs little of it. A porous surface built of substance like felt absorbs most of the sound and reflects little of it.

The reflection of sound in a hall poses a serious problem. It is now commonly agreed that sound suffers some 300 *reflections to and fro in a room of ordinary size before it actually becomes extinct. These successive reflections of sound result in a series of waves passing the listener's

* At every reflection, the sound loses a part of its energy, mostly due to friction, until finally it loses all energy and becomes inaudible.

ears, giving the impression of persistence or prolongation of sound. This considerably reduces the distinctiveness or clarity of the sound. It is evident that greater the absorption of sound in a hall, the lesser the reflection of sound and hence more is the clarity.

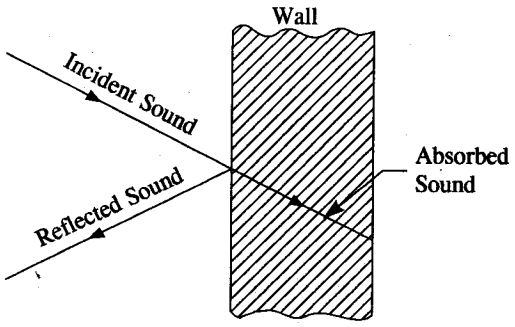


Fig. 26.37

26.46. ABSORBING POWER OF A SURFACE

The ratio of sound energy absorbed by a surface to the total sound energy incident on it is called absorbing power or *co-efficient of absorption of the surface.* It is represented by *'a' i.e.*

$$\text{Absorbing power, } a = \frac{\text{Sound energy absorbed by the surface}}{\text{Total sound energy incident on the surface}}$$

The absorbing power of a surface depends upon the character of the surface. Some surfaces absorb more sound than others.

(*i*) A porous surface has more absorbing power than that of a smooth surface.

(*ii*) *An open window is a perfect sound absorber.* It is because whole of the sound energy passes through it and none is reflected.

Practical definition of *a*. Since open window is a perfect sound absorber, it is a usual practice to express the co-efficient of absorption or absorbing power of a surface in terms of *open window unit* (O.W.U.) ; one square metre of an open window being taken as the unit of sound absorption.

The **co-efficient of absorption** *of a surface is the ratio of sound energy absorbed by $1m^2$ of the surface to that absorbed by $1m^2$ of an open window i.e.,*

$$a = \frac{\text{Sound energy absorbed by 1 m}^2 \text{ the surface}}{\text{Sound energy absorbed by 1m}^2 \text{ of open window}}$$

Clearly, the value of co-efficient of absorption will vary between 0 and 1 ; being 1 for open window. For example, the co-efficient of absorption of plastered wall is 0.03 O.W.U. It means that sound energy absorbed by 1 m^2 of plastered wall will be 0.03 times that absorbed by 1 m^2 of an open window. Below is given the values of co-efficient of absorption of some commonly used surfaces.

S.No.	Surface	Co-efficient of absorption (O.W.U.)
1.	Plastered wall	0.03
2.	Wooden floor	0.06
3.	Plastered ceiling	0.04
4.	Wooden door	0.06
5.	Cushioned seat	1.0/seat
6.	Audience	4.7/head

Note. The clothes of the audience act as sound absorbers. Women are better sound absorbers due to the nature and extent of their clothing.

26.47. REVERBERATION

The persistence or prolongation of sound in a room or hall (due to successive reflections from surfaces) after the original sound has ceased is called **reverberation**.

When sound is produced in an open space, it is heard by the listener only once, as the waves travel across him. However, when sound is produced in a room or hall, it is observed that sound persists even after the original sound has ceased. This is due to the repeated reflection of sound from the walls, floor, ceiling etc. The reflected sound waves go on traversing the room (or hall) a number of times before they become too weak to be heard. This persistence of sound after the original sound has ceased is called reverberation. Reverberation tends to prolong the sound and considerably impairs its distinctness or clarity.

Reverberation time. *The time for which the sound persists in a room or hall until it dies off or falls below the threshold of audibility is called reverberation time.*

Reverberation time is measured from the instant of generation (in case of short sounds) or stopping of the source (in case of continuous sounds). In order to produce best sound effects, the reverberation time should be as small as possible. The reverberation time of a hall or room can be decreased by increasing sound absorption in it. When sound falls on the walls, seats or audience, it is either reflected or absorbed. If a great part of it is absorbed, the reverberation time will be small. On the other hand, if most of the sound is reflected by the walls, floor or audience, the sound will continue for some time after the source has been discontinued. In such a case, the reverberation time will be large.

26.48. STANDARD REVERBERATION TIME

The **standard reverberation time** *of a room or hall is the time required for the intensity of sound to fall one-millionth (i.e. 10^{-6}th) of its original intensity after the original sound is cut off.*

The standard reverberation time t of a room or hall, as deduced by **Wallace C. Sabine**, is given by ;

$$\text{Standard reverberation time, } t = \frac{0.16V}{\sum as} \text{ seconds}$$

where

V = Volume of the room or hall in m^3

s = Area of reflecting surface in m^2

a = Co-efficient of absorption

$\sum as$ = Total surface absorption

= Sum of the products of co-efficients of absorption and the areas of the corresponding surfaces

If $a_1, a_2, a_3, ...a_n$ are the co-efficients of absorption of various surfaces inside the room or hall and $s_1, s_2, s_3, ... s_n$ are the respective areas of the surfaces, then,

$$\sum as = a_1 s_1 + a_2 s_2 + a_3 s_3 + ... + a_n s_n$$

26.49. METHODS OF REDUCING REVERBERATION TIME

The reverberation time of a room or hall can be decreased by increasing the absorption of sound in it. This can be done by having :

(*i*) a few open windows in the room or hall.

(*ii*) sound absorbing materials in the room or hall *e.g.* perforated cardboards, heavy curtains, pictures and maps on the walls, carpet on the floor etc.

(*iii*) a good number of audience in the room or hall. Each member of the audience has absorption of 4.7 O.W.U.

(*iv*) cushioned seats. Such seats not only provide greater comfort but also act as good absorbers of sound.

(*v*) no large concave, spherical or cylindrical surfaces on the walls or ceiling of the hall or auditorium. This will avoid excessive reflections of sound.

It may be noted that reverberation time should be reduced only to a reasonable limit. If the reverberation time is too much reduced, it may result in **deadening effect** (*i.e.* dullness). In fact, reverberation time should not be allowed to fall below a certain value, called the **optimum time**. This value is found to be 1 to 2 seconds for music and 0.5 to 1 second for speech.

26.50. FACTORS AFFECTING ACOUSTICS OF BUILDINGS

While designing a hall or auditorium, the following factors must be taken into account for best sound effects :

1. Reverberation time
2. Intensity
3. Interference
4. Focussing
5. Extraneous noise
6. Resonance

1. Reverberation time. Reverberation time is the most important single factor affecting the acoustics of a hall or auditorium. It should be small in order to achieve best sound effects. This can be done by the ways explained in Art. 26.49. However, reverberation time should not be allowed to fall below the optimum value—1 to 2 seconds for music and 0.5 to 1 second for speech.

2. Intensity. The intensity of sound in the auditorium or hall should be sufficiently high and uniform. In order to achieve this objective, the following points should be kept in mind :

(*i*) The length of the hall or auditorium should be more as compared to its breadth.

(*ii*) The height of the ceiling should be low.

(*iii*) The wall surface at speaker's end should be given a parabolic shape as shown in Fig. 26.38. This ensures approximately uniform sound intensity in every part of the hall or auditorium.

(*iv*) We should use large sounding boards arranged close to the speaker and facing the audience.

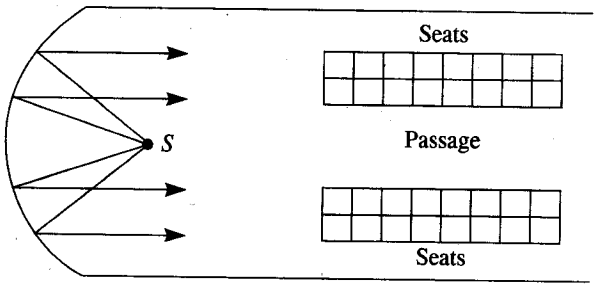


Fig. 26.38

3. Interference. When sound is produced in a hall or auditorium, the listener may receive the sound in two ways :

(*i*) sound directly from the source.

(*ii*) sound after reflection from walls, ceiling or floor.

If the two waves reach the listener in the same phase, constructive interference occurs and the intensity of sound will be increased. If, on the other hand, the two waves reach the listener in the opposite phase, destructive interference occurs and the intensity of sound will be decreased. There may be certain regions in the hall or auditorium where no sound is heard during destructive interference. Such regions are called *silence zones.*

Both constructive and destructive interference in a hall or auditorium result in uneven intensity distribution and are, therefore, undesirable from acoustic point of view. The problem of interference can be reduced to a considerable extent by using suitable absorbing materials inside the hall or auditorium.

4. Focussing. The presence of large concave, spherical or cylindrical surfaces on the walls or ceiling of a hall or auditorium may give rise to undesirable focussing effects. Fig. 26.39 shows the concave surface *AB* of the wall of a hall. A person at *L* will receive sound waves both directly from the speaker at *S* and by focussing from the concave surface. Consequently, the intensity of sound at *L* is increased. Not only that, there will be phase difference in the sound waves reaching *L*, causing interference. The result is that focussing effect may cause uneven distribution of intensity.

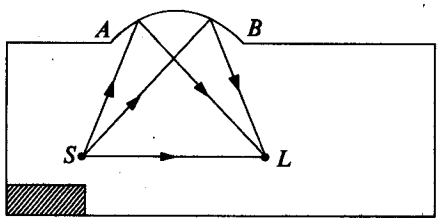


Fig. 26.39

In order to avoid focussing effects in a hall or auditorium, curved walls or corners should be avoided.

5. Extraneous noise. The extraneous noise entering a hall or auditorium may be either (*i*) air-borne or (*ii*) structure-borne

The air-borne noise originates in air and travels through the air to the boundaries of the hall or auditorium. The air-borne noise cannot be eliminated completely unless the hall is sealed off to all ventilation. This, of course, is absurd and absolutely out of question. However, we can minimise air-borne noise by (*a*) using double windows or doors, each with its own framework (*b*) baffles in the ventilator duct (*c*) controlling the speed of fans.

Structure-borne noise originates in the framework of the hall or auditorium by motors, elevators etc. This type of noise can be eliminated to a considerable extent by breaking the continuity of the walls by interposing absorbing materials like felt, cork etc., at different places along the length of the wall.

6. Resonance. Resonance in a hall or auditorium may produce undesirable effects. If resonance occurs for any note of audio frequency, the intensity of that note will be increased. It has been found that resonant frequency of a large room is roughly inversely proportional to the square root of the volume of the room. Consequently, in halls of ordinary size, the resonant frequency lies below the limit of audibility and is thus quite harmless.

Example 26.45. **A cinema hall has a volume of 3000 m^3 and total surface absorption is 200 O.W.U. Find the total reverberation time.**

Solution. Reverberation time, $t = \dfrac{0.16V}{\Sigma as}$

Here, $V = 3000\ m^3$; $\Sigma as = 200$ O.W.U.

$$\therefore \qquad t = \frac{0.16 \times 3000}{200} = 2.4 \text{ seconds}$$

CONCEPTUAL QUESTIONS

Q.1. If a water wave oscillates up and down three times each second and the distance between wave crests is 2 m, what is its frequency? Its wavelength? Its wave speed?

Ans. The frequency of the wave is 3 Hz, its wavelength is 2m and its wave speed = frequency × wavelength = $3 \times 2 = 6\ ms^{-1}$.

Q.2. When a source moves towards you, is there an increase or a decrease in wave speed?

Ans. Neither! It is the frequency of a wave that undergoes a change where there is a motion of the source, not the wave speed. Be clear about the distinction between frequency and speed. How frequently a wave vibrates is altogether different from how fast it moves from one place to another.

Q.3. What is the approximate distance of a thunderstorm when you note a 3 s delay between the flash of lightning and the sound of thunder?

Ans. Assuming the speed of sound in air is about 340 m s^{-1}, in 3 s it will travel (340 × 3) = 1020 m. We can assume no time delay for light so that storm is slightly more than 1 km away.

Q.4. What is the beat frequency when a 262 Hz and 266 Hz tuning forks are sounded together?

Ans. For the 262 Hz and 266 Hz forks, the ear will hear 264 Hz which will beat at 4 Hz (266 – 262). Note that beat frequencies greater than 10 Hz are too rapid to be heard normally.

Q.5. Is it correct to say that in every case, without exception, any radio wave travels faster than any sound wave?

Ans. Yes, because any radio wave travels at the speed of light. A radio wave is an electromagnetic wave – in a very real sense a low frequency light wave. On the other hand, a sound wave is a mechanical disturbance propagated through a material medium by material particles that vibrate against one another. In air, the speed of sound is about 340 ms^{-1}, about one-millionth of speed of a radio wave. Sound travels faster in other media but in no case at the speed of light. No sound can travel as fast as light.

Q.6. The velocity of a sound in a tube containing air at 27°C and a pressure of 76 cm of mercury is 330 ms^{-1} What will be its velocity if the pressure is increased to 100 cm of mercury and temperature is kept constant?

Ans. At a given temperature, the velocity of sound in a gas is independent of pressure. Therefore, the velocity of sound in the tube is 330 m s^{-1} even if the pressure is increased to 100 cm of mercury.

Q.7. Why is a diver under water unable to hear the sound produced in air?

Ans. The water surface reflects the sound back into the air and only a little fraction of incident intensity (= 0.1%) is refracted through water. For this reason, the diver under water cannot hear the sound produced in air.

Q.8. Why are bells made of metal and not of wood?

Ans. The medium required for the propagation of sound must have low damping. Since the wood has high damping, it rapidly dampens the vibrations. On the other hand, metal has low damping. For this reason, bells are made of metal and not of wood.

Q.9. Sound of maximum intensity is heard successively at an interval of 0.2 second on sounding two tuning forks together. What is the difference of frequencies of the two tuning forks?

Ans. The beat period is 0.2 second so that beat frequency is f_b = 1/0.2 = 5 Hz. Therefore, the difference of frequencies of the two tuning forks is 5 Hz.

Q.10. A vessel is placed below a water tap. We can estimate the height of water level reached in the vessel from a distance simply by listening the sound. How?

Ans. The frequency of a note emitted by an air column is inversely proportional to its length. Consequently, as the length of the air column decreases, the frequency increases *i.e.* the note becomes more shrill. In our case, as the water level rises, the length of air column

in the vessel goes on decreasing and the emitted sound becomes more and more shrill. Therefore, we can estimate the height of water level reached in the vessel from a distance simply by listening the sound.

Q.11. Why does an empty vessel produce more sound than a filled one?

Ans. When an empty vessel is struck, it begins to vibrate and the air column in it also sets in vibrations with a large amplitude. However, if the filled vessel is struck, the liquid molecules are set in vibrations (instead of air molecules) which have smaller amplitude than that of air molecules. Since intensity $\propto$ (amplitude)^2, the empty vessel produces more noise than the filled one.

Q.12. Why do the stages of large auditoriums have curved backs?

Ans. This is done so that when speaker stands at or near the focus of the curved surface, his voice is rendered parallel after reflection from the concave or parabolic surface. Consequently, his voice can be heard at large distances.

Q.13. Changes in pressure do not affect the velocity of sound. Explain.

Ans. The velocity v of sound in a gas (*e.g.* air) is given by ;

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

When there is a change in pressure of the medium, there takes place a corresponding change in the density of the medium so that the ratio P/ρ remains constant. Therefore, velocity remains constant during changes in pressure.

Q.14. How can we distinguish between the music of a sitar from that of a harmonium without looking at the instruments when the sounds are of the same pitch and loudness?

Ans. The quality of a note emitted by a certain instrument depends on the presence or absence of overtones and not on the pitch and loudness. Thus the sounds from various instruments may be of the same pitch and loudness but the quality of the notes emitted by them will be different depending upon the number of overtones present in the note.

Q.15. A train of freight cars, each 10 m long, pass by you at the rate of three cars each second. What is the speed of the train?

Ans. The answer is 30 ms^{-1}. We can find the answer in two ways. Firstly, $v = d/t = 3 \times 10/1 = 30\ ms^{-1}$ where d is the distance of train that passes you in t seconds. If we compare our train to wave motion, where wavelength corresponds to 10 m and frequency is 3 Hz, then speed = frequency × wavelength = $3 \times 10 = 30\ ms^{-1}$.

Q.16. We cannot hear echo in a room. Explain.

Ans. There are two basic conditions for an echo to be heard. First, the obstacle should be rigid and of large size. Secondly, the obstacle should be atleast at a distance of *17 m from the source. Since the length of a room is generally less than 17 m, no echo is heard in the room.

Q.17. What should be the properties of the medium for the propagation of a mechanical wave (transverse or longitudinal wave)?

Ans. For the propagation of a mechanical wave, the medium must meet two requirements. First,

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* The sensation of sound persists in our ear for about 0.1 second after the exciting stimulus ceases to act. Hence the echo is heard distinctly (separate from original sound) if it reaches the ear atleast 0.1 second later than the original sound is heard. If d is the distance between the observer and the obstalce and v is the speed of sound, then time t taken by sound to reach the obstacle and then to come back is

$$t = \frac{2d}{v} \quad \text{or} \quad 0.1 = \frac{2d}{332} \quad \therefore\ d \simeq 17m$$

the medium should have *elasticity* so that particles can return to equilibrium (or mean) position after having been disturbed. The medium must have *inertia* (or mass) so that particles can store energy and overshoot their mean position.

Q.18. What is the velocity of sound in solids, liquids and gases?

Ans. Since the elasticity of solids is greater than liquids and that of liquids is greater than gases, the velocity of sound is greater in solids than liquids and gases. For example,
Velocity of sound in air = 332 ms^{-1}
Velocity of sound in water = 1500 ms^{-1}
Velocity of sound in steel = 5000 ms^{-1}

Q.19. What is the effect of frequency or wavelength on the velocity of sound?

Ans. Since the formula of velocity of sound does not involve frequency or wavelength, the sound of any frequency or wavelength travels through a given material with the same velocity.

Q.20. Does the velocity of sound in a gas depend upon the atomicity of the gas?

Ans. Yes, the velocity v of sound in a gas is given by ;

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

The atomicity of the gas determines γ $(= C_p / C_v)$.
For monoatomic gases, $\gamma = 5/3$ and for diatomic gases, $\gamma = 7/5$.

Q.21. What is the effect of (*i*) loading the prong of a tuning fork (*ii*) filing the prong of a tuning fork?

Ans. (*i*) When one of the prongs of a tuning fork is loaded with wax, it becomes heavier. As a result, its frequency decreases.
(*ii*) When the prong of a tuning fork is filed a little, it becomes lighter. As a result, its frequency increases.

Q.22. Show that Doppler effect in sound is asymmetric.

Ans. The apparent frequency of sound when source is approaching the stationary listener (say with velocity v') is not the same as the apparent frequency of sound when the listener is approaching the stationary source with velocity v'. This shows that Doppler effect in sound is asymmetric. This is proved as follows :

Apparent frequency, $f' = \frac{V}{V - u_s} \times f$ .... source approaching stationary listener

Here V= Velocity of sound in air ; u_s = velocity of sound source = v'

$$\therefore \quad f' = \frac{V}{V - v'} \times f$$

Apparent frequency, $f'' = \frac{V + u_o}{V} \times f$ ...listener approaching stationary source

$$= \frac{V + v'}{V} \times f$$

Note that f' is not the same as f''.

Q.23. Give two cases in which there is no Doppler effect in sound.

Ans. The following are the two cases in which there is no Doppler effect in sound (*i.e.* no change in frequency) :

(*i*) When the source of sound as well as the listener moves in the same direction with the same speed.

(*ii*) When one of source/listener is at the centre of the circle and the other is moving on the circle with uniform speed.

Q.24. An observer standing at the sea coast observes 54 waves reaching the coast per minute. If the wavelength of the wave is 10 m, what is wave velocity?

Ans. Frequency, $f = \dfrac{\text{No. of waves}}{\text{Time taken}} = \dfrac{54}{1 \text{ min}} = \dfrac{54}{1 \times 60} = 0.9 \text{ s}^{-1}$

Wave velocity, v = Frequency × Wavelength = $0.9 \times 10 = 9 \text{ ms}^{-1}$

Q.25. The extension in a string (obeying Hooke's law) is x. The speed of sound in the stretched string is v. What will be the speed of sound in the string if the extension is increased to 1.5 x?

Ans. The speed of sound in a stretched string, $v = \sqrt{\dfrac{T}{m}}$ or $v \propto \sqrt{T}$.

Since the string obeys Hooke's law, $T \propto x$.

$$\therefore \quad \frac{v'}{v} = \frac{\sqrt{1.5x}}{\sqrt{x}} \quad \text{or} \quad v' = \sqrt{1.5}\, v = 1.22 \text{ v}$$

Q.26. In a stationary wave, where will a man hear a louder sound – at the node or at the antinode?

Ans. The sound is heard due to variation in pressure.

$$\Delta P = \text{Elasticity} \times \text{Strain}$$

At antinodes, the amplitude is maximum and strain is minimum. On the other hand, at nodes, the amplitude is minimum and strain is maximum. Therefore, one will hear a louder sound at the node than at the antinode.

Q.27. What is the speed of sound in a gas in which two waves of wavelengths 50 cm and 50.5 cm produce 6 beats per second?

Ans. Let v be the velocity of sound in the gas.

Frequency of first wave, $f_1 = \dfrac{v}{0.5}$; Frequency of second wave, $f_2 = \dfrac{v}{0.505}$

Clearly, $f_1 > f_2 \quad \therefore \quad f_1 - f_2 = 6$

$$\therefore \quad \frac{v}{0.5} - \frac{v}{0.505} = 6 \quad \text{or} \quad v = 303 \text{ ms}^{-1}$$

Q.28. What experimental evidence can be cited to show the speed of sound in a given medium is the same for all wavelengths?

Ans. Sound waves from the different instruments of the performers on the stage are of different wavelengths. However, they reach the audience simultaneously. This shows that speed of sound in a given medium is the same for all wavelengths.

Q.29. What is the working principle of musical instruments?

Ans. All musical instruments rely on the vibration of air to produce a note. The air is usually set in vibrations either by a moving string or by the oscillations of air enclosed in a pipe.

Q.30. What is the fundamental frequency in musical instruments?

Ans. It is that frequency (lowest frequency) for which the amplitude of vibrations is the largest. The relative amplitudes of the various overtones depend on the particular instrument being played and it is this that gives an instrument its characteristic sound.

Q.31. Why are all the stringed instruments provided with hollow boxes?

Ans. The stringed instruments are provided with a hollow box called sound box. When the strings are set into vibrations, forced vibrations are produced in the sound box. Since the sound box has a large area, it sets a large volume of air into vibrations. This produces a load sound of the same frequency as that of the string.

Q.32. What is loudness of a sound?

Ans. It is the property by virtue of which a loud sound can be distinguished from a faint one, both having the same pitch. It is a characteristic of all sounds whether musical or noise. Loudness depends on intensity or amplitude of wave. Note that loudness is not the same thing as intensity. Intensity is a measurable quantity while loudness is a sensation.

Q.33. What do you mean by intensity of sound?

Ans. The intensity of sound at any point of the medium is equal to the amount of sound energy passing per second normally through unit area at that point. It is of the order of microwatt per m^2. The greater the energy carried by a sound wave, the greater is the intensity of sound.

Q.34. Why is a given sound louder in a hall than in the open?

Ans. When a sound is produced in a hall, repeated reflections of sound take place from the walls and ceiling. The reflected sounds produce constructive interference with the direct sound. As a result, the intensity of sound is increased in the hall. However, in open, there are no such reflections and the intensity of sound is low.

VERY SHORT ANSWER QUESTIONS

Q.1. What is a wave?

Ans. A wave is a form of disturbance that transfers energy from one point to another point.

Q.2. A wave carries energy. Explain.

Ans. A wave in water passing under a boat is momentarily able to lift the boat—increasing its gravitational potential energy. The boat also gains a little upward kinetic energy as the wave passes. The wave could not momentarily increase the boat's energy if it did not carry energy.

Q.3. In which ways a wave can carry energy?

Ans. There are two ways of energy transfer by waves *viz* (*i*) mechanical waves (*ii*) electromagnetic waves.

Q.4. What is a mechanical wave?

Ans. A wave which requires material medium to transfer energy from one point to another is called a mechanical wave *e.g.* sound waves, waves on strings, waves on water surface etc. Clearly, mechanical waves cannot travel through vacuum.

Q.5. What is an electromagnetic wave?

Ans. A wave which does not require material medium to transfer energy from one point to another is called an electromagnetic wave *e.g.* light waves, X-rays etc.

Q.6. What do mechanical waves transfer : matter, energy, both or none?

Ans. The mechanical waves transfer only energy from one point to another.

Q.7. What types of waves are (*i*) sound waves (*ii*) light waves?

Ans. (*i*) Mechanical waves (*ii*) Electromagnetic waves.

Q.8. Are (*i*) sound waves (*ii*) light waves longitudinal or tranverse in nature?

Ans. (*i*) Sound waves are longitudinal waves (*ii*) Light waves are transverse waves.

Q.9. What is the direction of oscillations of the particles in a medium through which a (*i*) transverse wave (*ii*) longitudinal wave is propagating?

Ans. (*i*) In a transverse wave, the particles of the medium oscillate in a direction perpendicular to the direction of propagation of the wave (*ii*) In a longitudinal wave, the particles of the medium oscillate in the direction of propagation of the wave.

Q.10. What type of motion particles of a medium execute when a wave passes through the medium?

Ans. When a wave passes through a medium, the particles of the medium execute simple harmonic motion (S.H.M.).

Q.11. Which properties of a medium are responsible for propagation of a wave through it?

Ans. The properties of elasticity and inertia are responsible for it.

Q.12. Why are longitudinal waves called pressure waves?

Ans. It is because when longitudinal waves propagate through a medium, there are changes in pressure and volume of air as compressions and rarefactions are formed.

Q.13. Why cannot two astronauts talk on the moon as they do on earth?

Ans. For the propagation of sound, a medium is required. As there is no atmosphere (air) on the moon, the sound waves from one astronaut cannot travel to the other. Therefore, they cannot talk to each other.

Q.14. Explosions on other planets are not heard on earth. Why?

Ans. Since there is vast empty space (no medium) between the planets and earth, the sound waves of explosions cannot propagate upto earth.

Q.15. What is the range of frequency of audible sound?

Ans. 20Hz to 20kHz.

Q.16. What type of mechanical waves exists in (*i*) vacuum (*ii*) air (*iii*) on water surface (*iv*) inside water (*v*) on a string?

Ans. (*i*) No wave (*ii*) Longitudinal waves (*iii*) Ripples *i.e.* combined longitudinal and transverse waves (*iv*) Longitudinal waves (*v*) Transverse waves.

Q.17. Sometimes no sound is heard from a vibrating source. Why?

Ans. The sound from a vibrating source can only be heard if the frequency of vibrations lies in the audible range (*i.e.* 20Hz to 20kHz).

Q.18. Transverse waves are not produced in liquids and gases. Why?

Ans. Transverse waves propagate in the form of crests and troughs, thus changing the shape of the medium. Since liquids and gases do not possess the elasticity of shape, transverse waves cannot be produced in liquids and gases.

Q.19. Transverse waves cannot be produced in air. Why?

Ans. For answer refer to Q.18.

Q.20. When a stone is thrown on the surface of water, waves travel out. From where does the energy come?

Ans. The energy of water waves comes from the kinetic energy of the stone hitting the water surface.

Q.21. What is the distance between a compression and its nearest rarefaction in a longitudinal wave?

Ans. It is half of Wavelength (*i.e.* $\lambda/2$).

Q.22. What is the relation among wave velocity, frequency and wavelength of a wave?

Ans. Wave velocity = Frequency × Wavelength

This relation is true for all types of waves.

Q.23. The wavelength of a sound wave is 10m. What does it mean?

Ans. It means that distance between consecutive compressions (or rarefactions) is 10m.

Q.24. If tension of a wire is increased to four times, how is the wave speed affected?

Ans. Wave speed, $v \propto \sqrt{T}$. Therefore, if tension (T) is increased four times, the wave speed becomes twice.

Q.25. If the radius of a wire is reduced to half, how is wave speed affected?

Ans. Wave speed, $v = \sqrt{T/m}$. Now $m \propto r^2$ where r is the radius of the wire. Therefore, $v \propto 1/\sqrt{r^2}$. When radius of the wire is reduced to half, wave speed becomes 2 times the initial speed.

Q.26. Upon what factors the speed of transverse waves through a solid depend?

Ans. The speed of transverse waves through a solid depends upon (*i*) modulus of rigidity (*ii*) density of the solid.

Q.27. Why does sound travel faster in iron than air?

Ans. Sound travels faster in solids. It is because elasticity and density of solids are very large as compared to liquids and gases.

Q.28. Frequency is the most fundamental property of a wave. Why?

Ans. Wave velocity = Frequency × Wavelength

When a wave travels from one medium to another, its velocity as well as wavelength changes but its frequency remains the same. For this reason, the frequency is the most fundamental property of a wave.

Q.29. What will be the speed of sound in a perfectly rigid rod?

Ans. The speed of sound in a perfectly rigid rod will be infinite.

Q.30. What are infrasonics?

Ans. Infrasonics are longitudinal waves with frequencies below 20Hz. The human ear cannot hear infrasonics.

Q.31. What are ultrasonics?

Ans. Ultrasonics are longitudinal waves with frequencies above 20kHz. The human ear cannot hear ultrasonics.

Q.32. How much velocity of sound in air changes when temperature increases by 1°C?

Ans. The velocity of sound in air increases by 0.61 ms^{-1} for every 1°C rise in temperature.

Q.33. What fact supports that sound waves are longitudinal?

Ans. Sound waves cannot be polarised.

Q.34. Which factors of the medium determine the velocity of sound waves?

Ans. The velocity of sound in a medium is determined by (*i*) elasticity (*ii*) density of the medium.

Q.35. How does the speed of sound in air vary with temperature?

Ans. The speed of sound in air (gas) is directly proportional to the square root of the absolute temperature of air *i.e.* $v \propto \sqrt{T}$.

Q.36. What is the effect of pressure on the velocity of sound in air?

Ans. The velocity of sound in air (gas) is independent of pressure of air provided temperature remains constant.

Q.37. The density of oxygen is 16 times the density of hydrogen. What is the relation between the speeds of sound in the two gases?

Ans. The speed of sound in a gas is inversely proportional to the square root of the density of the gas *i.e.* $v \propto 1/\sqrt{\rho}$. Therefore, the speed of sound in hydrogen is four times that in oxygen.

Q.38. Why does sound travel faster on a rainy day than on a dry day?

Ans. Velocity of sound $v \propto 1/\sqrt{\rho}$. The presence of moisture on a rainy day decreases the density (ρ) of air. Therefore, velocity of sound increases on a rainy day.

Q.39. What is the nature of thermal changes in air, when a sound wave propagates through it?

Ans. When a sound wave travels through air, adiabatic changes take place in the medium. It is because temperature increases in the region of compression while temperature falls in the region of rarefaction.

Q.40. Why is thunder of lightning heard some moments after seeing the flash?

Ans. It is because the velocity of light (3×10^8 m/s) is much larger than the velocity of sound (332 m/s). For this reason, the flash of light reaches us earlier than the sound of thunder.

Q.41. One end of a long metal pipe is struck with a hammer. What does the listener hear at the other end of the pipe?

Ans. The listener at the other end of the pipe will hear two sounds ; one through the air and the other through the metal part of the pipe. The sound through the metal part will be heard earlier ($\because$ sound travels faster in solids than gases) and it will be more intense ($\because$ density of solids is large).

Q.42. What is a harmonic wave function?

Ans. A harmonic wave function is a periodic function whose functional form is sine or cosine.

Q.43. Represent a harmonic wave in terms of its wavelength (λ) and time period (T).

Ans. Harmonic wave in terms of wavelength is represented as :

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

Harmonic wave in terms of time period is represented as :

$$y = a \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right)$$

Q.44. What are the maximum values of particle (*i*) velocity (*ii*) acceleration of a simple harmonic wave?

Ans. (*i*) $V_{max} = \omega a = \frac{2\pi}{T} \times a$ (*ii*) $A_{max} = \omega^2 a = \frac{4\pi^2}{T^2} a$

Q.45. How are stationary waves produced?

Ans. When two progressive waves of the same frequency moving with same speed with equal amplitudes in the opposite directions in a medium superpose on each other, stationary waves are produced.

Q.46. Why are stationary waves called so?

Ans. In a stationary wave, the particles of the medium vibrate about their mean positions but disturbance does not travel in any direction. In other words, a stationary wave neither propagates in any direction nor there is any transfer of energy in the medium.

Q.47. What is the distance between (*i*) adjacent nodes (*ii*) a node and adjacent antinode?

Ans. (*i*) It is one-half of the wavelength *i.e.* $\lambda/2$ (*ii*) It is one-quarter of the wavelength *i.e.* $\lambda/4$.

Q.48. What is the minimum frequency with which a stretched string of length L can vibrate?

Ans. Minimum frequency, $f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$.

Q.49. A string is stretched between two fixed points. In how many loops will the string vibrate if the string is plucked (*i*) in the middle (*ii*) at one-fourth part of the string?

Ans. (*i*) The string will vibrate in one loop (*ii*) The string will vibrate in two loops.

Q.50. A string of length 1 m is stretched between two fixed points. If it is plucked at a point 25 cm from one end, how many nodes and antinodes will be formed?

Ans. The plucking point of the string will be an antinode and each end will be a node. Since nodes and antinodes are situated at equal distances, there will be three nodes and two antinodes.

Q.51. Standing waves are produced in 10m long stretched string. If the string vibrates in 5 segments and wave velocity is 20 m/s, what is the frequency?

Ans. Length of string $= 5 \times \frac{\lambda}{2}$ or $10 = 5 \times \frac{\lambda}{2}$ $\therefore$ $\lambda = 4$ m

$\therefore$ Frequency, $f = \frac{v}{\lambda} = \frac{20}{4} = 5$ Hz

Q.52. What are the frequencies of the notes produced in an open and closed pipe?

Ans. If v is wave velocity and L the length of the pipe,

In open pipe : $\frac{v}{2L}; \frac{2v}{2L}; \frac{3v}{2L} \cdots$

In closed pipe : $\frac{v}{4L}; \frac{3v}{4L}; \frac{5v}{4L} \cdots$

Q.53. Why are holes kept at different positions in a flute?

Ans. The frequency of sound produced by a flute (pipe) depends on its length (L). By opening or closing the holes of the flute, its effective length is changed. Consequently, sounds of different frequencies can be produced.

Q.54. What is beat frequency?

Ans. The number of beats (intense sounds) produced per second is called the beat frequency.

Q.55. Why are beats not heard if the difference in the frequency of the two sounding notes is more than 10?

Ans. Due to persistence of hearing, the impression of a sound heard by our ears persists on our mind for 1/10th of a second. Now $f = 1/T = 1/1/10 = 10$ Hz. Therefore, if beat frequency is more than 10, the rise and fall of intensity cannot be distinguished.

Q.56. Two sound sources produce 12 beats in 4 seconds. By how much do their frequencies differ ?

Ans. Number of beats per second = 12/4 = 3. Therefore, difference of frequencies of two sound sources is $f_1 - f_2 = 3$ Hz.

Q.57. What is Doppler effect?

Ans. The apparent change in the observed frequency of a wave due to the relative motion between the source of waves and the observer is called Doppler effect.

Q.58. A source of sound emits waves of frequency f and is moving with a speed u_s (*i*) towards the listener (*ii*) away from the listener. What will be the apparent frequency (f') of sound in each case?

Ans. (*i*) $f' = \dfrac{V}{V - u_s} f$ (*ii*) $f' = \dfrac{V}{V + u_s} f$

Q.59. Both the source of sound and the listener are moving with the same velocity and in the same direction. Will there be any Doppler effect?

Ans. No. In this case apparent frequency f' is

$$f' = \frac{V - u_0}{V - u_S} \times f$$

Since $u_0 = u_S$, $f' = f$ and no Doppler effect will be observed.

Q.60. What would a person hear if he moves away from a source of sound with the speed of sound?

Ans. The person will hear no sound. In this case, the apparent frequency (f') of sound is

$$f' = \frac{V - u_0}{V} \times f$$

Here velocity of sound $V = u_0$ (velocity of observer) so that $f' = 0$. Therefore, the person will hear nothing.

Q.61. The amplitude of a sound wave is doubled. What will be the change in its intensity?

Ans. Intensity $\propto$ (Amplitude)^2. Therefore, intensity will become four times.

Q.62. What is reverberation?

Ans. The persistance or prolongation of sound in a room or hall (due to successive reflections from surfaces) after the original sound has ceased is called reverberation.

SHORT ANSWER QUESTIONS

Q.1. How is energy transmitted in wave motion?

Ans. In wave motion, the particles of the medium vibrate about their mean position. A vibrating particle of the medium hands over its motion to a particle just ahead of it. This particle also starts vibrating about its mean position and transfers its motion to next particle and so on. In this way, disturbance (*i.e.* wave) is transferred from particle to particle while particles continue to vibrate about their mean positions.

Q.2. Liquids and gases cannot propagate transverse waves. Why?

Ans. Transverse waves travel in the form of crests and troughs involving change in shape of the medium. Since liquids and gases do not possess elasticity of shape, mechanical transverse waves cannot propagate through liquids and gases.

Q.3. Why only longitudinal waves can propagate in gases?

Ans. As explained in Q.2 above, transverse waves cannot propagate in gases and liquids. Longitudinal waves travel in the form of compressions and rarefactions involving changes in volume. Since gases (and liquids) possess volume elasticity, longitudinal waves can propagate in gases.

Q.4. The speed of a wave in a medium is 760 ms^{-1}. If 3600 waves are passing through a point in the medium in 2 minutes, what is its wavelength?

Ans. The frequency (f) of the wave is the number of waves (*i.e.* cycles) passing the point in one second.

$$\therefore \quad f = \frac{3600}{2 \times 60} = 30 \text{ Hz} \; ; \quad \text{Wavelength, } \lambda = \frac{v}{f} = \frac{760}{30} = 25.3 \text{ m}$$

Q.5. In a sinusoidal wave, the time required for a particular point to move from maximum displacement to zero displacement is 0.17 sec. What is the frequency of the wave?

Ans. If T is the time period of the wave, then time required to move from maximum displacement to zero displacement is $T/4$. It is given that $T/4 = 0.17$ or $T = 0.17 \times 4 = 0.68$ sec. Therefore, frequency $f = 1/T = 1/0.68 = 1.47$ Hz.

Q.6. Why the sound of a distant coming train easily detected by placing our ear near the rails?

Ans. The speed of sound is very large in solids as compared to that in gases. Moreover, due to high elasticity of solids, the loss of sound energy in solids is less than that in gases. That is why putting your ear to rails allows you to hear the approach of a distant train long before you can hear in the air.

Q.7. Two sounds have a phase difference of 60°. What is their path difference?

Ans. Phase difference, $\phi = 60° = \pi/3$ rad

Phase difference, $\phi = \frac{2\pi}{\lambda} \times$ Path difference or $\frac{\pi}{3} = \frac{2\pi}{\lambda} \times$ Path difference

$$\therefore \text{ Path difference } = \frac{\pi}{3} \times \frac{\lambda}{2\pi} = \frac{\lambda}{6}$$

Q.8. What is the nature of sound waves?

Ans. Sound waves travel in the form of longitudinal waves in gases and liquids. It is because gases and liquids possess volume elasticity and not elasticity of shape. However in solids, sound waves can travel both in the form of longitudinal and transverse waves.

Q.9. Does the sound of an explosion travel faster than the sound produced by a humming bee?

Ans. Both the sound of explosion and the sound produced by a humming bee will travel with the same speed if physical conditions do not change. But due to large amplitude of the sound of explosion, the compressions and rarefactions considerably increase the temperature of air. As the velocity of sound increases with temperature, therefore, the speed of sound of explosion will be greater than that produced by the humming bee.

Q.10. What are the salient differences between sound and light waves?

Ans. The salient differences between sound and light waves are :

(*i*) The sound waves are mechanical waves that propagate as longitudinal waves. However, light waves are electromagnetic waves that are transverse in nature.

(*ii*) Sound waves require material medium for propagation whereas light waves do not require any material medium.

(*iii*) The velocity of sound waves is very small compared to the velocity of light waves. For example, the velocity of sound in air at room temperature is about 340 ms^{-1} whereas velocity of light in air/vacuum is 3×10^8 ms^{-1}.

Q.11. Why does the intensity of sound increase at night?

Ans. At night, the amount of carbon dioxide in the atmosphere increases which increases the density of air. Since the intensity of sound is directly proportional to the density of air, the intensity of sound increases at night.

Q.12. Two waves of amplitude a produce a resultant wave of amplitude a. What is the phase difference between them?

Ans. Let ϕ be the phase difference between the two waves.

$\therefore$ (Resultant)$^2 = a^2 + a^2 + 2a^2\cos\phi$ or $a^2 = a^2 + a^2 + 2a^2\cos\phi$

$\therefore$ $\cos\phi = -1/2$ or $\phi = 120°$

Q.13. What is the evidence that (*i*) sound is a wave (*ii*) sound is a mechanical wave (*iii*) sound waves are longitudinal waves?

Ans. (*i*) Sound waves show the phenomenon of diffraction.

(*ii*) Sound waves require material medium for propagation.

(*iii*) Sound waves cannot be polarised.

Q.14. A wave is represented by the equation $y = a \sin (0.01x - 2t)$ where a and x are in cm. What is the velocity of the wave?

Ans. Given equation is $y = a \sin (0.01x - 2t)$. Compare it with the standard equation of progressive/ travelling wave : $y = a \sin (kx - \omega t)$. We find that $k = 0.01$ and $\omega = 2$. Therefore, wave velocity $v = \omega/k = 2/0.01 = 200$ cm s^{-1}.

Q.15. A stationary wave is represented by $y = A \sin (100t) \cos (0.01x)$ where A and y are in mm, t in sec and x in metres. What is the velocity of wave?

Ans. The given equation of stationary wave is $y = A \sin (100t) \cos (0.01x)$.

Comparing it with standard equation of stationary wave $y = 2a \sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi x}{\lambda}\right)$, we have,

$\frac{2\pi}{T} = 100$ and $\frac{2\pi}{\lambda} = 0.01$. Now $\frac{2\pi}{T} = \omega$ and $\frac{2\pi}{\lambda} = k$

$\therefore$ Velocity of wave, $v = \omega/k = 100/0.01 = 10^4$ ms^{-1}

Q.16. Is it necessary for beat production that amplitudes of the two waves should be exactly equal?

Ans. It is not necessary that the amplitudes of the two waves producing beats should be equal. However, if the amplitudes of the two waves are equal, we get zero sound at minima and high intensity sound at maxima. Therefore, in this case, the beats become more distinct.

Q.17. Why are strings of different thickness and materials used in a sitar or other such instruments?

Ans. For a vibrating string, frequency $f = \frac{1}{2L}\sqrt{\frac{T}{m}}$.

Here m is the mass of string per unit length. By using strings of different thickness and materials, the value of m and hence frequency of notes produced can be changed.

Q.18. There are two identical tubes A and B ; tube A has both ends open while tube B has one end closed. What is the ratio of fundamental frequency of the tubes A and B?

Ans. Let L be the length of each tube and v the velocity of sound. The fundamental frequencies in the two cases are :

For tube A : $f_A = \frac{v}{2L}$ ...open at both ends

For tube B : $f_B = \frac{v}{4L}$ ...closed at one end

$\therefore$ $\frac{f_A}{f_B} = \frac{v/2L}{v/4L} = 2$

Q.19. Whistle of approaching railway engine is shriller than the receding engine. Why?

Ans. This is due to Doppler effect. When the source of sound (engine) moves towards the stationary observer with velocity u_S, then apparent frequency is

$$f_1' = \frac{V}{V - u_s} \times f$$ ...V is the velocity of sound

When the source of sound (engine) is receding from a stationary observer with velocity u_S, then apparent frequency is

$$f_2' = \frac{V}{V + u_s} \times f$$ ...V is the velocity of sound

It is clear that $f_1' > f_2'$. Therefore, the whistle of the approaching railway engine is shriller than the receding engine.

Q.20. A listener is moving towards a stationary source of sound. What should be his velocity so that the apparent frequency of the sound is double of its actual value?

Ans. If the listener is moving with a velocity u_O towards the stationary source of sound, then apparent frequency f' of sound is

$$f' = \frac{V + u_O}{V} \times f \qquad \text{...}V \text{ is the velocity of sound}$$

$$\text{For } f' = 2f,\ 2f = \frac{V + u_O}{V} \times f \quad \therefore \quad u_O = V$$

LONG ANSWER QUESTIONS

1. What are transverse and longitudinal waves? Give examples. **[Refer to Art. 26.3]**
2. Explain the terms wavelength, frequency, time period and velocity of a wave. Establish relation between them. **[Refer to Arts. 26.5 and 26.6]**
3. Give differences between transverse waves and longitudinal waves. **[Refer to Art. 26.4]**
4. State Newton's formula for velocity of sound in air. **[Refer to Art. 26.12]**
5. Why and how Laplace corrected Newton's formula for sound in a gas? **[Refer to Art. 26.13]**
6. Discuss the effect of density, temperature and pressure on the velocity of sound in a gas. **[Refer to Art. 26.14]**
7. Explain the effect of humidity, frequency and wind on the velocity of sound in air? **[Refer to Art. 26.14]**
8. Derive the equation of simple harmonic travelling wave. **[Refer to Art. 26.19]**
9. Establish the relation between particle velocity and amplitude of a simple harmonic travelling wave. **[Refer to Art. 26.20]**
10. State and explain superposition principle. **[Refer to Art. 26.23]**
11. Disucss the phenomenon of reflection of waves. **[Refer to Art. 26.22]**
12. What are standing waves? Explain the formation of standing waves. **[Refer to Arts. 26.25 and 26.27]**
13. Derive an equation of a standing or stationary wave. **[Refer to Art. 26.29]**
14. Discuss the formation of standing waves in a string fixed at both ends and the different modes of vibration. **[Refer to Art. 26.30]**
15. Explain the formation of standing waves in case of a pipe closed at one end. Describe the various modes of vibration graphically. **[Refer to Art. 26.32]**
16. Explain the formation of stationary waves in case of a pipe open at both ends. Describe the various modes of vibration graphically. **[Refer to Art. 26.33]**
17. What are beats? Prove that the number of beats per second is equal to the difference between the frequencies of the two superposing waves. **[Refer to Arts. 26.35 and 26.36]**
18. Explain Doppler effect in sound. Derive an expression for apparent frequency of sound when (*i*) listener moves towards the stationary source (*ii*) source moves towards the stationary listener. **[Refer to Arts. 26.39 and 26.40]**
19. Explain the characteristics of musical sound. **[Refer to Art. 26.42]**
20. Write a note on acoustics of buildings. **[Refer to Arts. 26.44 and 26.45]**
21. Explain the factors that affect acoustics of a building. **[Refer to Art. 26.50]**

COMPETITION SUCCESS MATERIAL

Useful Concepts/Information

1. Mechanical waves (*e.g.* sound waves) require medium for their transmission. These waves travel in the medium through the vibrations of the medium particles about their mean positions.
2. A mechanical wave may be (*i*) transverse wave or (*ii*) longitudinal wave. If the medium particles vibrate perpendicular to the direction of propagation of the wave, the wave is called transverse wave. If the medium particles vibrate along the direction of propagation of the wave, the wave is called longitudinal wave.
3. A transverse wave travels forming crests and troughs whereas a longitudinal wave travels forming compressions and rarefactions.
4. When a wave is transmitted in a medium, energy is transferred without the actual physical transfer of matter or without the flow of the medium as a whole.
5. The velocity of sound in vacuum is zero.
6. Sound travels through air as longitudinal wave.
7. Transverse waves are formed only in that medium which has rigidity.
8. Transverse waves cannot be formed in gases and inside liquids.
9. Longitudinal waves can be set up in all media *i.e.* solid, liquid and gas.
10. The speed of sound in air at N.T.P. is 332 ms^{-1}.
11. The velocity of sound in a gas (*e.g.* air) is inversely proportional to the square root of the density of the gas. Since the density decreases with presence of moisture, the velocity of sound in moist air is more than in dry air.
12. Since density decreases with increase in temperature, the velocity of sound increases with increase in temperature.
13. The equation of plane progressive (travelling) wave along the positive direction of x-axis is

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

where a is the amplitude of wave, λ is the wavelength and v is the velocity of wave. If ϕ_0 is the initial phase of the wave, then,

$$y = a\sin\left[\frac{2\pi}{\lambda}(vt - x) + \phi_0\right]$$

14. The equation of a plane progressive wave travelling along the negative direction of x-axis is

$$y = a\sin\frac{2\pi}{\lambda}(vt + x)$$

or

$$y = a\sin\left[\frac{2\pi}{\lambda}(vt + x) + \phi_0\right]$$ ...if initial phase of wave is ϕ_0

15. A progressive or travelling wave may be transverse or longitudinal. The particle velocity varies harmonically with time but wave travels with a constant velocity in the medium.
16. When a transverse wave is reflected from a fixed boundary, a phase change of π occurs *i.e.* crest is reflected as a trough and *vice-versa*. When reflection takes place at a free boundary (*e.g.* open end of a pipe), the phase change is zero *i.e.* crest is reflected as crest and trough is reflected as a trough.
17. When a longitudinal wave is reflected from a fixed boundary, no phase change occurs *i.e.* compression is reflected as a compression and rarefaction is reflected as a rarefaction. When reflected from a free boundary, a phase change of π occurs *i.e.* a compression is reflected as a rarefaction and *vice-versa*.
18. A stationary (or standing) wave results when two waves which are travelling in opposite directions and which have the same speed and frequency and equal amplitudes are superposed.

19. If a transverse wave is caused to travel along a string, the wave is reflected on reaching the end of the string. The incident and reflected waves have the same speed, frequency and amplitude and, therefore, their superposition results in a stationary wave.

20. In a stationary wave, the amplitude varies from place to place along the wave. A *node* is a place of zero amplitude and *antinode* is a place of maximum amplitude.

21. The equation of a stationary wave is :

$$y = 2a\cos\left(\frac{2\pi x}{\lambda}\right)\sin\omega t$$

$$= A \sin \omega t \text{ where } A = 2a\cos\left(\frac{2\pi x}{\lambda}\right)$$

Note that this expression is composed of two terms :

(*i*) $\sin \omega t$ .... this shows a varying amplitude with *time* at a particular place.

(*ii*) $\cos\left(\frac{2\pi x}{\lambda}\right)$ ..... this shows a varying amplitude with *position* at a particular *time.*

22. Distance between two adjacent nodes is $\lambda/2$ and distance between adjacent antinodes is $\lambda/2$. The distance between a node and adjacent antinode is $\lambda/4$.

23. $y = 2a\cos\left(\frac{2\pi x}{\lambda}\right)\sin\omega t$ ...reference origin at antinode

$y = 2a\sin\left(\frac{2\pi x}{\lambda}\right)\cos\omega t$ ...reference origin at node

24. The frequency (f) of the note produced by a string of length l vibrating in p segments under a tension T is

$$f = \frac{p}{2l}\sqrt{\frac{T}{m}} \text{ where } m = \text{mass per unit length.}$$

25. In case of a string stretched between the two fixed points (*e.g.* in musical instruments such as sitar, guitar, violin etc.), the number of segments or loops in which the string vibrates depends at which point the string is plucked. If the string is plucked in the middle, the string vibrates in one loop (*i.e.* $p = 1$). If the string is plucked one-fourth part of the string, it vibrates in two loops ($p = 2$).

26. In a pipe closed at one end, the frequency of nth harmonic is given by :

$$f_n = n\left(\frac{v}{4l}\right) \text{ where } n = 1, 3, 5, 7 \ldots$$

where l = length of pipe ; v = velocity of sound in air.

27. In a pipe open at both ends, the frequency of nth harmonic is given by ;

$$f_n = \frac{nv}{2l} \quad \text{where } n = 1, 2\ 3, \ldots$$

28. The periodic variations in intensity at a given point due to the superposition of two sound waves of slightly different frequencies travelling in the same direction are called beats.

29. The frequency of the beats is the number of intense sounds heard per second. The beat frequency f_b of two frequencies f_1 and f_2 is given by :

$$f_b = f_1 - f_2 \quad (f_1 > f_2)$$

30. Beat period, $t_b = \frac{1}{f_b} = \frac{1}{f_1 - f_2}$

MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. An organ pipe P_1 closed at one end vibrating in its first overtone and another pipe P_2 open at both ends vibrating in its third overtone are in resonance with a given tuning fork. The ratio of lengths of P_1 and P_2 is **[AFMC Pune 2001]**

(a) 1 : 2 (b) 1 : 3
(c) 3 : 8 (d) 3 : 4

2. Two waves have equations:

$X_1 = a\sin(\omega t + \phi_1)$

$X_2 = a\sin(\omega t + \phi_2)$

If in the resultant wave, the amplitude remains equal to the amplitude of the superposing waves, then phase difference between them is **[CBSE PMT 2001]**

(a) $\pi/6$ (b) $2\pi/3$
(c) $\pi/4$ (d) $\pi/3$

3. The equation of a wave is represented by;

$$y = 10^{-4}\sin\left(100t - \frac{x}{10}\right)$$

The velocity of the wave will be **[CBSE PMT 2001]**

(a) 100 ms^{-1} (b) 250 ms^{-1}
(c) 750 ms^{-1} (d) 1000 ms^{-1}

4. A string of 7 m length has a mass of 0.035 kg. If the tension in the string is 60.5 N, then speed of the wave on the string is **[CBSE PMT 2001]**

(a) 77 ms^{-1} (b) 102 ms^{-1}
(c) 110 ms^{-1} (d) 165 ms^{-1}

5. A source and an observer move away from each other with a velocity of 10 m/s *w.r.t.* ground. If the observer finds the frequency of sound coming from the source as 1950 Hz, then original frequency of source is (velocity of sound in air = 340 m/s) **[CBSE PMT 2001]**

(a) 2068 Hz (b) 1950 Hz
(c) 2132 Hz (d) 2486 Hz

6. If the length of an organ pipe closed at one end is 1 m and velocity of sound is 330 m/s, then frequency for the second note is **[AFMC Pune 2001]**

(a) $4\times\frac{330}{4}$ Hz (b) $3\times\frac{330}{4}$ Hz
(c) $2\times\frac{330}{4}$ Hz (d) $2\times\frac{4}{330}$ Hz

7. A source of sound gives 5 beats per second when sounded with another source of frequency 100 Hz. The second harmonic of the source together with a source of 205 Hz gives 5 beats per second. What is the frequency of the source? **[PB PMT 2001]**

(a) 95 Hz (b) 100 Hz
(c) 205 Hz (d) 105 Hz

8. Two sound waves having a phase difference of 60° have path difference of **[AIIMS 2001]**

(a) $\frac{\lambda}{6}$ (b) 2λ
(c) $\frac{\lambda}{2}$ (d) $\frac{\lambda}{3}$

9. A wave is represented by the equation $y = a \sin (0.1x - 2t)$ where a and x are in cm. Velocity of propagation of the wave is **[AIIMS 2000]**

(a) 10 cm/s (b) 50 cm/s
(c) 100 cm/s (d) 200 cm/s

10. A tuning fork produces waves in a medium. If the temperature of the medium changes, then which of the following will change? **[Pb PMT 1999]**

(a) amplitude (b) frequency
(c) wavelength (d) time period

11. If the tension and diameter of a sonometer wire of fundamental frequency n are doubled and density is halved, then its fundamental frequency will become **[CBSE PMT 2001]**

(a) $n/4$ (b) $\sqrt{2}n$
(c) n (d) $n/\sqrt{2}$

12. A vehicle with a horn of frequency n is moving with velocity of 30 m/s in a direction perpendicular to the straight line joining the observer and the vehicle. The observer perceives the sound to have a frequency $n + n_1$. Then n_1 is equal to **[CBSE PMT 1998]**

(a) $n_1 = 0$ (b) $n_1 = 0.1\ n$
(c) $n_1 = 10\ n$ (d) $n_1 = -0.1\ n$

13. In a transverse progressive wave of amplitude A, the maximum particle velocity is four times its wave velocity. The wavelength of the wave is **[AFMC Pune 2000]**

(a) $\frac{\pi A}{4}$ (b) $\frac{\pi A}{2}$
(c) πA (d) $2\pi A$

14. A man fires bullet between two cliffs. First echo is heard after 3 seconds and second echo is heard after 5 seconds. If the velocity of sound is 330 m/s, then distance between the cliffs is **[AFMC Pune 2000]**
(*a*) 1320 m (*b*) 1650 m
(*c*) 990 m (*d*) 660 m

15. If vibrations of a string are to be increased by a factor of 2, then tension in the string must be made **[AIIMS 1999]**
(*a*) half (*b*) twice
(*c*) eight times (*d*) four times

16. A transverse wave is represented by the equation $y = y_0 \sin\frac{2\pi}{\lambda}(vt - x)$. For what value of λ is the maximum particle velocity equal to two times the wave velocity. **[CBSE PMT 1998]**
(*a*) $\lambda = 2\pi y_0$ (*b*) $\lambda = \frac{\pi y_0}{3}$
(*c*) $\lambda = \pi y_0$ (*d*) $\lambda = \frac{\pi y_0}{2}$

17. The intensity of sound increases at night due to **[AIIMS 1998]**
(*a*) increase in density of air
(*b*) decrease in density of air
(*c*) low temperature
(*d*) none of the above

18. The speed of a wave in a medium is 760 ms^{-1}. If 3600 waves are passing through a point in the medium in 2 minutes, then its wavelength is **[AFMC Pune 1998]**
(*a*) 13.8 m (*b*) 25.3 m
(*c*) 41.5 m (*d*) 57.2 m

19. Standing waves are produced in 10 m long stretched string. If the string vibrates in 5 segments and wave velocity is 20 m/s, the frequency is **[CBSE PMT 1997]**
(*a* 2 Hz (*b*) 4 Hz
(*c*) 10 Hz (*d*) 5 Hz

20. An air column in a pipe, which is closed at one end, will be in resonance with a vibrating tuning fork of frequency 264 Hz if the column is (v = 330 m/sec) **[CBSE PMT 1997]**
(*a*) 31.25 cm (*b*) 62.50 cm
(*c*) 93.75 cm (*d*) 125.75 cm

21. The equation of a sound wave is y = 0.0015 sin (62.4 x + 316 t). The wavelength of this wave is **[CBSE PMT 1996]**
(*a*) 0.2 unit (*b*) 0.1 unit
(*c*) 0.3 unit
(*d*) cannot be calculated

22. The length of a sonometer wire AB is 110 cm. Where should the two bridges be placed from A to divide the wire in 3 parts whose fundamental frequencies are in the ratio of 1 : 2 : 3? **[CBSE PMT 1995]**
(*a*) 30 cm and 90 cm (*b*) 40 cm and 80 cm
(*c*) 60 cm and 90 cm (*d*) 30 cm and 60 cm

23. A stationary wave is represented by $y = A$ sin (100 t) cos (0.01 x) where A and y are in millimetres, t in sec and x in metres. The velocity of wave is **[CBSE PMT 1994]**
(*a*) 1 m/s (*b*) 10^2 m/s
(*c*) 10^4 m/s (*d*) not derivable

24. A wave of frequency 100 Hz travels along a string towards its fixed end. When this wave travels back after reflection, a node is formed at a distance of 10 cm from the fixed end. The speed of the wave (incident and reflected) is **[CBSE PMT 1994]**
(*a*) 5 m/s (*b*) 10 m/s
(*c*) 20 m/s (*d*) 40 m/s

25. In a sinusoidal wave, the time required for a particular point to move from maximum displacement to zero displacement is 0.17 sec. The frequency of the wave is **[CBSE PMT 1998]**
(*a*) 1.47 Hz (*b*) 0.36 Hz
(*c*) 0.73 Hz (*d*) 2.94 Hz

ANSWERS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. (*c*)	**2.** (*b*)	**3.** (*d*)	**4.** (*c*)	**5.** (*a*)
6. (*b*)	**7.** (*d*)	**8.** (*a*)	**9.** (*d*)	**10.** (*c*)
11. (*c*)	**12.** (*a*)	**13.** (*b*)	**14.** (*a*)	**15.** (*d*)
16. (*c*)	**17.** (*a*)	**18.** (*b*)	**19.** (*d*)	**20.** (*a*)
21. (*b*)	**22.** (*c*)	**23.** (*c*)	**24.** (*c*)	**25.** (*a*)

HINTS TO MCQ FROM MEDICAL ENTRANCE EXAMINATIONS

1. Length of pipe closed at one end for first overtone, $l_1 = \frac{3\lambda}{4}$

Length of pipe open at both ends for third overtone, $l_2 = \frac{4\lambda}{2} = 2\lambda$

$\therefore \quad \frac{l_1}{l_2} = \frac{3\lambda/4}{2\lambda} = \frac{3}{8}$ or $l_1 : l_2 = \mathbf{3 : 8}$

2. Let the phase difference between the waves be ϕ.

Resultant amplitude = $\sqrt{a^2 + a^2 + 2a^2\cos\phi}$

or $\quad a = \sqrt{2a^2 + 2a^2\cos\phi}$ or $a = a\sqrt{2(1+\cos\phi)}$

$\therefore \quad a = 2a\cos\frac{\phi}{2}$ or $\cos\frac{\phi}{2} = \frac{1}{2}$ $\therefore \quad \frac{\phi}{2} = \cos^{-1}\left(\frac{1}{2}\right)$ or $\phi = 2 \times 60° = \mathbf{120°}$

3. $y = 10^{-4}\sin\left(100t - \frac{x}{10}\right)$. The standard equation of a progressive wave is

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) = a\sin\left(\frac{2\pi vt}{\lambda} - \frac{2\pi x}{\lambda}\right)$$

$\therefore \quad \frac{2\pi}{\lambda} = \frac{1}{10}$ or $\lambda = 2\pi \times 10 = 20\pi$

Also $\quad \frac{2\pi v}{\lambda} = 100$ or $\frac{2\pi v}{20\pi} = 100$ $\therefore \quad v = 100 \times 10 = \mathbf{1000\ ms^{-1}}$

4. Mass per unit length, $m = \frac{0.035}{7} = 0.005$ kg/m

Speed of wave, $v = \sqrt{\frac{T}{m}} = \sqrt{\frac{60.5}{0.005}} = \mathbf{110\ ms^{-1}}$

5. Velocity of sound, $V = 340\ ms^{-1}$; Velocity of observer, $u_O = 10\ ms^{-1}$; Velocity of source $u_S = 10\ ms^{-1}$.

According to Doppler's effect, the apparent frequency f' of the source when observer and source move away from each other is

$$f' = \left(\frac{V - u_o}{V + u_s}\right) \times f \text{ or } 1950 = \left(\frac{340 - 10}{340 + 10}\right) f \quad \therefore \quad f = \frac{350 \times 1950}{330} = \mathbf{2068\ Hz}$$

6. Length of closed organ pipe, l = 1m ; Velocity of sound, $v = 330\ ms^{-1}$

The frequencies of closed organ pipe are $\frac{v}{4l}, \frac{3v}{4l}, \frac{5v}{4l}\ldots$

$\therefore$ Frequency of second note = $\frac{3v}{4l} = \frac{3 \times 330}{4} = \mathbf{3 \times \frac{330}{4}\ Hz}$

7. The frequency of unknown source = 100 ± 5 = 105 Hz or 95 Hz. The frequency of the second harmonic of unknown source will be 210 Hz or 190 Hz. As the second harmonic gives 5 beats with the sound of frequency 205 Hz, the second harmonic should be 210 Hz or the frequency of source = **105 Hz.**

8. Phase difference, $\phi = 60° = \pi/3$ rad

Phase difference , $\phi = \frac{2\pi}{\lambda} \times$ Path difference or $\frac{\pi}{3} = \frac{2\pi}{\lambda} \times$ Path difference

$\therefore$ Path difference $= \frac{\pi}{3} \times \frac{\lambda}{2\pi} = \frac{\lambda}{6}$

9. Given wave equation is $y = a \sin (0.01\ x - 2t)$. Compare it with the standard equation of progressive wave: $y = a \sin (kx - \omega t)$. We find that $k = 0.01$ and $\omega = 2$. Therefore, wave velocity $v = \omega/k = 2/0.01 =$ **200 cm/s.**

10. Velocity of wave, $v \propto \sqrt{T}$ where T is the absolute temperature. Now $v = f\lambda$ so that $\lambda \propto \sqrt{T}$. Therefore , if temperature of the medium changes, then **wavelength** of the wave will also change.

11. Fundamental frequency, $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$

If d is the diameter of the wire and ρ is the density of wire material, then,

$$m = \text{mass per unit length} = \frac{\pi d^2}{4} \times 1 \times \rho$$

$$\therefore \quad n = \frac{1}{2l}\sqrt{\frac{4 \times T}{\pi d^2 \rho}} = \frac{1}{ld}\sqrt{\frac{T}{\pi\rho}}$$

When tension and diameter are doubled and density is halved, the fundamental frequency n' is given by ;

$$n' = \frac{1}{l \times 2d}\sqrt{\frac{2T}{\pi\rho/2}} = \frac{1}{ld}\sqrt{\frac{T}{\pi\rho}} = n$$

12. Doppler effect is not observed when source of sound and the observer are moving at right angles to each other.

13. Maximum particle velocity = $A\omega$. It is given that maximum particle velocity = $4v$ where v is wave velocity.

$$\therefore \quad 4v = A\omega \quad \text{or} \quad \omega = \frac{4v}{A}$$

$$\therefore \text{ Wavelength, } \lambda = \frac{v}{f} = \frac{v}{\omega/2\pi} = \frac{2\pi v}{4v/A} = \frac{\pi A}{2}$$

14. Time taken for the sound to travel from man to first cliff, $t_1 = 3/2 = 1.5$ sec
Time taken for the sound to travel from man to second cliff, $t_2 = 5/2 = 2.5$ sec
Distance of first cliff from man, d_1 = Velocity of sound $\times\ t_1 = 330 \times 1.5 = 495$ m
Distance of second cliff from man, $d_2 = 330 \times t_2 = 330 \times 2.5 = 825$ m
Distance between cliffs, $d = d_1 + d_2 = 495 + 825 =$ **1320 m**

15. Frequency, $f = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \therefore \quad f \propto \sqrt{T}$

$$\therefore \quad \frac{f_2}{f_1} = \sqrt{\frac{T_2}{T_1}} \quad \text{or} \quad 4 = \frac{T_2}{T_1} \quad \therefore\ T_2 = 4T_1$$

16. $y = y_0 \sin\frac{2\pi}{\lambda}(vt - x)$ where v is the wave velocity.

Particle velocity, $\frac{dy}{dt} = y_0 \frac{2\pi v}{\lambda}\cos(vt - x)$

$\therefore$ Maximum velocity of particle $= y_0 \frac{2\pi v}{\lambda}$

According to given condition, $y_0 \frac{2\pi v}{\lambda} = 2v \quad \therefore \quad \lambda = \pi\, y_0$

17. At night, the amount of carbon dioxide in atmosphere increases which increases the density of air. Since intensity of sound is directly proportional to the density of air, intensity of sound increases at night.

18. Frequency of wave, $f = \dfrac{3600}{2\times 60} = 30$ Hz $\quad \therefore \lambda = \dfrac{v}{f} = \dfrac{760}{30}$ = **25.3 m**

19. Length of string $= 5\times\dfrac{\lambda}{2}$ or $10 = 5\times\dfrac{\lambda}{2} \quad \therefore \lambda = 4$ m

Frequency, $f = \dfrac{v}{\lambda} = \dfrac{20}{4}$ = **5 Hz**

20. Fig. 26.40 shows the conditions of the problem.

$$l = \frac{\lambda}{4} \quad \text{or} \quad \lambda = 4l$$

$$\text{Frequency, } f = \frac{v}{\lambda} \quad \text{or} \quad 264 = \frac{330}{4l}$$

$$\therefore \quad l = \frac{330}{264\times 4} = 0.3125 \text{ m} = \textbf{31.25 cm}$$

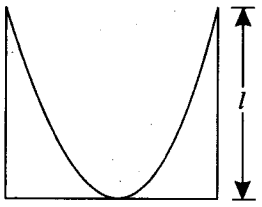


Fig. 26.40

21. $y = 0.0015 \sin (62.4x + 316t)$. Comparing it with standard equation of progressive wave $y = a \sin(\omega t + kx)$, we have, $a = 0.0015$, $\omega = 316$, $k = 62.4$.

$$\therefore \quad \frac{2\pi}{\lambda} = 62.4 \quad \text{or} \quad \lambda = \frac{2\pi}{62.4} = \textbf{0.1 unit}$$

22. Let the bridges be placed at points P and Q to get the condition given in the problem (See Fig. 26.41). Since the fundamental frequency of a stretched wire is inversely proportional to its length,

$$\therefore \quad AP : PQ : QB = \frac{1}{1} : \frac{1}{2} : \frac{1}{3} = 6:3:2$$

$$\therefore \quad AP = \frac{6}{6+3+2}\times 110 = \frac{6}{11}\times 110 = 60\text{cm}\,; \; PQ = \frac{3}{11}\times 110 = 30\text{ cm}\,;$$

$$AQ = AP + PQ = 60 + 30 = 90\text{cm}$$

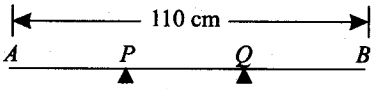


Fig. 26.41

Therefore, the two bridges should be placed at a distance of **60 cm** and **90 cm** from the end A of the wire.

23. The given equation of stationary wave is $y = A \sin(100\, t) \cos(0.01\, x)$. Comparing it with the standard equation of a stationary wave, $y = 2a\sin\left(\dfrac{2\pi t}{T}\right)\cos\left(\dfrac{2\pi x}{\lambda}\right)$, we have,

$$\frac{2\pi}{T} = 100 \quad \text{and} \quad \frac{2\pi}{\lambda} = 0.01. \text{ Now } \frac{2\pi}{T} = \omega \text{ and } \frac{2\pi}{\lambda} = k.$$

Velocity of waves, $v = \omega/k = 100/0.01 = \mathbf{10^4}$ **m/s**

24. The fixed end behaves as a node. Therefore, distance between two consecutive nodes = 10 cm or $\lambda/2 = 10$ or $\lambda = 20$ cm.

Wave velocity, $v = f\lambda$ = 100 × 20 = 2000cm/sec = **20 m/s**

25. If T is the time period of the wave, then time required to move from maximum displacement to zero displacement is $T/4$. Given : $T/4 = 0.17$ or $T = 0.17 \times 4 = 0.68$ sec. Therefore, $f = 1/T = 1/0.68 =$ **1.47 Hz.**

MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. A wave is represented by the equation :

$$y = A\sin(10\pi x + 15\pi t + \pi/3)$$

[IIT 1990]

where x is in metres and t is in seconds.

The expression represents :

(*a*) A wave travelling in the positive x-direction with a velocity of 1.5 m/s.

(*b*) A wave travelling in the negative x-direction with a velocity of 1.5 m/s.

(*c*) A wave travelling in the negative x-direction with a wavelength of 0.2 m.

(*d*) A wave travelling in the positive x-direction with a wavelength of 0.2 m.

2. String wires of the same material of lengths l and $2l$ vibrate with frequencies 100 Hz and 150 Hz respectively. The ratio of their tension is **[MP PET 1990]**

(*a*) 1 : 9 (*b*) 2 : 3

(*c*) 3 : 2 (*d*) 1 : 3

3. Two identical straight wires are stretched so as to produce 6 beats per second when vibrating simultaneously. On changing the tension slightly in one of them, the beat frequency remains unchanged. Denoting T_1 and T_2 the higher and lower initial tensions in the wires, then it could be said that while making the above changes in the tension **[IIT 1991]**

(*a*) T_2 was decreased (*b*) T_2 was increased

(*c*) T_1 was increased (*d*) T_1 was decreased

4. A tuning fork vibrating with a sonometer wire 20 cm long produces 5 beats per second. The beat frequency does not change if the length of the wire is changed to 21 cm. The frequency of the tuning fork is **[CEE 1991]**

(*a*) 200 Hz (*b*) 210 Hz

(*c*) 205 Hz (*d*) 215 Hz

5. If the velocity of sound in air is 336 m/s, the maximum length of a closed pipe that would produce just audible sound is **[CET Karnataka 2001]**

(*a*) 4.2 m (*b*) 1.6 cm

(*c*) 4.2 cm (*d*) 1.2 cm

6. When a wave travels in a medium, the particle displacements are given by : $y(x, t) = 0.03 \sin \pi (2t - 0.01x)$ where x and y are in metres and t in sec. The wavelength of the wave is **[CET Haryana 2001]**

(*a*) 200 m (*b*) 100 m

(*c*) 10 m (*d*) 20 m

7. A sound source of frequency 170 Hz is placed near a wall. A man walking normally towards the wall finds that there is a periodic rise and fall of intensity. If speed of sound in air is 340 m/s, the distance separating the two adjacent positions of minimum intensity is

(*a*) 0.5 m (*b*) 1.5 m

(*c*) 2 m (*d*) 1 m

8. Tuning forks A and B are sounded together and produce 2 beats. On loading the wax on A, the number of beats remains the same. If the frequency of B is 320 Hz, then the frequency of A will be **[CET Haryana 2001]**

(*a*) 324 Hz (*b*) 318 Hz

(*c*) 322 Hz (*d*) 316 Hz

9. A string under a tension of 129.6 N produces 10 beats/sec when it vibrates with a tuning fork. When the tension in the string is increased to 160 N, it vibrates in unison with the same tuning fork. Then frequency of tuning fork is **[CEE 1994]**

(*a*) 100 Hz (*b*) 110 Hz

(*c*) 90 Hz (*d*) 220 Hz

10. When two sound waves of almost equal frequencies n_1 and n_2 are produced simultaneously, then the interval between successive maxima is **[Pb CET 1996]**

(*a*) $\frac{1}{n_1} - \frac{1}{n_2}$ (*b*) $\frac{1}{n_1 - n_2}$

(*c*) $\frac{1}{n_1} + \frac{1}{n_2}$ (*d*) $\frac{1}{n_1 + n_2}$

11. A glass tube of 1.0 m length is filled with water. The water can be drained out slowly at the bottom of the tube. If a vibrating tuning fork of frequency 500 Hz is brought at the upper end of the tube and the velocity of sound is 330 m/s, then total number of resonances obtained will be **[CEE 1994]**

(*a*) 4 (*b*) 3

(*c*) 2 (*d*) 1

12. A whistle giving out 450 Hz approaches a stationary observer at a speed of 33 m/s. The frequency heard by the observer [velocity of sound in air = 333 ms^{-1}] is **[IIT 1997]**

(*a*) 409 Hz (*b*) 429 Hz

(*c*) 517 Hz (*d*) 500 Hz

13. Two waves produced displacements at a point given by ; **[CET Chandigarh 1997]**

$y_1 = a\sin\omega t$

$y_2 = a\sin\left(\omega t + \frac{\pi}{2}\right)$

The resultant amplitude is

(a) 0 (b) $2a$

(c) $\sqrt{2}\,a$ (d) $\frac{a}{\sqrt{2}}$

14. If two waves of amplitude a produce a resultant wave of amplitude a, then the phase difference between them will be **[MP PET 2000]**

(a) 60° (b) 90°

(c) 180° (d) 120°

15. In case of a moving source of sound approaching an observer, **[Pb CET 1996]**

(a) the wavelength of sound appears to be less

(b) the wavelength of sound appears to be more

(c) the frequency appears to be less

(d) none of the above

16. Two forks A and B when sounded together produce 4 beats/sec. The fork A is in unison with 30 cm length of a sonometer wire and B is in unison with 25 cm length of the same wire at the same tension. The frequencies of the forks are **[CET Karnataka 1997]**

(a) 24 Hz, 28 Hz (b) 20 Hz, 24 Hz

(c) 16 Hz, 20 Hz (d) 26 Hz, 30 Hz

17. The linear density of a vibrating string is 10^{-4} kg m^{-1}. A transverse wave is propagating on the string which is described by the equation : $y = 0.02 \sin (x + 30\ t)$ where x and y are in metres and t is in seconds. The tension in the string is **[MP PET 1997]**

(a) 0.09 N (b) 0.36 N

(c) 0.9 N (d) 3.6 N

18. A source of frequency f gives 5 beats/sec when sounded with a source of frequency 200 Hz. The second harmonic ($2f$) gives 10 beats/sec when sounded with a source of frequency 420 Hz. Then f is equal to **[CET Haryana 2000]**

(a) 195 Hz (b) 205 Hz

(c) 190 Hz (d) 220 Hz

19. The sonometer wire is vibrating in the second overtone. We may say that there are **[CET Karnataka 1997]**

(a) two nodes and two antinodes

(b) one node and two antinodes

(c) four nodes and three antinodes

(d) three nodes and three antinodes

20. A source of sound gives 5 beats per second when sounded with another source of frequency 100 Hz. The second harmonic of the source together with a source of frequency 205 Hz gives 5 beats/sec. What is the frequency of the source? **[BHU 1996]**

(a) 105 Hz (b) 205 Hz

(c) 95 Hz (d) 100 Hz

ANSWERS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. (b, c)	**2.** (a)	**3.** (b, d)	**4.** (c)	**5.** (a)
6. (a)	**7.** (d)	**8.** (c)	**9.** (a)	**10.** (b)
11. (b)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (a)
16. (b)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (a)

HINTS TO MCQ FROM ENGINEERING ENTRANCE EXAMINATIONS

1. Given equation is $y = A \sin (10\pi x + 15\pi t + \pi/3)$

Comparing it with standard equation $y = a \sin\left[\frac{2\pi x}{\lambda} + \frac{2\pi t}{T} + \phi_0\right]$,

we have, $\frac{2\pi}{\lambda} = 10\pi$ or $\lambda = \frac{2\pi}{10\pi} = 0.2$ m

Also $\frac{2\pi}{T} = 15\pi$ or $T = \frac{2\pi}{15\pi} = \frac{2}{15}$ s

$\therefore$ Wave velocity, $v = \frac{\lambda}{T} = \frac{0.2}{2/15} = 1.5$ m/s

In the given equation, x is positive. Therefore, the wave is travelling in the **negative** x-direction.

2. $f = \frac{1}{2l}\sqrt{\frac{T}{m}}$ $\therefore$ $100 = \frac{1}{2l}\sqrt{\frac{T_1}{m}}$ and $150 = \frac{1}{4l}\sqrt{\frac{T_2}{m}}$

$\therefore$ $\frac{100}{150} = 2\sqrt{\frac{T_1}{T_2}}$ or $\frac{1}{3} = \sqrt{\frac{T_1}{T_2}}$ or $\frac{T_1}{T_2} = \frac{1}{9}$ $\therefore$ $T_1 : T_2 = 1 : 9$

3. Frequency, $f \propto \sqrt{T}$. Since $T_1 > T_2$, $f_1 - f_2 = 6$. The beat frequency will remain the same when $f_2 - f_1 = 6$. To increase f_2, T_2 should be increased and to decrease f_1, T_1 should be decreased.

4. Frequency, $f \propto \frac{1}{l}$ $\therefore$ $\frac{f_1}{f_2} = \frac{21}{20}$ ...(*i*)

Let f be the frequency of tuning fork.

Also $f_1 - f = 5$ and $f - f_2 = 5$. $\therefore$ $f_1 - f_2 = 10$ ...(*ii*)

From eqs. (*i*) and (*ii*), we have, $f = 205$ Hz

5. The audible range is 20 Hz to 20 kHz. The fundamental frequency of a pipe closed at one end is $f_1 = v/4l$. To produce a sound of 20 Hz,

$$20 = \frac{v}{4l} \text{ or } l = \frac{v}{20\times 4} = \frac{336}{20\times 4} = 4.2 \text{ m}$$

6. The given eq. is $y\ (x, t) = 0.03 \sin \pi(2t - 0.01x)$. Comparing it with the standard eq., $y\ (x, t) = a \sin(\omega t - kx)$, we have,

$$k = \frac{2\pi}{\lambda} = 0.01\pi. \text{ Therefore, } \lambda = \frac{2\pi}{0.01\pi} = 200 \text{ m}$$

7. Due to reflection of sound from the wall, stationary waves are formed. In this case, wavelength is unaffected. Therefore, $\lambda = v/f = 340/170 = 2$ m. The distance between two adjacent nodes (positions of minimum intensity) is $\lambda/2 = 2/2 = 1$ m.

8. On loading the tuning fork A, its frequency decreases. Since the number of beats remains the same, it means $f_A > f_B$. Therefore, $f_A - f_B = 2$ or $f_A - 320 = 2$ or $f_A = 320 + 2 = 322$ Hz.

9. Frequency of stretched string, $f \propto \sqrt{T}$. The conditions of the problem suggest that initially, the frequency f of tuning fork is more than that of the string.

For the first case, $f - 10 \propto \sqrt{129.6}$; For the second case, $f \propto \sqrt{160}$

$\therefore$ $\frac{f-10}{f} = \sqrt{\frac{129.6}{160}} = 0.9$ or $f - 10 = 0.9f$ or $f = 100$ Hz

10. Time interval between two maxima is the beat period. Now beat frequency $= n_1 - n_2$.

$\therefore$ Beat period $= \frac{1}{\text{Beat Frequency}} = \frac{1}{n_1 - n_2}$

11. In a pipe closed at one end, the frequency of nth harmonic is given by ;

$$f_n = n\left(\frac{v}{4l_n}\right) \text{ or } l_n = n\left(\frac{v}{4f_n}\right) \text{ where } n = 1, 3, 5, \ldots\ldots$$

$\therefore$ $l_1 = 1\times\frac{330\times 100}{4\times 500} = 16.5$ cm ; $l_2 = 3l_1 = 16.5 \times 3 = 49.5$ cm

$l_3 = 5l_1 = 5 \times 16.5 = 82.5$ cm ; $l_4 = 7l_1 = 7 \times 16.5 = 115.5$ cm

Since the length of the tube is lm (= 100 cm), only **three** resonances will be obtained.

12. The apparent frequency f' heard when the source of sound of frequency f moves toward a stationary observer is

$$f' = \frac{V}{V - u_S} \times f = \frac{333}{333 - 33} \times 450 = 499.5\text{Hz} \simeq \mathbf{500\ Hz}$$

13. Phase difference = 90°. Therefore, resultant amplitude = $\sqrt{a^2 + a^2} = \sqrt{2}\,a$

14. Let ϕ be the phase difference between the two waves.

$$(\text{Resultant})^2 = a^2 + a^2 + 2a^2 \cos\phi \text{ or } a^2 = a^2 + a^2 + 2a^2\cos\phi$$

$$\therefore \quad \cos\phi = -\frac{1}{2} \text{ or } \phi = 120°$$

15. If the source of sound were stationary, then f waves sent out in one second toward the observer would occupy a distance V (= velocity of sound in air) and the wavelength would be $\lambda = V/f$. If the source of sound moves with a velocity u_S toward the observer, then f waves sent out occupy a distance $(V - u_S)$ because the source has moved a distance u_S toward the observer in one second. Therefore, apparent wavelength, $\lambda' = (V - u_S)/f$.

Clearly, $\lambda' < \lambda$.

16. As per the question, $f \propto \frac{1}{l}$. Therefore, $\frac{f_A}{f_B} = \frac{l_B}{l_A} = \frac{25}{30} = \frac{5}{6}$

$$\therefore \quad \frac{f_A}{f_B} = \frac{5}{6} \quad ...(i) \quad \text{Clearly } f_B > f_A. \text{ Also } f_B - f_A = 4 \quad ...(ii)$$

From equations (*i*) and (*ii*), we have, f_A = **20 Hz** and f_B = **24 Hz**

17. Given equation is $y = 0.02 \sin(x + 30t)$. Comparing it with the standard equation,

$$y = a \sin\frac{2\pi}{\lambda}(vt + x), \text{ we have,}$$

$$\frac{2\pi}{\lambda} = 1 \text{ and } \frac{2\pi v}{\lambda} = 30 \quad \therefore \quad v = 30 \text{ m/s}$$

Now $\quad v = \sqrt{\frac{T}{m}} \quad$ or $\quad T = v^2 \times m = (30)^2 \times 10^{-4} = \mathbf{0.09\ N}$

18. For the first case, $f = 200 \pm 5 = 205$ Hz or 195 Hz. For the second case, $2f = 420 \pm 10$ or $f = 215$ Hz or 205 Hz. Clearly, f = **205 Hz.**

19. Second overtone means third harmonic. Therefore, the wire vibrates in three loops/ segments. As a result, there are **four nodes and three antinodes.**

20. Let f be the frequency of the source. For the first case, $f = 100 \pm 5 = 105$ Hz or 95 Hz. For the second case, $2f = 205 \pm 5$ or $f = 105$ Hz or 100 Hz. Clearly, the frequency of source is **105 Hz.**

NUMERICAL PROBLEMS FOR COMPETITIVE EXAMINATIONS

1. A progressive wave of frequency 500 Hz is travelling with a velocity of 360 ms^{-1}. How far apart are two points 60° out of phase?

Hint. Velocity of wave, $v = 360\ ms^{-1}$; Frequency of wave, $f = 500$ Hz

Now $\quad v = f\lambda \quad \therefore \quad$ Wavelength, $\lambda = \frac{v}{f} = \frac{360}{500} = 0.72$ m

If Δx is the minimum distance between two points, then phase difference between them is

$$\Delta\phi = \frac{2\pi}{\lambda}\times\Delta x \quad \text{or} \quad \Delta x = \frac{\lambda}{2\pi}\times\Delta\phi = \frac{0.72}{2\pi}\times\frac{\pi}{3} = \mathbf{0.12\ m}$$

2. A simple harmonic wave has the equation :

$$y = 0.30 \sin (314t - 1.57x)$$

where t is in sec, x in metres and y in cm. Find the frequency and wavelength of this wave. Another wave has the equation :

$$y' = 0.1 \sin(314t - 1.57x + 1.57)$$

Deduce the phase difference and the ratio of intensities of the above two waves.

Hint. If y is expressed in metres, then equation becomes :

$$y = \frac{0.30}{100}\sin(314t - 1.57x) \qquad ...(i)$$

The standard equation of plane progressive wave is

$$y = a \sin (\omega t - kx) \qquad ...(ii)$$

Comparing eqs. (i) and (ii), we have,

$$\omega = 314 \ ; \ k = 1.57 \ ; \ a = \frac{0.30}{100} = 3\times10^{-3}\ \text{m}$$

$$\therefore \quad \text{Frequency, } f = \frac{\omega}{2\pi} = \frac{314}{2\pi} = \mathbf{50\ Hz}$$

$$\text{Wave velocity, } v = \frac{\omega}{k} = \frac{314}{1.57} = 200\ \text{ms}^{-1}$$

$$\therefore \quad \text{Wavelength, } \lambda = \frac{v}{f} = \frac{200}{50} = \mathbf{4\ m}$$

On inspection of the equations of the given two waves,

$$\text{Phase difference, } \Delta\phi = 1.57\ \text{rad} = 1.57\times\left(\frac{180^\circ}{\pi}\right) = \mathbf{90^\circ}$$

Ratio of amplitudes of two waves = a_1/a_2 = 0.3/0.1 = 3

$$\therefore \quad \text{Ratio of intensities} = \frac{I_1}{I_2} = \frac{a_1^2}{a_2^2} = \left(\frac{3}{1}\right)^2 = \frac{9}{1}$$

3. A simple harmonic wave is represented by ;

$$y = 10\sin\left(\frac{2\pi t}{T} + \phi\right)$$

The time period is 30 s. At t = 0, the displacement is 5 cm. Calculate (i) the phase angle at t = 7.5 s and (ii) phase difference between two positions at a time interval of 6 s.

Hint. $y = 10\sin\left(\frac{2\pi t}{T} + \phi\right)$; $T = 30$ s

At t = 0, y = 5 cm $\therefore$ 5 = 10 sin ϕ or sin ϕ = 0.5 or $\phi = \frac{\pi}{6}$ rad

(i) At t = 7.5 s, the phase angle

$$= \frac{2\pi t}{T} + \phi = \frac{2\pi\times7.5}{30} + \frac{\pi}{6} = \frac{2\pi}{3}\ \text{rad}$$

(*ii*) The phase difference for a time interval of 6 s

$$= \frac{2\pi t}{T} = \frac{2\pi \times 6}{30} = \frac{2\pi}{5} \text{ rad}$$

4. If the tension of a string be increased by 2%, what would be % change in its frequency?

Hint. $$f = k\sqrt{T}$$

$$\therefore \quad \log_e f = \log_e k + \frac{1}{2}\log_e T$$

Differentiating : $$\frac{df}{f} = 0 + \frac{1}{2}\frac{dT}{T}$$

$$\therefore \quad \% \text{ change in frequency} = \frac{df}{f} \times 100 = \frac{1}{2}\left(\frac{dT}{T}\right) \times 100 = \frac{1}{2} \times 2\% = 1\%$$

5. A wire of density 9 g/cm^3 is stretched between two clamps 1.0 m apart while subjected to an extension of 0.05 cm. What is the lowest frequency of transverse vibrations in the wire? Assume Young's modulus $Y = 9 \times 10^{10}$ N/m^2. **[I.I.T.]**

Hint. The lowest frequency of transverse vibrations is given by ;

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Here m = mass per unit length = area × density = $A\rho$

$$\therefore \quad f = \frac{1}{2L}\sqrt{\frac{T}{A\rho}} = \frac{1}{2L}\sqrt{\frac{T/A}{\rho}} = \frac{1}{2L}\sqrt{\frac{\text{stress}}{\rho}}$$

Now $$Y = \frac{\text{stress}}{\text{strain}} \quad \therefore \text{ stress} = Y \times \text{strain} = Y\frac{\Delta L}{L}$$

$$\therefore \quad f = \frac{1}{2L}\sqrt{\frac{Y \times \Delta L}{\rho \times L}} = \frac{1}{2 \times 1.0}\sqrt{\frac{9 \times 10^{10} \times 0.05 \times 10^{-2}}{9 \times 10^3 \times 1.0}} = 35.3 \text{ vib/sec}$$

6. A steel wire of length 1 m, mass 0.1 kg and uniform cross-sectional area 10^{-6} m^2 is rigidly fixed at both ends. The temperature of the wire is lowered by 20°C. The transverse waves are set up by plucking the string in the middle. Calculate the frequency of the fundamental mode of vibration. Young's modulus of steel $Y = 2 \times 10^{11}$ N/m^2 and coefficient of linear expansion of steel $\alpha = 1.21 \times 10^{-5}$ /°C . **[I.I.T.]**

Hint. The frequency of fundamental mode of vibration is given by ;

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

When temperature of the wire is changed by $\Delta\theta$ (=20°C), the tension developed in the wire is

$$T = YA\,\alpha\Delta\theta$$

and m = mass per unit length = $A\rho$

$$\therefore \quad f = \frac{1}{2L}\sqrt{\frac{YA\alpha\Delta\theta}{m}}$$

Here L = 1 m ; $Y = 2 \times 10^{11}$ N/m^2 ; $A = 10^{-6}$ m^2 ; $\alpha = 1.21 \times 10^{-5}$/°C ;

$\Delta\theta$ = 20°C ; $m = M/L = 0.1/1 = 0.1$ kg/m

$$\therefore \quad f = \frac{1}{2 \times 1}\sqrt{\frac{2 \times 10^{11} \times 10^{-6} \times 1.21 \times 10^{-5} \times 20}{0.1}} = 11 \text{ vib/sec}$$

7. A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope? [I.I.T.]

Hint. Tension at the lower end of the rope is

$$T_1 = 2 \times g = 2 \times 9.8 = 19.6 \text{ N}$$

Tension at the upper end of the rope is

$$T_2 = (2 + 6) \times g = (2 + 6) \times 9.8 = 78.4 \text{ N}$$

Let v_1 and v_2 be the speed of the pulse at the lower and upper ends respectively. Then,

$$v_1 = \sqrt{\frac{T_1}{m}} \quad \text{and} \quad v_2 = \sqrt{\frac{T_2}{m}}$$

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{78.4}{19.6}} = \sqrt{4} = 2$$

Let λ_1 and λ_2 be the wavelengths at the lower and upper ends respectively. Since the frequency of the wave is independent of the medium,

$$\therefore \quad v_1 = f\lambda_1 \quad \text{and} \quad v_2 = f\lambda_2$$

$$\therefore \quad \frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1} = 2 \quad \text{or} \quad \lambda_2 = 2\lambda_1 = 2 \times 0.06 = \mathbf{0.12 \text{ m}}$$

8. Two wires of the same material and radii r and $2r$ respectively are welded together end to end. The combination is used as a sonometer wire and kept under tension T. The welded point is midway between the two bridges. When the stationary waves are set up in the composite wire, the joint is a node. What is the ratio of the number of loops formed in the thinner to the thicker wire? [P.M.T. M.P.]

Hint. Let n_1 and n_2 be the number of loops in the thinner and thicker wires respectively. Since the wires are connected in series, the frequency of the two wires is the same, *i.e.*

$$f_1 = f_2$$

or $$\frac{n_1}{2L_1}\sqrt{\frac{T}{m_1}} = \frac{n_2}{2L_2}\sqrt{\frac{T}{m_2}} \quad \text{or} \quad \frac{n_1}{n_2} = \frac{L_1}{L_2}\sqrt{\frac{m_1}{m_2}}$$

The joint is midway between the bridges. Therefore, $L_1 = L_2 = L$. Also $m_1 = \pi r^2\rho$ and $m_2 = \pi(2r)^2\rho$.

$$\therefore \quad \frac{m_1}{m_2} = \frac{\pi r^2\rho}{\pi(2r)^2\rho} = \frac{1}{4}$$

$$\therefore \quad \frac{n_1}{n_2} = \frac{L}{L}\sqrt{\frac{1}{4}} = \frac{1}{2} \quad \text{or} \quad n_1 : n_2 = \mathbf{1 : 2}$$

9. The component waves producing a stationary wave have amplitude, frequency and velocity of 8 cm, 30 Hz and 180 cm/sec respectively. Write the equation of the stationary wave.

Hint. The equations of the component progressive waves which produce the stationary wave are :

$$y_1 = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad ; \quad y_2 = a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

The superposition of these component waves produces the stationary wave given by ;

$$y = 2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi t}{T}$$

Here a = 8 cm ; f = 30 Hz and v = 180 cm/sec. From this data, we get,

$$T = \frac{1}{30} \text{ sec} ; \lambda = \frac{v}{f} = vT = 180 \times \frac{1}{30} = 6 \text{ cm}$$

$$\therefore \quad y = 2 \times 8 \cos \frac{2\pi x}{6} \sin \frac{2\pi t}{1/30}$$

or $$y = 16 \cos \frac{\pi x}{3} \sin 60\pi t$$

10. A closed pipe and an open pipe produce simultaneously a sound with 5 beats per second. If the length of the open pipe is 30 cm, then determine the required change in length of the closed pipe if both the pipes are to produce the sound of the same frequency. The speed of sound is 330 ms^{-1}. [C.E.E.]

Hint. Frequency of open pipe, $f = \frac{v}{2L} = \frac{330 \times 10^2}{2 \times 30} = 550$ Hz

Since this pipe produces 5 beats per second with the closed pipe, the frequency of the closed pipe

$$= 550 \pm 5 = 555 \text{ or } 545 \text{ Hz}$$

Let l cm be the length of the closed pipe. Then its fundamental frequency = $v/4l$.

$$\therefore \quad \frac{v}{4l} = 555 \text{ or } 545$$

or $$\frac{330 \times 10^2}{4l} = 555 \text{ or } 545$$

$$\therefore \quad l = 14.86 \text{ cm or } 15.14 \text{ cm.}$$

In order that frequency of closed pipe must be the same as that of open pipe (*i.e.* 550 Hz), let its length be changed by x cm.

$$\therefore \quad \frac{330 \times 10^2}{4(l \pm x)} = 550 \quad \text{or} \quad l \pm x = 15$$

$$\therefore \quad 14.86 + x = 15 \quad \text{or} \quad 15.14 - x = 15$$

$$\therefore \quad x = \mathbf{0.14 \text{ cm}}$$

11. In an experiment, it was found that a tuning fork and a sonometer wire gave 5 beats per second, both when the length of the wire was 1 m and 1.05 m. Calculate the frequency of the tuning fork.

Hint. $\lambda_1 = 2L_1 = 2 \times 1 = 2$ m ; $\lambda_2 = 2L_2 = 2 \times 1.05 = 2.1$ m

$\therefore f_1 = v/\lambda_1 = v/2$; $f_2 = v/\lambda_2 = v/2.1$

Let the frequency of the tuning fork be f Hz.

$$\therefore \quad f_1 - f = 5 \qquad \text{(given)}$$

or $$\frac{v}{2} - f = 5 \qquad \text{...(i)}$$

Also $$f - f_2 = 5 \qquad \text{(given)}$$

or $$f - \frac{v}{2.1} = 5 \qquad \text{...(ii)}$$

Solving eqs. (*i*) and (*ii*), we get, f = **205 Hz**

12. Two tuning forks *A* and *B* give 5 beats/second. The frequency of *A* = 512 Hz. When *B* is filed, 5 beats/second are again produced. Find the frequency of *B* before and after filing.

Hint. Frequency of A = 512 Hz ; Beats per second = 5

$\therefore$ Frequency of B before filing

$$= 512 \pm 5 = 517 \text{ Hz or } 507 \text{ Hz}$$

Since after filing 5 beats/second are heard, the frequency of B after filing

$$= 512 \pm 5 = \mathbf{517 \text{ Hz or } 507 \text{ Hz}}$$

13. A person standing between two parallel hills fires a gun. He hears the first echo after 1.5 seconds and second after 2.5 seconds. If speed of sound is 332 ms^{-1}, calculate the distance between the hills. When will he hear the third echo? [Roorkee]

Hint. Figure 26.42 shows the conditions of the problem. Here P is the person standing between parallel hills A and B. Let the distance of the person be x and y metres from the hills A and B respectively. The first echo is heard after 1.5 seconds on reflection of sound from hill A.

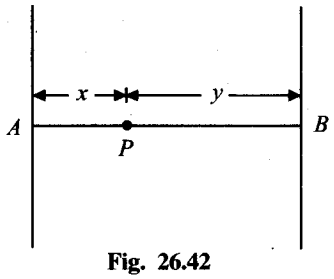


Fig. 26.42

$\therefore$ $2x = vt_1$ [v = velocity of sound]

or $$x = \frac{vt_1}{2} = \frac{332 \times 1.5}{2} = 249 \text{ m}$$

The second echo is heard after 2.5 seconds on reflection of sound from hill B.

$\therefore$ $2y = vt_2$

or $$y = \frac{vt_2}{2} = \frac{332 \times 2.5}{2} = 415 \text{ m}$$

$\therefore$ Distance between hills = $x + y = 249 + 415 = \mathbf{664 \text{ m}}$

The third echo will be heard when the sound reflected from hill A will strike the hill B and after reflection reaches the person.

$\therefore$ Time for third echo = $\frac{2x+2y}{v} = \frac{2(x+y)}{v} = \frac{2 \times 664}{332} = 4$ s

14. A tuning fork of frequency 300 Hz resonates with an air column closed at one end at 27°C. How many beats will be heard in the vibrations of the fork and the air column at 0°C? End-correction is negligible.

Hint. As per the conditions of the problem, the frequency of the air column at 27°C is 300 Hz. If L is the length of the air column and speed of sound at 27°C is v_{27}, then,

$$300 = \frac{v_{27}}{4L} \qquad ...(i)$$

Let the speed of sound at 0°C be v_0. Then,

$$\frac{v_0}{v_{27}} = \sqrt{\frac{0+273}{27+273}} = \sqrt{\frac{273}{300}} = 0.954 \quad \text{or} \quad v_0 = 0.954\, v_{27}$$

$\therefore$ Frequency of air column at 0°C is

$$f' = \frac{v_0}{4L} = \frac{0.954\, v_{27}}{4L}$$

Putting the value of $v_{27}/4L$ (= 300) from eq. (i), we get,

$$f' = 0.954 \times 300 = 286 \text{ Hz}$$

The frequency of tuning fork remains 300 Hz.

$\therefore$ Number of beats = 300 − 286 = **14 per second**

N.C.E.R.T. TEXTBOOK EXERCISES

NCERT CHAPTER 14 : OSCILLATIONS

Q.1. Which of the following examples represent periodic motion?

(*a*) A swimmer completing one (return) trip from one bank of a river to the other and back.

(*b*) A freely suspended bar magnet displaced from its N-S direction and released.

(*c*) A hydrogen molecule rotating about its centre of mass.

(*d*) An arrow released from a bow.

Ans. (*a*) It is not a periodic motion. Although the motion of the swimmer is to and fro but it does not have a definite time period.

(*b*) It is a periodic motion. Once the freely suspended magnet is displaced and is allowed to execute motion, it will oscillate about a fixed point with a definite time period. In fact, it will execute S.H.M.

(*c*) It is a periodic motion.

(*d*) It is not a periodic motion.

Q.2. Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?

(*a*) the rotation of earth about its axis.

(*b*) motion of an oscillating mercury column in a U-tube.

(*c*) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lowermost point.

(*d*) general vibrations of a polyatomic molecule about its equilibrium position.

Ans. (*a*) It is periodic but not S.H.M. because the motion is not to and fro about a fixed point.

(*b*) It is S.H.M. It is because acceleration of the liquid is directly proportional to displacement and its direction is opposite to the displacement.

(*c*) It is simple harmonic motion (S.H.M.).

(*d*) It is periodic but not S.H.M. A polyatomic molecule has a number of natural frequencies. Therefore, its general motion is due to the superposition of S.H.M.s of a number of natural frequencies. As a result, the resultant motion is periodic but not S.H.M.

Q.3. Fig. 14.01 depicts four *x-t* plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion)?

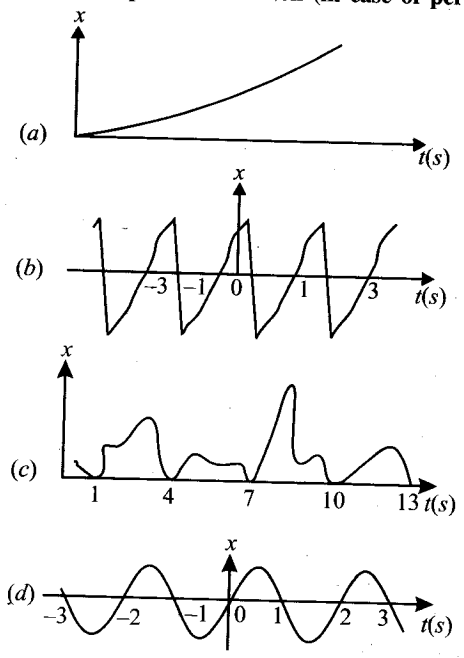


Fig. 14.01

Ans. (*a*) Fig. 14.01(*a*) does not represent periodic motion because the motion of the particle is not repeated.

(*b*) Fig. 14.01(*b*) represents periodic motion because motion is repeated after a definite period. This periodic motion has a period of 2s [1 – (–1) = 2s].

(*c*) Fig. 14.01(*c*) does not represent periodic motion. It is because if you see $x - t$ graph between the time 1 – 4, 4 – 7, 7 – 10 and 10 – 13 seconds, motions are not repeated in an identical manner.

(*d*) Fig. 14.01(*d*) represents periodic motion with a time period equal to 2s [1 – (–1) = 2s].

Q.4. Which of the following functions of time represent (*a*) simple harmonic (*b*) periodic but not simple harmonic and (*c*) non-periodic motion? Give period for each case of periodic motion [ω is any positive constant] :

(*a*) $\sin \omega t - \cos \omega t$ **(*b*) $\sin^3 \omega t$**

(*c*) $3 \cos (\pi/4 - 2\omega t)$ **(*d*) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$**

(*e*) $\exp [-\omega^2 t^2]$ **(*f*) $1 + \omega t + \omega^2 t^2$**

Ans. The function will represent a periodic motion if it is ideally repeated after a fixed interval of time. However, the function will represent S.H.M. if it can be written in the form $\sin (\omega t \pm \phi)$ or $\cos (\omega t \pm \phi)$.

(*a*) $$\sin \omega t - \cos \omega t = \sqrt{2}\left[\frac{1}{\sqrt{2}} \sin \omega t - \frac{1}{\sqrt{2}} \cos \omega t\right]$$

$$= \sqrt{2}\left[\sin \omega t \cos \frac{\pi}{4} - \cos \omega t \sin \frac{\pi}{4}\right] = \sqrt{2} \sin (\omega t - \pi/4)$$

$$\therefore \quad \sin \omega t - \cos \omega t = \sqrt{2} \sin(\omega t - \pi/4)$$

Therefore, the function represents S.H.M. and its period T is $2\pi/\omega$ [$\omega = 2\pi f = 2\pi/T$ or $T = 2\pi/\omega$].

(*b*) $$\sin^3 \omega t = \frac{1}{4}[3 \sin \omega t - \sin 3 \omega t]$$

Here $\sin \omega t$ and $\sin 3\omega t$ individually represent S.H.M. But the function $\sin^3 \omega t$ is a superposition of two S.H.M.s of different frequencies. Hence the resultant motion is not S.H.M. but periodic with a period equal to $2\pi/\omega$.

(*c*) $$3 \cos\left(\frac{\pi}{4} - 2\omega t\right) = 3 \cos\left(2\omega t - \frac{\pi}{4}\right) \qquad [\because \cos(-\theta) = \cos \theta]$$

It is obvious that the function represents S.H.M. with a period equal to $2\pi/2\omega$.

(*d*) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$

As explained in part (*b*), the function represents periodic motion but not S.H.M. Its time period is $2\pi/\omega$.

(*e*) $e^{-\omega^2 t^2}$. As the time t increases, the value of the function decreases and it becomes zero when $t \to \infty$. Therefore, the value of function never repeats. Hence the function is non-periodic.

(*f*) $1 + \omega t + \omega^2 t^2$. As the time t increases, the value of the function also increases and it approaches ∞ as $t \to \infty$. Therefore, the value of function never repeats. Hence the function is non-periodic.

Q.5. A particle is in linear simple harmonic motion between two points, *A* and *B*, 10 cm apart. Take the direction from *A* to *B* as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

(*a*) at the end *A*,

(*b*) at the end *B*,

(*c*) at the mid-point of *AB* going towards *A*,

(*d*) at 2 cm away from *B* going towards *A*,

(*e*) at 3 cm away from *A* going towards *B*, and

(*f*) at 4 cm away from *A* going towards *A*.

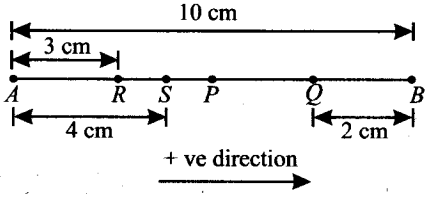


Fig. 14.02

Ans. Fig. 14.02 shows the conditions of the problem. Remember that for a particle executing S.H.M., the acceleration and force are always directed towards the mean position (point P in Fig. 14.02). It is given that direction from A to B is positive.

(a) **At the end A.** At the extreme end A, velocity is zero. The acceleration and force are directed towards the mean position P *i.e.* along AP—the positive direction. Hence acceleration as well as force is positive.

(b) **At the end B.** At the extreme end B, velocity is zero. The acceleration and force are directed towards the mean position P *i.e.* along BP—the negative direction. Hence acceleration and force are negative.

(c) **At mid point of AB going towards A.** The particle is at mean position P and going towards A *i.e.* it has a tendency to move along PA *i.e.* negative direction. Hence velocity is negative. Both acceleration and force are zero.

(d) **At 2 cm away from B going towards A.** The particle is at point Q. Since the motion is directed along BP (negative direction), velocity, acceleration and force are negative.

(e) **At 3 cm away from A going towards B.** The particle is at point R. Since the motion is directed along AP, velocity, acceleration and force are positive.

(f) **At 4 cm away from A going towards A.** The particle is at point S. The particle is moving along SA, the velocity is negative. However, acceleration and force are directed towards the mean position P *i.e.* along SP. Hence acceleration and force are positive.

Q.6. **Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?**

(a) $a = 0.7x$ (b) $a = -200\,x^2$ (c) $a = -10x$ (d) $a = 100\,x^2$

Ans. For a particle executing S.H.M., $a = -kx$. Only option (c) satisfies this condition.

Q.7. **The motion of a particle executing simple harmonic motion is described by the displacement function :**

$$x(t) = A\cos(\omega t + \phi)$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is π s^{-1}. If instead of the cosine function, we choose the sine function to describe the S.H.M. : $x = B\sin(\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions?

Ans. We know that S.H.M. can be represented by cosine function or sine function.

(i) $$x = A\cos(\omega t + \phi)$$

Here at $t = 0$, $x = 1$ cm and $V = \omega$ cms^{-1} ; $\omega = \pi$ s^{-1}

$$\therefore \quad 1 = A\cos(\pi \times 0 + \phi) = A\cos\phi \qquad ...(i)$$

Now, Velocity, $V = \dfrac{dx}{dt} = -A\,\omega\sin(\omega t + \phi)$

or $$\omega = -A\,\omega\sin(\pi \times 0 + \phi) \text{ or } 1 = -A\sin\phi \qquad ...(ii)$$

Squaring and adding eqs. (i) and (ii), we have

$$A^2(\sin^2\phi + \cos^2\phi) = 1 + 1 = 2 \quad \therefore \quad A = \sqrt{2}\text{ cm}$$

(ii) Dividing eq. (ii) by eq. (i), $\tan\phi = -1 \quad \therefore \quad \phi = 3\pi/4$ or $7\pi/4$

$$x = B\sin(\omega t + \alpha)$$

Again at $t = 0$, $x = 1$ cm and $V = \omega$ cm s^{-1} ; $\omega = \pi$ s^{-1}

$$\therefore \quad 1 = B\sin(\pi \times 0 + \alpha) = B\sin\alpha \qquad ...(iii)$$

$$\text{Now velocity, } V = \frac{dx}{dt} = B\,\omega\cos(\omega t + \alpha)$$

or $$\omega = B\,\omega\cos(\pi \times 0 + \alpha) \text{ or } 1 = B\cos\alpha \qquad ...(iv)$$

Squaring and adding eqs. (iii) and (iv), we have,

$$B^2(\sin^2\alpha + \cos^2\alpha) = 1 + 1 = 2 \quad \therefore \quad B = \sqrt{2}\text{ cm}$$

Dividing eq. (iii) by (iv), $\tan\alpha = 1 \quad \therefore \quad \alpha = \pi/4$ or $5\pi/4$

Q.8. A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

Ans. The conditions of the problem suggest that when a mass of 50 kg is suspended from the spring balance, the extension produced is 20 cm.

$\therefore$ Force constant of the spring is

$$k = \frac{F}{y} = \frac{mg}{y} = \frac{50 \times 9.8}{0.2} = 2450 \text{ N m}^{-1}$$

Let M be the mass of the suspended body. The period of oscillations is given by ;

$$T = 2\pi\sqrt{\frac{M}{k}}$$

Here $T = 0.60$ s ; $k = 2450$ N m^{-1}

$$\therefore \quad 0.60 = 2\pi\sqrt{\frac{M}{2450}} \text{ or } M = 22.34 \text{ kg}$$

$\therefore$ Weight of the body $= Mg = 22.34 \times 9.8 = \mathbf{218.93}$ N

Q.9. A spring having a spring constant 1200 Nm^{-1} is mounted on a horizontal table as shown in Fig. 14.03. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

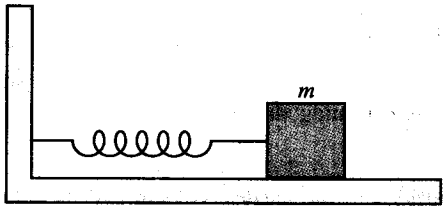


Fig. 14.03

Determine (*i*) the frequency of oscillations (*ii*) maximum acceleration of the mass and (*iii*) the maximum speed of the mass.

Ans. Here $k = 1200$ Nm^{-1} ; $m = 3.0$ kg ; $a = 2.0$ cm $= 2.0 \times 10^{-2}$ m

(*i*) The frequency of oscillations of the mass is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{1200}{3.0}} = \mathbf{3.18} \text{ s}^{-1}$$

(*ii*) Angular frequency, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3.0}} = 20 \text{ s}^{-1}$

Maximum acceleration of the mass is

$$A_{max} = \omega^2 a = (20)^2 \times 2.0 \times 10^{-2} = \mathbf{8.0} \text{ m s}^{-2}$$

(*iii*) Maximum speed of the mass is

$$V_{max} = \omega a = 20 \times 2.0 \times 10^{-2} = \mathbf{0.40} \text{ m s}^{-1}$$

Q.10. In Q.9, let us take the position of mass when the spring is unstreched as $x = 0$ and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch ($t = 0$), the mass is

(*a*) at the mean position

(*b*) at the maximum stretched position and

(*c*) at the maximum compressed position.

In what way do these functions for S.H.M. differ from each other, in frequency, in amplitude or the initial phase?

Ans. (*a*) When the mass starts from the mean position, the displacement of S.H.M. is given by ;

$$x = a \sin \omega t = \mathbf{2 \sin 20\, t} \qquad (\because\ a = 2 \text{ cm} ;\ \omega = 20\ \text{s}^{-1})$$

(*b*) When the mass starts from extreme position, the displacement of S.H.M. is given by ;

$$x = \pm a \cos \omega t$$

When the mass is at the maximum stretched position, the displacement of S.H.M. is given by ;

$$x = a \cos \omega t = \mathbf{2 \cos 20\, t}$$

(*c*) When the mass is at the maximum compressed position, the displacement of S.H.M. is given by ; $x = -a \cos \omega t = \mathbf{-2 \cos 20\, t}$

These functions have the same amplitude and frequency. They only differ in initial phase.

Q.11. Figures 14.04 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (*i.e.* clockwise or anti-clockwise) are indicated on each figure.

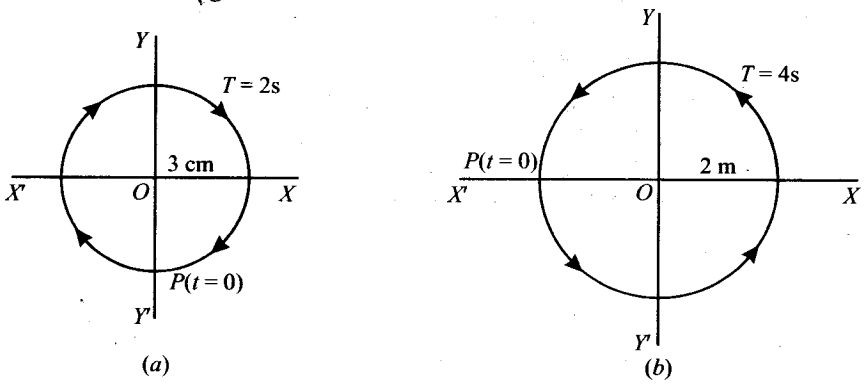


Fig. 14.04

Obtain the corresponding simple harmonic motions of the *x*-projection of the radius vector of the revolving particle *P* in each case.

Ans. Referring to Fig. 14.04(*a*), $T = 2$ s ; $a = 3$ cm. The particle P is executing clockwise motion. At $t = 0$, the phase of the particle is $\phi = +\pi/2$ radians. Therefore, the x-projection of OP at any time t will give the equation of S.H.M. as :

$$x = a\cos(\omega t + \phi) = a\cos\left(\frac{2\pi t}{T} + \frac{\pi}{2}\right) = 3\cos\left(\frac{2\pi t}{2} + \frac{\pi}{2}\right)$$

$$\therefore \qquad x = -3 \sin \pi t \ \text{(in cm)}$$

Referring to Fig. 14.04(*b*), $T = 4$ s ; $a = 2$ m. The particle P is executing anticlockwise motion. At $t = 0$, the phase of the particle is $\phi = +\pi$. Therefore, the x-projection of OP at any time t will give the equation of S.H.M. as :

$$x = a\cos(\omega t + \phi) = a\cos\left(\frac{2\pi t}{T} + \pi\right) = 2\cos\left(\frac{2\pi t}{4} + \pi\right)$$

$$\therefore \qquad x = -2\cos\frac{\pi}{2}t \ \text{(in m)}$$

Q.12. Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial ($t = 0$) position of the particle, the radius of the circle and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case : (x is in cm and t is in s).

(*a*) $x = -2 \sin (3t + \pi/3)$ **(*b*) $x = \cos (\pi/6 - t)$**

(*c*) $x = 3 \sin (2\pi t + \pi/4)$ **(*d*) $x = 2 \cos \pi t$**

Ans. The standard equation of S.H.M. is $x = a \cos (\omega t + \phi)$. Here ϕ is the initial phase. We shall express each given function in the form of standard equation of S.H.M. in order to find the various quantities.

(*a*)
$$x = -2\sin\left(3t + \frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{2} + 3t + \frac{\pi}{3}\right)$$

$$\therefore \quad x = 2\cos\left(3t + \frac{5\pi}{6}\right)$$

Comparing it with standard S.H.M. equation $x = a\cos(\omega t + \phi)$,

$$a = 2 \text{ cm} ; \ \omega = 3 \text{ rad s}^{-1} ; \ \phi = 5\pi/6$$

Therefore, the reference circle for the given S.H.M. will be as shown in Fig. 14.05(*a*). Note that at $t = 0$, the particle is at point P such that $\angle POX = \phi = 5\pi/6$.

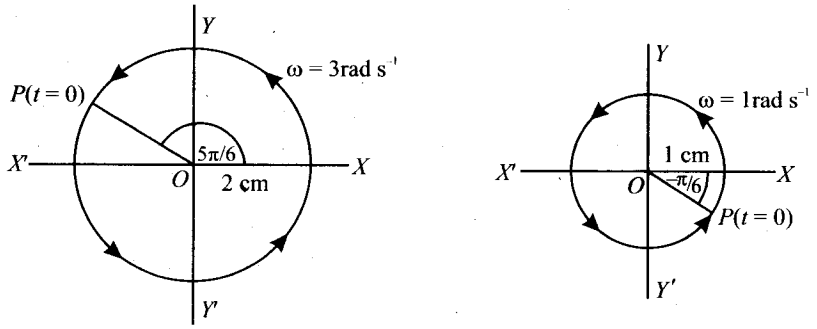


Fig. 14.05

(*b*) $$x = \cos\left(\frac{\pi}{6} - t\right) = \cos\left(t - \frac{\pi}{6}\right) \qquad [\because \cos(-\theta) = \cos\theta]$$

$$\therefore \quad x = \cos\left(t - \frac{\pi}{6}\right)$$

Comparing it with standard S.H.M. equation $x = a\cos(\omega t + \phi)$,

$$a = 1 \text{ cm} ; \ \omega = 1 \text{ rad s}^{-1} ; \ \phi = -\pi/6$$

Therefore, the reference circle for the given S.H.M. will be as shown in Fig. 14.05(*b*). Note that at $t = 0$, the particle is at point P such that $\angle POX = \phi = -\pi/6$.

(*c*) $$x = 3\sin\left(2\pi t + \frac{\pi}{4}\right) = 3\cos\left(2\pi t + \frac{3\pi}{2} + \frac{\pi}{4}\right) = 3\cos\left(2\pi t + \frac{7\pi}{4}\right)$$

$$\therefore \quad x = 3\cos\left(2\pi t + \frac{7\pi}{4}\right)$$

$$\therefore \quad a = 3 \text{ cm} ; \ \omega = 2\pi \text{ rad s}^{-1} ; \ \phi = 7\pi/4$$

Therefore, the reference circle for the given S.H.M. will be as shown in Fig. 14.06(*a*). At $t = 0$, the particle is at point P such that $\angle POX = \phi = 7\pi/4$.

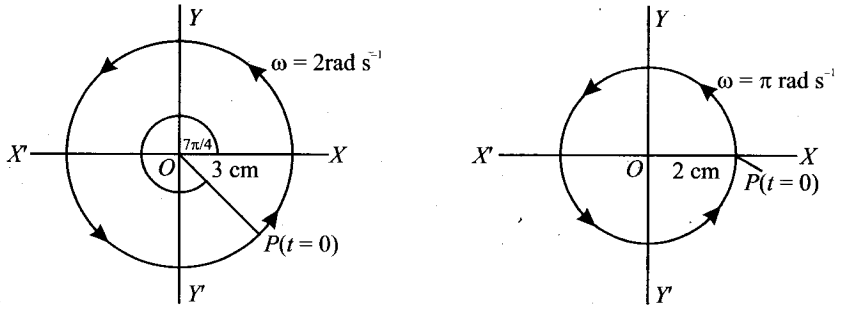


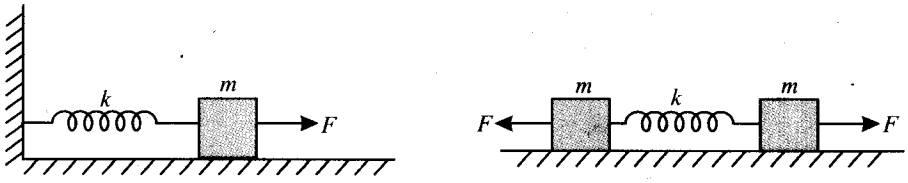
Fig. 14.06

(*d*) $$x = 2\cos\pi t$$

$$\therefore \quad a = 2 \text{ cm} ; \ \omega = \pi \text{ rad s}^{-1} ; \ \phi = 0$$

Therefore, the reference circle for the given S.H.M. will be as shown in Fig. 14.06(*b*). At $t = 0$, the particle will be at point P on the right extreme position.

Q.13. Fig. 14.07(*a*) shows a spring of force constant *k* clamped rigidly at one end and a mass *m* attached to its free end. A force *F* applied at the free end stretches the spring. Fig. 14.07(*b*) shows the same spring with both ends free and attached to a mass *m* at either end. Each end of the spring in Fig. 14.07(*b*) is stretched by the same force *F*.



(*a*) (*b*)

Fig. 14.07

(*a*) What is the maximum extension of the spring in the two cases?

(*b*) If the mass in Fig. (*a*) and the two masses in Fig. (*b*) are released, what is the period of oscillations in each case?

Ans. (*a*) In Fig. 14.07(*a*), the maximum extension of the spring is $x = F/k$ where k is the spring constant of the spring. In Fig. 14.07(*b*), each mass relative to the other behaves as if the other mass is fixed. Therefore, in this case also, maximum extension of the spring is $x = F/k$.

(*b*) In Fig. 14.07(*a*), the restoring force $F = -kx$ or $F \propto -x$. Therefore, the spring will execute S.H.M. and the time period (T) of the oscillations is given by ;

$$T = 2\pi\sqrt{\frac{m}{k}}$$

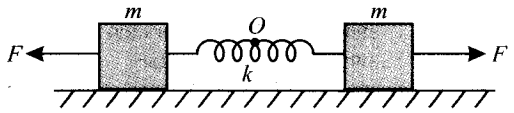


Fig. 14.08

In Fig. 14.07(*b*), we can consider the system to be equivalent to two springs, each of length $l/2$ and joined at point O, the centre of the spring as shown in Fig. 14.08. Therefore, the effective spring constant of the system becomes $2k$.

$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{2k}}$$

Q.14. The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed?

Ans. Here $\omega = 200$ rad min^{-1} $= 200/60$ rad s^{-1} ; $2a = 1$ m or $a = 0.5$ m

$\therefore$ Maximum speed, $V_{max} = a\,\omega = 0.5 \times 200/60 = 1.67$ ms^{-1}

Q.15. The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s? (g on the surface of earth is 9.8 ms^{-2}).

Ans. Let l be the length of the pendulum.

$$\textit{On earth} : \; T = 2\pi\sqrt{\frac{l}{g}} \; ; \; \textit{On moon} : \; T' = 2\pi\sqrt{\frac{l}{g'}}$$

$$\therefore \qquad \frac{T'}{T} = \sqrt{\frac{g}{g'}} \quad \text{or} \quad T' = T \times \sqrt{\frac{g}{g'}} = 3.5 \times \sqrt{\frac{9.8}{1.7}} = 8.4 \text{ s}$$

Q.16. Answer the following questions :

(*a*) Time period of a particle in S.H.M. depends on the force constant *k* and mass *m* of the particle : $T = 2\pi\sqrt{m/k}$

A simple pendulum executes S.H.M. approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

(*b*) The motion of a simple pendulum is approximately simple harmonic for small angle of oscillations. For larger angles of oscillations, a more involved analysis shows that T is greater than $2\pi\sqrt{l/g}$. Think of a qualitative argument to appreciate this result.

(*c*) A man with a wrist watch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?

(*d*) What is the frequency of oscillations of a simple pendulum mounted in a cabin that is freely falling under gravity?

Ans. (*a*) Time period, $T = 2\pi\sqrt{\frac{m}{k}}$ ...(*i*)

In case of a simple pendulum, the restoring force (F) is

$$F = \frac{-mg}{l}y \quad \therefore \quad \text{Force constant, } k = \frac{mg}{l}$$

In we put the value of k in eq. (*i*), the term m cancels out. For this reason, the time period of a simple pendulum is independent of mass of the pendulum.

(*b*) The restoring force tending to bring the pendulum to its mean position is $mg\sin\theta$. In arriving at the formula $T = 2\pi\sqrt{l/g}$, we take $\sin\theta \simeq \theta$ *i.e.* restoring force = $mg\theta$. For large values of θ, $\sin\theta < \theta$. Therefore, the restoring force decreases from $mg\theta$ to $mg\sin\theta$. As a result, the pendulum takes a longer time to complete one vibration.

(*c*) Yes. A wrist watch is spring wound and its working is based on spring action *i.e.* it is based on the potential energy stored in the spring. Therefore, the time period of wrist watch is independent of acceleration due to gravity. For this reason, the wrist watch will give correct time during the free fall.

(*d*) The time period of a simple pendulum falling with acceleration a is

$$T = 2\pi\sqrt{\frac{l}{g-a}}$$

For a freely falling cabin, $a = g$ so that $T = \infty$. Hence frequency $f = 1/T = 0$.

Q.17. A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?

Ans. When the car moves along a circular track, the centripetal acceleration (a_c) acting on the bob is

$$a_c = v^2/R \quad \text{...radially inward along horizontal}$$

The acceleration due to gravity (g) acts vertically downwards.

$\therefore$ Effective acceleration due to gravity is

$$g' = \sqrt{g^2 + a_c^2} = \sqrt{g^2 + (v^2/R)^2} = \sqrt{g^2 + v^4/R^2}$$

$\therefore$ Time period of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g^2 + v^4/R^2}}$$

Q.18. A cylindrical piece of cork of base area A and height h floats in a liquid of density ρ_1. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period :

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$$

where ρ is the density of rork. (Ignore damping due to viscosity of the liquid).

Ans. Refer to Art. 25.15.

Q.19. One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Ans. Refer to Art. 25.14

Q.20. An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig. 14.09). Show that when the ball is pressed down a little and released, it executes S.H.M. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal [See Fig. 14.09].

Ans. Fig. 14.09 shows the air chamber. If the ball is depressed by distance y, decrease in volume, $\Delta V = A \times y$.

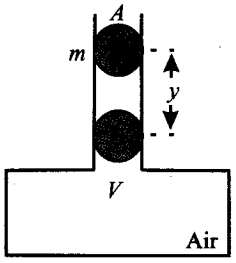


Fig. 14.09

If Δp is the increase in pressure applied on the ball, then, bulk modulus of elasticity of air is

$$K = -\frac{\Delta p}{\Delta V / V} = -\frac{\Delta p}{Ay / V}$$

$$\therefore \quad \Delta p = -\frac{KA}{V} y$$

Restoring force, $F = \Delta p \times A = -\frac{KAy}{V} \times A = -\frac{KA^2}{V} y$

Since F is directly proportional to y and acts in opposite direction, the ball executes S.H.M.

Now $F = -k\,y$ where $k = \frac{KA^2}{V}$

Period of oscillations of the ball is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{KA^2/V}} = 2\pi\sqrt{\frac{mV}{KA^2}}$$

The P–V variations are isothermal so that $K = \Delta p$.

$$\therefore \quad T = 2\pi\sqrt{\frac{mV}{\Delta p A^2}}$$

Note. If P–V variations are adiabatic, $K = \gamma \Delta p$.

$$\therefore \quad T = 2\pi\sqrt{\frac{mV}{\gamma \Delta p A^2}}$$

Q.21. You are riding in an automobile of mass 3000 kg. Assume that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillations decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg. Take $g = 9.8$ m s^{-2}.

Ans. (a) Suppose k is the spring constant of the spring of one wheel. When the spring is compressed through a distance x, the restoring force developed is

$$F = -kx$$

If M is the mass supported by each wheel, then, $F = -Mg$.

$$\therefore \quad -k\,x = -Mg \text{ or } k = \frac{Mg}{x}$$

Here, $M = 750$ kg ; $g = 9.8$ ms^{-2} ; $x = 15$ cm $= 0.15$ m

$$\therefore \quad k = \frac{750 \times 9.8}{0.15} = 4.9 \times 10^4 \text{ N/m}$$

(b) If b is the damping constant for the spring and shock absorber system, then damped amplitude (x) of oscillations is

$$x = x_0\, e^{-bT/2M} \qquad ...(i)$$

Here x_0 = Initial amplitude of oscillations

T = Time period of the spring

M = Mass supported by spring = 750 kg

Since amplitude of oscillations decreases by 50% during one complete oscillation, $x = x_0/2$. Therefore, eq. (*i*) becomes :

$$\frac{x_0}{2} = x_0 e^{-bT/2M} \text{ or } \frac{1}{2} = e^{-bT/2M}$$

$$\therefore \quad 2 = e^{bT/2M} \text{ or } \log_e 2 = \frac{bT}{2M} \log_e e = \frac{bT}{2M}$$

$$\therefore \quad \text{Damping constant, } b = \frac{2M \log_e 2}{T}$$

Here
$$T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{750}{4.9\times10^4}} = 0.77 \text{ s}$$

$$\therefore \quad b = \frac{2\times750\times\log_e 2}{0.77} = 1350.3 \text{ kg s}^{-1}$$

Q.22. **Show that for a particle in linear S.H.M., the average kinetic energy over a period of oscillation equals the average potential energy over the same period.**

Ans. Consider a particle of mass m executing S.H.M. with amplitude a and angular frequency ω. Then displacement y of the particle from the mean position at any time t is given by ;

$$y = a \sin \omega t$$

$$\therefore \quad \text{Velocity, } V = \frac{dy}{dt} = a\,\omega\cos\omega t$$

$$\text{K.E., } E_K = \frac{1}{2}mV^2 = \frac{1}{2}m a^2\omega^2\cos^2\omega t$$

$$\text{P.E., } E_P = \frac{1}{2}k y^2 = \frac{1}{2}m\omega^2 a^2\sin^2\omega t \qquad (\because k = m\omega^2)$$

Average K.E. over one cycle. The average K.E. over one cycle is given by ;

$$E_{K_{av}} = \frac{1}{T}\int_0^T E_K\,dt = \frac{1}{T}\int_0^T \frac{1}{2}m a^2\omega^2\cos^2\omega t\,dt$$

$$= \frac{1}{2T}m a^2\omega^2\int_0^T \frac{1+\cos 2\omega t}{2}dt$$

$$= \frac{1}{4T}m a^2\omega^2\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T = \frac{1}{4T}m a^2\omega^2(T)$$

$$\therefore \quad E_{K_{av}} = \frac{1}{4}m a^2\omega^2 \qquad ...(i)$$

Average P.E. over one cycle. The average P.E. over one cycle is given by ;

$$E_{P_{av}} = \frac{1}{T}\int_0^T E_P\,dt = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 a^2\sin^2\omega t\,dt$$

$$= \frac{1}{2T}m\omega^2 a^2\int_0^T \frac{1-\cos 2\omega t}{2}dt = \frac{1}{4T}m\omega^2 a^2\left[1 - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T}m\omega^2 a^2[T] = \frac{1}{4}m a^2\omega^2$$

$$\therefore \quad E_{P_{av}} = \frac{1}{4}m a^2\omega^2 \qquad ...(ii)$$

From eqs. (*i*) and (*ii*), $E_{K_{av}} = E_{P_{av}}$ Q.E.D.

Q.23. **A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation $J = -\alpha\,\theta$, where J is the restoring couple and θ the angle of twist).**

Ans. Here m = 10 kg ; R = 15 cm = 0.15 m ; T = 1.5 s ; α = ?

Moment of inertia of disc, $I = \frac{1}{2}mR^2 = \frac{1}{2}\times 10\times(0.15)^2 = 0.1125 \text{ kg m}^2$

Now Time period, $T = 2\pi\sqrt{\frac{I}{\alpha}}$ or $T^2 = 4\pi^2\times\frac{I}{\alpha}$

$\therefore$ $\alpha = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2\times 0.1125}{(1.5)^2} = 1.97$ Nm/rad

Q.24. A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (*a*) 5 cm (*b*) 3 cm and (*c*) 0 cm.

Ans. Here a = 5 cm = 0.05 m ; T = 0.2 s ; $\omega = 2\pi/T = 2\pi/0.2 = 10\ \pi$ rad/s

If a particle executing S.H.M. has displacement y, then,

Acceleration, $A = -\omega^2 y$; Velocity, $V = \omega\sqrt{a^2 - y^2}$

(*a*) *When y = 5 cm = 0.05 m.*

$A = -\omega^2 y = -(10\pi)^2 \times 0.05 = -5\pi^2$ m/s^2

$V = \omega\sqrt{a^2 - y^2} = 10\pi\times\sqrt{(0.05)^2 - (0.05)^2} = 0$

(*b*) *When y = 3 cm = 0.03 m.*

$A = -\omega^2 y = -(10\pi)^2 \times 0.03 = -3\pi^2$ m/s^2

$V = \omega\sqrt{a^2 - y^2} = 10\pi\times\sqrt{(0.05)^2 - (0.03)^2} = 0.4\ \pi$ m/s

(*c*) *When y = 0 cm.*

$A = -\omega^2 y = -(10\pi)^2 \times 0 = 0$

$V = \omega\sqrt{a^2 - y^2} = 10\pi\sqrt{(0.05)^2 - (0)^2} = 0.5\ \pi$ m/s

Q.25. A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time t = 0. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0 and v_0. [Hint. Start with the equation $x = a\cos(\omega t + \theta)$ and note that the initial velocity is negative].

Ans. The general equation of S.H.M. is

$x = a\cos(\omega t + \phi)$ ...a is the amplitude

It is given that at t = 0, $x = x_0$.

$\therefore$ $x_0 = a\cos(\omega\times 0 + \phi)$ or $a\cos\phi = x_0$ ...(*i*)

Velocity, $V = \frac{dx}{dt} = \frac{d}{dt}[a\cos(\omega t + \phi)] = -a\,\omega\sin(\omega t + \phi)$

At t = 0, $V = -v_0$ so that :

$-v_0 = -a\omega\sin(\omega\times 0 + \phi)$ or $a\sin\phi = v_0/\omega$ ...(*ii*)

Squaring and adding eqs. (*i*) and (*ii*), we have,

$a^2(\cos^2\phi + \sin^2\phi) = x_0^2 + (v_0/\omega)^2$

$\therefore$ Amplitude, $a = \sqrt{x_0^2 + v_0^2/\omega^2}$

NCERT CHAPTER 15 : WAVES

Q.1. A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does the disturbance take to reach the other end?

Ans. Tension, T = 200 N ; Mass per unit length, $m = \frac{2.5}{20.0} = 0.125$ kg m^{-1}

Now Velocity, $v = \sqrt{\frac{T}{m}} = \sqrt{\frac{200}{0.125}} = 40 \text{ ms}^{-1}$

$\therefore$ Time taken by the disturbance to reach the other end is

$$t = \frac{l}{v} = \frac{20}{40} = \mathbf{0.5 \text{ s}}$$

Q.2. A stone dropped from the top of a tower of height 300 m high splashes into the water of a pond near the base of the tower. When is the splash heard at the top given that the speed of sound in air is 340 m s^{-1}? (g = 9.8 m s^{-2}).

Ans. Suppose t_1 is the time taken by the stone to reach the surface of water and t_2 is the time taken by the splash to reach from water surface to the top. Then splash will be heard after time $t = t_1 + t_2$. Time t_1 taken by stone to reach water surface is given by the equation :

$$S = v_0 t_1 + \frac{1}{2} a t_1^2$$

Here S = 300 m ; v_0 = 0 ; $a = g$ = 9.8 m s^{-2} ; t_1 = ?

$\therefore$ $$300 = \frac{1}{2} \times 9.8 t_1^2$$

or $$t_1 = \sqrt{\frac{300 \times 2}{9.8}} = 7.82 \text{ s}$$

Time t_2 taken by the splash to reach from water surface to the top is

$$t_2 = \frac{\text{distance}}{\text{sound speed}} = \frac{300}{340} = 0.88 \text{ s}$$

$\therefore$ $$t = t_1 + t_2 = 7.82 + 0.88 = \mathbf{8.7 \text{ s}}$$

Q.3. A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wave on the wire equals the speed of sound in dry air at 20°C = 343 m s^{-1}?

Ans. The speed of a transverse wave on the wire is given by ;

$$v = \sqrt{\frac{T}{m}} \quad \text{or} \quad T = v^2 \times m$$

Here v = 343 m s^{-1} ; m = 2.10/12.0 kg m^{-1}

$\therefore$ $$T = (343)^2 \times \frac{2.10}{12.0} = \mathbf{2.06 \times 10^4 \text{ N}}$$

Q.4. Use the formula $v = \sqrt{\frac{\gamma P}{\rho}}$ to explain why the speed of sound in air

(*a*) is independent of pressure (*b*) increases with temperature (*c*) increases with humidity.

Ans. Here $$v = \sqrt{\frac{\gamma P}{\rho}} \qquad ...(i)$$

(*a*) Let V be the volume of 1 mole of air. Then,

$$\rho = \frac{M}{V} \text{ or } V = \frac{M}{\rho}$$

For 1 mole of air, $PV = RT$ or $P \times \frac{M}{\rho} = RT$ $\therefore$ $\frac{P}{\rho} = \frac{RT}{M}$

Putting the value $P/\rho = RT/M$ in eq. (*i*), we have,

$$v = \sqrt{\frac{\gamma RT}{M}} \qquad ...(ii)$$

Since, γ, R and M are constant, the speed of sound is independent of pressure provided temperature remains constant.

(*b*) From eq. (*ii*), it is clear that $v \propto \sqrt{T}$. Therefore, speed of sound increases with the increase in temperature.

(*c*) From eq. (*i*), $v \propto \sqrt{\frac{\gamma}{\rho}}$. For moist air, the values of both ρ and γ are less than that for dry air. However, the effect of ρ is more than that of γ. Therefore, the speed of sound increases with humidity.

Q.5. You have learnt that a travelling wave in one dimension is represented by a function $y = f(x, t)$ where x and t must appear in the combination $x - v\,t$ or $x + v\,t$, i.e. $y = f(x \pm v\,t)$. Is the converse true? Examine if the following functions for y can possibly represent a travelling wave :

(*a*) $(x - v\,t)^2$ (*b*) $\log[(x + v\,t)/x_0]$ (*c*) $1/(x + v\,t)$

Ans. No, the converse is not true. It is because the basic requirement of a wave function to represent a travelling wave is that for all values of x and t, the wave function must have finite value.

The functions (*a*), (*b*) and (*c*) are not finite for all values of x and t. Therefore, none can represent a travelling wave.

Q.6. A bat emits ultrasonic sound of frequency 1000 kHz in air. If the sound meets a water surface, what is the wavelength of (*a*) the reflected sound (*b*) the transmitted sound? Speed of sound in air is 340 m s^{-1} and in water 1486 m s^{-1}.

Ans. Speed of sound in air, v_a = 340 ms^{-1} ; Speed of sound in water, v_w = 1486 ms^{-1}

Since the source of sound is the same, the frequency (= 1000 kHz = 10^6 Hz) of sound in the two cases will be the same.

(*a*) Now $v_a = f\lambda_a$ $\therefore$ Wavelength of reflected sound, $\lambda_a = \frac{v_a}{f} = \frac{340}{10^6} = 3.4 \times 10^{-4}$ m

(*b*) Also $v_w = f\lambda_w$ $\therefore$ Wavelength of transmitted sound, $\lambda_w = \frac{v_w}{f} = \frac{1486}{10^6} = 1.49 \times 10^{-3}$ m

Q.7. A hospital uses an ultrasonic scanner to locate tumours in a tissue. What is the wavelength of sound in the tissue in which the speed of sound is 1.7 km s^{-1}? The operating frequency of the scanner is 4.2 MHz.

Ans. Here v = 1.7 kms^{-1} = 1700 ms^{-1} ; f = 4.2 MHz = 4.2 × 10^6 Hz

$\therefore$ Wavelength, $\lambda = \frac{v}{f} = \frac{1700}{4.2 \times 10^6} = 0.405 \times 10^{-3}$ m = **0.405 mm**

Q.8. A transverse harmonic wave on a string is described by :

$$y(x, t) = 3.0 \sin (36\,t + 0.018\,x + \pi/4)$$

where x and y are in cm and t in s. The positive direction of x is from left to right.

(*a*) Is this a travelling wave or a stationary wave?
If it is travelling, what are the speed and direction of its propagation?

(*b*) What are its amplitude and frequency?

(*c*) What is the initial phase at the origin?

(*d*) What is the least distance between two successive crests in the wave?

Ans.
$$y\,(x, t) = 3.0 \sin (36t + 0.018\,x + \pi/4) \quad ...(i)$$

The general equation of a plane progressive wave of amplitude a travelling with a velocity v from right to left is

$$y\,(x, t) = a \sin\left[\frac{2\pi}{\lambda}(vt + x) + \phi_0\right] \quad ...(ii)$$

(*a*) Comparing eqs. (*i*) and (*ii*), we note that the given equation represents a *transverse harmonic wave travelling from right to left.* It is not a stationary wave. The eq. (*i*) can be written as :

$$y\,(x, t) = 3.0 \sin\left[0.018\left(\frac{36}{0.018}t + x\right) + \frac{\pi}{4}\right] \quad ...(iii)$$

Equating the coefficients of t in eqs. (*ii*) and (*iii*), we have,

$$v = \frac{36}{0.018} = \mathbf{2000\ cm\ s^{-1}}$$

(*b*) Amplitude, $a = \mathbf{3.0\ cm}$

Also $\frac{2\pi}{\lambda} = 0.018$ or $\lambda = \frac{2\pi}{0.018}$

$\therefore$ Frequency, $f = \frac{v}{\lambda} = \frac{2000}{2\pi} \times 0.018 = \mathbf{5.73\ s^{-1}}$

(*c*) Initial phase, $\phi_0 = \boldsymbol{\pi/4}$ **rad**

(*d*) Least distance between two successive crests is equal to the wavelength of the wave.

$\therefore$ Wavelength, $\lambda = \frac{2\pi}{0.018} = \mathbf{349\ cm}$

Q.9. For the wave described in Q.8, plot the displacement (y) versus (t) graphs for x = 0, 2 and 4 cm. What are the shapes of these graphs? In which aspects does the oscillatory motion in travelling wave differ from one point to another : amplitude, frequency or phase?

Ans. The transverse harmonic wave on the string is

$$y\ (x,\ t) = 3.0 \sin\left(36t + 0.018x + \frac{\pi}{4}\right)$$

***y-t* graph when x = 0.**

For x = 0, the equation becomes :

$$y\ (0,\ t) = 3.0 \sin\left(36t + \frac{\pi}{4}\right) \qquad ...(i)$$

Here a = 3 cm ; ω = 36 rad s^{-1} ; $\phi = \pi/4$; $T = \frac{\pi}{18}$ sec

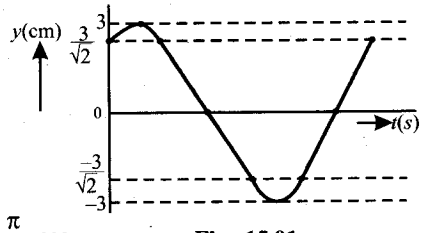


Fig. 15.01

For different values of t, we can find the value of y using eq. (*i*). The displacement (y) at different times will be as tabulated below.

t	0	$T/8$	$2T/8$	$3T/8$	$4T/8$	$5T/8$	$6T/8$	$7T/8$	T
y	$3/\sqrt{2}$	3	$3/\sqrt{2}$	0	$-3/\sqrt{2}$	−3	$-3/\sqrt{2}$	0	$3/\sqrt{2}$

On plotting y-t graph, we get sinusoidal curve as shown in Fig. 15.01.

***y-t* graph when x = 2 cm and x = 4 cm.** Following the procedure described above, the y-t graph for the two cases can be obtained. It will be found that y-t graph in the two cases will also be sinusoidal curve.

The oscillatory motion in travelling wave differs from one point to another only in initial phase. The amplitude and frequency remain the same in the three cases.

Q.10. For the travelling harmonic wave :

$$\boldsymbol{y(x,\ t) = 2.0 \cos 2\pi\ (10t - 0.0080\ x + 0.35)}$$

where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of (*a*) 4 m (*b*) 0.5 m (*c*) $\lambda/2$ (*d*) $3\lambda/4$.

Ans.

$$y = 2.0 \cos 2\pi\ (10t - 0.0080\ x + 0.35)$$

$$= 2.0 \cos\ [2\pi\ (10t - 0.0080\ x) + 2\pi \times 0.35]$$

$\therefore$

$$y = 2.0 \cos\left[2\pi \times 0.0080\left(\frac{10t}{0.0080} - x\right) + 0.7\,\pi\right] \qquad ...(i)$$

The general equation of a travelling harmonic wave is

$$y = a \cos\left[\frac{2\pi}{\lambda}(vt - x) + \phi_0\right] \qquad ...(ii)$$

Comparing eqs. (*i*) and (*ii*), we have, $\frac{2\pi}{\lambda} = 2\pi \times 0.0080$

We know that phase difference, $\phi = \frac{2\pi}{\lambda} \times x$

(*a*) **When x = 4 m = 400 cm.**

$$\phi = \frac{2\pi}{\lambda} \times x = 2\pi \times 0.0080 \times 400 = 6.4\ \pi \text{ rad}$$

(*b*) **When x = 0.5 m = 50 cm.**

$$\phi = \frac{2\pi}{\lambda} \times x = 2\pi \times 0.0080 \times 50 = 0.8\ \pi \text{ rad}$$

(*c*) **When $x = \lambda/2$.**

$$\phi = \frac{2\pi}{\lambda} \times x = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} = \pi \text{ rad}$$

(*d*) **When $x = \frac{3\lambda}{4}$.**

$$\phi = \frac{2\pi}{\lambda} \times x = \frac{2\pi}{\lambda} \times \frac{3\lambda}{4} = \frac{3\pi}{2} \text{ rad}$$

Q.11. The transverse displacement of a string (clamped at its both ends) is given by :

$$y(x, t) = 0.06 \sin\left(\frac{2\pi}{3}x\right)\cos(120\pi t)$$

where x and y are in m and t in s. The length of the string is 1.5 m and its mass is 3.0×10^{-2} kg. Answer the following :

(*a*) Does the function represent a travelling wave or a stationary wave?

(*b*) Interpret the wave as a superposition of two waves travelling in opposite directions. What is the wavelength, frequency and speed of each wave?

(*c*) Determine the tension in the string.

Ans. The given equation is

$$y(x, t) = 0.06 \sin\frac{2\pi}{3}x \cos(120\pi t) \quad ...(i)$$

The general equation of a standing wave is

$$y(x, t) = 2a \sin kx \cos \omega t \quad ...(ii)$$

(*a*) Comparing eqs. (*i*) and (*ii*), it is clear that the given equation is the equation of a standing wave.

Here $k = \frac{2\pi}{3}$; $2a = 0.06$ or $a = 0.03$; $\omega = 120\pi$

(*b*) The given standing wave is the result of superposition of two waves travelling in the opposite directions. The individual wave functions are :

$$y_1 = a \sin(kx - \omega t) \ ; \quad y_2 = a \sin(kx + \omega t)$$

or

$$y_1 = 0.03 \sin\left(\frac{2\pi}{3}x - 120\pi t\right); \ y_2 = 0.03 \sin\left(\frac{2\pi}{3}x + 120\pi t\right)$$

Now

$$k = \frac{2\pi}{3} \quad \text{or} \quad \frac{2\pi}{\lambda} = \frac{2\pi}{3} \quad \therefore \quad \lambda =$$

Also

$$\omega = 120\pi \quad \text{or} \quad \frac{2\pi}{T} = 120\pi \quad \text{or} \quad T = \frac{1}{60}$$

$\therefore$

$$f = 60 \text{ Hz}$$

Speed of propagation of each wave is

$$v = f\lambda = 60 \times 3 = 180$$

(*c*) Velocity of transverse wave in a string is given by ;

$$v = \sqrt{\frac{T}{m}} \quad \text{or} \quad T = v^2 m$$

Here

$$m = \frac{3.0 \times 10^{-2}}{1.5} = 2 \times 10^{-2} \text{ kg m}^{-1}$$

$\therefore$

$$T = (180)^2 \times 2 \times 10^{-2} = 648 \text{ N}$$

Q.12. (*i*) For the wave on a string described in Q.11, do all the points on the string oscillate with the same (*a*) frequency (*b*) phase (*c*) amplitude? Explain your answers. (*ii*) What is the amplitude of a point 0.375 m away from one end?

Ans. (*i*) All the points on the string

(*a*) have the same frequency of oscillation except at the nodes where frequency is zero.

(*b*) have the same phase in one loop except at nodes.

(*c*) have different amplitudes at different points.

(*ii*) The amplitude at point x from one end is

$$a = 0.06 \sin \frac{2\pi}{3} x$$

$$\text{At } x = 0.375 \text{ m}, \; a = 0.06 \sin \frac{2\pi}{3} \times 0.375 = 0.06 \sin \frac{\pi}{4} = 0.042 \text{ m}$$

Q.13. Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represent (*i*) a travelling wave (*ii*) a stationary wave or (*iii*) none at all :

(*a*) $y = 2 \cos (3x) \sin (10t)$ (*b*) $y = 2\sqrt{x - vt}$

(*c*) $y = 3 \sin (5x - 0.5t) + 4 \cos (5x - 0.5t)$ (*d*) $y = \cos x \sin t + \cos 2x \sin 2t$

Ans. (*a*) As the equation contains harmonic functions of x and t separately, it represents a stationary wave.

(*b*) It does not represent any type of wave.

(*c*) It represents a harmonic travelling/progressive wave.

(*d*) The equation is the sum of two functions, each representing a stationary wave. Therefore, the given equation represents superposition of two stationary waves.

Q.14. A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is 3.5×10^{-2} kg and its linear mass density is 4.0×10^{-2} kg m^{-1}. What is (*a*) the speed of a transverse wave on the string and (*b*) the tension in the string?

Ans. $$\text{Length of wire, } L = \frac{\text{Mass of wire}}{\text{Linear denstiy}} = \frac{3.5 \times 10^{-2}}{4.0 \times 10^{-2}} = 0.875 \text{ m}$$

(*a*) When the wire vibrates in its fundamental mode, then,

$$L = \frac{\lambda}{2} \quad \text{or} \quad \lambda = 2L = 2 \times 0.875 = 1.75 \text{ m}$$

$\therefore$ Speed of transverse waves on the string is

$$v = f\lambda = 45 \times 1.75 = 78.75 \text{ m s}^{-1}$$

(*b*) Now $$v = \sqrt{\frac{T}{m}} \quad \text{or} \quad T = v^2 m$$

$\therefore$ $$T = (78.75)^2 \times 4.0 \times 10^{-2} = 248.1 \text{ N}$$

Q.15. A metre-long tube open at one end, with a movable piston at the other end, shows resonance with a fixed frequency source (a tuning fork of frequency 340 Hz) when the tube length is 25.5 cm or 79.3 cm. Estimate the speed of sound in air at the temperature of the experiment. The edge effects may be neglected.

Ans. Suppose n is the mode of vibration when the tuning fork of frequency 340 Hz is in resonance with a tube length l_1 (= 25.5 cm).

$\therefore$ $$340 = \frac{(2n-1)v}{4l_1} \qquad ...(i)$$

When the length of the tube is increased to l_2 (= 79.3 cm), the next mode [*i.e.* $(n + 1)$th mode] is in resonance with the same tuning fork.

$$\therefore \quad 340 = \frac{[2(n+1)-1]v}{4l_2} = \frac{(2n+1)v}{4l_2} \qquad ...(ii)$$

$$\therefore \quad \frac{(2n-1)v}{4l_1} = \frac{(2n+1)v}{4l_2}$$

or $$\frac{2n-1}{2n+1} = \frac{l_1}{l_2} = \frac{25.5}{79.3} = \frac{1}{3}$$

or $$6n - 3 = 2n + 1 \quad \therefore \quad n = 1$$

Putting $n = 1$ in eq. (i), we have,

$$340 = \frac{(2\times1-1)v}{4\times25.5} \quad \therefore \quad v = \mathbf{346.8\ m\ s^{-1}}$$

Q.16. A steel rod 100 cm long is clamped at its middle. The fundamental frequency of longitudinal vibrations of the rod is given to be 2.53 kHz. What is the speed of sound in steel?

Ans. $L = 100$ cm $= 1$ m ; $f = 2.53$ kHz $= 2.53 \times 10^3$ Hz

When the rod is clamped at the middle, then in the fundamental mode of vibration, a node is formed at the middle and an antinode is fomed at each end.

$$\therefore \quad L = \frac{\lambda}{2} \quad \text{or} \quad \lambda = 2L = 2 \times 1 = 2 \text{ m}$$

$\therefore$ Speed of sound waves, $v = f\lambda = 2.53 \times 10^3 \times 2 = \mathbf{5.06 \times 10^3\ m\ s^{-1}}$

Q.17. A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is resonantly excited by a 430 Hz source? Will the same source be in resonance with the pipe if both ends are open? (speed of sound in air is 340 m s^{-1}).

Ans. The frequency of the nth mode of vibration for a closed pipe is

$$f_n = \frac{(2n-1)v}{4L} \qquad \text{where } n = 1, 2, 3, ...$$

Here $L = 20$ cm $= 0.2$ m ; $v = 340$ m s^{-1} ; $f_n = 430$ Hz

$$\therefore \quad 430 = \frac{(2n-1)340}{4\times0.2}$$

or $$2n - 1 = \frac{430\times4\times0.2}{340} = 1.02 \quad \therefore \quad n = 1.01$$

Therefore, it will be in the first mode of vibration.

The frequency of the nth mode of vibration of an open pipe is

$$f_n = \frac{nv}{2L}$$

or $$430 = \frac{n\times340}{2\times0.2} \quad \therefore \quad n = 0.5$$

Since n must be an integer, the source cannot be in resonance with the pipe in this case.

Q.18. Two sitar strings *A* and *B* playing the note 'Ga' are slightly out of tune and produce beats of frequency 6 Hz. The tension in the string *A* is slightly reduced and the beat frequency is found to reduce to 3 Hz. If the original frequency of *A* is 324 Hz, what is the frequency of *B*?

Ans. Frequency of $A = 324$ Hz

Beat frequency $= 6$ Hz

$\therefore$ Possible frequencies of $B = 324 \pm 6 = 330$ Hz or 318 Hz

On reducing the tension in the string A, its frequency should decrease ($\because f \propto \sqrt{T}$). If the frequency of B is 330 Hz, then on reducing the tension in A, the number of beats produced per second will increase. But according to the question, the beat frequency decreases from 6 Hz to 3 Hz.

$\therefore$ Frequency of $B = \mathbf{318\ Hz}$

Q.19. **Explain why (or how) :**

(*a*) in a sound wave, a displacement node is a pressure antinode and *vice-versa.*

(*b*) bats can ascertain distances, directions, nature and sizes of the obstacles without any "eyes".

(*c*) a violin note and sitar note may have the same frequency, yet we can distinguish between the two notes.

(*d*) solids can support both longitudinal and transverse waves but only longitudinal waves can propagate in gases, and

(*e*) the shape of a pulse gets distorted during propagation in a dispersive medium.

Ans. (*a*) A **node** (*N*) is a point where the amplitude of oscillation is minimum (zero) and pressure is maximum. At this point, pressure variation is maximum so that it is a pressure antinode. An **antinode** (*A*) is a point where amplitude of oscillation is maximum and pressure is minimum (zero). At this point, pressure variation is minimum so that it is a pressure node.

(*b*) Bats emit ultrasonics of high frequencies (very small wavelengths). When these waves are reflected from obstacles in their path, they give them the idea about the distances, directions, nature and sizes of the obstacles.

(*c*) Every instrument emits a sound of fundamental frequency (called fundamental note) and a number of overtones. Therefore, although the violin note and sitar note have the same frequency (fundamental), yet the number and relative strengths of overtones produced in the two cases are different. This enables us to distinguish between the two notes.

(*d*) Transverse waves travel in the form of crests and troughs involving change in shape of the medium. Longitudinal waves travel in the form of compressions and rarefactions involving change in pressure (volume change). Since solids have both elasticity of shape and elasticity of volume, they can support both transverse and longitudinal waves. On the other hand, gases possess only volume elasticity (gases do not possess elasticity of shape), only longitudinal waves can propagate through gases.

(*e*) A sound pulse is a combination of a number of waves of different wavelengths. As a pulse travels through a dispersive medium, its constituent waves travel with different speeds. As a result, the shape of the pulse gets distorted.

Q.20. **A train, standing at the outer signal of a railway station blows a whistle of frequency 400 Hz in still air. (*i*) What is the frequency of the whistle for a platform observer when the train (*a*) approaches the platform with a speed of 10 m s^{-1} (*b*) recedes from the platform with a speed of 10 m s^{-1}? (*ii*) What is the speed of sound in each case? The speed of sound in still air can be taken as 340 m s^{-1}.**

Ans. $V = 340$ m s^{-1}; $u_S = 10$ m s^{-1}; $f = 400$ Hz

(*i*) (*a*) **When the train is approaching platform.**

Velocity of sound relative to passenger, $V' = V = 340$ m s^{-1}

Wavelength of waves reaching passenger, $\lambda' = \dfrac{V - u_S}{f} = \dfrac{340 - 10}{400} = \dfrac{330}{400}$ m

$\therefore$ Apparent frequency, $f' = \dfrac{V'}{\lambda'} = \dfrac{340 \times 400}{330} = 412.1$ Hz

(*b*) **When the train recedes from platform.**

Velocity of sound relative to passenger, $V' = V = 340$ m s^{-1}

Wavelength of waves reaching passenger, $\lambda' = \dfrac{V + u_S}{f} = \dfrac{340 + 10}{400} = \dfrac{350}{400}$ m

$\therefore$ Apparent frequency, $f' = \dfrac{V'}{\lambda'} = \dfrac{340 \times 400}{350} = 388.6$ Hz

(*ii*) Speed of sound in each case = 340 ms^{-1}

Q.21. **A train, standing in a station-yard blows a whistle of frequency 400 Hz in still air. The wind starts blowing in the direction from the yard to the station with a speed of 10 m s^{-1}. What are the frequency, wavelength and speed of sound for an observer standing on the station's**

platform? Is the situation exactly identical to the case when the air is still and the observer runs towards the yard at a speed of 10 m s^{-1}? The speed of sound in still air can be taken as 340 m s^{-1}.

Ans. Speed of sound in still air, $V = 340$ m s^{-1}

True frequency, $f = 400$ Hz

Speed of wind, $u_w = 10$ m s^{-1}

Since the wind blows in the direction of sound,

$\therefore$ Sound speed, $V' = V + u_w = 340 + 10 = 350$ ms^{-1}

There is no relative motion between source of sound and the observer. Therefore, the frequency remains unaltered.

$\therefore$ Observerd frequency = 400 Hz

$$\text{Wavelength of sound, } \lambda' = \frac{V'}{f} = \frac{350}{400} = 0.875 \text{ m}$$

This is a case in which the source is stationary and the observer is moving towards it.

Velocity of sound relative to the observer is

$$V' = V + u_O = 340 + 10 = 350 \text{ m s}^{-1}$$

Wavelength of the waves reaching the observer

$$\lambda' = \frac{V}{f}$$

$\therefore$ Apparent frequency, $f' = \frac{V'}{\lambda'} = \frac{V + u_O}{V} \times f$

$$= \frac{340 + 10}{340} \times 400 = 411.8 \text{ Hz}$$

Since the motion of the observer has no effect on the wavelength, the wavelength remains *unchanged.* Note that situations (*i*) and (*ii*) are not equivalent.

Q.22. A travelling harmonic wave on a string is described by :

$$y(x, t) = 7.5 \sin (0.0050\, x + 12t + \pi/4)$$

(*a*) What are the displacement and velocity of oscillation of a point at $x = 1$ cm and $t = 1$ s? Is this velocity equal to the velocity of wave propagation?

(*b*) Locate the points of the string which have the same transverse displacements and velocity as the $x = 1$ cm point at $t = 2$ s, 5 s and 11s.

Ans. The equation of the travelling harmonic wave is

$$y(x, t) = 7.5 \sin (0.0050\, x + 12t + \pi/4) \quad ...(i)$$

$$= 7.5 \sin\left[0.0050\left(\frac{12}{0.0050}t + x\right) + \frac{\pi}{4}\right]$$

The general equation of a travelling harmonic wave is

$$y(x, t) = a \sin\left[\frac{2\pi}{\lambda}(vt + x) + \phi_0\right] \quad ...(ii)$$

Comparing eqs. (*i*) and (*ii*), we have,

$$a = 7.5 \text{ cm} \; ; \; v = \frac{12}{0.0050} \text{ cm s}^{-1} \; ; \; \frac{2\pi}{\lambda} = 0.0050 \text{ cm}^{-1}$$

(*a*) At $x = 1$cm and $t = 1$s, the displacement (y) can be found by putting these values in eq. (*i*).

$\therefore$ $y(1\text{cm}, 1\text{s}) = 7.5 \sin\left(0.0050 \times 1 + 12 \times 1 + \frac{\pi}{4}\right)$

$= 7.5 \sin 12.79 = 7.5 \sin *732.8° = \mathbf{1.66\ cm}$

Velocity of oscillation of particle is

$$V = \frac{dy}{dt} = \frac{d}{dt}\left[7.5 \sin\left(0.0050\,x + 12t + \frac{\pi}{4}\right)\right]$$

$$= 7.5 \cos\left(0.0050\,x + 12t + \frac{\pi}{4}\right) \times 12$$

$$\therefore \quad V = 90 \cos\left(0.0050\,x + 12t + \frac{\pi}{4}\right)$$

Therefore, velocity of the particle at x = 1cm and t = 1s is

$$V(1\text{cm},\ 1\text{s}) = 90 \cos\left(0.0050 \times 1 + 12 \times 1 + \frac{\pi}{4}\right)$$

$$= 90 \cos 12.79 = 90 \cos 732.8° = \mathbf{87.76\ cm\ s^{-1}}$$

As calculated above, velocity of wave, $v = \dfrac{12}{0.0050} = 2400\ \text{cm s}^{-1}$

Therefore, velocity of oscillation of particle is not equal to the velocity of wave.

(*b*) As calculated above, $\dfrac{2\pi}{\lambda} = 0.0050 \quad \therefore \quad \lambda = \dfrac{2\pi}{0.0050} = 1256\ \text{cm} = 12.56\ \text{m}$

Now all points which are at a distance λ, 2λ, 3λ from x = 1 cm will have the same transverse displacement and velocity.

Since λ = 12.56 m, all the points at a distance of 12.6 m, 25.2 m, 37.8 m .... from x = 1 cm will have the same transverse displacement and velocity as x = 1 cm point at t = 2 s, 5 s and 11 s.

Q.23. A narrow sound pulse (for example, a short pip by a whistle) is sent across a medium. (*a*) Does the pulse have a definite (*i*) frequency (*ii*) wavelength (*iii*) speed of propagation? (*b*) If the pulse rate is 1 after every 20 s (that is the whistle is blown for a split of second after every 20 s), is the frequency of the note produced by the whistle equal to 1/20 or 0.05 Hz?

Ans. (*a*) A narrow sound pulse such as a short pip by a whistle does not have a definite wavelength or frequency. However, it is a sound wave and, therefore, its speed is fixed, being equal to the speed of sound in air.

(*b*) If the short pip is produced after every 20 s, its frequency is not 1/20 = 0.05 Hz. We can say that 0.05 Hz is the frequency of repetition of the short pip of the whistle.

Q.24. One end of a long string of linear mass density 8.0×10^{-3} kg m^{-1} is connected to an electrically driven tuning fork of frequency 256 Hz. The other end passes over a pulley and is tied to a pan containing a mass of 90 kg. The pulley end absorbs all the incoming energy so that reflected waves at this end have negligible amplitude. At t = 0, the left end (fork end) of the string x = 0 has zero transverse displacement (y = 0) and is moving along positive y-direction. The amplitude of the wave is 5.0 cm. Write down the transverse displacement y as function of x and t that describes the wave on the string.

Ans. Since the wave is travelling along the positive x direction, its equation is

$$y(x, t) = a \sin \frac{2\pi}{\lambda}(vt - x) = a \sin\left(\frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x\right)$$

$$= a \sin (2\pi f\, t - kx) \qquad \left[\because \frac{v}{\lambda} = f \text{ and } k = \frac{2\pi}{\lambda}\right]$$

$$= a \sin (\omega t - kx) \qquad [\because \omega = 2\pi f]$$

$$\therefore \quad y(x, t) = a \sin (\omega t - kx) \qquad \ldots(i)$$

---

* 12.79 rad $= \dfrac{180}{\pi} \times 12.79$ degrees $= 732.8°$

Now a = 5.0 cm = 0.05 m ; $\omega = 2\pi f = 2\pi \times 256 = 1.61 \times 10^3$ rad s^{-1}

The velocity v of the wave along the string is

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{90 \times 9.8}{8.0 \times 10^{-3}}} = 332 \text{ m s}^{-1}$$

$$\therefore \quad \lambda = \frac{v}{f} = \frac{332}{256} = 1.297 \text{ m and } k = \frac{2\pi}{\lambda} = \frac{2\pi}{1.297} = 4.84 \text{ m}^{-1}$$

Putting the values of a, ω and k in eq. (i), we have,

$$y(x, t) = 0.05 \sin (1.61 \times 10^3 t - 4.84 x)$$

Q.25. A SONAR system fixed in a submarine operates at a frequency 40.0 kHz. An enemy submarine moves towards the SONAR with a speed of 360 km h^{-1}. What is the frequency of sound reflected by the submarine? Take the speed of sound in water to be 1450 m s^{-1}.

Ans. Here f = 40.0 kHz ; V = 1450 m s^{-1} ; u_O = 360 km/h = 100 ms^{-1}

The frequency of the waves from the SONAR will undergo a change in frequency in two steps.

Step 1. Velocity of sound relative to the enemy submarine is

$$V' = V + u_O = 1450 + 100 = 1550 \text{ m s}^{-1}$$

Wavelength of the waves reaching enemy submarine

$$\lambda' = \frac{V}{f}$$

$$\therefore \quad \text{Apparent frequency, } f' = \frac{V'}{\lambda'} = \frac{V + u_O}{Vf} = \frac{V + u_O}{V} \times f$$

$$= \frac{1550}{1450} \times 40.0 = 42.758 \text{ kHz}$$

Step 2. The enemy submarine will reflect waves of frequency 42.758 kHz and will act as a source of waves. Now the source (enemy submarine) is moving with a speed u_S = 100 m s^{-1} toward the SONAR system (observer).

Velocity of sound relative to the SONAR system is

$$V'' = V$$

Wavelength of waves reaching the SONAR system is

$$\lambda'' = \frac{V - u_S}{f'}$$

$$\therefore \quad \text{Apparent frequency, } f'' = \frac{V''}{\lambda''} = \frac{V}{V - u_S} \times f'$$

$$= \frac{1450}{1450 - 100} \times 42.758 = 45.92 \text{ kHz}$$

Q.26. Earthquakes generate sound waves inside the earth. Unlike a gas, the earth can experience both transverse (S) and longitudinal (P) sound waves. Typically the speed of S wave is about 4.0 km s^{-1} and that of P wave is 8.0 km s^{-1}. A seismograph records P and S waves from an earthquake. The first P wave arrives 4 min before the first S wave. Assuming the waves travel in straight line, at what distance does the earthquake occur?

Ans. Here V_S = 4.0 km s^{-1}; V_P = 8.0 km s^{-1}

Let d be the distance of the seisimograph from the point of occurrence of earthquake. If t_S and t_P be the time taken by the S-waves and P-waves respectively, then,

$$t_S = \frac{d}{V_S} \quad \text{and} \quad t_P = \frac{d}{V_P}$$

$$\therefore \quad t_S - t_P = \frac{d}{V_S} - \frac{d}{V_P} = d\left(\frac{V_P - V_S}{V_P V_S}\right)$$

$$\therefore \quad d = \frac{V_P V_S}{V_P - V_S} \times (t_S - t_P)$$

Here $t_S - t_P = 4 \text{ min} = 4 \times 60 = 240 \text{ s}$

$$\therefore \quad d = \frac{8.0 \times 4.0}{8.0 - 4.0} \times 240 = \mathbf{1920 \text{ km}}$$

Therefore, earthquake occurs 1920 km from the seismograph.

Q.27. A bat is flying about in a cave, navigating via ultrasonic bleeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall?

Ans. Frequency of sound emitted by the bat, $f = 40$ kHz and velocity of bat, $V_S = 0.03\ V$ where V is the velocity of sound. We shall solve the problem in two steps.

Step 1. The source of sound (bat) is moving towards stationary wall with velocity $V_S = 0.03\ V$.

$\therefore$ Apparent frequency of sound striking the wall is

$$f' = \frac{V}{V - V_S} \times f = \frac{V}{V - 0.03V} \times 40 \text{ kHz} = 41.24 \text{ kHz}$$

Step 2. The sound of frequency 41.24 kHz is reflected by the wall and travels towards the bat (listener). Now source of sound (wall) is stationary and the listener (bat) is moving towards it with velocity $V_L = 0.03\ V$.

$\therefore$ Frequency heard by the bat is

$$f'' = \frac{V + V_L}{V} \times f' = \frac{V + 0.03V}{V} \times f' = 1.03 \times 41.24 = \mathbf{42.47 \text{ kHz}}$$

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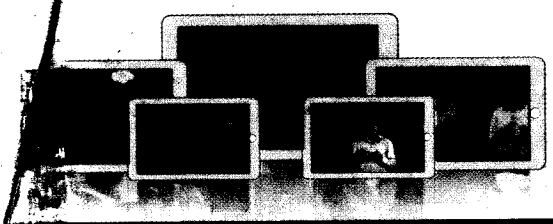

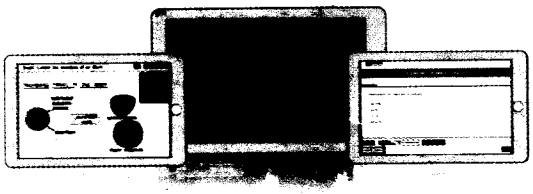

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